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Zero Problems of the Bergman Kernel Function on the First Type of Cartan-Hartogs Domain*

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Abstract The authors give the condition that the Bergman kernel function on the first type of Cartan-Hartogs domain exists zeros. If the Bergman kernel function of this type of domain has zeros, the zero set is composed of several path-connected branches, and there exists a continuous curve to connect any two points in the non-zero set.

Keywords Cartan-Hartogs domain, Zeros of Bergman kernel function, Path connectivity
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1 Introduction

The biholomorphic equivalence classification of bounded domains is one of the central problems in the study of several complex variables. The Riemann mapping theorem resolves the problem about the biholomorphic equivalence classification of the simply connected bounded domains in $\mathbb{C}_\infty$. At the beginning of 20th century, Poincaré pointed out that the unit ball and the polydisc are not biholomorphically equivalent in $\mathbb{C}^n$ $(n > 1)$, which indicates that Riemann mapping theorem fails to hold in higher dimensional case. In 1936, Cartan gave the whole conclusion about the biholomorphic equivalence classification of the bounded symmetric domains in $\mathbb{C}^n$, i.e., any bounded symmetric domain is the topological product of irreducible bounded symmetric domains, and irreducible bounded symmetric domains consist of domains of four classical types and two exceptional ones. However, the biholomorphic equivalence classification of general bounded domains has not been resolved. It is an important research direction of several complex variables.

In 1930, Bergman introduced the concept of representative coordinates and representative domain when he studied the biholomorphic equivalence classification of bounded domains in $\mathbb{C}^n$ (see [5]). Let G be a bounded domain in $\mathbb{C}^n$. $K(z,\overline{w})$ denotes the Bergman kernel function of G, and $(T_{jk}(z,\overline{z}))$ denotes the Bergman metric matrix of G. For any fixed $w_0 \in G$ and any

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$z \in G$, φ_{w_0} maps z to $\varphi_{w_0}(z)$:

$$\varphi_{w_0}(z) = \frac{\partial}{\partial \overline{w}} \log \frac{K(z,\overline{w})}{K(w,\overline{w})}\Big|_{w=w_0} (T_{jk}(w_0,\overline{w_0}))^{-1},$$

then φ_{w_0} is called the representative mapping, and $\varphi_{w_0}(z)$ is the representative coordinates of z. If $\varphi_{w_0}(G)$ is a domain, $\varphi_{w_0}(G)$ is called the representative domain of G. Since domains which are biholomorphically equivalent have identical representative domain, the representative domain can be used as the representative element of biholomorphically equivalence class of domains. Moreover, two domains must not be biholomorphically equivalent if their representative domains are not identical. Therefore, the representative domain could be the tool for studying the biholomorphic equivalence classification of bounded domains. In 1966, when Lu was studying that bounded domain with complete Bergman metric of negative constant holomorphic sectional curvature is biholomorphically equivalent to the unit ball, he needed to define the representative mapping globally on the bounded domain. About the above definition of representative mapping, he pointed out that φ_{w_0} is well defined on a sufficiently small neighborhood $U(w_0)$ of w_0 and at the point where $K(z,\overline{w}) \neq 0$ for any $z \in G$. However, defining representative mapping globally on a general bounded domain depends on that the Bergman kernel function of this domain is zero-free (see [15]). The Bergman kernel functions of some specific domains do not have zeros, such as unit ball, polydisc and bounded symmetric domains, but there is no definite conclusion as to whether the Bergman kernel functions of general bounded domains have zeros or not. Therefore, Lu raised an open question of whether the Bergman kernel functions of general bounded domains are zero-free. In 1969, Skwarczyński formally called the question raised by Lu as Lu Qi-Keng conjecture, i.e., the Bergman kernel function of a general bounded domain is zero-free. Moreover, he proved Lu Qi-Keng conjecture does not hold with counterexample: The Bergman kernel function of an annulus has zeros in the complex plane $\{z \in \mathbb{C} : r < |z| < 1, 0 < r < \mathrm{e}^{-2}\}$ (see [21]). It shows that the bounded domains on which Bergman kernel functions have zeros exist.

Although Lu Qi-Keng conjecture fails, the problem of determining whether the Bergman kernel function of a bounded domain has zeros is of great significance to discuss the properties of Bergman kernel functions, the properties of bounded domains, the biholomorphic equivalence classification of bounded domains and so on. Therefore, this research has received extensive attention in the area of several complex variables, and the problem about studying whether the Bergman kernel function of a domain has zeros or not is called the Lu Qi-Keng problem. Since Skwarczyński, many scholars have studied Lu Qi-Keng problem and achieved fruitful results. In 1966, Rosenthal proved that the Bergman kernel function of an annulus in the complex plane has zeros, moreover there exists k ($k > 2$) connected domain whose Bergman kernel function has zeros in $\mathbb{C}$ (see [20]). In the same year, Bell showed that the Bergman kernel function of every bounded, homogeneous, complete circular domain is zero-free (see [4]). In 1976, Suita and Yamada proved that the Bergman kernel function of non-simply connected finite Riemann surface has zeros, and obtained a general conclusion, i.e., the Bergman kernel function of bounded, multiple connected, planar domain with smooth boundary has zeros (see [22]). In 1981, Greene and Krantz proved that the sufficiently small smooth perturbations of the ball has zero-free Bergman kernel function (see [14]). In 1986, Boas gave a specific example to show that the Bergman kernel function of a bounded, strongly pseudo-convex, contractible domain with C^∞ regular boundary has zeros (see [6]). In 1996, Boas proved that the bounded holomorphic domains whose Bergman kernel functions have no zeros, form a nowhere dense set in the topological space composed of all bounded holomorphic domains (see [7]). This conclusion indicates that it is a normal situation for the Bergman kernel function of a holomorphic domain to have zeros. In 1999, Boas, Fu and Straube further proved that the Bergman kernel function of the following domain $\{z \in \mathbb{C}^2 : |z_1| + |z_2|^{\frac{2}{p}} < 1\}$ has zeros when $p > 2$, and gave concrete examples that the Bergman kernel functions of strongly convex domains have zeros in [9]. In 2000, Boas obtained a more general conclusion that the Bergman kernel function of the domain defined as

$$\{z = (z_1, z_2, \cdots, z_n) \in \mathbb{C}^n : |z_1| + |z_2|^{\frac{2}{p_2}} + \cdots + |z_n|^{\frac{2}{p_n}} < 1\}$$

has zeros when $p_2 + p_3 + \cdots + p_n > 2$ (see [8]). In 1998, Pflug and Youssfi proved that the Bergman kernel function of the minimal ball exists zeros when $n \geq 4$ in $\mathbb{C}^n$ (see [19]). In 2000, Engliš proved that the Bergman kernel function of a kind of Hartogs domain has zeros (see [13]). In 2005, Edigarian and Zwonek got that the Bergman kernel function of symmetric double-disc G_2 is zero-free (see [12]), while Nikolov and Zwonek later proved that the Bergman kernel function of symmetric polydisc $G_n (n \geq 3)$ has zeros (see [18]). In 2006, Chen studied the Lu Qi-Keng problem on a pseudo-convex domain applying the asymptotic behavior of the weighted Bergman kernel (see [10]). In 2009, Zhang and Yin discussed the conditions that the Bergman kernel function of the generalized Thullen domain has zeros (see [28]). In 2012, Ahn and Park presented the algorithmic procedure to determine the condition that the Bergman kernel function is zero-free on a type of generalized Cartan-Hartogs domain (see [1]). In 2016, Beberok investigated the Lu Qi-Keng problem of the intersection of two complex ellipsoids (see [3]). We refer to [2, 11, 16, 23–24, 27] for more research results about Lu Qi-Keng problem.

In the study on Lu Qi-Keng problem, many research achievements show that the Bergman kernel functions of bounded domains have zeros. In fact, if the Bergman kernel function of a domain has zeros, these zeros are not isolated and form a lower dimensional manifolds. It is significant for discussing the geometry properties of domains and the properties of Bergman kernel functions to study the structure and topological properties of the zero-sets of the Bergman kernel functions on bounded domains. The aim of this present paper is to study the Lu Qi-Keng problem on the first type of Cartan-Hartogs domain, and discuss the topological properties of the zero set when the Bergman kernel function of this type of domain has zeros. The form of the first type of Cartan-Hartogs domain is as following (see [25]):

$$\Omega = \{(\xi, z) \in \mathbb{C} \times R_I(m, n) : |\xi|^{2\mu} < N_I(z, z), \mu > 0\},$$

where $R_I(m, n)$ denotes the first type of Cartan domain, $N_I(z, z)$ denotes the generic norm of $R_I(m, n)$ in Jordan triple system. By constructing a biholomorphic invariant x on $\Omega \times \Omega$, we write the Bergman kernel function of Ω into the product of a polynomial $F(x)$ and a non-vanishing function. Then the research on the zero problem of the Bergman kernel function with multidimensional complex variable is transformed into the research on the zero problem of a polynomial function with single complex variable. From the condition that $F(x)$ has zeros in the unit disc, we obtain that there exists $\mu_0(m,n) > 0$ such that the Bergman kernel function of Ω has zeros when the parameter μ satisfies $0 < \mu < \mu_0(m,n)$ for any m and n. Furthermore, we show that the zero set of the Bergman kernel function of Ω is composed of some path-connected branches, and a continuous curve is constructed to connect any two points in the non-zero set.

2 Preliminaries

In this section, we are going to outline some basic conclusions about Ω, and introduce a type of special self-reciprocal polynomial.

Theorem 2.1 (see [26]) *The transforms $\psi(\xi,z) = (\psi_1(\xi,z), \psi_2(\xi,z))$ constitute the holomorphic automorphism group of Ω, and the specific forms of ψ_1 and ψ_2 are as below*:

$$\begin{cases} \psi_1(\xi,z) = \mathrm{e}^{\sqrt{-1}\theta}\xi N_I(z_0,z_0)^{\frac{1}{2\mu}} N_I(z,z_0)^{-\frac{1}{\mu}}, \\ \psi_2(\xi,z) = \phi(z), \quad \phi \in \mathrm{Aut}(R_I(m,n)), \end{cases}$$

where $\theta \in \mathbb{R}$, $z_0 \in R_I(m,n), z_0 = \phi^{-1}(0)$. $\mathrm{Aut}(\Omega)$ denotes the holomorphic automorphism group of Ω.

Now, we construct a complex value function x on $\Omega \times \Omega$, and it is a invariant under the holomorphic transformation of $\Omega \times \Omega$.

Theorem 2.2 *Let*

$$x = x((\xi,z),(\eta,w)) = \xi\overline{\eta} N_I(z,w)^{-\frac{1}{\mu}}$$

for any $((\xi,z),(\eta,w)) \in \Omega \times \Omega$. Then

$$x((\xi,z),(\eta,w)) = x(\psi(\xi,z),\psi(\eta,w)) \quad \textit{for } \forall\psi \in \mathrm{Aut}(\Omega),$$

and $|x| < 1$.

Proof For any $z = (z_{11}, z_{12}, \cdots, z_{1n}, \cdots, z_{m1}, z_{m2}, \cdots, z_{mn}) \in R_I(m,n)$, the components of z can be arranged into a matrix

$$Z = \begin{pmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{21} & z_{22} & \cdots & z_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{m1} & z_{m2} & \cdots & z_{mn} \end{pmatrix},$$

where z is corresponding to Z, and the first type of Cartan domain is represented as $R_I(m,n) := \{Z \in \mathbb{C}^{mn} : I - Z\overline{Z}^t > 0\}$. So the point in $R_I(m,n)$ is denoted by Z in the proof of the theorem.

For any $((\xi,Z),(\eta,W)) \in \Omega \times \Omega$, (ξ,Z) and (η,W) are mapped to (ξ^*,Z^*) and (η^*,W^*) respectively by any $\psi \in \mathrm{Aut}(\Omega)$. Applying Theorem 2.1, we have

$$\begin{aligned} \xi^*\overline{\eta}^* &= \xi\overline{\eta} N_I(Z_0,Z_0)^{\frac{1}{\mu}} N_I(Z,Z_0)^{-\frac{1}{\mu}} N_I(W,Z_0)^{-\frac{1}{\mu}}, \\ Z^* &= P(Z-Z_0)(I-\overline{Z}_0^t Z)^{-1} Q^{-1}, \end{aligned}$$

$$W^* = P(W - Z_0)(I - \overline{Z}_0^t W)^{-1} Q^{-1},$$

where $Z_0 = \phi^{-1}(0)$ ($\phi \in \mathrm{Aut}(R_I(m,n))$), $\overline{P}^t P = (I - Z_0 \overline{Z}_0^t)^{-1}$, $\overline{Q}^t Q = (I - \overline{Z}_0^t Z_0)^{-1}$.

Since

$$N_I(Z,W) = \det(I - Z\overline{W}^t)$$

and

$$I - Z^*(\overline{W^*})^t = (\overline{P}^t)^{-1}(I - Z\overline{Z}_0^t)^{-1}(I - Z\overline{W}^t)(I - Z_0\overline{W}^t)^{-1}P^{-1}$$

for any $Z, W \in R_I(m,n)$, we obtain

$$\begin{aligned} x((\xi^*, Z^*), (\eta^*, W^*)) &= \xi^* \overline{\eta}^* N_I(Z^*, W^*)^{-\frac{1}{\mu}} \\ &= \xi\overline{\eta} N_I(Z_0, Z_0)^{\frac{1}{\mu}} N_I(Z, Z_0)^{-\frac{1}{\mu}} N_I(Z_0, W)^{-\frac{1}{\mu}} \\ &\quad \det(\overline{P}^t P)^{\frac{1}{\mu}} N_I(Z, Z_0)^{\frac{1}{\mu}} N_I(Z, W)^{-\frac{1}{\mu}} N_I(Z_0, W)^{\frac{1}{\mu}} \\ &= \xi\overline{\eta} N_I(Z, W)^{-\frac{1}{\mu}} = x((\xi, Z), (\eta, W)), \end{aligned}$$

i.e., x is a invariant under the holomorphic transformation of $\Omega \times \Omega$.

We can prove

$$N_I(Z,Z) N_I(W,W) \le |N_I(Z,W)|^2$$

for any $((\xi, Z), (\eta, W)) \in \Omega \times \Omega$ using the conclusion in [17]. Then based on the inequality, we obtain

$$\begin{aligned} |x|^2 &= |\xi|^2 |\eta|^2 |N_I(Z,W)|^{-\frac{2}{\mu}} \\ &< (N_I(Z,Z) N_I(W,W) |N_I(Z,W)|^{-2})^{\frac{1}{\mu}} \le 1. \end{aligned}$$

A special kind of self-reciprocal polynomial plays an important role in investigating the Lu Qi-Keng problem on the first type of Cartan-Hartogs domain, here we give the definition of a general self-reciprocal polynomial and discuss the properties of the special kind of self-reciprocal polynomial.

Definition 2.1 *Let $f_k(x)$ be a real coefficient polynomial of degree k. If $f_k(0) \ne 0$ and $f_k(x) = x^k f_k\left(\frac{1}{x}\right)$, then $f_k(x)$ is called self-reciprocal polynomial of degree k.*

Remark 2.1 $f_k(x) = \sum\limits_{l=0}^{k} a_l x^l$ is a real coefficient self-reciprocal polynomial of degree k if and only if its coefficients satisfy $a_0 \ne 0$ and $a_l = a_{k-l}$ $(l = 0, 1, \cdots, k)$.

Theorem 2.3 *Let $f_k(x)$ be a real coefficient self-reciprocal polynomial of degree k. If x_0 is a zero of $f_k(x)$, then $\frac{1}{x_0}$ is also its zero.*

Proof From Definition 2.1, this theorem can be proved straightforwardly.

Lemma 2.1 *Let*

$$\sigma_r(\alpha_1, \cdots, \alpha_l) = \sum_{1 \le j_1 < j_2 < \cdots < j_r \le l} \alpha_{j_1} \alpha_{j_2} \cdots \alpha_{j_r} \quad (\alpha_1, \alpha_2, \cdots, \alpha_l \in \mathbb{R}),$$

then $\sigma_r(\alpha_1, \cdots, \alpha_l) > 0$ $(r = 1, 2, \cdots, l)$ if and only if $\alpha_j > 0$, $j = 1, \cdots, l$.

Proof Since the sufficiency of the lemma is obvious, we need only to prove its necessity.

We can construct an equation of degree l with roots of $-\alpha_1, -\alpha_2, \cdots, -\alpha_l$:

$$(x+\alpha_1)(x+\alpha_2)\cdots(x+\alpha_l) = 0. \tag{2.1}$$

Expand the left side of (2.1), it is changed into the following form

$$x^l + \sigma_1(\alpha_1, \cdots, \alpha_l)x^{l-1} + \cdots + \sigma_{l-1}(\alpha_1, \cdots, \alpha_l)x + \sigma_l(\alpha_1, \cdots, \alpha_l) = 0.$$

Because α_j $(j = 1, \cdots, l)$ are real numbers, and $\sigma_r(\alpha_1, \cdots, \alpha_l) > 0$ $(r = 1, 2, \cdots, l)$, the roots of (2.1) can not be 0 or positive real numbers, and they must be negative real numbers, i.e., $\alpha_j > 0$ $(j = 1, \cdots, l)$.

Theorem 2.4 *Let*

$$f_k(x) = \sum_{l=0}^{k} a_{k,l}x^l \quad (k \in \mathbb{N})$$

be a polynomial of degree k with respect to x, where $a_{k,l}$ denotes the coefficient of the l-th order term of the polynomial of degree k:

$$a_{k,l} = \frac{1}{l!}\sum_{j=1}^{k+1-l}(-1)^{k+1-l-j}S(k+1,j)\frac{(k+1-j)!j!}{(k+1-l-j)!} \quad (l = 0, 1, \cdots, k), \tag{2.2}$$

$S(k+1, j)$ denotes the Stirling number of the second kind, i.e.,

$$S(k+1,j) = \frac{1}{j!}\sum_{r=1}^{j}(-1)^{j-r}\frac{j!}{(j-r)!r!}r^{k+1}.$$

Set $a_{k,l} = 0$ for $l > k$ and $l < 0$, then $f_k(x)$ satisfies the following properties:

(1) $a_{k,l} = (k+1-l)a_{k-1,l-1} + (l+1)a_{k-1,l}$ $(k \geq 1,\ 0 \leq l \leq k)$.

(2) $f_k(x)$ $(k \in \mathbb{N})$ *is a self-reciprocal polynomial, i.e.,* $a_{k,l} = a_{k,k-l}$ $(0 \leq l \leq k)$, *and* $a_{k,0} = a_{k,k} = 1$.

(3) $a_{k,l} > 0$ $(0 \leq l \leq k)$.

(4) $a_{k,l-1} < a_{k,l}$ $\left(k \geq 2,\ l = 1, \cdots, \left[\frac{k}{2}\right]\right)$.

(5) *When k is odd number,* $f_k(x) = (x+1)h_{k-1}(x)$ $(k \geq 1)$, *and $h_{k-1}(x)$ is also a self-reciprocal polynomial with positive real coefficients.*

(6) $f_k(x) = [(k-1)^2x^2 + (3k-2)x + 1]f_{k-2}(x) + x(1-x)[(2k-3)x+3]f'_{k-2}(x) + (1-x)^2x^2f''_{k-2}(x)$ $(k \geq 2)$.

(7) $f_k(x)$ $(k \geq 2)$ *has at least one zero in the unit disc.*

Proof (1) According to (2.2) and the property of $S(k+1, j)$, for any fixed l, we have

$$\begin{aligned} a_{k,l} &= \frac{1}{l!}\sum_{j=1}^{k+1-l}(-1)^{k+1-l-j}[S(k,j-1) + jS(k,j)]\frac{(k+1-j)!j!}{(k+1-j-l)!} \\ &= \frac{1}{l!}\sum_{j=1}^{k-l}(-1)^{k-l-j}S(k,j)\frac{(k-j)!(j+1)!}{(k-j-l)!} \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{l!}\sum_{j=1}^{k+1-l}(-1)^{k+1-l-j}S(k,j)\frac{(k+1-j)!j!}{(k+1-j-l)!}j\\
&=\frac{1}{l!}\sum_{j=1}^{k-l}(-1)^{k-l-j}S(k,j)\frac{(k-j)!j!}{(k-j-l)!}\\
&+\frac{l}{l!}\sum_{j=1}^{k-l}(-1)^{k+1-l-j}S(k,j)\frac{(k-j)!j!}{(k+1-j-l)!}[(k+1-l)-(k+1-l-j)]\\
&+S(k,k+1-l)(k+1-l)^2(k-l)!\\
&=a_{k-1,l}+\frac{l}{l!}\sum_{j=1}^{k-l}(-1)^{k-l-j}S(k,j)\frac{(k-j)!j!}{(k-j-l)!}\\
&+(k+1-l)\frac{1}{(l-1)!}\sum_{j=1}^{k+1-l}(-1)^{k+1-l-j}S(k,j)\frac{(k-j)!j!}{(k+1-j-l)!}\\
&=(l+1)a_{k-1,l}+(k+1-l)a_{k-1,l-1}.
\end{aligned}
$$

Therefore, we obtain the recurrent formula about the coefficients of $f_k(x)$:

$$a_{k,l}=(k+1-l)a_{k-1,l-1}+(l+1)a_{k-1,l}\quad (0\le l\le j). \tag{2.3}$$

About (2)–(4), these properties can be proved by applying (2.3) and mathematical induction.

(5) Because $f_k(x)$ is a self-reciprocal polynomial, i.e., $a_{k,l}=a_{k,k-l}$ $(0\le l\le k)$, -1 must be the zero of $f_k(x)$ for odd number k, and

$$f_k(x)=(x+1)h_{k-1}(x)\quad (k\ge 1).$$

We are going to prove that $h_{k-1}(x)$ is also a self-reciprocal polynomial with positive real coefficients.

Put $h_{k-1}(x):=\sum\limits_{l=0}^{k-1}b_{k-1,l}x^l$, then

$$f_k(x)=(x+1)\sum_{l=0}^{k-1}b_{k-1,l}x^l=\sum_{l=0}^{k-1}(b_{k-1,l}x^{l+1}+b_{k-1,l}x^l).$$

By comparing the above expression with $f_k(x)=\sum\limits_{l=0}^{k}a_{k,l}x^l$, we obtain the relationship between $b_{k-1,l}$ and $a_{k,l}$ as following:

$$b_{k-1,l}=\sum_{r=1}^{k-l}(-1)^{r+1}a_{k,l+r}=\sum_{r=0}^{l}(-1)^r a_{k,l-r}. \tag{2.4}$$

From (2.4) and property (2) of $f_k(x)$, it follows that

$$b_{k-1,l}=\sum_{r=0}^{k-l-1}(-1)^r a_{k,(k-l-1)-r}=b_{k-1,k-1-l},$$

which means that $h_{k-1}(x)$ is a self-reciprocal polynomial.

Next, we will show that the coefficients of $h_{k-1}(x)$ are all positive. As $h_{k-1}(x)$ is a self-reciprocal polynomial, it suffices to prove $b_{k-1,l} > 0$ for $l > \frac{k-1}{2}$.

For $a_{k,l+1} > a_{k,l+2} > \cdots > a_{k,k}$ $(l > \frac{k-1}{2})$, if l is odd, then

$$b_{k-1,l} = (a_{k,l+1} - a_{k,l+2}) + \cdots + (a_{k,k-1} - a_{k,k}) > 0;$$

if l is even, then

$$b_{k-1,l} = (a_{k,l+1} - a_{k,l+2}) + \cdots + (a_{k,k-2} - a_{k,k-1}) + a_{k,k} > 0.$$

In conclusion, when k is odd number, $h_{k-1}(x)$ is a self-reciprocal polynomial with positive real coefficients.

(6) To prove the property (6) of $f_k(x)$, we first need to verify

$$f_k(x) = (kx+1)f_{k-1}(x) + x(1-x)f'_{k-1}(x) \quad (k \geq 1). \tag{2.5}$$

From $f_k(x) = \sum\limits_{l=0}^{k} a_{k,l}x^l$, it is easy to know that $f_{k-1}(x) = \sum\limits_{l=0}^{k-1} a_{k-1,l}x^l$, and $f'_{k-1}(x) = \sum\limits_{l=0}^{k-1} l a_{k-1,l}x^{l-1}$. Plugging these equations back into the right of (2.5), we get

$$\begin{aligned}(kx+1)f_{k-1}(x) + x(1-x)f'_{k-1}(x) &= \sum_{l=0}^{k-1}(k-l)a_{k-1,l}x^{l+1} + \sum_{l=0}^{k-1}(1+l)a_{k-1,l}x^l \\ &= \sum_{l=0}^{k}[(k+1-l)a_{k-1,l-1} + (1+l)a_{k-1,l}]x^l \\ &= \sum_{l=0}^{k} a_{k,l}x^l = f_k(x).\end{aligned}$$

According to (2.5), if $k \geq 2$, we have

$$f_{k-1}(x) = [(k-1)x+1]f_{k-2}(x) + x(1-x)f'_{k-2}(x). \tag{2.6}$$

Differentiating the both sides of (2.6) with respect to x gives the following formula:

$$f'_{k-1}(x) = (k-1)f_{k-2}(x) + [(k-3)x+2]f'_{k-2}(x) + (x-x^2)f''_{k-2}(x). \tag{2.7}$$

By substituting (2.6) and (2.7) back into (2.5), it follows that

$$\begin{aligned}f_k(x) = {} & [(k-1)^2x^2 + (3k-2)x + 1]f_{k-2}(x) + x(1-x)[(2k-3)x+3]f'_{k-2}(x) \\ & + (1-x)^2x^2f''_{k-2}(x).\end{aligned} \tag{2.8}$$

(7) We first show that $f_k(x)$ has at least one zero in the unit disc for $k = 2p$ $(p \in \mathbb{Z}^+)$ by negation.

If the assertion fails to hold, $f_{2p}(x)$ would have no zeros in the unit disc. As $f_{2p}(x)$ is a self-reciprocal polynomial, according to Theorem 2.3, if x_0 is a zero of $f_{2p}(x)$, then $\frac{1}{x_0}$ is also the zero of $f_{2p}(x)$. Thereby, based on the hypothesis, $f_{2p}(x)$ has no zeros out of the unit disk either, which means that all of the zeros of $f_{2p}(x)$ are on the boundary of the unit disc. Moreover, the reciprocal of every zero is just its conjugate zero. Set the zeros of $f_{2p}(x)$ as following:

$$\mathrm{e}^{\sqrt{-1}\theta_1}, \mathrm{e}^{-\sqrt{-1}\theta_1}, \cdots, \mathrm{e}^{\sqrt{-1}\theta_p}, \mathrm{e}^{-\sqrt{-1}\theta_p}, \quad \theta_q \in \mathbb{R}\ (q = 1, 2, \cdots, p).$$

With the zeros of $f_{2p}(x)$, we can write $f_{2p}(x)$ as the product of $2p$ factors:

$$\begin{aligned} f_{2p}(x) &= (x - \mathrm{e}^{\sqrt{-1}\theta_1})(x - \mathrm{e}^{-\sqrt{-1}\theta_1})(x - \mathrm{e}^{\sqrt{-1}\theta_2})(x - \mathrm{e}^{-\sqrt{-1}\theta_2}) \\ &\quad \cdots (x - \mathrm{e}^{\sqrt{-1}\theta_p})(x - \mathrm{e}^{-\sqrt{-1}\theta_p}) \\ &= (x^2 - 2\cos\theta_1 x + 1)(x^2 - 2\cos\theta_2 x + 1)\cdots(x^2 - 2\cos\theta_p x + 1) \\ &= x^p\Big(x + \frac{1}{x} - 2\cos\theta_1\Big)\Big(x + \frac{1}{x} - 2\cos\theta_2\Big)\cdots\Big(x + \frac{1}{x} - 2\cos\theta_p\Big) \\ &= x^p(y + \alpha_1)(y + \alpha_2)\cdots(y + \alpha_p), \end{aligned}$$

where $y = x + \frac{1}{x} + 2$, $\alpha_q = -2\cos\theta_q - 2$ and $\alpha_q \le 0$ $(q = 1, 2, \cdots, p)$.

In the above expression of $f_{2p}(x)$, we put

$$G_p(y) := (y + \alpha_1)(y + \alpha_2)\cdots(y + \alpha_p) = \sum_{r=0}^{p} \sigma_{p-r}(\alpha_1, \cdots, \alpha_p) y^r, \tag{2.9}$$

where

$$\begin{aligned} &\sigma_0 = 1, \\ &\sigma_{p-r}(\alpha_1, \cdots, \alpha_p) = \sum_{1 \le q_1 < q_2 < \cdots < q_{p-r} \le p} \alpha_{q_1}\alpha_{q_2}\cdots\alpha_{q_{p-r}} \quad (0 \le r \le p-1). \end{aligned} \tag{2.10}$$

However, according to the property of $f_k(x)$ and Lemma 2.1, it can be inferred that $\alpha_q > 0$ $(q = 1, 2, \cdots, p)$ by showing that $\sigma_{p-r}(\alpha_1, \cdots, \alpha_p) > 0$ $(0 \le r \le p-1)$.

In fact, because $f_{2p}(x) = x^p G_p(y)$ and $y = x + \frac{1}{x} + 2$, we have

$$\begin{aligned} &f_{2(p-1)}(x) = x^{p-1}G_{p-1}(y), \\ &f'_{2(p-1)}(x) = (p-1)x^{p-2}G_{p-1}(y) + x^{p-3}(x^2 - 1)G'_{p-1}(y), \\ &f''_{2(p-1)}(x) = (p-1)(p-2)x^{p-3}G_{p-1}(y) + [(2p-2)x^2 + 4 - 2p]x^{p-4}G'_{p-1}(y) \\ &\qquad\qquad + (x^2 - 1)^2 x^{p-5} G''_{p-1}(y). \end{aligned}$$

By substituting $f_{2p}(x)$, $f_{2(p-1)}(x)$, $f'_{2(p-1)}(x)$ and $f''_{2(p-1)}(x)$ into (2.8), it follows that

$$\begin{aligned} x^p G_p(y) &= x^{p-1}G_{p-1}(y)[p^2x^2 + (2p^2 + 2p)x + p^2] \\ &\quad + x^{p-2}G'_{p-1}(y)[(1-2p)x^4 - 2x^3 + (4p+2)x^2 - 2x + (1-2p)] \\ &\quad + x^{p-3}G''_{p-1}(y)(x^6 - 2x^5 - x^4 + 4x^3 - x^2 - 2x + 1). \end{aligned}$$

Dividing both sides of the above equation by x^p, and substituting y for $x + \frac{1}{x} + 2$, we obtain

$$\begin{aligned} G_p(y) &= G_{p-1}(y)(p^2 y + 2p) \\ &\quad + G'_{p-1}(y)[(1-2p)y^2 + (8p-6)y + 8] + G''_{p-1}(y)(y^3 - 8y^2 + 16y). \end{aligned}$$

Furthermore, by calculating the r-order derivative of $G_p(y)$, we acquire

$$\begin{aligned} G_p^{(r)}(0) &= 8(2r+1)G_{p-1}^{(r+1)}(0) + r[p-(r-1)]^2 G_{p-1}^{(r-1)}(0) \\ &\quad + 2[(p-r)(4r+1)+2r]G_{p-1}^{(r)}(0) \quad (0 \le r \le p) \end{aligned} \tag{2.11}$$

with setting $G_{p-1}^{(-1)}(y) = 0$.

From $f_2(x) = x^2+4x+1$, it follows that $G_1(y) = y+2$, $G_1(0) = 2$ and $G_1'(0) = 1$. According to (2.11), we have $\frac{G_p^{(r)}(0)}{r!} > 0$ $(0 \le r \le p)$ by recurrence method. From (2.9) and (2.10), it follows that

$$\frac{G_p^{(r)}(0)}{r!} = \sigma_{p-r}(\alpha_1, \cdots, \alpha_p) > 0 \quad (0 \le r \le p-1).$$

Finally, we obtain $\alpha_q > 0$ $(q = 1, 2, \cdots, p)$ from Lemma 2.1. But this conclusion is contrary to $\alpha_q = -2\cos\theta_q - 2 \le 0$ $(q = 1, 2, \cdots, p)$. Therefore, the zeros of $f_{2p}(x)$ are not all on the boundary of the unit disc. Based on Theorem 2.3, $f_{2p}(x)$ has at least one zero in the unit disk.

For $k = 2p+1$ $(p \in \mathbb{Z}^+)$, according to the property (5) of $f_k(x)$: $f_{2p+1}(x) = (x+1)h_{2p}(x)$, because $h_{2p}(x)$ is also a self-reciprocal polynomial, we can prove that $h_{2p}(x)$ has at least one zero in the unit disc by the same method.

Consequently, we infer that $f_k(x)$ $(k \ge 2)$ has at least one zero in the unit disc.

3 Lu Qi-Keng Problem on Ω

In this section, we discuss that the Bergman kernel function of Ω has zeros in $\Omega \times \Omega$ if the parameter μ of Ω satisfies certain condition, and obtain the following conclusion.

Theorem 3.1 *Let the Bergman kernel function of* Ω *be*

$$K((\xi, z), (\overline{\eta}, \overline{w})) = F(x)[\mu^{mn}\pi^{mn+1}N_I(z,w)^{m+n+\frac{1}{\mu}}(1-x)^{mn+2}]^{-1}, \tag{3.1}$$

where

$$x = \xi\overline{\eta}N_I(z,w)^{-\frac{1}{\mu}},$$

$$F(x) = \sum_{j=1}^{mn+1}\sum_{k=1}^{j} \frac{(-1)^k(-k)\prod\limits_{l=0}^{m-1}\prod\limits_{q=0}^{n-1}[-k+(n+l-q)\mu]}{(j+1)B(k+1, j-k+1)}(1-x)^{mn+1-j}, \tag{3.2}$$

and $B(\alpha, \beta)$ *denotes the Beta function* (see [26, 28]). *Then for any* m *and* n*, there exists* $\mu_0(m,n) > 0$ *such that* $K((\xi, z), (\overline{\eta}, \overline{w}))$ *has zeros in* $\Omega \times \Omega$ *when* $0 < \mu < \mu_0(m,n)$*.*

Proof In the expression of the Bergman kernel function of Ω, we have

$$[\mu^{mn}\pi^{mn+1}N_I(z,w)^{m+n+\frac{1}{\mu}}(1-x)^{mn+2}]^{-1} \neq 0,$$

so $K((\xi, z), (\overline{\eta}, \overline{w}))$ has zeros in $\Omega \times \Omega$ if and only if $F(x)$ has zeros in the unit disc. We will get the condition that $K((\xi, z), (\overline{\eta}, \overline{w}))$ has zeros by studying the condition that $F(x)$ has zeros in the unit disc.

In (3.2), the part containing μ can be rewritten as following:

$$(-k)\prod_{l=0}^{m-1}\prod_{q=0}^{n-1}[-k+(n+l-q)\mu]=(-k)^{mn+1}+\sum_{l=1}^{mn}c_l(-k)^{mn+1-l}\mu^l, \tag{3.3}$$

where

$$\sum_{l=1}^{mn}c_l=\Big[\prod_{l=0}^{m-1}\prod_{q=0}^{n-1}(u+(n+l-q))\Big]_{u=1}-1. \tag{3.4}$$

By substituting (3.3) into $F(x)$, $F(x)$ can be expressed as below:

$$F(x)=f_{mn}(x)+\mu g_{mn}(x),$$

where

$$f_{mn}(x):=\sum_{j=1}^{mn+1}\sum_{k=0}^{j}\frac{(-1)^k(-k)^{mn+1}}{(j+1)B(k+1,j-k+1)}(1-x)^{mn+1-j}$$

and

$$g_{mn}(x):=\sum_{j=1}^{mn+1}\sum_{k=1}^{j}\frac{(-1)^k\sum\limits_{l=1}^{mn}c_l(-k)^{mn+1-l}\mu^{l-1}}{(j+1)B(k+1,j-k+1)}(1-x)^{mn+1-j}. \tag{3.5}$$

By rearranging the expression of $f_{mn}(x)$, we get

$$f_{mn}(x)=\sum_{l=0}^{mn}\frac{1}{l!}\sum_{j=1}^{mn+1-l}S(mn+1,j)(-1)^{mn+1-l-j}\frac{(mn+1-j)!j!}{(mn+1-j-l)!}x^l,$$

so it is exactly a real coefficient polynomial of degree mn as the one defined in Theorem 2.4.

From Theorem 2.4(7), we have known that $f_{mn}(x)$ has at least one zero in the unit disc. According to the relationship between $F(x)$ and $f_{mn}(x)$, we will prove that for any m and n, there exists $\mu_0(m,n)>0$ such that $F(x)$ has zeros in the unit disc when $0<\mu<\mu_0(m,n)$.

Let x_0 be a zero of $f_{mn}(x)$ in the unit disc, because the zero of a non-constant holomorphic function with a single complex variable is isolated, there exists $0<\rho<1$ such that x_0 is the unique zero of $f_{mn}(x)$ in the closed disc $\overline{D(x_0,\rho)}=\{x\in\mathbb{C}:|x-x_0|\le\rho\}$.

According to (3.5), for $0<\mu<1$, we get

$$\begin{aligned}|g_{mn}(x)|_{|x-x_0|=\rho}&\le\sum_{j=1}^{mn+1}\Big|\sum_{k=1}^{j}\frac{(-1)^k\sum\limits_{l=1}^{mn}c_l(-k)^{mn+1-l}\mu^{l-1}}{(j+1)B(k+1,j-k+1)}(1-x)^{mn+1-j}\Big|_{|x-x_0|=\rho}\\&<\sum_{j=1}^{mn+1}2^{mn+1-j}\sum_{k=1}^{j}\frac{\sum\limits_{l=1}^{mn}c_lj^{mn}}{(j+1)B(k+1,j-k+1)}\\&<\sum_{j=1}^{mn+1}2^{mn+1-j}2^j\sum_{l=1}^{mn}c_lj^{mn}.\end{aligned}$$

It follows from (3.4) that

$$\sum_{l=1}^{mn} c_l < \frac{(n+1)!}{1!}\frac{(n+2)!}{2!}\cdots\frac{(n+m)!}{m!} < [(n+m)!]^m.$$

Therefore,

$$|g_{mn}(x)|_{|x-x_0|=\rho} < 2^{mn+1}[(n+m)!]^m \sum_{j=1}^{mn+1} j^{mn} < [2(mn+1)]^{mn+1}[(m+n)!]^m.$$

Taking $\mu_0(m,n) = \min\left\{1, \frac{\min\limits_{|x-x_0|=\rho}|f_{mn}(x)|}{[2(mn+1)]^{mn+1}[(m+n)!]^m}\right\}$, when $0 < \mu < \mu_0(m,n)$, we have

$$\mu|g_{mn}(x)|_{|x-x_0|=\rho} < \mu_0|g_{mn}(x)|_{|x-x_0|=\rho} < \min_{|x-x_0|=\rho}|f_{mn}(x)| < |f_{mn(x)}|_{|x-x_0|=\rho}.$$

Thereby, $F(x) = f_{mn}(x) + \mu g_{mn}(x)$ has one zero in $D(x_0,\rho)$ from Rouché theorem. Namely, $F(x)$ has at least one zero in the unit disc when μ satisfies the above condition.

Consequently, for any fixed m, n, when $0 < \mu < \mu_0(m,n)$, the Bergman kernel function of Ω has zeros in $\Omega \times \Omega$.

4 The Topological Properties of the Zero Set of the Bergman Kernel Function on Ω

According to Theorem 3.1, there exists $\mu_0(m,n) > 0$ such that $F(x)$ as (3.2) has zeros in the unit disc when $0 < \mu < \mu_0(m,n)$, and the set of these zeros is denoted by

$$\mathcal{A} = \{\lambda_j \in D(0,1) : F(\lambda_j) = 0,\ j = 1,2,\cdots,M, M < mn\}. \tag{4.1}$$

Under the same condition, the Bergman kernel function of Ω has zeros in $\Omega \times \Omega$. Put

$$\Lambda_j := \{P = ((\xi,z),(\eta,w)) \in \Omega\times\Omega : x(P) = \xi\overline{\eta}N_I(z,w)^{-\frac{1}{\mu}} = \lambda_j,\ \lambda_j \in \mathcal{A}\}, \tag{4.2}$$

$j = 1,2,\cdots,M,\ M < mn$, then the zero set of the Bergman kernel function of Ω is the union of these sets.

We are going to discuss the topological properties of the zero set of the Bergman kernel function on Ω, and give the main results and proofs about the connectivity of the zero set.

Theorem 4.1 *If the Bergman kernel function of Ω has zeros, and the zero set is denoted by $\Lambda = \bigcup\limits_{j=1}^{M} \Lambda_j$, where the form of Λ_j is as* (4.2)*, then Λ_j $(j = 1,2,\cdots,M, M < mn)$ is a path-connected subset.*

Proof Without loss of generality, we may prove Λ_1 is a path-connected subset. Set $Q^* = ((|\lambda_1|^{\frac{1}{2}},0),(\overline{\lambda}_1|\lambda_1|^{-\frac{1}{2}},0))$, because $|\lambda_1| < 1$ and $x(Q^*) = \lambda_1$, it is easy to see that Q^* is a fixed point in Λ_1. Let $P = ((\xi,z),(\eta,w))$ be any point in Λ_1, by proving that there exists a continuous curve to connect P and Q^*, we can obtain that any two points can be connected with a continuous curve in Λ_1, i.e., Λ_1 is a path-connected subset.

According to Theorem 2.1, we select $\psi_{z_0,\theta} \in \mathrm{Aut}(\Omega)$, where $z_0 = z$ and $\theta = -\arg\xi$, such that

$$\Psi(P) = (\psi_{z,-\arg\xi}(\xi,z),\psi_{z,-\arg\xi}(\eta,w)) = P^*.$$

Since x is an invariant under the action of holomorphic transformation Ψ of $\Omega\times\Omega$, i.e., $x(P)=x(P^*)=\lambda_1$, it follows $P^*\in\Lambda_1$, and

$$P^*=((|\xi|N_I(z,z)^{-\frac{1}{2\mu}},0),(\overline{\lambda}_1|\xi|^{-1}N_I(z,z)^{\frac{1}{2\mu}},w^*)).$$

Furthermore, putting $Q:=\Psi^{-1}(Q^*)$, we have $Q\in\Lambda_1$ from $x(Q^*)=x(Q)=\lambda_1$, and

$$\begin{aligned}Q&=(\psi^{-1}_{z,-\arg\xi}(|\lambda_1|^{\frac12},0),\psi^{-1}_{z,-\arg\xi}(\overline{\lambda}_1|\lambda_1|^{-\frac12},0))\\&=((|\lambda_1|^{\frac12}N_I(z,z)^{\frac{1}{2\mu}}\mathrm{e}^{\sqrt{-1}\arg\xi},z),(\overline{\lambda}_1|\lambda_1|^{-\frac12}N_I(z,z)^{\frac{1}{2\mu}}\mathrm{e}^{\sqrt{-1}\arg\xi},z)).\end{aligned}$$

Because P and Q are respectively mapped to P^* and Q^* by Ψ, we will show that P and Q can be connected with a continuous curve by proving that P^* and Q^* can be connected with a continuous curve in Λ_1. Then, a continuous curve can be constructed to connect Q and Q^*. So there is a continuous curve to join P and Q^* in Λ_1.

Firstly, there exists

$$\widetilde{P}^*=((|\xi|N_I(z,z)^{-\frac{1}{2\mu}},0),(\overline{\lambda}_1|\xi|^{-1}N_I(z,z)^{\frac{1}{2\mu}},0))$$

corresponding to P^* in Λ_1. We can construct a continuous curve as following to connect P^* and $\widetilde{P}^*$:

$$\varphi_1(s)=\{((|\xi|N_I(z,z)^{-\frac{1}{2\mu}},0),(\overline{\lambda}_1|\xi|^{-1}N_I(z,z)^{\frac{1}{2\mu}},sw^*)),\ s\in[0,1]\}.$$

In fact,

$$\begin{aligned}|\lambda_1|^{2\mu}|\xi|^{-2\mu}N_I(z,z)<\det(I-w^*\overline{w^*}^t)<\det(I-s^2w^*\overline{w^*}^t)\quad(s\in[0,1]),\\x(\varphi_1(s))=\lambda_1,\end{aligned}$$

so $\varphi_1(s)\subset\Lambda_1$, and $\varphi_1(0)=\widetilde{P}^*,\varphi_1(1)=P^*$.

Next, we construct a continuous curve

$$\varphi_2(s)=\{((s,0),(\overline{\lambda}_1s^{-1},0)),\ s\in[|\lambda_1|^{\frac12},|\xi|N_I(z,z)^{-\frac{1}{2\mu}}]\},$$

and set $|\lambda_1|^{\frac12}<|\xi|N_I(z,z)^{-\frac{1}{2\mu}}$ without loss of generality. Obviously, $\varphi_2(s)\subset\Omega\times\Omega$. It follows from $x(\varphi_2(s))=\lambda_1$ that $\varphi_2(s)\subset\Lambda_1$. Moreover, $\varphi_2(|\lambda_1|^{\frac12})=Q^*$ and $\varphi_2(|\xi|N_I(z,z)^{-\frac{1}{2\mu}})=\widetilde{P}^*$. So $\varphi_2(s)$ is a continuous curve to connect Q^* and $\widetilde{P}^*$.

Summarizing the above discussion, φ_1 is a continuous curve connecting P^* and $\widetilde{P}^*$, and φ_2 is a continuous curve connecting Q^* and $\widetilde{P}^*$, therefore there exists a continuous curve to connect P^* and Q^*, which is denoted by φ. Obviously, we obtain that

$$\Psi^{-1}(\varphi)=(\psi^{-1}_{z,-\arg\xi},\psi^{-1}_{z,-\arg\xi})(\varphi)$$

is a continuous curve connecting P and Q, which is denoted by γ_1.

Furthermore, we construct a continuous curve

$$\begin{aligned}\gamma_2(s)=\{((|\lambda_1|^{\frac12}N_I(sz,sz)^{\frac{1}{2\mu}}\mathrm{e}^{s\sqrt{-1}\arg\xi},sz),\\(\overline{\lambda}_1|\lambda_1|^{-\frac12}N_I(sz,sz)^{\frac{1}{2\mu}}\mathrm{e}^{s\sqrt{-1}\arg\xi},sz)),\ s\in[0,1]\}.\end{aligned}$$

We have

$$||\lambda_1|^{\frac12}N_I(sz,sz)^{\frac{1}{2\mu}}\mathrm{e}^{s\sqrt{-1}\arg\xi}|^{2\mu}=|\lambda_1|^{\mu}N_I(sz,sz)<N_I(sz,sz),$$

$$
\begin{aligned}
&|\overline{\lambda}_1|\lambda_1|^{-\frac{1}{2}}N_I(sz,sz)^{\frac{1}{2\mu}}\mathrm{e}^{s\sqrt{-1}\arg\xi}|^{2\mu}=|\overline{\lambda}_1|\lambda_1|^{-\frac{1}{2}}|^{2\mu}N_I(sz,sz)<N_I(sz,sz),\\
&x(\gamma_1(s))=|\lambda_1|^{\frac{1}{2}}N_I(sz,sz)^{\frac{1}{2\mu}}\mathrm{e}^{s\sqrt{-1}\arg\xi}\\
&\qquad\qquad\cdot\lambda_1|\lambda_1|^{-\frac{1}{2}}N_I(sz,sz)^{\frac{1}{2\mu}}\mathrm{e}^{-s\sqrt{-1}\arg\xi}N_I(sz,sz)^{-\frac{1}{\mu}}\\
&\qquad\quad=\lambda_1.
\end{aligned}
$$

Moreover, $\gamma_2(0)=Q^*$, $\gamma_2(1)=Q$. Hence, $\gamma_2(s)\subset\Lambda_1$ and $\gamma_2(s)$ is a continuous curve connecting Q and Q^*.

In conclusion, γ_1 is a continuous curve connecting P and Q, and γ_2 is a continuous curve connecting Q and Q^*, so there exists a continuous curve connecting any point P and the fixed point Q^*. So Λ_1 is a path-connected subset. Using the same method, we can show that every Λ_j $(j=2,\cdots,M,M<mn)$ is a path-connected subset of the zero set of the Bergman kernel function on Ω.

Theorem 4.2 *If the Bergman kernel function of Ω has zeros, let the zero set be $\Lambda=\bigcup\limits_{j=1}^{M}\Lambda_j$, where the form of Λ_j is as (4.2), then for any $P\in\Lambda_l$ and $Q\in\Lambda_k$ $(l\neq k)$, there is not a continuous curve to connect P and Q in Λ.*

Proof We will prove the theorem by negation.

For any $P\in\Lambda_l$ and $Q\in\Lambda_k$ $(l\neq k)$, suppose that there is $\varphi(s)$ $(s\in[0,1])\subset\Lambda$ to connect P and Q, $\varphi(0)=P$, $\varphi(1)=Q$, then $x(\varphi(0))=\lambda_l$, $x(\varphi(1))=\lambda_k$, and $\lambda_l\neq\lambda_k$. Because $x(\varphi(s))$ is uniformly continuous on $[0,1]$, for any $s_0\in[0,1]$ and for $\varepsilon=\frac{1}{2}\min\{|\lambda'-\lambda''|,\forall\lambda',\lambda''\in\mathcal{A}\}$, where $\mathcal{A}$ is defined as (4.1), there exists $\delta>0$ such that

$$|x(\varphi(s))-x(\varphi(s_0))|<\varepsilon,\quad s\in U(s_0,\delta)\cap[0,1].$$

If s_0 takes all the values in $[0,1]$, there is a set of open intervals $\{U(s_0,\delta),s_0\in[0,1]\}$ covering $[0,1]$. Moreover, there are following finite open intervals to cover $[0,1]$ in the set of open intervals:

$$U_\alpha(s_\alpha,\delta_0),\quad \alpha=1,2,\cdots,q.$$

Without loss of generality, let $s_1<s_2<\cdots<s_q$. Every interval $U_\alpha(s_\alpha,\delta_0)$ satisfies

$$|x(\varphi(s))-x(\varphi(s_\alpha))|<\varepsilon\quad\text{for any } s\in U(s_\alpha,\delta)\cap[0,1].$$

Because $0\in U(s_1,\delta_0)$, $x(\varphi(0))=\lambda_l$ and $|x(\varphi(s_1))-x(\varphi(0))|<\varepsilon$, we have $x(\varphi(s_1))=\lambda_l$. Next, as any $s\in U(s_1,\delta)\cap[0,1]$, $|x(\varphi(s))-x(\varphi(s_1))|<\varepsilon$, we acquire $x(\varphi(s))=\lambda_l$ for any $s\in U(s_1,\delta)$. Since any $s\in U(s_1,\delta)\cap U(s_2,\delta)$, $|x(\varphi(s))-x(\varphi(s_2))|<\varepsilon$, it is turned out that $x(\varphi(s_2))=\lambda_l$. Moreover, any $s\in U(s_2,\delta)\cap[0,1]$, $|x(\varphi(s))-x(\varphi(s_2))|<\varepsilon$, we can show that $x(\varphi(s))=\lambda_l$ for any $s\in U(s_2,\delta)$. Proceeding the derivation in a similar manner, it follows that $x(\varphi(1))=\lambda_l$, which is contrary to $x(\varphi(1))=\lambda_k$.

Consequently, the hypothesis fails to hold, i.e., P and Q which belong to different subset Λ_l and Λ_k respectively, can not be joined with a continuous curve in Λ.

According to Theorems 4.1–4.2, if the Bergman kernel function of Ω has zeros, the zero set is composed of M $(M<mn)$ path-connected branches. It is easy to know that the complement set of the zero set in $\Omega\times\Omega$ is connected. Furthermore, we can construct a continuous curve to connect any two points in the non-zero set.

Theorem 4.3 *If the Bergman kernel function of* Ω *has zeros, let the zero set be* $\Lambda = \bigcup_{j=1}^{M} \Lambda_j$, *where the form of* Λ_j *is as* (4.2), *and* $\Omega \times \Omega \setminus \Lambda$ *denote the complement set of* Λ *in* $\Omega \times \Omega$, *then for any* $P, Q \in \Omega \times \Omega \setminus \Lambda$, *there exists a continuous curve to connect the two points.*

Proof Let $O = ((0,0),(0,0))$. Since $x(O) = 0$ and $0 \notin \mathcal{A}$, where $\mathcal{A}$ is defined as (4.1), we have $O \in \Omega \times \Omega \setminus \Lambda$. For any $P = ((\xi, z), (\eta, w)) \in \Omega \times \Omega \setminus \Lambda$, if we show that P and O can be joined with a continuous curve in $\Omega \times \Omega \setminus \Lambda$, the continuous curve can be constructed to connect P and another any point Q.

We are going to prove that there exists a continuous curve to connect P and O according to the character of P in two cases:

(i) Let $P = ((\xi, z), (\eta, w)) \in \Omega \times \Omega \setminus \Lambda$. If $x(P) = 0 \notin \mathcal{A}$, then $\xi = 0$ or $\eta = 0$. Construct a continuous curve

$$\gamma_0(s) = \{((s\xi, sz), (s\eta, sw)),\ s \in [0,1]\}.$$

It is easy to verify that $\gamma_0(s) \subset \Omega \times \Omega$ for $\forall s \in [0,1]$. Moreover, $x(\gamma_0(s)) = 0$, and $\gamma_0(0) = O$, $\gamma_0(1) = P$. Thereby, $\gamma_0(s)$ is a continuous curve connecting P and O in $\Omega \times \Omega \setminus \Lambda$.

(ii) Let $P = ((\xi, z), (\eta, w)) \in \Omega \times \Omega \setminus \Lambda$, if $x(P) \notin \mathcal{A}$ and $x(P) \neq 0$, then $\xi \neq 0$ and $\eta \neq 0$. There is $\widetilde{P} = ((0, z), (\eta, w))$ corresponding to P in $\Omega \times \Omega \setminus \Lambda$. We are going to first prove that P and $\widetilde{P}$ can be joined with a continuous curve in $\Omega \times \Omega \setminus \Lambda$.

Construct a cluster of curves, which are disjoint curves with P and $\widetilde{P}$ as endpoints:

$$\gamma_k(s) = \Big\{\Big(\Big(\frac{sa + \sqrt{-1}s^k b}{a + \sqrt{-1}b}\xi, z\Big), (\eta, w)\Big),\ s \in [0,1]\Big\}, \quad k = 1, 2, \cdots, M, M+1,$$

where $a, b \in \mathbb{R}, a \neq 0$ and $b \neq 0$. Because

$$\Big|\frac{sa + \sqrt{-1}s^k b}{a + \sqrt{-1}b}\xi\Big|^{2\mu} \leq |\xi|^{2\mu} < N_I(z, z),$$

we have $\gamma_k(s) \subset \Omega \times \Omega$, and $\gamma_k(0) = \widetilde{P}$, $\gamma_k(1) = P$. Moreover,

$$x(\gamma_k(s)) = \frac{sa + \sqrt{-1}s^k b}{a + \sqrt{-1}b}\xi\overline{\eta}N_I(z, w)^{-\frac{1}{\mu}} = x(Q)\frac{sa + \sqrt{-1}s^k b}{a + \sqrt{-1}b} \quad (s \in [0,1]),$$

so $x(\gamma_1(s)), x(\gamma_2(s)), \cdots, x(\gamma_{M+1}(s))$ are disjoint continuous curves connecting $x(P)$ and origin in $\mathbb{C}$. Since $\mathcal{A}$ is a finite set containing M elements at most, there exists k_0 such that $x(\gamma_{k_0}(s)) \bigcap \mathcal{A} = \emptyset$. Namely, there exists at least one continuous curve $\gamma_{k_0}(s)$ to join P and $\widetilde{P}$ in $\Omega \times \Omega \setminus \Lambda$.

Next, from (i), $\gamma_0(s) = \{((0, sz), (s\eta, sw)),\ s \in [0,1]\}$ is a continuous curve to join $\widetilde{P}$ and O in $\Omega \times \Omega \setminus \Lambda$. Hence, joining $\gamma_{k_0}(s)$ and $\gamma_0(s)$, we get a continuous curve to connect $P = ((\xi, z), (\eta, w))$ $(\xi \neq 0, \eta \neq 0)$ and O.

In the same way, for another $Q \in \Omega \times \Omega \setminus \Lambda$, we also get a continuous curve to connect Q and O. So by connecting P and Q with O, we construct a continuous curve to connect any two points in $\Omega \times \Omega \setminus \Lambda$.

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