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Hardy-Rellich Type Inequalities Associated with Dunkl Operators*

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Abstract In this paper, the authors obtain the Dunkl analogy of classical L^p Hardy inequality for $p > N + 2\gamma$ with sharp constant $\left(\frac{p-N-2\gamma}{p}\right)^p$, where 2γ is the degree of weight function associated with Dunkl operators, and L^p Hardy inequalities with distant function in some G-invariant domains. Moreover they prove two Hardy-Rellich type inequalities for Dunkl operators.

Keywords Hardy inequalities, Hardy-Rellich inequalities, Best constant, Dunkl operators
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1 Introduction

The classical Hardy inequality

$$\int_{\mathbb{R}^N} |\nabla u|^p \mathrm{d}x \geq \Big|\frac{N-p}{p}\Big|^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} \mathrm{d}x \tag{1.1}$$

holds for $u \in C_0^\infty(\mathbb{R}^{\mathbb{N}})$ when $1 < p < N$ and for $u \in C_0^\infty(\mathbb{R}^{\mathbb{N}}\backslash\{0\})$ when $N < p < \infty$. It has extensive applications in analysis, partial differential equation and physical research. In [11], Hardy firstly proved this inequality in the case of one dimension. Since then, many researchers devoted themselves to it and made great progress, not only in Euclidean spaces, there are counterparts in Riemannian manifolds and Carnot groups, see [2, 5, 7–8, 12–16] and the references therein.

If $\mathbb{R}^N$ is replaced by a bounded convex domain Ω, the following sharp inequality holds for $1 < p < \infty$,

$$\int_{\Omega} |\nabla u|^p \mathrm{d}x \geq \Big(\frac{p-1}{p}\Big)^p \int_{\Omega} \frac{|u|^p}{\delta^p} \mathrm{d}x, \tag{1.2}$$

where $\delta(x) := \mathrm{dist}(x, \partial\Omega)$ (see [18]). Mazya proved in [19] that (1.2) can be characterized in terms of p-capacity. When Ω is non-convex, the problem is more complicated. For domains such that $-\Delta\delta$ is nonnegative in the distributional sense, some results were obtained by Barbatis, Filippas and Tertikas in [4]. It is equivalent between non-negativity of $-\Delta\delta$ in the distributional sense and the mean-convexity of the domain when the boundary is smooth enough, see [9–10,

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17, 20]. Ancona [1] obtained some results in planar simply connected domains by using Koebe one-quarter theorem; some other Hardy inequalities for special domains see [8].

The Rellich inequality

$$\int_{\mathbb{R}^N} |\Delta u|^2 \mathrm{d}x \geq \frac{N^2(N-4)^2}{16} \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^4} \mathrm{d}x \tag{1.3}$$

is a generalization of Hardy inequality, which holds for $u \in C_0^\infty(\mathbb{R}^N)$ and the constant $\frac{N^2(N-4)^2}{16}$ is sharp when $N \geq 5$. In [22], Tertikas and Zographopoulos obtained a Hardy-Rellich type inequality which reads as

$$\int_{\mathbb{R}^N} |\Delta u|^2 \mathrm{d}x \geq \frac{N^2}{4} \int_{\mathbb{R}^N} \frac{|\nabla u|^2}{|x|^2} \mathrm{d}x, \tag{1.4}$$

where $N \geq 5$, the constant $\frac{N^2}{4}$ is also sharp.

In the setting of Dunkl operators, the author in [23] proved a sharp analogical inequality of (1.1) for Dunkl operators

$$\int_{\mathbb{R}^N} |\nabla_k u|^2 \mathrm{d}\mu_k \geq \Big(\frac{N+2\gamma-2}{2}\Big)^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} \mathrm{d}\mu_k, \tag{1.5}$$

and the following inequality for $1 < p < \frac{N+2\gamma}{1+2\gamma}$,

$$\int_{\mathbb{R}^N} |\nabla_k u|^p \mathrm{d}\mu_k \geq \Big(\frac{N+2\gamma-2p\gamma-p}{p}\Big)^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} \mathrm{d}\mu_k, \tag{1.6}$$

however the sharpness of the constant for $p \neq 2$ in (1.6) is not known. They also obtained an analogical inequality of (1.3) for Dunkl Laplacian

$$\int_{\mathbb{R}^N} |\Delta_k u|^2 \mathrm{d}\mu_k \geq \frac{(N+2\gamma)^2(N+2\gamma-4)^2}{16} \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^4} \mathrm{d}\mu_k, \tag{1.7}$$

where the constant $\frac{(N+2\gamma)^2(N+2\gamma-4)^2}{16}$ is sharp.

The plan of this paper is as follows: We introduce some definitions and basic facts of Dunkl operators in the second section. Then, in section three, we obtain some L^p Hardy inequalities associated with distant function for Dunkl operators by choosing specific vector fields, especially a Hardy inequality on a non-convex domain $\Omega = B(0,R)^c$, which leads to classical L^p Hardy inequality associated with Dunkl operators for $p > N + 2\gamma$. In the last section, we obtain two Hardy-Rellich type inequalities for Dunkl operators by the method of spherical h-harmonic decomposition.

2 Preliminaries

Dunkl theory is a generalization of Fourier analysis and special function theory about root system. It generalizes Bessel functions on flat symmetric spaces, also Macdonald polynomials on affine buildings. Moreover, Dunkl theory has extensive applications in algebra (double affine Hecke algebras), probability theory (Feller processes with jump) and mathematical physics (quantum many body problems, Calogero-Moser-Sutherland molds).

In this section, we will introduce some fundamental concepts and notations of Dunkl operators, see also [6, 21] for more details.

If a finite set $R \subset \mathbb{R}^N \setminus \{0\}$ such that $R \cap \alpha\mathbb{R} = \{-\alpha, \alpha\}$ and $\sigma_\alpha(R) = R$ for all $\alpha \in R$, then we call R a root system. Denote σ_α as the reflection on the hyperplane which is orthogonal to the root α, written as

$$\sigma_\alpha x = x - 2\frac{\langle \alpha, x\rangle}{\langle \alpha, \alpha\rangle}\alpha.$$

We write G as the group generated by all the reflections σ_α for $\alpha \in R$, it is a finite group. Let $k : R \to [0, \infty)$ be a G-invariant function, i.e., $k(\alpha) = k(v\alpha)$ for all $v \in G$ and all $\alpha \in R$, simply written as $k_\alpha = k(\alpha)$. R can be denoted as $R = R_+ \cup (-R_+)$, when $\alpha \in R_+$, then $-\alpha \in -R_+$, and R_+ is called a positive subsystem. We fix a positive subsystem R_+ in a root system R. Without loss of generality, we assume that $|\alpha|^2 = 2$ for all $\alpha \in R$.

Definition 2.1 *For $i = 1, \cdots, N$, the Dunkl operators on $C^1(\mathbb{R}^N)$ is defined as follows*

$$T_i u(x) = \partial_i u(x) + \sum_{\alpha\in R_+} k_\alpha \alpha_i \frac{u(x) - u(\sigma_\alpha x)}{\langle \alpha, x\rangle}.$$

By this definition, we can see that even if the decomposition of R is not unique, the different choices of positive subsystems make no difference in the definitions due to the G-invariance of k. Denote by $\nabla_k = (T_1, \cdots, T_N)$ the Dunkl gradient, $\Delta_k = \sum_{i=1}^{N} T_i^2$ the Dunkl-Laplacian. Especially, for $k = 0$ we have $\nabla_0 = \nabla$ and $\Delta_0 = \Delta$. The Dunkl-Laplacian can be written in terms of the usual gradient and Laplacian as follows:

$$\Delta_k u(x) = \Delta u(x) + 2\sum_{\alpha\in R_+} k_\alpha \Big[\frac{\langle \nabla u(x), \alpha\rangle}{\langle \alpha, x\rangle} - \frac{u(x) - u(\sigma_\alpha x)}{\langle \alpha, x\rangle^2}\Big].$$

The weight function naturally associated to Dunkl operators is

$$\omega_k(x) = \prod_{\alpha\in R_+} |\langle \alpha, x\rangle|^{2k_\alpha}.$$

This is a homogeneous function of degree 2γ, where

$$\gamma := \sum_{\alpha\in R_+} k_\alpha.$$

We will work in spaces $L^p(\mu_k)$, where $\mathrm{d}\mu_k = \omega_k \mathrm{d}x$ is the weighted measure. About this weighted measure we have the formula of integration by parts

$$\int_{\mathbb{R}^N} T_i(u) v \mathrm{d}\mu_k = -\int_{\mathbb{R}^N} u T_i(v) \mathrm{d}\mu_k.$$

If at least one of the functions u, v is G-invariant, the following Leibniz rule

$$T_i(uv) = uT_i v + vT_i u$$

holds. In general, we have

$$T_i(uv)(x) = v(x)T_i u(x) + u(x)T_i v(x) - \sum_{\alpha\in R_+} k_\alpha \alpha_i \frac{(u(x) - u(\sigma_\alpha x))(v(x) - v(\sigma_\alpha x))}{\langle \alpha, x\rangle}.$$

3 L^p Hardy Inequalities

In this section we prove a general Hardy inequality with remainder terms for Dunkl operators in G-invariant domains, then we get the Dunkl analogy of Hardy inequality (1.1) for $p > N+2\gamma$.

Firstly, we review some basic facts of distant function.

Lemma 3.1 (see [3]) *Let $\Omega \subset \mathbb{R}^N$ be an open set such that $\partial\Omega \neq \emptyset$. The following propositions hold true.*

(i) *The function $\delta(x)$ is differentiable at a point $x \in \Omega$ if and only if there exists a unique point $N(x) = y \in \partial\Omega$ such that $\delta(x) = |x-y|$. If $\delta(x)$ is differentiable, then $\nabla\delta(x) = \frac{x-y}{|x-y|}$ and $|\nabla\delta| = 1$.*

(ii) *Denote $\Sigma(\Omega)$ as the set of points where $\delta(x)$ is not differentiable. If Ω is bounded with $C^{2,1}$ boundary, then $|\Sigma(\Omega)| = 0$.*

(iii) *Assume that Ω is convex. Then $\Delta\delta \leq 0$ in the sense of distributions, i.e.,*

$$\int_\Omega \delta(x)\Delta\varphi(x)\mathrm{d}x \leq 0, \quad \varphi \in C_0^\infty(\Omega),\ \varphi \geq 0.$$

For $\Omega \subset \mathbb{R}^N$, if for all $x \in \Omega$, $g \in G$, we have $gx \in \Omega$, then Ω is called a G-invariant domain.

Lemma 3.2 *If $\Omega \subset \mathbb{R}^N$ is G-invariant, $g \in G$, $x \in \Omega \setminus \Sigma(\Omega)$, then*

$$(\nabla\delta \circ g)(x) = (g \circ \nabla\delta)(x). \tag{3.1}$$

Proof From the proof of Theorem 5.2 in [23], the function $\delta(x)$ is G-invariant. For any $x \in \Omega$, we have $y = N(x) \in \partial\Omega$, $\delta(x) = |x-y|$ and

$$\delta(gx) = \delta(x) = |x-y| = |g(x-y)| = |gx - gy|.$$

Due to the uniqueness of $N(x)$, we get that $N(gx) = gy$. Therefore

$$\nabla\delta(gx) = \frac{gx-gy}{|gx-gy|} = \frac{g(x-y)}{|x-y|} = g(\nabla\delta(x)).$$

Remark 3.1 If $F = h_1 x + h_2 \nabla\delta$, where h_1, h_2 are G-invariant functions, then by Lemma 3.2, we have that $\langle\alpha, F(\sigma_\alpha x)\rangle = -\langle\alpha, F\rangle$.

Theorem 3.1 *Let $\Omega \subset \mathbb{R}^N$ be a G-invariant domain with $|\Sigma(\Omega)| = 0$. Then for all $u \in C_0^\infty(\Omega)$, we have the inequality*

$$\begin{aligned}\int_\Omega |\nabla_k u|^2 \mathrm{d}\mu_k \geq{}& \Big(\frac{p-1}{p}\Big)^p \int_\Omega \frac{|u|^p}{\delta^p}\mathrm{d}\mu_k \\ &+ \Big(\frac{p-1}{p}\Big)^{p-1}\int_\Omega \Big[-\Delta\delta + \Big(\frac{p}{2}-1\Big)\langle\rho,\nabla\delta\rangle - \frac{p}{2}|\langle\rho,\nabla\delta\rangle|\Big]\frac{|u|^p}{\delta^{p-1}}\mathrm{d}\mu_k,\end{aligned} \tag{3.2}$$

where $\rho := 2\sum\limits_{\alpha\in R_+} k_\alpha \frac{\alpha}{\langle\alpha,x\rangle}$.

Proof If F satisfies that $\langle\alpha, F(\sigma_\alpha x)\rangle = -\langle\alpha, F\rangle$, then

$$\int_\Omega (\nabla_k \cdot F)|u|^p \mathrm{d}\mu_k = -\int_\Omega F \cdot \nabla_k(|u|^p)\mathrm{d}\mu_k$$

$$= -\int_\Omega F\cdot\nabla(|u|^p)\mathrm{d}\mu_k - \int_\Omega \sum_{\alpha\in R_+} k_\alpha \frac{|u|^p - |u(\sigma_\alpha x)|^p}{\langle\alpha, x\rangle}\langle\alpha, F\rangle\mathrm{d}\mu_k. \tag{3.3}$$

Let $x = \sigma_\alpha y$. Then

$$\int_\Omega |u|^p \frac{\langle\alpha, F\rangle}{\langle\alpha, x\rangle}\mathrm{d}\mu_k = \int_\Omega |u(\sigma_\alpha y)|^p \frac{\langle\alpha, F(\sigma_\alpha y)\rangle}{\langle\alpha, \sigma_\alpha y\rangle}\mathrm{d}\mu_k(\sigma_\alpha y). \tag{3.4}$$

Because of $\langle\alpha, F(\sigma_\alpha y)\rangle = -\langle\alpha, F(y)\rangle$, $\langle\alpha, \sigma_\alpha y\rangle = -\langle\alpha, y\rangle$, $\mathrm{d}\mu_k(\sigma_\alpha y) = \omega_k(\sigma_\alpha y)\mathrm{d}(\sigma_\alpha y) = \omega_k(y)|J|\mathrm{d}y$, where

$$J = \Big|\frac{\partial(\sigma_\alpha y)}{\partial(y)}\Big| = \begin{vmatrix} 1-\alpha_1^2 & -\alpha_1\alpha_2 & -\alpha_1\alpha_3 & \cdots & -\alpha_1\alpha_n \\ -\alpha_2\alpha_1 & 1-\alpha_2^2 & -\alpha_2\alpha_3 & \cdots & -\alpha_2\alpha_n \\ -\alpha_3\alpha_1 & -\alpha_3\alpha_2 & 1-\alpha_3^2 & \cdots & -\alpha_3\alpha_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\alpha_n\alpha_1 & -\alpha_n\alpha_2 & -\alpha_n\alpha_3 & \cdots & 1-\alpha_n^2 \end{vmatrix}.$$

Straightforward calculation shows that $J = -1$.

Thus $\mathrm{d}\mu_k(\sigma_\alpha y) = \mathrm{d}\mu_k(y)$, and we have

$$\int_\Omega |u|^p \frac{\langle\alpha, F\rangle}{\langle\alpha, x\rangle}\mathrm{d}\mu_k = \int_\Omega |u(\sigma_\alpha y)|^p \frac{\langle\alpha, F(y)\rangle}{\langle\alpha, y\rangle}\mathrm{d}\mu_k(y). \tag{3.5}$$

Putting (3.5) into (3.3), we get

$$\begin{aligned}\int_\Omega (\nabla_k\cdot F)|u|^p\mathrm{d}\mu_k &= -\int_\Omega p|u|^{p-2}u(F\cdot\nabla u)\mathrm{d}\mu_k \\ &= -\int_\Omega p|u|^{p-2}u\Big((F\cdot\nabla_k u) - \sum_{\alpha\in R_+} k_\alpha \frac{u(x)-u(\sigma_\alpha x)}{\langle\alpha, x\rangle}\langle\alpha, F\rangle\Big)\mathrm{d}\mu_k \\ &= -\int_\Omega p|u|^{p-2}u(F\cdot\nabla_k u)\mathrm{d}\mu_k + p\sum_{\alpha\in R_+} k_\alpha \int_\Omega \frac{\langle\alpha, F\rangle}{\langle\alpha, x\rangle}|u|^p\mathrm{d}\mu_k \\ &\quad - p\sum_{\alpha\in R_+} k_\alpha \int_\Omega \frac{\langle\alpha, F\rangle}{\langle\alpha, x\rangle}|u|^{p-2}u\cdot u(\sigma_\alpha x)\mathrm{d}\mu_k \\ &\le p\Big(\frac{p-1}{p}\epsilon^{-\frac{p}{p-1}}\int_\Omega |F|^{\frac{p}{p-1}}|u|^p\mathrm{d}\mu_k + \frac{\epsilon^p}{p}\int_\Omega |\nabla_k u|^p\mathrm{d}\mu_k\Big) \\ &\quad + \frac{1}{2}p\int_\Omega \langle\rho, F\rangle|u|^p\mathrm{d}\mu_k - \frac{1}{2}p\int_\Omega \langle\rho, F\rangle|u|^{p-2}u\cdot u(\sigma_\alpha x)\mathrm{d}\mu_k, \end{aligned} \tag{3.6}$$

we used Hölder inequality and Young inequality in the last inequality above. Then,

$$\begin{aligned}\int_\Omega \epsilon^p|\nabla_k u|^p\mathrm{d}\mu_k \ge &\int_\Omega \Big(\nabla_k\cdot F - (p-1)\epsilon^{-\frac{p}{p-1}}|F|^{\frac{p}{p-1}} - \frac{1}{2}p\langle\rho, F\rangle\Big)|u|^p\mathrm{d}\mu_k \\ &+ \frac{1}{2}p\int_\Omega \langle\rho, F\rangle|u|^{p-2}u\cdot u(\sigma_\alpha x)\mathrm{d}\mu_k. \end{aligned} \tag{3.7}$$

Let $F = -\frac{\nabla\delta}{\delta^{p-1}}$. Since δ is G-invariant, $\nabla_k\delta = \nabla\delta$, thus $\nabla_k\cdot F = -\frac{\Delta_k\delta}{\delta^{p-1}} + (p-1)\frac{|\nabla\delta|^2}{\delta^p}$. By (3.7), we have

$$\int_\Omega |\nabla_k u|^p\mathrm{d}\mu_k \ge \Big(\frac{p-1}{\epsilon^p} - \frac{p-1}{\epsilon^{p+\frac{p}{p-1}}}\Big)\int_\Omega \frac{|u|^p}{\delta^p}\mathrm{d}\mu_k$$

$$
\begin{aligned}
&+\frac{1}{\epsilon^p}\int_\Omega\Big(-\Delta_k\delta+\frac{p}{2}\langle\rho,\nabla\delta\rangle\Big)\frac{|u|^p}{\delta^{p-1}}\mathrm{d}\mu_k\\
&-\frac{p}{2\epsilon^p}\int_\Omega\langle\rho,\nabla\delta\rangle\frac{|u|^{p-2}u\cdot u(\sigma_\alpha x)}{\delta^{p-1}}\mathrm{d}\mu_k\\
\geq&\Big(\frac{p-1}{\epsilon^p}-\frac{p-1}{\epsilon^{p+\frac{p}{p-1}}}\Big)\int_\Omega\frac{|u|^p}{\delta^p}\mathrm{d}\mu_k\\
&+\frac{1}{\epsilon^p}\int_\Omega\Big(-\Delta_k\delta+\frac{p}{2}\langle\rho,\nabla\delta\rangle-\frac{p}{2}|\langle\rho,\nabla\delta\rangle|\Big)\frac{|u|^p}{\delta^{p-1}}\mathrm{d}\mu_k. \qquad (3.8)
\end{aligned}
$$

The last inequality above is obtained by using Hölder inequality

$$
\begin{aligned}
\int_\Omega\frac{\langle\rho,\nabla\delta\rangle}{\delta^{p-1}}|u|^{p-2}u\cdot u(\sigma_\alpha x)\mathrm{d}\mu_k&\leq\Big(\int_\Omega\frac{|\langle\rho,\nabla\delta\rangle|}{\delta^{p-1}}|u|^p\mathrm{d}\mu_k\Big)^{\frac{p-1}{p}}\Big(\int_\Omega\frac{|\langle\rho,\nabla\delta\rangle|}{\delta^{p-1}}|u(\sigma_\alpha x)|^p\mathrm{d}\mu_k\Big)^{\frac{1}{p}}\\
&=\int_\Omega\frac{|\langle\rho,\nabla\delta\rangle|}{\delta^{p-1}}|u|^p\mathrm{d}\mu_k.
\end{aligned}
$$

The $\frac{p-1}{\epsilon^p}-\frac{p-1}{\epsilon^{p+\frac{p}{p-1}}}$ takes the maximum value $(\frac{p-1}{p})^p$ when $\epsilon=(\frac{p}{p-1})^{\frac{p-1}{p}}$. Also,

$$
-\Delta_k\delta+\frac{p}{2}\langle\rho,\nabla\delta\rangle=-\Delta\delta+\Big(\frac{p}{2}-1\Big)\langle\rho,\nabla\delta\rangle,
$$

we thus complete the proof of Theorem 3.1.

Remark 3.2 If the root system $\widetilde{R}$ satisfies $\mathrm{span}(\widetilde{R})\subset\mathbb{R}^{N-1}$. Then the following inequality holds for any $u\in C_0^\infty(\mathbb{R}^{N-1}\times\mathbb{R}_+)$,

$$
\int_{\mathbb{R}^{N-1}\times\mathbb{R}_+}|\nabla_k u|^p\mathrm{d}\mu_k\geq\Big(\frac{p-1}{p}\Big)^p\int_{\mathbb{R}^{N-1}\times\mathbb{R}_+}\frac{|u|^p}{x_n^p}\mathrm{d}\mu_k.
$$

Let S_N denote the symmetric group in N elements. A root system of S_N is given by $R=\{\pm(e_i-e_j),1\leq i<j\leq N\}$ and

$$
(\mathrm{span}(R))^\perp=e_1+\cdots+e_N=:\eta,
$$

see [3] for more details. Let the domain $\Omega=\mathrm{span}(R)\times\eta_+$, where η_+ is the positive direction of the straight line coinciding with η. Then Ω is G-invariant, $\delta(x)=\mathrm{dist}(x,\mathrm{span}R)$ and

$$
\nabla\delta=\frac{e_1+\cdots+e_N}{|e_1+\cdots+e_N|}=\frac{\eta}{\sqrt{N}}.
$$

Fix $R_+=\{e_i-e_j,1\leq i<j\leq N\}$, then $-\Delta\delta=0$ and $\langle\rho,\nabla\delta\rangle=0$, by Theorem 3.1, we have the following corollary.

Corollary 3.1 *For $R=\{\pm(e_i-e_j),1\leq i<j\leq N\}$, $u\in C_0^\infty(\mathrm{span}(R)\times\eta_+)$, the following inequality holds*

$$
\int_{\mathrm{span}(R)\times\eta_+}\Big|\nabla u+k\sum_{1\leq i<j\leq N}\frac{u(x)-u(\widetilde{x}_{ij})}{x_i-x_j}(e_i-e_j)\Big|^p\mathrm{d}\mu_k\geq\Big(\frac{p-1}{p}\Big)^p\int_{\mathrm{span}(R)\times\eta_+}\frac{|u|^p}{\delta^p}\mathrm{d}\mu_k,
$$

where $k=k_\alpha=k_\beta,\forall\alpha,\beta\in R$, $\widetilde{x}_{ij}=(x_1,\cdots,x_{i-1},x_j,x_{i+1},\cdots,x_{j-1},x_i,x_{j+1},\cdots,x_N)$.

Proof It is easy to prove $v\circ\sigma_\alpha\circ v^{-1}=\sigma_{v\alpha}$ for all $v\in G$, as there is one conjugate class in R, so $k_\alpha=k_\beta$ for all $\alpha,\beta\in R$, see also [6]. Straightforward computation shows $\sigma_{e_i-e_j}(x)=\widetilde{x}_{ij}$.

By (3.8) in the proof of Theorem 3.1, it is easy to see that the following theorem holds.

Theorem 3.2 *If $\Omega \subset \mathbb{R}^N$ satisfies $|\Sigma(\Omega)| = 0$, $\langle \rho, \nabla\delta\rangle \geq 0$. The following inequality holds for all $u \in C_0^\infty(\Omega)$,*

$$\int_\Omega |\nabla_k u|^p \mathrm{d}\mu_k \geq (p-1)(\epsilon^{-p} - \epsilon^{-\frac{p^2}{p-1}}) \int_\Omega \frac{|u|^p}{\delta^p} \mathrm{d}\mu_k - \epsilon^{-p} \int_\Omega \Delta_k \delta \frac{|u|^p}{\delta^{p-1}} \mathrm{d}\mu_k, \tag{3.9}$$

where ϵ is a positive constant.

Remark 3.3 If a domain Ω satisfies that $|\Sigma(\Omega)| = 0$, $\langle \rho, \nabla\delta\rangle \geq 0$ and $\delta\Delta_k\delta \leq \theta < p-1$, where θ is a positive constant, i.e., then there is a positive constant $C = C(\theta, p)$ such that

$$\int_\Omega |\nabla_k u|^p \mathrm{d}\mu_k \geq C \int_\Omega \frac{|u|^p}{\delta^p} \mathrm{d}\mu_k.$$

Corollary 3.2 *Suppose that $\Omega = B(0,r)^c$, $p > N + 2\gamma$, the following inequality holds for all $u \in C_0^\infty(\Omega)$,*

$$\int_\Omega |\nabla_k u|^p \mathrm{d}\mu_k \geq \Big(\frac{p-N-2\gamma}{p}\Big)^p \int_\Omega \frac{|u|^p}{\delta^p} \mathrm{d}\mu_k. \tag{3.10}$$

Proof When $\Omega = B(0,r)^c$, then $|\Sigma(\Omega)| = 0$, $\delta = |x| - r$, $\nabla\delta = \frac{x}{|x|}$, $\Delta_k\delta = \frac{N+2\gamma-1}{|x|} < \frac{p-1}{\delta}$, and

$$\begin{aligned}(p-1)(\epsilon^{-p} - \epsilon^{-\frac{p^2}{p-1}})\frac{1}{\delta} - \epsilon^{-p}\Delta_k\delta &= (p-1)(\epsilon^{-p} - \epsilon^{-\frac{p^2}{p-1}})\frac{1}{|x|-r} - \epsilon^{-p}\frac{N+2\gamma-1}{|x|} \\ &\geq [(p-N-2\gamma)\epsilon^{-p} - (p-1)\epsilon^{-\frac{p^2}{p-1}}]\frac{1}{|x|-r},\end{aligned}$$

we note $(p-N-2\gamma)\epsilon^{-p} - (p-1)\epsilon^{-\frac{p^2}{p-1}}$ takes the maximum value $\big(\frac{p-N-2\gamma}{p}\big)^p$ when $\epsilon = \big(\frac{p}{p-N-2\gamma}\big)^{\frac{p-1}{p}}$. Note that $\langle \rho, \nabla\delta\rangle = \frac{2\gamma}{|x|} \geq 0$, we complete the proof by Theorem 3.2.

Let r tend to zero, the following sharp inequality follows from Corollary 3.2.

Corollary 3.3 *Suppose that $p > N + 2\gamma$. The following inequality holds for all $u \in C_0^\infty(\mathbb{R}^N \backslash \{0\})$,*

$$\int_{\mathbb{R}^N} |\nabla_k u|^p \mathrm{d}\mu_k \geq \Big(\frac{p-N-2\gamma}{p}\Big)^p \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} \mathrm{d}\mu_k. \tag{3.11}$$

Proof There only remains to prove the optimality of the constant $\big(\frac{p-N-2\gamma}{p}\big)^p$. For any $\epsilon > 0$ we choose

$$u_\epsilon = \begin{cases} r, & r \leq 1, \\ r^{\frac{p-N-2\gamma-\epsilon}{p}}, & r > 1. \end{cases}$$

We can write $\mathrm{d}\mu_k = r^{N+2\gamma-1}\omega_k(\xi)\mathrm{d}r\mathrm{d}\nu(\xi)$, where ν is the surface measure on the sphere $\mathbb{S}^N$. Thus by directly computing, we have

$$\lim_{\epsilon\to 0} \frac{\int_{\mathbb{R}^N} |\nabla_k u_\epsilon|^p \mathrm{d}\mu_k}{\int_{\mathbb{R}^N} \frac{|u_\epsilon|^p}{|x|^p} \mathrm{d}\mu_k} = \Big(\frac{p-N-2\gamma}{p}\Big)^p.$$

4 Hardy-Rellich Type Inequality

Spherical h-harmonics We will introduce some concepts and fundermental facts for spherical h-harmonic theory, see [6] for more details. If a homogeneous polynomial p of degree n that satisfies

$$\Delta_k p = 0,$$

then we call it an h-harmonic polynomial of degree n. Spherical h-harmonics (or just h-harmonics) of degree n are defined as the restrictions of h-harmonic polynomials of degree n to the unit sphere $\mathbb{S}^{N-1}$. Denote $\mathcal{P}_n$ the space of h-harmonics of degree n. Denote $d(n)$ the dimension of $\mathcal{P}_n$, it is finite and given by following formula:

$$d(n) = \tbinom{n+N-1}{N-1} - \tbinom{n+N-3}{N-1}.$$

Moreover, the space $L^2(\mathbb{S}^{N-1}, \omega_k(\xi)\mathrm{d}\xi)$ can be decomposed as the orthogonal direct sum of the spaces $\mathcal{P}_n$, for $n = 0, 1, 2, \cdots$.

Let Y_i^n, $i = 1, \cdots, d(n)$ be an orthogonal basis of $\mathcal{P}_n$. In spherical polar coordinates $x = r\xi$ for $r \in [0, \infty)$ and $\xi \in \mathbb{S}^{N-1}$, we can write the Dunkl Laplacian as

$$\Delta_k = \frac{\partial^2}{\partial r^2} + \frac{N + 2\gamma - 1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{k,0},$$

where $\Delta_{k,0}$ is an analogue of the classical Laplace-Beltrami operator on the sphere, and it only acts on the ξ variable. Then the spherical h-harmonics Y_i^n are eigenfunctions of $\Delta_{k,0}$, and its eigenvalues are given by

$$\Delta_{k,0} Y_i^n = -n(n + N + 2\gamma - 2) Y_i^n =: \lambda_n Y_i^n.$$

The h-harmonic expansion of a function $u \in L^2(\mu_k)$ can be expressed as

$$u(r\xi) = \sum_{n=0}^{\infty} \sum_{i=1}^{d(n)} u_{n,i}(r) Y_i^n(\xi),$$

where

$$u_{n,i}(r) = \int_{\mathbb{S}^{N-1}} u(r\xi) Y_i^n(\xi) \omega_k(\xi) \mathrm{d}\nu(\xi), \tag{4.1}$$

and ν is the surface measure on the sphere $\mathbb{S}^{N-1}$.

Theorem 4.1 *Let $\overline{N} \neq 2$. Then we have the inequality*

$$\int_{\mathbb{R}^N} |x|^2 |\Delta_k u|^2 \mathrm{d}\mu_k \geq \frac{(\overline{N} - 2)^2}{4} \int_{\mathbb{R}^N} |\nabla_k u|^2 \mathrm{d}\mu_k, \tag{4.2}$$

where $\overline{N} := N + 2\gamma$, and the constant $\frac{(\overline{N}-2)^2}{4}$ is sharp.

Proof Our goal is to find best constant C satisfying

$$\int_{\mathbb{R}^N} |x|^2 |\Delta_k u|^2 \mathrm{d}\mu_k - C \int_{\mathbb{R}^N} |\nabla_k u|^2 \mathrm{d}\mu_k \geq 0.$$

Using spherical decomposition:

$$\int_{\mathbb{R}^N} |x|^2|\Delta_k u|^2 \mathrm{d}\mu_k = \sum_{n=0}^{\infty}\sum_{i=1}^{d(n)} \int_0^{+\infty} \Big(u''_{n,i} + \frac{\overline{N}-1}{r}u'_{n,i} + \frac{\lambda_n}{r^2}u_{n,i}\Big)^2 r^{\overline{N}+1}\mathrm{d}r,$$

$$\int_{\mathbb{R}^N} |\nabla_k u|^2 \mathrm{d}\mu_k = -\sum_{n=0}^{\infty}\sum_{i=1}^{d(n)} \int_0^{+\infty} \Big(u''_{n,i} + \frac{\overline{N}-1}{r}u'_{n,i} + \frac{\lambda_n}{r^2}u_{n,i}\Big)u_{n,i} r^{\overline{N}-1}\mathrm{d}r.$$

By integration by parts, we have

$$\begin{aligned}&\int_{\mathbb{R}^N} |x|^2|\Delta_k u|^2 \mathrm{d}\mu_k - C\int_{\mathbb{R}^N} |\nabla_k u|^2 \mathrm{d}\mu_k \\ =&\sum_{n=0}^{\infty}\sum_{i=1}^{d(n)} \int_0^{+\infty} (|u''_{n,i}|^2 r^{\overline{N}+1} - [(\overline{N}-1)+2\lambda_n + C]|u'_{n,i}|^2 r^{\overline{N}-1} + (C\lambda_n + \lambda_n^2)u_{n,i}^2 r^{\overline{N}-3})\mathrm{d}r.\end{aligned}$$

Let

$$\begin{aligned}I_{n,i} = &\int_0^{+\infty} |u''_{n,i}|^2 r^{\overline{N}+1}\mathrm{d}r - [(\overline{N}-1)+2\lambda_n + C]\int_0^{+\infty} |u'_{n,i}|^2 r^{\overline{N}-1}\mathrm{d}r \\ &+ (C\lambda_n + \lambda_n^2)\int_0^{+\infty} u_{n,i}^2 r^{\overline{N}-3}\mathrm{d}r.\end{aligned}$$

By using the following two weighted Hardy inequalities,

$$\int_0^{+\infty} |u'|^2 r^{\overline{N}+1}\mathrm{d}r \geq \frac{\overline{N}^2}{4}\int_0^{+\infty} u^2 r^{\overline{N}-1}\mathrm{d}r, \tag{4.3}$$

$$\int_0^{+\infty} |u'|^2 r^{\overline{N}-1}\mathrm{d}r \geq \frac{(\overline{N}-2)^2}{4}\int_0^{+\infty} u^2 r^{\overline{N}-3}\mathrm{d}r, \tag{4.4}$$

we get

$$\begin{aligned}I_{n,i} &\geq \Big(\frac{\overline{N}^2}{4} - (\overline{N}-1) - 2\lambda_n - C\Big)\int_0^{+\infty} |u'_{n,i}|^2 r^{\overline{N}-1}\mathrm{d}r + (\lambda_n^2 + C\lambda_n)\int_0^{+\infty} u_{n,i}^2 r^{\overline{N}-3}\mathrm{d}r \\ &= \Big(\frac{(\overline{N}-2)^2}{4} - C - 2\lambda_n\Big)\int_0^{+\infty} |u'_{n,i}|^2 r^{\overline{N}-1}\mathrm{d}r + \lambda_n(\lambda_n + C)\int_0^{+\infty} u_{n,i}^2 r^{\overline{N}-3}\mathrm{d}r.\end{aligned}$$

Let $C \leq \frac{(\overline{N}-2)^2}{4} - 2\lambda_n$. Then we have

$$\begin{aligned}I_{n,i} &\geq \Big[\Big(\frac{(\overline{N}-2)^2}{4} - C - 2\lambda_n\Big)\frac{(\overline{N}-2)^2}{4} + \lambda_n(\lambda_n + C)\Big]\int_0^{+\infty} u_{n,i}^2 r^{\overline{N}-3}\mathrm{d}r \\ &= \Big[\Big(\frac{(\overline{N}-2)^2}{4} - C\Big)\frac{(\overline{N}-2)^2}{4} + \lambda_n\Big(\lambda_n + C - \frac{(\overline{N}-2)^2}{2}\Big)\Big]\int_0^{+\infty} u_{n,i}^2 r^{\overline{N}-3}\mathrm{d}r \geq 0.\end{aligned}$$

Because $C \leq \frac{(\overline{N}-2)^2}{4} - 2\lambda_n$ and $C_{max} = \min\limits_n \big\{\frac{(\overline{N}-2)^2}{4} - 2\lambda_n\big\} = \frac{(\overline{N}-2)^2}{4}$, we derive

$$\Big(\frac{(\overline{N}-2)^2}{4} - C\Big)\frac{(\overline{N}-2)^2}{4} + \lambda_n\Big(\lambda_n + C - \frac{(\overline{N}-2)^2}{2}\Big) \geq 0.$$

Thus (4.1) holds. Finally, we show the optimality of $\frac{(\overline{N}-2)^2}{4}$. For any $\epsilon > 0$,

$$u_\epsilon = \begin{cases} 1, & r \leq 1, \\ r^{-\frac{\overline{N}-2+\epsilon}{2}}, & r > 1. \end{cases}$$

Straightforward calculation shows

$$\lim_{\epsilon\to 0}\frac{\int_{\mathbb{R}^N}|x|^2|\Delta_k u_\epsilon|^2\mathrm{d}\mu_k}{\int_{\mathbb{R}^N}|\nabla_k u_\epsilon|^2\mathrm{d}\mu_k}=\frac{(\overline{N}-2)^2}{4}.$$

Theorem 4.2 *Assume $N\geq 5+2\gamma$. Then, for any $u\in C_0^\infty(\mathbb{R}^N)$, we have the inequality*

$$\int_{\mathbb{R}^N}|\Delta_k u|^2\mathrm{d}\mu_k\geq\frac{\overline{N}^2}{4}\int_{\mathbb{R}^N}\frac{|\nabla_k u|^2}{|x|^2}\mathrm{d}\mu_k, \tag{4.5}$$

where the constant $\frac{\overline{N}^2}{4}$ is sharp.

Proof By integration by parts,

$$\int_{\mathbb{R}^N}\frac{|\nabla_k u|^2}{|x|^2}\mathrm{d}\mu_k=-\int_{\mathbb{R}^N}\frac{\Delta_k u\cdot u}{|x|^2}\mathrm{d}\mu_k+2\int_{\mathbb{R}^N}u\frac{x\cdot\nabla_k u}{|x|^4}\mathrm{d}\mu_k,$$

where

$$\begin{aligned}\int_{\mathbb{R}^N}u\frac{x\cdot\nabla_k u}{|x|^4}\mathrm{d}\mu_k&=-\int_{\mathbb{R}^N}u\cdot\nabla_k\Big(\frac{xu}{|x|^4}\Big)\mathrm{d}\mu_k\\&=-\int_{\mathbb{R}^N}u\Big(\frac{N-4}{|x|^4}u+\frac{x}{|x|^4}\nabla_k u+\frac{2}{|x|^4}\sum_{\alpha\in R_+}k_\alpha u(\sigma_\alpha x)\Big)\mathrm{d}\mu_k.\end{aligned}\tag{4.6}$$

Then

$$\int_{\mathbb{R}^N}\frac{x\cdot\nabla_k u}{|x|^4}\mathrm{d}\mu_k=-\frac{N-4}{2}\int_{\mathbb{R}^N}\frac{u^2}{|x|^4}\mathrm{d}\mu_k-\sum_{\alpha\in R_+}k_\alpha\int_{\mathbb{R}^N}\frac{u(\sigma_\alpha x)u}{|x|^4}\mathrm{d}\mu_k.$$

Therefore

$$\begin{aligned}\int_{\mathbb{R}^N}\frac{|\nabla_k u|^2}{|x|^2}\mathrm{d}\mu_k&=-\int_{\mathbb{R}^N}\frac{\Delta_k u\cdot u}{|x|^2}\mathrm{d}\mu_k-(N-4)\int_{\mathbb{R}^N}\frac{u^2}{|x|^4}\mathrm{d}\mu_k-2\sum_{\alpha\in R_+}k_\alpha\int_{\mathbb{R}^N}\frac{u(\sigma_\alpha x)}{|x|^4}\mathrm{d}\mu_k\\&=-\int_{\mathbb{R}^N}\frac{\Delta_k u\cdot u}{|x|^2}\mathrm{d}\mu_k-(\overline{N}-4)\int_{\mathbb{R}^N}\frac{u^2}{|x|^4}\mathrm{d}\mu_k\\&\quad+2\sum_{\alpha\in R_+}k_\alpha\int_{\mathbb{R}^N}\frac{(u-u(\sigma_\alpha x))u}{|x|^4}\mathrm{d}\mu_k.\end{aligned}$$

Let

$$u=\sum_{n=0}^{+\infty}\sum_{i=1}^{d(n)}u_{n,i}(r)Y_i^n(\xi),$$

$$u(\sigma_\alpha x)=\sum_{n=0}^{+\infty}\sum_{i=1}^{d(n)}\widetilde{u}_{n,i}(r)Y_i^n(\xi),$$

where

$$u_{0,1}(r)=\frac{1}{\omega_d^k}\int_{\mathbb{S}^{N-1}}u(r\xi)\omega_k(\xi)\mathrm{d}\nu(\xi),$$

$$\widetilde{u}_{0,1}(r)=\frac{1}{\omega_d^k}\int_{\mathbb{S}^{N-1}}u(r\cdot\sigma_\alpha(\xi))\omega_k(\xi)\mathrm{d}\nu(\xi),$$

where $\omega_d^k := \int_{\mathbb{S}^{N-1}} \omega_k(\xi)\mathrm{d}\nu(\xi)$ is the spherical measure. Note that $\omega_k(\xi)\mathrm{d}\nu(\xi)$ is G-invariant, by a change of variables $\sigma_\alpha\xi \to \xi$, we obtain

$$\begin{aligned}\widetilde{u}_{0,1}(r) &= u_{0,1}(r),\\ u - u(\sigma_\alpha x) &= \sum_{n=1}^{+\infty}\sum_{i=1}^{d(n)}(u_{n,i}(r) - \widetilde{u}_{n,i}(r))Y_i^n(\xi).\end{aligned}$$

From Parseval identity, we have

$$\begin{aligned}\int_{\mathbb{R}^N}\frac{1}{|x|^4}(u - u(\sigma_\alpha x))u\mathrm{d}\mu_k &= \sum_{n=1}^{+\infty}\sum_{i=1}^{d(n)}\int_0^{+\infty}(u_{n,i}(r) - \widetilde{u}_{n,i}(r))\cdot u_{n,i}r^{\overline{N}-5}\mathrm{d}r\\ &= \int_{\mathbb{R}^N}\frac{1}{|x|^4}[(u - u_{0,1}) - (u(\sigma_\alpha x) - \widetilde{u}_{0,1})](u - u_{0,1})\mathrm{d}\mu_k.\end{aligned}$$

Also

$$\begin{aligned}&-\int_{\mathbb{R}^N}\frac{1}{|x|^4}(u(\sigma_\alpha x) - \widetilde{u}_{0,1})(u - u_{0,1})\mathrm{d}\mu_k\\ &\le \Big(\int_{\mathbb{R}^N}\frac{1}{|x|^4}(u(\sigma_\alpha x) - \widetilde{u}_{0,1})^2\mathrm{d}\mu_k\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{R}^N}\frac{1}{|x|^4}(u - u_{0,1})^2\mathrm{d}\mu_k\Big)^{\frac{1}{2}}\\ &= \int_{\mathbb{R}^N}\frac{1}{|x|^4}(u - u_{0,1})^2\mathrm{d}\mu_k.\end{aligned}$$

Then we have

$$\int_{\mathbb{R}^N}\frac{1}{|x|^4}(u - u(\sigma_\alpha x))u\mathrm{d}\mu_k \le 2\int_{\mathbb{R}^N}\frac{1}{|x|^4}(u - u_{0,1})^2\mathrm{d}\mu_k. \tag{4.7}$$

By using spherical decomposition,

$$\begin{aligned}&\int_{\mathbb{R}^N}\frac{|\nabla_k u|^2}{|x|^2}\mathrm{d}\mu_k\\ &\le -\int_{\mathbb{R}^N}\frac{u\cdot\Delta_k u}{|x|^2}\mathrm{d}\mu_k - (\overline{N} - 4)\int_{\mathbb{R}^N}\frac{u^2}{|x|^4}\mathrm{d}\mu_k + 4\gamma\int_{\mathbb{R}^N}\frac{1}{|x|^4}(u - u_{0,1})^2\mathrm{d}\mu_k\\ &= -\sum_{n=0}^{\infty}\sum_{i=1}^{d(n)}\int_0^{+\infty}\Big[u_{n,i}\Big(u''_{n,i} + \frac{\overline{N}-1}{r}u'_{n,i} + \frac{\lambda_n}{r^2}u_{n,i}\Big)r^{\overline{N}-3} + (\overline{N} - 4)\cdot u_{n,i}^2 r^{\overline{N}-5}\Big]\mathrm{d}r\\ &\quad + 4\gamma\sum_{n=1}^{\infty}\sum_{i=1}^{d(n)}\int_0^{+\infty}u_{n,i}^2 r^{\overline{N}-5}\mathrm{d}r\\ &= \sum_{n=0}^{\infty}\sum_{i=1}^{d(n)}\int_0^{+\infty}[|u'_{n,i}|^2 r^{\overline{N}-3} - \lambda_n u_{n,i}^2 r^{\overline{N}-5}]\mathrm{d}r + 4\gamma\sum_{n=1}^{\infty}\sum_{i=1}^{d(n)}\int_0^{+\infty}u_{n,i}^2 r^{\overline{N}-5}\mathrm{d}r.\end{aligned}$$

So

$$\begin{aligned}&\int_{\mathbb{R}^N}|\Delta_k u|^2\mathrm{d}\mu_k - C\int_{\mathbb{R}^N}\frac{|\nabla_k u|^2}{|x|^2}\mathrm{d}\mu_k\\ &\ge \sum_{n=0}^{\infty}\sum_{i=1}^{d(n)}\int_0^{+\infty}\Big[\Big(u''_{n,i} + \frac{\overline{N}-1}{r}u'_{n,i} + \frac{\lambda_n}{r^2}u_{n,i}\Big)^2 r^{\overline{N}-1} - C|u'_{n,i}|^2 r^{\overline{N}-3} + \lambda_n C u_{n,i}^2 r^{\overline{N}-5}\Big]\mathrm{d}r\end{aligned}$$

$$
\begin{aligned}
&-4C\gamma\sum_{n=1}^{\infty}\sum_{i=1}^{d(n)}\int_0^{+\infty}u_{n,i}^2r^{\overline{N}-5}\mathrm{d}r\\
=&\sum_{n=0}^{\infty}\sum_{i=1}^{d(n)}\int_0^{+\infty}[|u_{n,i}''|^2r^{\overline{N}-1}+A_n|u_{n,i}'|^2r^{\overline{N}-3}+B_nu_{n,i}^2r^{\overline{N}-5}]\mathrm{d}r.
\end{aligned}
$$

By integration by parts, we obtain

$$
\begin{aligned}
&A_n=\overline{N}-2\lambda_n-1-C,\\
&B_n=\begin{cases}\lambda_0(\lambda_0-2(\overline{N}-4)+C), & n=0,\\ \lambda_n(\lambda_n-2(\overline{N}-4)+C)-4C\gamma, & n\geq 1,\end{cases}
\end{aligned}
$$

since $\lambda_0=0$, one has $B_0=0$.

Using the following weighted Hardy inequality

$$
\int_0^{+\infty}|u'|^2r^{\overline{N}-1}\mathrm{d}r\geq\frac{(\overline{N}-2)^2}{4}\int_0^{+\infty}u^2r^{\overline{N}-3}\mathrm{d}r, \tag{4.8}
$$

$$
\int_0^{+\infty}|u'|^2r^{\overline{N}-3}\mathrm{d}r\geq\frac{(\overline{N}-4)^2}{4}\int_0^{+\infty}u^2r^{\overline{N}-5}\mathrm{d}r, \tag{4.9}
$$

and denoting

$$
I_{n,i}=\int_0^{+\infty}[|u_{n,i}''|^2r^{\overline{N}-1}+A_n|u_{n,i}'|^2r^{\overline{N}-3}+B_nu_{n,i}^2r^{\overline{N}-5}]\mathrm{d}r,
$$

we have

$$
I_{n,i}\geq\Big[A_n+\frac{(\overline{N}-2)^2}{4}\Big]\int_0^{+\infty}|u_{n,i}'|^2r^{\overline{N}-3}\mathrm{d}r+B_n\int_0^{+\infty}u_{n,i}^2r^{\overline{N}-5}\mathrm{d}r. \tag{4.10}
$$

For $n=0$,

$$
I_{0,1}\geq\Big(\frac{\overline{N}^2}{4}-C\Big)\int_0^{+\infty}|u_{0,1}'|^2r^{\overline{N}-3}\mathrm{d}r,
$$

so we get $C\leq\frac{\overline{N}^2}{4}$.

For $n\geq 1$, take $C=\frac{\overline{N}^2}{4}$, we get

$$
\begin{aligned}
I_{n,i}&\geq-2\lambda_n\int_0^{+\infty}|u_{n,i}'|^2r^{\overline{N}-3}\mathrm{d}r+B_n\int_0^{+\infty}u_{n,i}^2r^{\overline{N}-5}\mathrm{d}r\\
&\geq\Big[-2\lambda_n\frac{(\overline{N}-4)^2}{4}+\lambda_n\Big(\lambda_n-2(\overline{N}-4)+\frac{\overline{N}^2}{4}\Big)-\overline{N}^2\gamma\Big]\int_0^{+\infty}u_{n,i}^2r^{\overline{N}-5}\mathrm{d}r\\
&=D_n\int_0^{+\infty}u_{n,i}^2r^{\overline{N}-5}\mathrm{d}r,
\end{aligned}
$$

here

$$
D_n:=\lambda_n\Big(\lambda_n-\frac{\overline{N}^2-8\overline{N}}{4}\Big)-\overline{N}^2\gamma.
$$

So

$$
D_1=\frac{(N-5-2\gamma)\overline{N}^2+4}{4}.
$$

When $N \geq 5 + 2\gamma$, $D_1 \geq 0$,

$$D_2 = \frac{2N\overline{N}^2}{4} \geq 0,$$

$D_n \geq D_2 \geq 0$ $(n = 3, 4, \cdots)$, so (4.5) holds.

Next we prove the optimality of the constant $\frac{\overline{N}^2}{4}$. For any $\epsilon > 0$, take

$$u_\epsilon = \begin{cases} 1, & r \leq 1, \\ r^{-\frac{\overline{N}-4+\epsilon}{2}}, & r > 1. \end{cases}$$

By directly computing, it follows that

$$\lim_{\epsilon \to 0} \frac{\int_{\mathbb{R}^N} |\Delta_k u_\epsilon|^2 \mathrm{d}\mu_k}{\int_{\mathbb{R}^N} \frac{|\nabla_k u_\epsilon|^2}{|x|^2} \mathrm{d}\mu_k} = \frac{\overline{N}^2}{4}.$$

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