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# Almansi-Type Decomposition Theorem for Bi-$k$-regular Functions in the Clifford Algebra $Cl_{2n+2,0}(\mathbb{R})$*

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**Abstract** Almansi-type decomposition theorem for bi-$k$-regular functions defined in a star-like domain $\Omega \subseteq \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ centered at the origin with values in the Clifford algebra $Cl_{2n+2,0}(\mathbb{R})$ is proved. As a corollary, Almansi-type decomposition theorem for biharmonic functions of degree $k$ is given.

**Keywords** Real Clifford analysis, Biregular functions, Bi-$k$-regular functions, Almansi-type decomposition theorem
**2000 MR Subject Classification** 17B40, 17B50

## 1 Introduction

Clifford algebra is an associative and noncommutative algebra introduced in 1878 by Clifford [2]. In 1982, Brackx et al. [1] established the theoretical basis of Clifford analysis. Eriksson [3–4], Huang [6], Ren [7, 9], Sakakibara [10], Garcia [5], Qiao [8], Xie [11–12] and Yang [13] have done a lot of work in Clifford analysis.

In 2002, Malonek and Ren [7] gave the Almansi-type theorem for polymonogenic functions defined in a star-like domain $\Omega \subseteq \mathbb{R}^n$ with values in the Clifford algebra $Cl_{0,n}(\mathbb{R})$. In 2006, Ren and Kahler [9] gave Almansi decomposition for polyharmonic, polyheat, and polywave functions. In 2017, Sakakibara [10] gave the method of fundamental solutions for biharmonic equation based on Almansi-type decomposition. In 2020, Garcia et al. [5] gave the decomposition of inframonogenic functions with applications in elasticity theory.

Based on the above, Almansi-type decomposition theorem for bi-$k$-regular functions defined in a star-like domain $\Omega \subseteq \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ centered at the origin O with values in the Clifford algebra $Cl_{2n+2,0}(\mathbb{R})$ is proved. As a corollary, Almansi-type decomposition theorem for biharmonic functions of degree $k$ in the Clifford algebra $Cl_{2n+2,0}(\mathbb{R})$ is given. It generalizes the work of Reference [7] from the Clifford algebra $Cl_{0,n}(\mathbb{R})$ to $Cl_{2n+2,0}(\mathbb{R})$ and from one variable to two variables.

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## 2 Preliminaries

Real Clifford algebra $Cl_{n+1,0}(\mathbb{R})$ is generated by $\{e_0, e_1, \cdots, e_n\}$, whose identity is $e_\varnothing = 1$ and whose basis $e_0, e_1, \cdots, e_n; e_0e_1, \cdots, e_{n-1}e_n; \cdots; e_0e_1\cdots e_n$ satisfies $e_ie_j + e_je_i = 2\delta_{ij}$, $i, j = 0, 1, \cdots, n$, where $\delta_{ij}$ is the Kronecker sign. For any element $a \in Cl_{n+1,0}(\mathbb{R})$, $a = \sum\limits_A a_Ae_A$, where $A = \{\alpha_1, \alpha_2, \cdots, \alpha_h\}$, the integers $\alpha_l$ $(l = 1, 2, \cdots, h)$ satisfy $0 \le \alpha_1 < \alpha_2 < \cdots < \alpha_h \le n$, $a_A \in \mathbb{R}$, $e_A = e_{\alpha_1}e_{\alpha_2}\cdots e_{\alpha_h}$ or $e_\varnothing = 1$. Define the norm of the element $a \in Cl_{n+1,0}(\mathbb{R})$ as $|a| = \big(\sum\limits_A |a_A|^2\big)^{\frac{1}{2}}$.

Let $\Omega_0$ be a nonempty connected open set in $\mathbb{R}^{n+1}$. Denote the function $f : \Omega_0 \to Cl_{n+1,0}(\mathbb{R})$ by $f(x) = \sum\limits_A f_A(x)e_A$, where $f_A \in \mathbb{R}$. $f : \Omega_0 \to Cl_{n+1,0}(\mathbb{R})$ is continuous on $\Omega_0$ means each component $f_A(x)$ is continuous on $\Omega_0$. Suppose $C^r(\Omega_0, Cl_{n+1,0}(\mathbb{R}))$=$\{f | f : \Omega_0 \to Cl_{n+1,0}(\mathbb{R}), f(x) = \sum\limits_A f_A(x)e_A$, where $f_A$ is $r$-time continuously differentiable on $\Omega_0$, $r \in \mathbb{N}^*\}$.

In this paper, we suppose that $\Omega = \Omega_1\times\Omega_2$ is a nonempty connected open set in $\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}$.

If $f \in C^2(\Omega, Cl_{2n+2,0}(\mathbb{R}))$, define some operators as follows:

$$D_xf = \sum_{i=0}^{n} e_i\frac{\partial f}{\partial x_i}, \quad fD_y = \sum_{j=0}^{n} \frac{\partial f}{\partial y_j}e_j,$$
$$\Delta_xf = \sum_{i=0}^{n} \frac{\partial^2 f}{\partial x_i^2}, \quad f\Delta_y = \sum_{j=0}^{n} \frac{\partial^2 f}{\partial y_j^2}.$$

**Lemma 2.1** *Suppose $\Omega \subseteq \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ is a domain, $f, g \in C^2(\Omega, Cl_{2n+2,0}(\mathbb{R}))$, then for any $(x, y) \in \Omega$, we have*

$$D_x(fg) = (D_xf)g + \sum_{i=0}^{n} e_if\frac{\partial g}{\partial x_i}, \quad (fg)D_y = f(gD_y) + \sum_{j=0}^{n} \frac{\partial f}{\partial y_j}ge_j.$$

**Definition 2.1** *Suppose $\Omega \subseteq \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ is a domain, if $t(x, y) \in \Omega$ holds for any $(x, y) \in \Omega$ and $t \in (0, 1)$, we say that $\Omega$ is a star-like domain centered at the origin* O.

**Definition 2.2** *Suppose $\Omega \subseteq \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ is a domain, $k \in N^*$, $f \in C^{2k}(\Omega, Cl_{2n+2,0}(\mathbb{R}))$ satisfies*

$$\begin{cases} D_x^kf(x, y) = 0, \\ f(x, y)D_y^k = 0 \end{cases}$$

*on $\Omega$, then $f$ is called a bi-k-regular function on $\Omega$. When $k = 1$ it is called a bi-regular function on $\Omega$ for short.*

**Definition 2.3** *Suppose $\Omega \subseteq \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ is a domain, $k \in N^*$, $f \in C^{2k}(\Omega, Cl_{2n+2,0}(\mathbb{R}))$ satisfies*

$$\begin{cases} \Delta_x^kf(x, y) = 0, \\ f(x, y)\Delta_y^k = 0 \end{cases}$$

*on $\Omega$, then $f$ is called a bi-k-harmonic function on $\Omega$. When $k = 1$ it is called a bi-harmonic function on $\Omega$ for short.*

## 3 Almansi-Type Decomposition Theorem for Bi-*k*-regular Functions

For any $\alpha \geq 0$, we define the operator $S_\alpha$ by

$$S_\alpha = \alpha I + E_1 + E_2,$$

where $I$ is a unit operator, $E_1 = \sum\limits_{i=0}^{n} x_i \frac{\partial}{\partial x_i}$ and $E_2 = \sum\limits_{j=0}^{n} y_j \frac{\partial}{\partial y_j}$ are Euler operators.

If $\Omega \subseteq \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ is a star-like domain centered at the origin O, $\beta \geq 1$ and $f \in C^2(\Omega, Cl_{2n+2,0}(\mathbb{R}))$, we define the operator $T_\beta : C^2(\Omega, Cl_{2n+2,0}(\mathbb{R})) \to C^2(\Omega, Cl_{2n+2,0}(\mathbb{R}))$ by

$$T_\beta f(x,y) = \int_0^1 f(tx,ty) t^{\beta-1} \mathrm{d}t.$$

**Theorem 3.1** *If* $\Omega \subseteq \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ *is a star-like domain centered at the origin* O*,* $\beta \geq 1$ *and* $f \in C^2(\Omega, Cl_{2n+2,0}(\mathbb{R}))$*, then for any* $(x,y) \in \Omega$*, we have*

$$f(x,y) = T_\beta S_\beta f(x,y) = S_\beta T_\beta f(x,y). \tag{3.1}$$

**Proof** By some straightforward calculations, we have

$$\begin{aligned}
&f(x,y) \\
&= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}(t^\beta f(tx,ty))\mathrm{d}t \\
&= \int_0^1 \Big(\beta t^{\beta-1} f(tx,ty) + t^\beta \frac{\mathrm{d}f(tx,ty)}{\mathrm{d}t}\Big)\mathrm{d}t \\
&= \int_0^1 \Big(\beta t^{\beta-1} f(tx,ty) + t^{\beta-1}\Big(\sum_{i=0}^{n} m_i \frac{\partial f}{\partial m_i}\Big)(tx,ty) + t^{\beta-1}\Big(\sum_{j=0}^{n} k_j \frac{\partial f}{\partial k_j}\Big)(tx,ty)\Big)\mathrm{d}t \\
&= \int_0^1 \Big(\beta f + \Big(\sum_{i=0}^{n} m_i \frac{\partial f}{\partial m_i}\Big) + \Big(\sum_{j=0}^{n} k_j \frac{\partial f}{\partial k_j}\Big)\Big)(tx,ty) t^{\beta-1}\mathrm{d}t \\
&= T_\beta S_\beta f(x,y),
\end{aligned}$$

where $m_i = tx_i, k_j = ty_j$.

On the other hand,

$$\begin{aligned}
&f(x,y) \\
&= \int_0^1 \Big(\beta t^{\beta-1} f(tx,ty) + t^{\beta-1}\Big(\sum_{i=0}^{n} m_i \frac{\partial f}{\partial m_i}\Big)(tx,ty) + t^{\beta-1}\Big(\sum_{j=0}^{n} k_j \frac{\partial f}{\partial k_j}\Big)(tx,ty)\Big)\mathrm{d}t \\
&= \beta \int_0^1 f(tx,ty) t^{\beta-1}\mathrm{d}t + \Big(\sum_{i=0}^{n} m_i \frac{\partial}{\partial m_i}\Big) \int_0^1 f(tx,ty) t^{\beta-1}\mathrm{d}t \\
&\quad + \Big(\sum_{j=0}^{n} k_j \frac{\partial}{\partial k_j}\Big) \int_0^1 f(tx,ty) t^{\beta-1}\mathrm{d}t \\
&= S_\beta T_\beta f(x,y).
\end{aligned}$$

we complete the proof.

**Theorem 3.2** *If $\Omega \subseteq \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ is a star-like domain centered at the origin* O *and $f \in C^2(\Omega, Cl_{2n+2,0}(\mathbb{R}))$, $\alpha_1 \geq 0, \alpha_2 \geq 0$, then for any $(x, y) \in \Omega$, we have*

$$S_{\alpha_1} S_{\alpha_2} = S_{\alpha_2} S_{\alpha_1}. \tag{3.2}$$

**Proof** As $S_\alpha = \alpha I + \sum\limits_{i=0}^{n} x_i \frac{\partial}{\partial x_i} + \sum\limits_{j=0}^{n} y_j \frac{\partial}{\partial y_j}$, we get

$$\begin{aligned}
&S_{\alpha_1}(S_{\alpha_2} f) = \alpha_1 (S_{\alpha_2} f) + \sum_{i=0}^{n} x_i \frac{\partial (S_{\alpha_2} f)}{\partial x_i} + \sum_{j=0}^{n} y_j \frac{\partial (S_{\alpha_2} f)}{\partial y_j} \\
&= \alpha_1 \Big( \alpha_2 f + \sum_{i=0}^{n} x_i \frac{\partial f}{\partial x_i} + \sum_{j=0}^{n} y_j \frac{\partial f}{\partial y_j} \Big) + \sum_{i=0}^{n} x_i \frac{\partial \big( \alpha_2 f + \sum\limits_{k=0}^{n} x_k \frac{\partial f}{\partial x_k} + \sum\limits_{l=0}^{n} y_l \frac{\partial f}{\partial y_l} \big)}{\partial x_i} \\
&\quad + \sum_{j=0}^{n} y_j \frac{\partial \big( \alpha_2 f + \sum\limits_{k=0}^{n} x_k \frac{\partial f}{\partial x_k} + \sum\limits_{l=0}^{n} y_l \frac{\partial f}{\partial y_l} \big)}{\partial y_j}, \\
&S_{\alpha_2}(S_{\alpha_1} f) = \alpha_2 (S_{\alpha_1} f) + \sum_{i=0}^{n} x_i \frac{\partial (S_{\alpha_1} f)}{\partial x_i} + \sum_{j=0}^{n} y_j \frac{\partial (S_{\alpha_1} f)}{\partial y_j} \\
&= \alpha_2 \Big( \alpha_1 f + \sum_{i=0}^{n} x_i \frac{\partial f}{\partial x_i} + \sum_{j=0}^{n} y_j \frac{\partial f}{\partial y_j} \Big) + \sum_{i=0}^{n} x_i \frac{\partial \big( \alpha_1 f + \sum\limits_{k=0}^{n} x_k \frac{\partial f}{\partial x_k} + \sum\limits_{l=0}^{n} y_l \frac{\partial f}{\partial y_l} \big)}{\partial x_i} \\
&\quad + \sum_{j=0}^{n} y_j \frac{\partial ( \alpha_1 f + \sum\limits_{k=0}^{n} x_k \frac{\partial f}{\partial x_k} + \sum\limits_{l=0}^{n} y_l \frac{\partial f}{\partial y_l} )}{\partial y_j}.
\end{aligned}$$

Thus (3.2) holds.

**Theorem 3.3** *If $\Omega \subseteq \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ is a star-like domain centered at the origin* O*, $\alpha \geq 0$, $f \in C^3(\Omega, Cl_{2n+2,0}(\mathbb{R}))$, then for any $(x, y) \in \Omega$, we have*

$$D_x(S_\alpha f) = S_{\alpha+1}(D_x f), \tag{3.3}$$

$$(S_\alpha f) D_y = S_{\alpha+1}(f D_y), \tag{3.4}$$

$$D_x (S_\alpha f) D_y = S_{\alpha+2}(D_x f D_y). \tag{3.5}$$

**Proof**

$$\begin{aligned}
D_x(S_0 f) &= D_x \Big( \sum_{i=0}^{n} x_i \frac{\partial f}{\partial x_i} + \sum_{j=0}^{n} y_j \frac{\partial f}{\partial y_j} \Big) \\
&= \Big( \sum_{i=0}^{n} e_i \frac{\partial f}{\partial x_i} + \sum_{i,k} e_k x_i \frac{\partial^2 f}{\partial x_k \partial x_i} + \sum_{j,k} e_k y_j \frac{\partial^2 f}{\partial x_k \partial y_j} \Big) \\
&= D_x f + E_1(D_x f) + E_2(D_x f) \\
&= S_1(D_x f), \\
D_x(S_\alpha f) &= D_x(\alpha f + E_1 f + E_2 f)
\end{aligned}$$

$$
\begin{aligned}
&= \alpha D_x f + D_x(S_0 f) \\
&= \alpha D_x f + S_1(D_x f) \\
&= \alpha D_x f + D_x f + E_1(D_x f) + E_2(D_x f) \\
&= S_{\alpha+1}(D_x f),
\end{aligned}
$$

that is, (3.3) is true. Similarly, (3.4) is true.

$$
\begin{aligned}
D_x(S_0 f)D_y &= D_x\Big(\sum_{i=0}^{n} x_i \frac{\partial f}{\partial x_i} + \sum_{j=0}^{n} y_j \frac{\partial f}{\partial y_j}\Big) D_y \\
&= \Big(\sum_{i=0}^{n} e_i \frac{\partial f}{\partial x_i} + \sum_{i,k} e_k x_i \frac{\partial^2 f}{\partial x_k \partial x_i} + \sum_{j,k} e_k y_j \frac{\partial^2 f}{\partial x_k \partial y_j}\Big) D_y \\
&= \sum_{l=0}^{n} \frac{\partial}{\partial y_l}\Big(\sum_{i=0}^{n} e_i \frac{\partial f}{\partial x_i} + \sum_{i,k} e_k x_i \frac{\partial^2 f}{\partial x_k \partial x_i} + \sum_{j,k} e_k y_j \frac{\partial^2 f}{\partial x_k \partial y_j}\Big) e_l \\
&= \sum_{i,l} e_i \frac{\partial^2 f}{\partial y_l \partial x_i} e_l + \sum_{i,k,l} e_k x_i \frac{\partial^3 f}{\partial y_l \partial x_k \partial x_i} e_l + \sum_{j,k} e_k \frac{\partial^2 f}{\partial x_k \partial y_j} e_j \\
&\quad + \sum_{j,k,l} e_k y_j \frac{\partial^3 f}{\partial y_l \partial x_k \partial y_j} e_l \\
&= 2 D_x f D_y + \sum_{i,k,l} e_k x_i \frac{\partial^3 f}{\partial y_l \partial x_k \partial x_i} e_l + \sum_{j,k,l} e_k y_j \frac{\partial^3 f}{\partial y_l \partial x_k \partial y_j} e_l.
\end{aligned}
$$

Since

$$
\begin{aligned}
E_1(D_x f D_y) &= \sum_{i=0}^{n} x_i \frac{\partial}{\partial x_i}\Big(\sum_{k,l} e_k \frac{\partial^2 f}{\partial y_l \partial x_k} e_l\Big) = \sum_{i,k,l} x_i e_k \frac{\partial^3 f}{\partial x_i \partial y_l \partial x_k} e_l, \\
E_2(D_x f D_y) &= \sum_{j,k,l} y_j e_k \frac{\partial^3 f}{\partial y_l \partial x_k \partial y_j} e_l,
\end{aligned}
$$

we get $D_x(S_0 f)D_y = 2D_x f D_y + E_1(D_x f D_y) + E_2(D_x f D_y) = S_2(D_x f D_y)$.

Similarly,

$$
\begin{aligned}
D_x(S_\alpha f)D_y &= D_x(\alpha f + E_1 f + E_2 f)D_y \\
&= \alpha D_x f D_y + D_x(S_0 f)D_y \\
&= \alpha D_x f D_y + S_2(D_x f D_y) \\
&= \alpha D_x f D_y + 2D_x f D_y + E_1(D_x f D_y) + E_2(D_x f D_y) \\
&= S_{\alpha+2}(D_x f D_y),
\end{aligned}
$$

that is, (3.5) is true.

**Theorem 3.4** *If* $\Omega \subseteq \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ *is a star-like domain centered at the origin* O, $\beta \geq 1, f \in C^2(\Omega, Cl_{2n+2,0}(\mathbb{R}))$, *then for any* $(x, y) \in \Omega$, *we have*

$$D_x(T_\beta f) = T_{\beta+1}(D_x f), \tag{3.6}$$

$$(T_\beta f)D_y = T_{\beta+1}(fD_y), \tag{3.7}$$

$$D_x(T_\beta f)D_y = T_{\beta+2}(D_x f D_y). \tag{3.8}$$

**Proof**

$$D_x(T_\beta f(x,y)) = D_x\Big(\int_0^1 f(tx,ty)t^{\beta-1}\mathrm{d}t\Big)$$

$$= \int_0^1 \sum_{i=0}^n e_i \frac{\partial f(tx,ty)}{\partial x_i} t^{\beta-1}\mathrm{d}t$$

$$= \int_0^1 \sum_{i=0}^n e_i \frac{\partial f}{\partial x_i}(tx,ty)t^{\beta}\mathrm{d}t,$$

$$T_{\beta+1}(D_x f(x,y)) = \int_0^1 \sum_{i=0}^n e_i \frac{\partial f}{\partial x_i}(tx,ty)t^{\beta}\mathrm{d}t,$$

hence $D_x(T_\beta f) = T_{\beta+1}(D_x f)$. Similarly, $(T_\beta f)D_y = T_{\beta+1}(fD_y)$.

$$D_x(T_\beta f(x,y))D_y = D_x\Big(\int_0^1 f(tx,ty)t^{\beta-1}\mathrm{d}t\Big)D_y$$

$$= \Big(\int_0^1 \sum_{i=0}^n e_i \frac{\partial f}{\partial x_i}(tx,ty)t^{\beta}\mathrm{d}t\Big)D_y$$

$$= \int_0^1 \sum_{i,j} e_i \frac{\partial^2 f}{\partial y_j \partial x_i}(tx,ty)e_j t^{\beta+1}\mathrm{d}t,$$

$$T_{\beta+2}(D_x f(x,y)D_y) = \int_0^1 \sum_{i,j} e_i \frac{\partial^2 f}{\partial y_j \partial x_i}(tx,ty)e_j t^{\beta+1}\mathrm{d}t,$$

hence $D_x(T_\beta f)D_y = T_{\beta+2}(D_x f D_y)$.

**Remark 3.1** If $\Omega \subseteq \mathbb{R}^{n+1}\times\mathbb{R}^{n+1}$ is a star-like domain centered at the origin O, $\beta \geq 1$, $f \in C^3(\Omega, Cl_{2n+2,0}(\mathbb{R}))$ is a biregular function on $\Omega$, then $S_\beta f$ and $T_\beta f$ are biregular functions on $\Omega$.

**Theorem 3.5** *If $\Omega \subseteq \mathbb{R}^{n+1}\times\mathbb{R}^{n+1}$ is a star-like domain centered at the origin* O, *$f \in C^\infty(\Omega, Cl_{2n+2,0}(\mathbb{R}))$ is a biregular function on $\Omega$ and $E_2E_1 f = 0$, then for any $k \in N^*$, we have*

$$D_x(x^{2k} f y^{2k})D_y = (2k)^2 x^{2k-1} f y^{2k-1}, \tag{3.9}$$

$$D_x(x^{2k-1} f y^{2k-1})D_y = 4\Big(\frac{n+1}{2}+k-1\Big)x^{2(k-1)}(S_{\frac{n+1}{2}+k-1}f)y^{2(k-1)}, \tag{3.10}$$

$$D_x^{2k}(x^{2k} f y^{2k})D_y^{2k} = 4^{2k}(k!)^2\Big(\frac{n+1}{2}+k-1\Big)\cdots \Big(\frac{n+1}{2}\Big)S_{\frac{n+1}{2}+k-1}\cdots S_{\frac{n+1}{2}}f, \tag{3.11}$$

$$D_x^{2k-1}(x^{2k-1} f y^{2k-1})D_y^{2k-1} = 4^{2k-1}((k-1)!)^2\Big(\frac{n+1}{2}+k-1\Big)\cdots \Big(\frac{n+1}{2}\Big)S_{\frac{n+1}{2}+k-1}\cdots S_{\frac{n+1}{2}}f. \tag{3.12}$$

**Proof** Firstly, we prove (3.9).

Because $f$ is a biregular function on $\Omega$, by Lemma 2.1 we have

$$D_x(x^{2k}fy^{2k})D_y = \Big((D_x x^{2k})fy^{2k} + \sum_{i=0}^{n} e_i x^{2k}\frac{\partial f}{\partial x_i}y^{2k}\Big)D_y$$

$$= (D_x x^{2k})\Big(f(y^{2k}D_y) + \sum_{j=0}^{n}\frac{\partial f}{\partial y_j}y^{2k}e_j\Big) + \sum_{i=0}^{n} e_i x^{2k}\frac{\partial f}{\partial x_i}(y^{2k}D_y) + \sum_{i,j} e_i x^{2k}\frac{\partial^2 f}{\partial y_j \partial x_i}y^{2k}e_j$$

$$= 2kx^{2k-1}\Big(2kfy^{2k-1} + \sum_{j=0}^{n}\frac{\partial f}{\partial y_j}e_j y^{2k}\Big) + 2kx^{2k}\sum_{i=0}^{n} e_i\frac{\partial f}{\partial x_i}y^{2k-1} + x^{2k}\sum_{i,j} e_i\frac{\partial^2 f}{\partial y_j\partial x_i}e_j y^{2k}$$

$$= 2kx^{2k-1}\Big(2kfy^{2k-1} + (fD_y)y^{2k}\Big) + 2kx^{2k}(D_x f)y^{2k-1} + x^{2k}(D_x f D_y)y^{2k}$$

$$= (2k)^2 x^{2k-1}fy^{2k-1}.$$

Secondly, we prove (3.10).

By Lemma 2.1 we have

$$D_x(x^{2k-1}fy^{2k-1})D_y = D_x(x^{2(k-1)}(xfy)y^{2(k-1)})D_y$$

$$= \Big((D_x x^{2(k-1)})xfy^{2k-1} + \sum_{i=0}^{n} e_i x^{2(k-1)}\frac{\partial(xfy)}{\partial x_i}y^{2(k-1)}\Big)D_y$$

$$= \Big(2(k-1)x^{2(k-1)}fy^{2k-1} + (n+1)x^{2(k-1)}fy^{2k-1} + \sum_{i=0}^{n} e_i x^{2(k-1)}x\frac{\partial f}{\partial x_i}y^{2k-1}\Big)D_y$$

$$= (2k-2+n+1)x^{2(k-1)}(fy^{2k-1})D_y + \Big(\sum_{i=0}^{n} e_i x^{2(k-1)}x\frac{\partial f}{\partial x_i}y^{2k-1}\Big)D_y.$$

By $\begin{cases} e_i x + xe_i = 2x_i & (i=0,1,2,\cdots,n), \\ ye_j + e_j y = 2y_j & (j=0,1,2,\cdots,n), \end{cases}$ we have

$$(2k-2+n+1)x^{2(k-1)}(fy^{2k-1})D_y = (2k-2+n+1)x^{2(k-1)}((fy)y^{2(k-1)})D_y$$

$$= (2k-2+n+1)x^{2(k-1)}\Big(fy(y^{2(k-1)}D_y) + \sum_{j=0}^{n}\frac{\partial(fy)}{\partial y_j}y^{2(k-1)}e_j\Big)$$

$$= (2k-2+n+1)x^{2(k-1)}\Big((2k-2)fy^{2(k-1)} + \sum_{j=0}^{n}\frac{\partial f}{\partial y_j}yy^{2(k-1)}e_j + (n+1)fy^{2(k-1)}\Big)$$

$$= (2k-2+n+1)x^{2(k-1)}\Big((2k-2+n+1)fy^{2(k-1)} + \sum_{j=0}^{n}\frac{\partial f}{\partial y_j}(2y_j - e_j y)y^{2(k-1)}\Big)$$

$$= (2k-2+n+1)x^{2(k-1)}((2k-2+n+1)fy^{2(k-1)} + 2(E_2 f)y^{2(k-1)} - (fD_y)y^{2k-1})$$

$$= (2k-2+n+1)^2 x^{2(k-1)}fy^{2(k-1)} + 2(2k-2+n+1)x^{2(k-1)}(E_2 f)y^{2(k-1)}.$$

$$\Big(\sum_{i=0}^{n} e_i x^{2(k-1)}x\frac{\partial f}{\partial x_i}y^{2k-1}\Big)D_y = x^{2(k-1)}\Big(\sum_{i=0}^{n}(2x_i - xe_i)\frac{\partial f}{\partial x_i}y^{2k-1}\Big)D_y$$

$$= x^{2(k-1)}(2(E_1 f)y^{2k-1} - x(D_x f)y^{2k-1})D_y = 2x^{2(k-1)}((E_1 f)yy^{2(k-1)})D_y$$

$$= 2x^{2(k-1)}\Big((E_1 f)y(y^{2(k-1)}D_y) + \sum_{j=0}^{n}\frac{\partial((E_1 f)y)}{\partial y_j}y^{2(k-1)}e_j\Big)$$

$$
\begin{aligned}
&= 2x^{2(k-1)}\Big((2k-2)(E_1 f)y^{2(k-1)} + \sum_{j=0}^{n} \frac{\partial(E_1 f)}{\partial y_j} y y^{2(k-1)} e_j + (n+1)(E_1 f)y^{2(k-1)}\Big)\\
&= 2x^{2(k-1)}\Big((2k-2+n+1)(E_1 f)y^{2(k-1)} + \sum_{j=0}^{n} \frac{\partial(E_1 f)}{\partial y_j}(2y_j - e_j y)y^{2(k-1)}\Big)\\
&= 2x^{2(k-1)}((2k-2+n+1)(E_1 f)y^{2(k-1)} + 2(E_2E_1 f)y^{2(k-1)} - E_1(fD_y)y^{2k-1})\\
&= 2(2k-2+n+1)x^{2(k-1)}(E_1 f)y^{2(k-1)}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&D_x(x^{2k-1}fy^{2k-1})D_y\\
&= (2k-2+n+1)^2 x^{2(k-1)} f y^{2(k-1)} + 2(2k-2+n+1)x^{2(k-1)}(E_2 f)y^{2(k-1)}\\
&\quad + 2(2k-2+n+1)x^{2(k-1)}(E_1 f)y^{2(k-1)}\\
&= 4\Big(\frac{n+1}{2} + k - 1\Big)x^{2(k-1)}(S_{\frac{n+1}{2}+k-1} f)y^{2(k-1)}.
\end{aligned}
$$

By (3.10) we get

$$
D_x(xfy)D_y = 2(n+1)S_{\frac{n+1}{2}} f. \tag{3.13}
$$

Next, we prove (3.11) by induction.

When $k = 1$, by (3.9) and (3.13) we have

$$
D_x^2(x^2 f y^2)D_y^2 = D_x(D_x(x^2 f y^2)D_y)D_y = D_x(4xfy)D_y = 8(n+1)S_{\frac{n+1}{2}} f.
$$

Suppose (3.11) holds for $k = l$, that is,

$$
D_x^{2l}(x^{2l} f y^{2l})D_y^{2l} = 4^{2l}(l!)^2\Big(\frac{n+1}{2} + l - 1\Big)\cdots\Big(\frac{n+1}{2}\Big)S_{\frac{n+1}{2}+l-1}\cdots S_{\frac{n+1}{2}} f. \tag{3.14}
$$

Then when $k = l+1$, by (3.2), (3.9)–(3.10) and (3.14), we have

$$
\begin{aligned}
&D_x^{2(l+1)}(x^{2(l+1)} f y^{2(l+1)})D_y^{2(l+1)}\\
&= D_x^{2l+1}(D_x(x^{2(l+1)} f y^{2(l+1)})D_y)D_y^{2l+1}\\
&= D_x^{2l+1}((2(l+1))^2 x^{2l+1} f y^{2l+1})D_y^{2l+1}\\
&= (2(l+1))^2 D_x^{2l}(D_x(x^{2l+1} f y^{2l+1})D_y)D_y^{2l}\\
&= (2(l+1))^2 D_x^{2l}\Big(4\Big(\frac{n+1}{2} + l\Big)x^{2l}(S_{\frac{n+1}{2}+l} f)y^{2l}\Big)D_y^{2l}\\
&= 4(2(l+1))^2\Big(\frac{n+1}{2} + l\Big)D_x^{2l}(x^{2l}(S_{\frac{n+1}{2}+l} f)y^{2l})D_y^{2l}\\
&= 4^{2(l+1)}((l+1)!)^2\Big(\frac{n+1}{2} + l\Big)\cdots\Big(\frac{n+1}{2}\Big)S_{\frac{n+1}{2}+l-1}\cdots S_{\frac{n+1}{2}}S_{\frac{n+1}{2}+l} f\\
&= 4^{2(l+1)}((l+1)!)^2\Big(\frac{n+1}{2} + l\Big)\cdots\Big(\frac{n+1}{2}\Big)S_{\frac{n+1}{2}+l}S_{\frac{n+1}{2}+l-1}\cdots S_{\frac{n+1}{2}} f.
\end{aligned}
$$

Finally, we prove (3.12).

By (3.2), (3.10) and (3.11), we have

$$
D_x^{2k-1}(x^{2k-1} f y^{2k-1})D_y^{2k-1}
$$

$$
\begin{aligned}
&= D_x^{2(k-1)}(D_x(x^{2k-1}fy^{2k-1})D_y)D_y^{2(k-1)}\\
&= 4\Big(\frac{n+1}{2}+k-1\Big)D_x^{2(k-1)}(x^{2(k-1)}(S_{\frac{n+1}{2}+k-1}f)y^{2(k-1)})D_y^{2(k-1)}\\
&= 4^{2k-1}((k-1)!)^2\Big(\frac{n+1}{2}+k-1\Big)\cdots\Big(\frac{n+1}{2}\Big)S_{\frac{n+1}{2}+k-2}\cdots S_{\frac{n+1}{2}}S_{\frac{n+1}{2}+k-1}f\\
&= 4^{2k-1}((k-1)!)^2\Big(\frac{n+1}{2}+k-1\Big)\cdots\Big(\frac{n+1}{2}\Big)S_{\frac{n+1}{2}+k-1}\cdots S_{\frac{n+1}{2}}f.
\end{aligned}
$$

**Remark 3.2** If $\Omega\subseteq\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}$ is a star-like domain centered at the origin O, $f\in C^\infty(\Omega,Cl_{2n+2,0}(\mathbb{R}))$ is a biregular function on $\Omega$ and $E_2E_1f=0$, then $D_x^l(x^kfy^k)D_y^l=0$ $(l,k\in\mathbf{N}^*,l>k)$.

Let

$$\lambda_k=\frac{1}{C_k}T_{\frac{n+1}{2}}T_{\frac{n+1}{2}+1}\cdots T_{\frac{n+1}{2}+[\frac{k-1}{2}]},$$

where $C_k=4^k([\frac{k}{2}]!)^2\big(\frac{n+1}{2}+\big[\frac{k-1}{2}\big]\big)\big(\frac{n+1}{2}+\big[\frac{k-1}{2}\big]-1\big)\cdots\big(\frac{n+1}{2}\big)$, $k\in\mathbf{N}^*$, $\big[\frac{k}{2}\big]$ is the integral function of $\frac{k}{2}$.

**Theorem 3.6** *If* $\Omega\subseteq\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}$ *is a star-like domain centered at the origin* O, $k>1$, $k\in\mathbf{N}^*$, $f\in C^\infty(\Omega,Cl_{2n+2,0}(\mathbb{R}))$ *is a bi-k-regular function on* $\Omega$ *and* $E_2E_1f=0$, *then there exist uniquely* $f_1,f_2,\cdots,f_k$ *where* $f_i$ $(i=1,2,\cdots,k)$ *satisfies* $D_xf_iD_y=0$ *on* $\Omega$ *such that*

$$f(x,y)=f_1(x,y)+xf_2(x,y)y+\cdots+x^{k-1}f_k(x,y)y^{k-1},\tag{3.15}$$

*where*

$$
\begin{aligned}
f_k&=\lambda_{k-1}(D_x^{k-1}fD_y^{k-1}),\\
f_{k-1}&=\lambda_{k-2}(D_x^{k-2}(f-x^{k-1}f_ky^{k-1})D_y^{k-2}),\\
f_{k-2}&=\lambda_{k-3}(D_x^{k-3}(f-x^{k-1}f_ky^{k-1}-x^{k-2}f_{k-1}y^{k-2})D_y^{k-2}),\\
&\vdots\\
f_2&=\lambda_1(D_x(f-x^{k-1}f_ky^{k-1}-x^{k-2}f_{k-1}y^{k-2}-\cdots-x^2f_3y^2)D_y),\\
f_1&=f-x^{k-1}f_ky^{k-1}-x^{k-2}f_{k-1}y^{k-2}-\cdots-x^2f_3y^2-xf_2y.
\end{aligned}
$$

**Proof** Let $G_k=\{f\in C^\infty(\Omega,Cl_{2n+2,0}(\mathbb{R})):D_x^kfD_y^k=0\}$.

**Step 1** We prove that for $k>1,k\in\mathbf{N}^*$, we have

$$G_k=G_{k-1}+x^{k-1}G_1y^{k-1}.$$

On one hand, it is obvious that $G_{k-1}\subseteq G_k$, and it follows from Remark 3.2 that $D_x^k(x^{k-1}fy^{k-1})D_y^k=0$, that is, $x^{k-1}G_1y^{k-1}\subseteq G_k$. Thus $G_{k-1}+x^{k-1}G_1y^{k-1}\subseteq G_k$.

On the other hand, for any $f\in G_k$, we have

$$f=(fI-x^{k-1}(\lambda_{k-1}(D_x^{k-1}fD_y^{k-1}))y^{k-1})+x^{k-1}(\lambda_{k-1}(D_x^{k-1}fD_y^{k-1}))y^{k-1}.$$

Since $f \in G_k$, we have $D_x^k f D_y^k = D_x(D_x^{k-1} f D_y^{k-1}) D_y = 0$, then $D_x^{k-1} f D_y^{k-1} \in G_1$, by Remark 3.1, we have $\lambda_{k-1}(D_x^{k-1} f D_y^{k-1}) \in G_1$. From (3.11)–(3.12), we have

$$\begin{aligned}
&D_x^{k-1}(fI - x^{k-1}(\lambda_{k-1}(D_x^{k-1} f D_y^{k-1}))y^{k-1})D_y^{k-1}\\
&= D_x^{k-1} f D_y^{k-1} - D_x^{k-1}(x^{k-1}(\lambda_{k-1}(D_x^{k-1} f D_y^{k-1}))y^{k-1})D_y^{k-1}\\
&= D_x^{k-1} f D_y^{k-1} - D_x^{k-1} f D_y^{k-1}\\
&= 0,
\end{aligned}$$

that is, $fI - x^{k-1}(\lambda_{k-1}(D_x^{k-1} f D_y^{k-1}))y^{k-1} \in G_{k-1}$, hence $G_k \subseteq G_{k-1} + x^{k-1}G_1 y^{k-1}$.

Therefore, for any $k > 1$, $G_k = G_{k-1} + x^{k-1}G_1 y^{k-1}$.

**Step 2** We prove that for $k > 1, k \in \mathbf{N}^*$, we have

$$G_k = G_1 + xG_1 y + \cdots + x^{k-1}G_1 y^{k-1}.$$

By Step 1, we obtain

$$\begin{aligned}
G_k &= G_{k-1} + x^{k-1}G_1 y^{k-1}\\
&= G_{k-2} + x^{k-2}G_1 y^{k-2} + x^{k-1}G_1 y^{k-1}\\
&\ \vdots\\
&= G_1 + xG_1 y + \cdots + x^{k-1}G_1 y^{k-1}.
\end{aligned}$$

**Step 3** We prove that for any $f \in G_k$, the decomposition $f = g + x^{k-1} f_k y^{k-1}$ $(g \in G_{k-1}, f_k \in G_1)$ is unique.

Suppose $f = g + x^{k-1} f_k y^{k-1} = g^* + x^{k-1} f_k^* y^{k-1}$, where $g, g^* \in G_{k-1}, f_k, f_k^* \in G_1$, then

$$f - f = (g - g^*) + (x^{k-1}(f_k - f_k^*)y^{k-1}) = 0.$$

By Theorem 3.5, we have

$$\begin{aligned}
&D_x^{k-1}((g - g^*) + (x^{k-1}(f_k - f_k^*)y^{k-1}))D_y^{k-1}\\
&= D_x^{k-1}(x^{k-1}(f_k - f_k^*)y^{k-1})D_y^{k-1}\\
&= 4^{k-1}\Big(\Big[\frac{k-1}{2}\Big]!\Big)^2\Big(\frac{n+1}{2} + \Big[\frac{k-2}{2}\Big]\Big)\cdots\Big(\frac{n+1}{2}\Big) S_{\frac{n+1}{2}+[\frac{k-2}{2}]} \cdots S_{\frac{n+1}{2}}(f_k - f_k^*)\\
&= 0.
\end{aligned}$$

As $T_\beta 0 = 0$ and $S_\beta T_\beta = I$, $f_k - f_k^* = 0$. Hence

$$f_k = f_k^*, \quad g = f - x^{k-1} f_k y^{k-1} = f - x^{k-1} f_k^* y^{k-1} = g^*,$$

i.e., the decomposition $f = g + x^{k-1} f_k y^{k-1}$ $(g \in G_{k-1}, f_k \in G_1)$ is unique.

**Step 4** By Theorem 3.5 and $f = g + x^{k-1} f_k y^{k-1}$ $(g \in G_{k-1}, f_k \in G_1)$, we have

$$D_x^{k-1} f D_y^{k-1} = D_x^{k-1}(g + x^{k-1} f_k y^{k-1})D_y^{k-1} = D_x^{k-1}(x^{k-1} f_k y^{k-1})D_y^{k-1} = \lambda_{k-1}^{-1} f_k,$$

hence $f_k = \lambda_{k-1}(D_x^{k-1} f D_y^{k-1})$.

Moreover, as $g \in G_{k-1},\ g = g_1 + x^{k-2} f_{k-1} y^{k-2}$ $(g_1 \in G_{k-2}, f_{k-1} \in G_1)$, then by Theorem 3.5,

$$D_x^{k-2} g D_y^{k-2} = D_x^{k-2}(g_1 + x^{k-2} f_{k-1} y^{k-2}) D_y^{k-2} = D_x^{k-2}(x^{k-2} f_{k-1} y^{k-2}) D_y^{k-2} = \lambda_{k-2}^{-1} f_{k-1},$$

hence

$$\begin{aligned} f_{k-1} &= \lambda_{k-2}(D_x^{k-2} g D_y^{k-2}) \\ &= \lambda_{k-2}(D_x^{k-2}(f - x^{k-1} f_k y^{k-1}) D_y^{k-2}). \end{aligned}$$

Let $g_1 = g_2 + x^{k-3} f_{k-2} y^{k-3}$ $(g_2 \in G_{k-3}, f_{k-2} \in G_1)$. By Theorem 3.5,

$$D_x^{k-3} g_1 D_y^{k-3} = D_x^{k-3}(g_2 + x^{k-3} f_{k-2} y^{k-3}) D_y^{k-3} = D_x^{k-3}(x^{k-3} f_{k-2} y^{k-3}) D_y^{k-3} = \lambda_{k-3}^{-1} f_{k-2},$$

hence

$$\begin{aligned} f_{k-2} &= \lambda_{k-3}(D_x^{k-3} g_1 D_y^{k-3}) \\ &= \lambda_{k-3}(D_x^{k-3}(f - x^{k-1} f_k y^{k-1} - x^{k-2} f_{k-1} y^{k-2}) D_y^{k-3}). \end{aligned}$$

In the same way, we can get the expression of $f_{k-3}, f_{k-4}, \cdots, f_2, f_1$.

Let

$$H_k = \{f \in C^\infty(\Omega, Cl_{2n+2,0}(\mathbb{R})) : \Delta_x^k f \Delta_y^k = 0\}.$$

**Remark 3.3** If $\Omega \subseteq \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ is a star-like domain centered at the origin O, $k \in \mathbf{N}^*$, $f \in C^\infty(\Omega, Cl_{2n+2,0}(\mathbb{R}))$ is a bi-$k$-harmonic function on $\Omega$ and $E_2 E_1 f = 0$, then

$$H_k = H_1 \oplus x^2 H_1 y^2 \oplus \cdots \oplus x^{2(k-1)} H_1 y^{2(k-1)}. \tag{3.16}$$

**Proof** It is obviously true for $k = 1$.

It can be observed that

$$H_k = \{f \in C^\infty(\Omega, Cl_{2n+2,0}(\mathbb{R})) : D_x^{2k} f D_y^{2k} = 0\} = G_{2k}.$$

For $k = 1, k \in \mathbf{N}^*$, from Theorem 3.6 it follows that

$$\begin{aligned} &G_{2k} = G_1 \oplus x G_1 y \oplus x^2 G_1 y^2 \oplus \cdots \oplus x^{2(k-1)} G_1 y^{2(k-1)} \oplus x^{2k-1} G_1 y^{2k-1}, \\ &H_1 = G_2 = G_1 \oplus x G_1 y. \end{aligned}$$

Hence

$$\begin{aligned} H_k = G_{2k} &= G_1 \oplus x G_1 y \oplus x^2 G_1 y^2 \oplus \cdots \oplus x^{2(k-1)} G_1 y^{2(k-1)} \oplus x^{2k-1} G_1 y^{2k-1} \\ &= H_1 \oplus x^2 H_1 y^2 \oplus \cdots \oplus x^{2(k-1)} H_1 y^{2(k-1)}. \end{aligned}$$

## References

[1] Brackx, F., Delanghe, R. and Sommen, F., Clifford Analysis, Pitman Books Limits, Boston, 1982.

[2] Clifford, W. K., Applications of Grassman's extensive algebra, American Journal of Mathematics, **1**(4), 1878, 350–358.

[3] Eriksson, S. L., Hyperbolic extensions of integral formulas, *Adv. Appl. Clifford Algebras*, **20**(3–4), 2010, 575–586.

[4] Eriksson, S. L. and Orelma, H., Hyperbolic function theory in the Clifford algebra $Cl_{n+1,0}$, *Adv. Appl. Clifford Algebras*, **19**(2), 2009, 283–301.

[5] Garcia, A. M., Garcia, T. M., Blaya, R. A. and Reyes, J. B., Decomposition of inframonogenic functions with applications in elasticity theory, *Math. Methods Appl. Sci.*, **43**(4), 2020, 1915–1924.

[6] Huang, S., Qiao, Y. Y. and Wen, G. C., Real and Complex Clifford Analysis, Springer-Verlag, Berlin, 2006.

[7] Malonek, H. and Ren, G. B., Almansi-type theorems in Clifford analysis, *Math. Methods Appl. Sci.*, **25**(15–18), 2002, 1541–1552.

[8] Qiao, Y. Y., A boundary value problem for hypermonogenic functions in Clifford analysis, *Science in China*, **48**A, 2005, 324–332.

[9] Ren, G. B. and Kahler, U., Almansi decompositions for polyharmonic, polyheat, and polywave functions, *Studia Math.*, **172**(1), 2006, 91–100.

[10] Sakakibara, K., Method of fundamental solutions for biharmonic equation based on Almansi-type decomposition, *App. Math.*, **62**(4), 2017, 297–317.

[11] Xie, Y. H., Boundary properties of hypergenic-Cauchy type integrals in real Clifford analysis, *Complex Var. Ell. Eq.*, **59**(5), 2014, 599–615.

[12] Xie, Y. H., Zhang, X. F. and Tang, X. M., Some properties of k-hypergenic functions in Clifford analysis, *Complex Var. Ell. Eq.*, **61**(12), 2016, 1614–1626.

[13] Yang, H. J., Qiao, Y. Y. and Xie, Y. H., Cauchy integral formula for k-monogenic function with $\alpha$-weight, *Adv. Appl. Clifford Algebras*, **28**(2), 2018, 1–11.