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Ground States of K-component Coupled Nonlinear Schrödinger Equations with Inverse-square Potential*

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Abstract In this paper, the authors study ground states for a class of K-component coupled nonlinear Schrödinger equations with a sign-changing potential which is periodic or asymptotically periodic. The resulting problem engages three major difficulties: One is that the associated functional is strongly indefinite, the second is that, due to the asymptotically periodic assumption, the associated functional loses the $\mathbb{Z}^N$-translation invariance, many effective methods for periodic problems cannot be applied to asymptotically periodic ones. The third difficulty is singular potential $\frac{\mu_i}{|x|^2}$, which does not belong to the Kato's class. These enable them to develop a direct approach and new tricks to overcome the difficulties caused by singularity and the dropping of periodicity of potential.

Keywords Schrödinger equations, Ground states, Strongly indefinite functionals, Non-Nehari manifold method.

2000 MR Subject Classification 35J10, 35J20, 58E05

1 Introduction

In this paper, we study standing waves for the following system of time-dependent nonlinear Schrödinger equations:

$$\begin{cases} \mathrm{i}\dfrac{\partial \phi_1}{\partial t} + \Delta\phi_1 + \Big(V_1(x) - \dfrac{\mu_1}{|x|^2} + \omega_1\Big)\phi_1 = f_1(x,\Phi), \\ \mathrm{i}\dfrac{\partial \phi_2}{\partial t} + \Delta\phi_2 + \Big(V_2(x) - \dfrac{\mu_2}{|x|^2} + \omega_2\Big)\phi_2 = f_2(x,\Phi), \\ \cdots\cdots \\ \mathrm{i}\dfrac{\partial \phi_K}{\partial t} + \Delta\phi_K + \Big(V_K(x) - \dfrac{\mu_K}{|x|^2} + \omega_K\Big)\phi_K = f_K(x,\Phi), \end{cases} \tag{1.1}$$

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where $\Phi = (\phi_1, \phi_2, \cdots, \phi_K), \mu_i$ $(i = 1, 2, \cdots, K)$ are non-negative constants. ϕ_j $(t, x)(j = 1, 2, \cdots, K)$ are the complex valued envelope functions. Suppose that $f(x, e^{i\theta}\Phi) = f(x, \Phi)$ for $\theta \in [0, 2\pi],\ x \in \mathbb{R}^N \setminus \{0\},\ N \geq 3$. We will look for standing waves of the form

$$\phi_j(t, x) = e^{-i\omega_j t} u_j(x), \quad j = 1, 2, \cdots, K,$$

which propagate without changing their shape and thus have a soliton-like behavior. It is well known that solutions of (1.1) are related to the solitary waves of the Gross-Pitaevskii equations, which have applications in many physical models, such as in nonlinear optics and in Bose-Einstein condensates for multi-species condensates (see [4, 26]) and the references therein. In general, the above coupled nonlinear Schrödinger system leads to the elliptic system

$$\begin{cases} -\Delta u_1 + \Big(V_1(x) - \dfrac{\mu_1}{|x|^2}\Big) u_1 = f_1(x, u) & \text{in } \mathbb{R}^N, \\ -\Delta u_2 + \Big(V_2(x) - \dfrac{\mu_2}{|x|^2}\Big) u_2 = f_2(x, u) & \text{in } \mathbb{R}^N, \\ \cdots\cdots \\ -\Delta u_K + \Big(V_K(x) - \dfrac{\mu_K}{|x|^2}\Big) u_K = f_K(x, u) & \text{in } \mathbb{R}^N, \end{cases} \tag{1.2}$$

where $N \geq 3, f_i(x, u) = \partial_{u_i} F(x, u)$ with $u = (u_1, u_2, \cdots, u_K) : \mathbb{R}^N \to \mathbb{R}^K$.

When $\mu_i = 0$ $(i = 1, 2, \cdots, K)$, (1.2) reduces to

$$\begin{cases} -\Delta u_1 + V_1(x) u_1 = f_1(x, u) & \text{in } \mathbb{R}^N, \\ -\Delta u_2 + V_2(x) u_2 = f_2(x, u) & \text{in } \mathbb{R}^N, \\ \cdots\cdots \\ -\Delta u_K + V_K(x) u_K = f_K(x, u) & \text{in } \mathbb{R}^N. \end{cases} \tag{1.3}$$

In the past fifteen years, the two-coupled case of (1.3) (i.e., $k = 2$) has been studied extensively in the literature [5, 11–12, 14–15, 17, 21–25, 28–29, 37–40] and the references therein. By using variational methods, Lyapunov-Schmidt reduction methods or bifurcation methods, various theorems, about the existence, multiplicity and qualitative properties of nontrivial solutions of the two-coupled elliptic systems similar to (1.3), have been established in the literature under various assumptions. However, there are very few works about $k \geq 3$ in the context [18–19, 35–36]. It is worth to mention that most of them focused on the case that V is non-negative constant or function, compared to this case, it is more difficult to consider the case that V is a sign-changing function to which the energy functional corresponding has the strongly indefinite structure. Very recently, Mederski [20] considered (1.3) and obtained the existence of ground state solution for the case of periodic potential by applying a new linking-type result involving the Nehari-Pankov manifold.

For $\mu_i \neq 0$, $\frac{\mu_i}{|x|^2}$ is called inverse square potential or Hardy potential which arises in many other areas such as quantum mechanics, nuclear, molecular physics and quantum cosmology. From the mathematical point of view, the inverse square potential is critical: Indeed, it has the same homogeneity as the Laplacian and does not belong to the Kato's class, hence it cannot be regarded as a lower order perturbation term of second order operator, which may result in the change of the essential spectrum of the operator. Moreover, any nontrivial solutions of system (1.2) are singular at $x = 0$ if $\mu \neq 0$. Since the appearance of inverse square potential, compared with system (1.3), system (1.2) becomes more complicated to deal with and therefore we have to face more difficulties. As far as we know, it seems that there are no existence results of solution for system (1.2), hence, it makes sense for us to investigate system (1.2) thoroughly. Due to the special physical importance and the above facts, in the present paper, we will study the existence and some properties of solutions of system (1.2).

As a motivation, we recall that there are many of articles concerning the nonlinear Schrödinger equations with the inverse square potentials

$$-\triangle u + \Big(V(x) - \frac{\mu}{|x|^2}\Big)u = f(x,u), \tag{1.4}$$

see for example, [1–4, 6–9, 16, 27–28] and the references therein. These authors studied the existence of positive solutions, nodal solutions, multiple solutions and ground state solutions under suitable assumptions. Most of them focused on the case that V is non-negative constant or function in which the energy functional corresponding to (1.4) has the mountain pass structure. Only very recently, Guo and Mederski [10] studied the case that V is a general periodic function, possibly sign-changing, and the corresponding energy functional may be strongly indefinite. Combining Nehari manifold technique (see [22–23, 29]) and linking argument, they proved the existence of ground state solutions for the case $\mu \geq 0$ and the non-existence of ground state solutions was explored for the case $\mu_i < 0$. Furthermore, some asymptotical behavior of ground state solutions are derived.

Inspired by the aforementioned works, we are going to consider two situations in the present paper: Periodic case and asymptotically periodic case. Our aim is to find ground states for (1.2) on some suitable manifold, one difficulty is that the associated functional is strongly indefinite, i.e., its quadratic part is respectively coercive and anti-coercive in infinitely dimensional subspace of the energy space. To tackle this difficulty, we adapt the properties of the spectrum of the corresponding opertor which had been analysed in [40], it is convenient to decompose the functional space L^2 into a direct sum of two subspaces E^+ and E^- (E is defined in Section 2), one of which is infinite dimensional.

Another difficulty is lack of periodic assumption on potential. As a result, neither the periodic translation technique nor the compact inclusion method can be adapted. In this case, the functional loses the $\mathbb{Z}^N$-translation invariance. For the above reasons, many effective methods for periodic problems cannot be applied to asymptotically periodic ones. To the best of our knowledge, there are no results on the existence of ground state solutions to (1.2) when V_i is not periodic. In this paper, we find new tricks to overcome the difficulties caused by getting rid of periodicity condition.

The last difficulty is singular potential $\frac{\mu}{|x|^2}$, which does not belong to the Kato's class. This enables us to develop a direct approach and new tricks to overcome the difficulties caused by singularity. We find a new method to overcome the difficulty caused by the non-compactness of the embedding $H^1(\mathbb{R}^N) \hookrightarrow L^2(B_{R(0)}, |x|^{-2}\mathrm{d}x)$. Our treatments presented in the paper differ from those in [10] and other existing literature.

In this paper, we further develop the non-Nehari method in [32–33] which is completely different from the one of Szulkin-Weth [31] and Mederski [20] to find ground state solution of Nehari-Pankov type for (1.2). For the asymptotically periodic case, a nontrivial solution is obtained by using a generalized linking theorem and comparing with a ground state solution of the periodic problem associated with (1.2). More precisely, we will prove that system (1.2) possesses a ground state solution via variational methods for sufficiently small $\mu \geq 0$, and provide the comparison of the energy of ground state solutions for the case $\mu > 0$ and $\mu = 0$.

Moreover, we also give the convergence property of ground state solutions as $\mu \to 0^+$.

To simplify notation, we set

$$\mu = (\mu_1, \mu_2, \cdots, \mu_K), \quad \underline{\mu} := \min\{\mu_i\}_{i=1}^K, \quad \overline{\mu} := \max\{\mu_i\}_{i=1}^K.$$

For the sake of convenience, let E be the Hilbert spaces with an orthogonal decomposition $E = E^- \oplus E^+$, and let $\mathcal{I}_\mu$ denote the energy functional of system (1.2), where E and $\mathcal{I}_\mu$ will be defined in Section 2. A ground state solution stands for a critical point being a minimizer of $\mathcal{I}_\mu$ on the Nehari-Pankov manifold introduced in [22–23],

$$\mathcal{N}_\mu := \{u \in E \setminus E^- : \langle \mathcal{I}'_\mu(u), u\rangle = \langle \mathcal{I}'_\mu(u), w\rangle = 0, \ \forall\, w \in E^-\},$$

the set $\mathcal{N}_\mu$ is a natural constraint and it contains all nontrivial critical points, any ground state solution is a nontrivial critical point with the least energy of $\mathcal{I}_\mu$.

Let l_0 be a positive constant (l_0 will be given later in (2.3)). Now, we are ready to state the main results of the present paper as follows.

1.1 Periodic potential

(V1) For $i = 1, 2, \cdots, K$, $V_i \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is $\mathbb{Z}^N$-periodic and

$$\sup[\sigma(-\Delta + V_i) \cap (-\infty, 0)] := \underline{\Lambda}_i < 0 < \overline{\Lambda}_i := \inf[\sigma(-\Delta + V_i) \cap (0, \infty)]$$

for all $x \in \mathbb{R}^N$;

(F1) $f_i : \mathbb{R}^N \times \mathbb{R}^K \to \mathbb{R}$ is measurable, $\mathbb{Z}^N$-periodic in $x \in \mathbb{R}^N$ and continuous in $u \in \mathbb{R}^K$ for a.e. $x \in \mathbb{R}^N$. Moreover $f = (f_1, f_2, \cdots, f_K) = \partial_u F$, where $F : \mathbb{R}^N \times \mathbb{R}^K \to \mathbb{R}$ is differentiable with respect to the second variable $u \in \mathbb{R}^K$ and $F(x, 0) = 0$ for a.e. $x \in \mathbb{R}^N$;

(F2) there exist constants $C > 0$ and $2 < p < 2^* = \frac{2N}{N-2}$ such that

$$|f(x, u)| \le C(1 + |u|^{p-1}), \quad \forall\, (x, u) \in \mathbb{R}^N \times \mathbb{R}^K;$$

(F3) $f(x, u) = o(u)$ as $|u| \to 0$ uniformly in $x \in \mathbb{R}^N$;

(F4) $\lim\limits_{|u|\to\infty} \frac{|F(x,u)|}{|u|^2} = \infty$ uniformly in $x \in \mathbb{R}^N$;

(F5) for all $\kappa \ge 0, u, v \in \mathbb{R}^K$,

$$F(x, \kappa u + v) - F(x, u) + \frac{1 - \kappa^2}{2} F_u(x, u) \cdot u - \kappa F_u(x, u) \cdot v \ge 0;$$

(F6) $\partial_u F(x, \cdot)$ is of C^1 class for a.e. $x \in \mathbb{R}^N$ and there exist $b_1, b_2 > 0$ and $1 < q \le 2$ such that for all $x \in \mathbb{R}^N$,

$$f(x, u) \cdot u - 2F(x, u) \ge \begin{cases} b_1|u|^2 & \text{for } |u| \le 1, \\ b_2|u|^q & \text{for } |u| > 1. \end{cases}$$

Theorem 1.1 *Assume that* (V1), (F1)–(F5) *are satisfied and* $0 \le \underline{\mu} \le \overline{\mu} < \frac{(N-2)^2}{4} l_0^2$, *then system* (1.2) *has a ground state, i.e., it has at least a solution* $u_\mu \in E$ *such that* $\mathcal{I}_\mu(u_\mu) = \inf\limits_{\mathcal{N}_\mu} \mathcal{I}_\mu > 0$.

Theorem 1.2 *Assume that* (V1), (F1)–(F5) *are satisfied and* $0 \le \underline{\mu} \le \overline{\mu} < \frac{(N-2)^2}{4} l_0^2$. *Let* u_μ *be a ground state solution of* $\mathcal{I}_\mu$ *and* u_0 *be a ground state solution of* $\mathcal{I}_0$. *Then*

(i) *there exist $t>0$ and $w \in E^-$ such that $tu_\mu + w \in \mathcal{N}_0$ and*

$$\inf_{\mathcal{N}_0} \mathcal{I}_0 \le \inf_{\mathcal{N}_\mu} \mathcal{I}_\mu + \frac{\overline{\mu}}{2} \int_{\mathbb{R}^N} \frac{|tu_\mu + w|^2}{|x|^2} \mathrm{d}x;$$

(ii) *there exist $t>0$ and $w \in E^-$ such that $tu_0 + w \in \mathcal{N}_\mu$ and*

$$\inf_{\mathcal{N}_\mu} \mathcal{I}_\mu \le \inf_{\mathcal{N}_0} \mathcal{I}_0 - \frac{\underline{\mu}}{2} \int_{\mathbb{R}^N} \frac{|tu_0 + w|^2}{|x|^2} \mathrm{d}x.$$

Theorem 1.3 *Assume that* (V1), (F1)–(F5) *and* (F6) *are satisfied, let u_μ be a ground state solution of $\mathcal{I}_\mu$ and u_0 be a ground state solution of $\mathcal{I}_0$. Then*

(i) *there holds*

$$\lim_{\mu\to 0^+} \inf_{\mathcal{N}_\mu} \mathcal{I}_\mu = \inf_{\mathcal{N}_0} \mathcal{I}_0;$$

(ii) *for any sequence $\{\mu^{(n)}\}$, there exists a subsequence $u_{\mu^{(n)}}$ such that*

$$\lim_{\mu^{(n)}\to 0^+} u_{\mu^{(n)}} = u_0.$$

1.2 Asymptotically periodic potential

(V1') $V_i(x) = U_i(x) + W_i(x)$, $i = 1, 2, \cdots, K$, where $U_i \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is $\mathbb{Z}^N$-periodic and

$$\sup[\sigma(-\Delta + U_i) \cap (-\infty, 0)] := \underline{\Lambda}_i < 0 < \overline{\Lambda}_i := \inf[\sigma(-\Delta + U_i) \cap (0, \infty)]$$

for all $x \in \mathbb{R}^N$, $W_i \in C(\mathbb{R}^N)$ and $\lim\limits_{|x|\to\infty} W_i(x) = 0$. Moreover,

$$0 \le -W_i(x) \le \sup_{\mathbb{R}^N} [-W_i(x)]_{i=1}^K \le \frac{\min\{\overline{\Lambda}_i\}_{i=1}^K}{2} \Big[1 - \frac{4\overline{\mu}}{(N-2)^2 l_0^2}\Big];$$

(F1') $f_i(x,u) = g_i(x,u) + h_i(x,u)$, $g_i : \mathbb{R}^N \times \mathbb{R}^K \to \mathbb{R}$ is measurable, $\mathbb{Z}^N$-periodic in $x \in \mathbb{R}^N$ and continuous in $u \in \mathbb{R}^K$ for a.e. $x \in \mathbb{R}^N$, $g_i(x,u) = o(u)$ as $|u| \to 0$, uniformly in $x \in \mathbb{R}^N$;

(F5') $\partial_u G = (g_1, g_2, \cdots, g_K)$, where G satisfies that for all $\kappa \ge 0, u, v \in \mathbb{R}^K$,

$$G(x, \kappa u + v) - G(x,u) + \frac{1-\kappa^2}{2} G_u(x,u) \cdot u - \kappa G_u(x,u) \cdot v \ge 0.$$

Furthermore, $\partial_u H = (h_1, h_2, \cdots, h_K)$ and H satisfies that

$$0 \le h_i(x,u) \le a_i(x)(|u|^2 + |u|^p),$$

where $a_i \in C(\mathbb{R}^N)$, $\lim\limits_{|x|\to\infty} a_i(x) = 0$ $(i = 1, 2, \cdots, K)$ and $2 < p < 2^* = \frac{2N}{N-2}$. Moreover,

$$H(x,u) - \sum_{i=1}^K W_i(x) u_i^2 > 0, \quad \forall (x,u) \in B_{1+\sqrt{N}}(0) \times \mathbb{R}^K \setminus \{0\}.$$

Theorem 1.4 *Assume that* (V1'), (F1'), (F2)–(F4), (F5') *are satisfied and $0 \le \underline{\mu} \le \overline{\mu} < \frac{(N-2)^2}{4} l_0^2$, then system* (1.2) *has a ground state, i.e., it has at least a solution $u_\mu \in E$ such that $\mathcal{I}_\mu(u_\mu) = \inf\limits_{\mathcal{N}_\mu} \mathcal{I}_\mu > 0$.*

Theorem 1.5 *Assume that* (V1'), (F1'), (F2)–(F4), (F5') *are satisfied and* $0 \le \underline{\mu} \le \overline{\mu} < \frac{(N-2)^2}{4} l_0^2$. *Let* u_μ *be a ground state solution of* $\mathcal{I}_\mu$ *and* u_0 *be a ground state solution of* $\mathcal{I}_0$. *Then*

(i) *there exist* $t > 0$ *and* $w \in E^-$ *such that* $tu_\mu + w \in \mathcal{N}_0$ *and*

$$\inf_{\mathcal{N}_0} \mathcal{I}_0 \le \inf_{\mathcal{N}_\mu} \mathcal{I}_\mu + \frac{\overline{\mu}}{2} \int_{\mathbb{R}^N} \frac{|tu_\mu + w|^2}{|x|^2} \mathrm{d}x;$$

(ii) *there exist* $t > 0$ *and* $w \in E^-$ *such that* $tu_0 + w \in \mathcal{N}_\mu$ *and*

$$\inf_{\mathcal{N}_\mu} \mathcal{I}_\mu \le \inf_{\mathcal{N}_0} \mathcal{I}_0 - \frac{\underline{\mu}}{2} \int_{\mathbb{R}^N} \frac{|tu_0 + w|^2}{|x|^2} \mathrm{d}x.$$

Theorem 1.6 *Assume that* (V1'), (F1'), (F2)–(F4), (F5'), (F6) *are satisfied, let* u_μ *be a ground state solution of* $\mathcal{I}_\mu$ *and* u_0 *be a ground state solution of* $\mathcal{I}_0$. *Then*

(i) *there holds*

$$\lim_{\mu \to 0^+} \inf_{\mathcal{N}_\mu} \mathcal{I}_\mu = \inf_{\mathcal{N}_0} \mathcal{I}_0;$$

(ii) *for any sequence* $\{\mu^{(n)}\}$, *there exists a subsequence* $u_{\mu^{(n)}}$ *such that*

$$\lim_{\mu^{(n)} \to 0^+} u_{\mu^{(n)}} = u_0.$$

The present paper is organized as follows. Section 2 is dedicated to the variational form associated with problem (1.2). We recall the abstract linking theorem in [13], which is going to be used to prove the existence of solution in periodic case, as well as in the asymptotically periodic case. Some preliminaries are introduced in Section 3. Section 4 is dedicated to the periodic case. In order to do so, exploiting the profile of spectrum presented by the associated operator, we decompose the space E in appropriate subspaces for the linking structure. Subsequently, the requirements of the abstract result are verified: compactness, linking geometry and boundedness of Cerami sequences for the functional associated with problem (1.2). Section 5 is dedicated to asymptotically periodic case. Our greatest challenge in the asymptotically periodic case is the functional loses the $\mathbb{Z}^N$-translation invariance, many effective methods for periodic problems cannot be applied to asymptotically periodic ones.

2 Variational Structure

Let $\mathcal{A}_i = -\Delta + V_i$, here and in what follows $i = 1, 2, \cdots, K$. Then $\mathcal{A}_i$ are self-adjoint in $L^2(\mathbb{R}^N)$ with domain $\mathscr{D}(\mathcal{A}_i) = H^2(\mathbb{R}^N)$. Let $\{\mathcal{E}_i(\lambda) : -\infty \le \lambda \le \infty\}$ and $|\mathcal{A}_i|$ be the spectral family and the absolute value of $\mathcal{A}_i$, respectively, and $|\mathcal{A}_i|^{\frac{1}{2}}$ be the square root of $\mathcal{A}_i$. Set $\mathcal{U}_i = id - \mathcal{E}_i(0) - \mathcal{E}_i(0-)$. Then $\mathcal{U}_i$ commutes with $\mathcal{A}_i, |\mathcal{A}_i|$ and $|\mathcal{A}_i|^{\frac{1}{2}}$, and $\mathcal{A}_i = \mathcal{U}_i |\mathcal{A}_i|$ is the polar decomposition of $\mathcal{A}_i$.

Let

$$H_i = \mathscr{D}(|\mathcal{A}_i|^{\frac{1}{2}}), \quad H_i^- = \mathcal{E}_i(0) H_i, \quad H_i^+ = [id - \mathcal{E}_i(0)] H_i.$$

Define

$$\langle u, v \rangle_{H_i} = (|\mathcal{A}_i|^{\frac{1}{2}} u, |\mathcal{A}_i|^{\frac{1}{2}} v)_{L^2}, \quad \forall u, v \in H_i$$

and the corresponding norm

$$\|u\|_{H_i} = \||\mathcal{A}_i|^{\frac{1}{2}}u\|_{L^2}, \quad u \in H_i.$$

For any $u \in H$, fixing $i = 1, 2, \cdots, K$, it is easy to see that

$$\begin{aligned} &u = u_i^- + u_i^+, \quad u_i^- := \mathcal{E}_i(0)u \in H_i^-, \quad u_i^+ := [id - \mathcal{E}_i(0)]u \in H_i^+, \\ &\mathcal{A}_i u_i^- = -|\mathcal{A}_i|u_i, \quad \mathcal{A}_i u_i^+ = |\mathcal{A}_i|u_i^+, \quad \forall u = u_i^- + u_i^+ \in H \cap \mathscr{D}(\mathcal{A}_i). \end{aligned}$$

Since $0 \notin \sigma(-\Delta + V_i)$, the spectral theory asserts that we may find continuous projections P_i^+ and P_i^- of $H^1(\mathbb{R}^N)$ onto H_i^+ and H_i^-, respectively, such that $H^1(\mathbb{R}^N) = H_i^+ \oplus H_i^-$, then

$$\begin{aligned} \langle u, v\rangle_{H_i} = &\int_{\mathbb{R}^N} \langle \nabla P_i^+(u), \nabla P_i^+(v)\rangle + V_i(x)\langle P_i^+(u), P_i^+(v)\rangle \mathrm{d}x \\ &- \int_{\mathbb{R}^N} \langle \nabla P_i^-(u), \nabla P_i^-(v)\rangle + V_i(x)\langle P_i^-(u), P_i^-(v)\rangle \mathrm{d}x, \end{aligned}$$

and norms are given by

$$\|u\|_{H_i} := (\langle u, u\rangle_i)^{\frac{1}{2}}.$$

Let

$$\begin{aligned} E^+ &: = H_1^+ \times H_2^+ \times \cdots \times H_K^+, \\ E^- &: = H_1^- \times H_2^- \times \cdots \times H_K^-. \end{aligned}$$

Note that any $u \in E := H^1(\mathbb{R}^N)^K$ admits a unique decomposition $u = u^+ + u^-$, where

$$u^+ = (P_1^+(u_1), P_2^+(u_2), \cdots, P_K^+(u_K)) \in E^+, \quad u^- = (P_1^-(u_1), P_2^-(u_2), \cdots, P_K^-(u_K)) \in E^-.$$

We introduce a new norm on E given by

$$\|u\|^2 = \sum_{i=1}^K (\|P_i^+(u_i)\|_i^2 + \|P_i^-(u_i)\|_i^2) = \sum_{i=1}^K \|u_i\|_i^2.$$

Then

$$\begin{aligned} \mathcal{I}_\mu(u) = &\frac{1}{2}\sum_{i=1}^K \int_{\mathbb{R}^N} (|\nabla u_i|^2 + U_i(x)u_i^2)\mathrm{d}x + \frac{1}{2}\sum_{i=1}^K \int_{\mathbb{R}^N} W_i(x)u_i^2 \mathrm{d}x \\ &- \frac{1}{2}\sum_{i=1}^K \int_{\mathbb{R}^N} \frac{\mu_i u_i^2}{|x|^2}\mathrm{d}x - \int_{\mathbb{R}^N} F(x,u)\mathrm{d}x \\ = &\frac{1}{2}\sum_{i=1}^K (\|P_i^+(u_i)\|_i^2 - \|P_i^-(u_i)\|_i^2) + \frac{1}{2}\sum_{i=1}^K \int_{\mathbb{R}^N} W_i(x)u_i^2 \mathrm{d}x \\ &- \frac{1}{2}\sum_{i=1}^K \int_{\mathbb{R}^N} \frac{\mu_i u_i^2}{|x|^2}\mathrm{d}x - \int_{\mathbb{R}^N} F(x,u)\mathrm{d}x \\ = &\frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) + \frac{1}{2}\sum_{i=1}^K \int_{\mathbb{R}^N} W_i(x)u_i^2 \mathrm{d}x \end{aligned}$$

$$-\frac{1}{2}\sum_{i=1}^{K}\int_{\mathbb{R}^N}\frac{\mu_i u_i^2}{|x|^2}\mathrm{d}x-\int_{\mathbb{R}^N}F(x,u)\mathrm{d}x \tag{2.1}$$

and

$$\begin{aligned}\langle \mathcal{I}_\mu'(u),v\rangle=&\sum_{i=1}^{K}\int_{\mathbb{R}^N}(\nabla u_i\cdot\nabla v_i+(U_i(x)+W_i(x))u_iv_i)\mathrm{d}x\\&-\sum_{i=1}^{K}\int_{\mathbb{R}^N}\frac{\mu_i u_i\cdot v_i}{|x|^2}\mathrm{d}x-\int_{\mathbb{R}^N}f(x,u)\cdot v\mathrm{d}x.\end{aligned} \tag{2.2}$$

Our hypotheses imply that $\mathcal{I}_\mu\in C^1(E,\mathbb{R})$ and a standard argument shows that critical points of $\mathcal{I}_\mu$ are solutions of (1.2).

Lemma 2.1 *E is continuously embedded in $L^q(\mathbb{R}^N,\mathbb{R}^K)$ and compactly embedded in $L^{q'}_{\mathrm{loc}}(\mathbb{R}^N,\mathbb{R}^K)$, where $2\le q\le 2^*, 2\le q'<2^*$, 2^* is defined in* (F2).

By Lemma 2.1, there exist positive constants l_0,l_1 such that

$$l_0\|z\|_{H^1}\le\|z\|\le l_1\|z\|_{H^1}\quad\text{for all } z\in E. \tag{2.3}$$

Observe that, in view of the Hardy inequality

$$\frac{(N-2)^2}{4}\int_{\mathbb{R}^N}\frac{u_i^2}{|x|^2}\mathrm{d}x\le\int_{\mathbb{R}^N}|\nabla u_i|^2\mathrm{d}x\quad\text{for all } u_i\in H_i,$$

then we have

$$\begin{aligned}\frac{(N-2)^2}{4}\sum_{i=1}^{K}\int_{\mathbb{R}^N}\frac{\mu_i u_i^2}{|x|^2}\mathrm{d}x&\le\overline{\mu}\sum_{i=1}^{K}\int_{\mathbb{R}^N}|\nabla u_i|^2\mathrm{d}x\\&\le\overline{\mu}\sum_{i=1}^{K}\int_{\mathbb{R}^N}(|\nabla u_i|^2+|u_i|^2)\mathrm{d}x\le\frac{\overline{\mu}}{l_0^2}\|u\|^2.\end{aligned} \tag{2.4}$$

To get the ground state solutions of (1.2), we define the generalized Nehari manifold

$$\mathcal{N}_\mu:=\{u\in E\setminus E^-:\langle\mathcal{I}_\mu'(u),u\rangle=\langle\mathcal{I}_\mu'(u),w\rangle=0,\ \forall w\in E^-\}. \tag{2.5}$$

This type of manifold was first introduced by Pankov [22–23]. As is well known that if $u_\mu\ne0$ is a critical point of $\mathcal{I}_\mu$, then $u_\mu\in\mathcal{N}_\mu$. The ground state solution will be obtained as a nontrivial critical point of $\mathcal{I}_\mu$ in $\mathcal{N}_\mu$. The next section will be used to get such points.

3 Preliminaries

Lemma 3.1 (see [13]) *Let X be a real Hilbert space, $X=X^-\oplus X^+$ and $X^-\perp X^+$, and let $\varphi\in C^1(X,\mathbb{R})$ be of the form*

$$\varphi(u)=\frac{1}{2}(\|u^+\|^2-\|u^-\|^2)-\psi(u),\quad u=u^-+u^+\in X^-\oplus X^+.$$

Suppose that the following assumptions are satisfied:

(A1) *$\psi\in C^1(X,\mathbb{R})$ is bounded from below and weakly sequentially lower semi-continuous*;

(A2) ψ' *is weakly sequentially continuous*;

(A3) *there exist* $r > \rho > 0$, $e \in X^+$ *with* $\|e\| = 1$ *such that*

$$\kappa := \inf \varphi(S_\rho^+) > \sup \varphi(\partial Q),$$

where

$$S_\rho^+ = \{u \in X^+ : \|u\| = \rho\}, \quad Q = \{v + se : v \in X^-, s \geq 0, \|v + se\| \leq r\}.$$

Then for some $c \in [\kappa, \sup \varphi(Q)]$, *there exists a sequence* $\{u_n\} \subset X$ *satisfying*

$$\varphi(u_n) \to c, \quad \|\varphi'(u_n)\|(1 + \|u_n\|) \to 0.$$

Set

$$\mathcal{F}_\mu(u) = \frac{1}{2}\sum_{i=1}^{K}\int_{\mathbb{R}^N} \frac{\mu_i u_i^2}{|x|^2}\mathrm{d}x + \int_{\mathbb{R}^N} F(x, u)\mathrm{d}x.$$

Employing a standard argument, one can check easily the following lemma.

Lemma 3.2 *Assume that* (V1'), (F1') *and* (F2)–(F4), (F5') *are satisfied, then* $\mathcal{F}_\mu$ *is non-negative, weakly sequentially lower semicontinuous, and* $\mathcal{F}'_\mu$ *is weakly sequentially continuous.*

Lemma 3.3 *Assume that* (V1'), (F1') *and* (F2)–(F4), (F5') *are satisfied. Then for all* $\kappa \geq 0, u \in E, \zeta = (\zeta_1, \zeta_2, \cdots, \zeta_K) \in E^-$,

$$\begin{aligned}\mathcal{I}_\mu(u) \geq{}& \mathcal{I}_\mu(\kappa u + \zeta) + \frac{1}{2}\|\zeta\|^2 - \frac{1}{2}\sum_{i=1}^{K}\int_{\mathbb{R}^N} W_i(x)\zeta_i^2\mathrm{d}x \\ &+ \frac{\underline{\mu}}{2}\int_{\mathbb{R}^N}\frac{|\zeta|^2}{|x|^2}\mathrm{d}x + \frac{1-\kappa^2}{2}\langle \mathcal{I}'_\mu(u), u\rangle - \kappa\langle \mathcal{I}'_\mu(u), \zeta\rangle. \end{aligned} \tag{3.1}$$

Proof From (2.1)–(2.2) and (F5') we have

$$\begin{aligned}&\mathcal{I}_\mu(u) - \mathcal{I}_\mu(\kappa u + \zeta) + \frac{1}{2}\sum_{i=1}^{K}\int_{\mathbb{R}^N} W_i(x)\zeta_i^2\mathrm{d}x \\ ={}& \frac{1}{2}\|\zeta\|^2 + \frac{1}{2}\sum_{i=1}^{K}\int_{\mathbb{R}^N}\frac{\mu_i\zeta_i^2}{|x|^2}\mathrm{d}x + \frac{1-\kappa^2}{2}(\|u^+\|^2 - \|u^-\|^2) \\ &- \int_{\mathbb{R}^N}[F(x,u) - F(x, \kappa u + \zeta)]\mathrm{d}x + \kappa(u, \zeta) \\ ={}& \frac{1}{2}\|\zeta\|^2 + \frac{1-\kappa^2}{2}\langle \mathcal{I}'_\mu(u), u\rangle - \kappa\langle \mathcal{I}'_\mu(u), \zeta\rangle \\ &+ \int_{\mathbb{R}^N}\Big[\frac{1-\kappa^2}{2}F_u(x,u)\cdot u - \kappa F_u(x,u)\cdot\zeta\Big]\mathrm{d}x \\ &+ \int_{\mathbb{R}^N}[F(x, \kappa u + \zeta) - F(x,u)]\mathrm{d}x + \frac{\underline{\mu}}{2}\int_{\mathbb{R}^N}\frac{|\zeta|^2}{|x|^2}\mathrm{d}x \\ \geq{}& \frac{1}{2}\|\zeta\|^2 + \frac{1-\kappa^2}{2}\langle \mathcal{I}'_\mu(u), u\rangle - \kappa\langle \mathcal{I}'_\mu(u), \zeta\rangle, \quad \forall \kappa \geq 0, u \in E, \zeta \in E^-. \end{aligned}$$

Using Lemma 3.3, some important corollaries are given as follows, the proof process will be omitted.

Corollary 3.1 *Assume that* (V1'), (F1') *and* (F2)–(F4), (F5') *are satisfied. Then for* $u \in \mathcal{N}_\mu$, *we have*

$$\mathcal{I}_\mu(u) \geq \mathcal{I}_\mu(\kappa u + \zeta) + \frac{1}{2}\|\zeta\|^2 + \frac{\mu}{2}\int_{\mathbb{R}^N} \frac{|\zeta|^2}{|x|^2}\mathrm{d}x - \frac{1}{2}\sum_{i=1}^{K}\int_{\mathbb{R}^N} W_i(x)\zeta_i^2\mathrm{d}x, \quad \forall \kappa \geq 0,\ \zeta \in E^-.$$

Corollary 3.2 *Assume that* (V1'), (F1') *and* (F2)–(F4), (F5') *are satisfied. Then for all* $u \in E,\ \kappa \geq 0$,

$$\begin{aligned}\mathcal{I}_\mu(u) \geq{}& \frac{\kappa^2}{2}\|u\|^2 + \frac{1-\kappa^2}{2}\langle \mathcal{I}'_\mu(u), u\rangle + \kappa^2\langle \mathcal{I}'_\mu(u), u^-\rangle - \int_{\mathbb{R}^N} F(x, \kappa u^+)\mathrm{d}x \\ &+ \frac{\kappa^2}{2}\sum_{i=1}^{K}\int_{\mathbb{R}^N} W_i(x)[(u_i^+)^2 - (u_i^-)^2]\mathrm{d}x - \frac{\overline{\mu}\kappa^2}{2}\int_{\mathbb{R}^N} \frac{|u^+|^2 - |u^-|^2}{|x|^2}\mathrm{d}x. \end{aligned} \tag{3.2}$$

Lemma 3.4 *Assume that* (V1'), (F1') *and* (F2)–(F4), (F5') *are satisfied. Then*
(i) *there exists* $\rho > 0$ *such that*

$$m_\mu := \inf_{\mathcal{N}_\mu} \mathcal{I}_\mu \geq \Lambda_\mu := \inf\{\mathcal{I}_\mu(u) : u \in E^+,\ \|u\| = \rho\} > 0;$$

(ii) $\|u^+\| \geq \max\{\|u^-\|, \sqrt{2m_\mu}\}$ *for all* $u \in \mathcal{N}_\mu$.

Proof Set $\gamma_0 = \sup_{\mathbb{R}^N}[-W_i(x)]_{i=1}^K,\ \Lambda_0 = \min\{\overline{\Lambda}_i\}_{i=1}^K$. It follows from (V1') that

$$\Lambda_0\|u\|_2^2 \leq \|u\|^2, \quad \forall u \in E^+.$$

By (V1') and (F2), there exist $p \in (2, 2^*)$ and $C_1 > 0$ such that

$$F(x,u) \leq \Big[\frac{\Lambda_0}{4}\Big(1 - \frac{4\mu}{(N-2)^2 l_0^2}\Big) - \frac{\gamma_0}{2}\Big]|u|^2 + C_1|u|^p, \quad \forall u \in \mathbb{R}^K.$$

From Corollary 3.1, we have for $u \in \mathcal{N}_\mu$,

$$\begin{aligned}\mathcal{I}_\mu(u) &= \frac{1}{2}\|u\|^2 + \frac{1}{2}\sum_{i=1}^{K}\int_{\mathbb{R}^N} W_i(x)u_i^2\mathrm{d}x - \frac{1}{2}\sum_{i=1}^{K}\int_{\mathbb{R}^N} \frac{\mu_i u_i^2}{|x|^2}\mathrm{d}x - \int_{\mathbb{R}^N} F(x,u)\mathrm{d}x \\ &\geq \frac{1}{2}\|u\|^2 - \frac{\gamma_0}{2}\|u\|_2^2 - \frac{2\overline{\mu}}{(N-2)^2 l_0^2}\|u\|^2 \\ &\quad - \Big[\frac{\Lambda_0}{4}\Big(1 - \frac{4\overline{\mu}}{(N-2)^2 l_0^2}\Big) - \frac{\gamma_0}{2}\Big]\|u\|_2^2 - C_1\|u\|_p^p \\ &\geq \frac{1}{4}\Big(1 - \frac{4\overline{\mu}}{(N-2)^2 l_0^2}\Big)\|u\|^2 - C_2\|u\|^p > 0, \quad \forall u \in E^+.\end{aligned}$$

This shows that there exists a $\rho > 0$ such that (i) holds.

By Lemma 3.2, $\mathcal{F}_\mu(u) > 0$ for all $(x,u) \in \mathbb{R}^N \times \mathbb{R}^K$, so we have for $u \in \mathcal{N}_\mu$,

$$\begin{aligned} m_\mu \leq{}& \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 + \frac{1}{2}\sum_{i=1}^{K}\int_{\mathbb{R}^N} W_i(x)u_i^2\mathrm{d}x - \frac{1}{2}\sum_{i=1}^{K}\int_{\mathbb{R}^N} \frac{\mu_i u_i^2}{|x|^2}\mathrm{d}x \\ &- \int_{\mathbb{R}^N} F(x,u)\mathrm{d}x \leq \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 \leq \frac{1}{2}\|u^+\|^2, \end{aligned}$$

which implies that $\|u^+\| \geq \max\{\|u^-\|, \sqrt{2m_\mu}\}$.

With the help of the preceding two corollaries, an argument similar to the one used in [34] shows that we can now prove the following lemma in the same way as [34].

Lemma 3.5 *Assume that* (V1'), (F1)–(F4) *are satisfied. Then for every* $e \in E^+$, *we have* $\sup \mathcal{I}_\mu(E^- \oplus \mathbb{R}^+ e) < \infty$, *and there exists* $R_e > 0$ *such that*

$$\mathcal{I}_\mu(u) \leq 0, \quad \forall z \in E^- \oplus \mathbb{R}^+ e, \ \|z\| \geq R_e.$$

Proof Let $e \in E^+, t \geq 0$ and $u = te + u^- \in E^- \oplus \mathbb{R}^+ e$. Since $\mu_i \geq 0$, we have

$$\mathcal{I}_\mu(u) \leq \mathcal{I}_0(u) = \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \int_{\mathbb{R}^N} F(x, u) \mathrm{d}x.$$

For the proof of the functional $\mathcal{I}_0$ is standard, see [34]. So we omit its details here.

Corollary 3.3 *Assume that* (V1'), (F1)–(F4) *are satisfied. Let* $e \in E^+$ *with* $\|e\| = 1$. *Then there exists* $r_0 > \rho$ *such that* $\sup \mathcal{I}_\mu(\partial Q) \leq 0$ *as* $r \geq r_0$, *where*

$$Q = \{\zeta + se : \zeta \in E^-, \ s \geq 0, \ \|\zeta + se\| \leq r\}. \tag{3.3}$$

Lemma 3.6 *Assume that* (V1'), (F1)–(F4) *are satisfied and* $0 \leq \underline{\mu} \leq \overline{\mu} < \frac{(N-2)^2 l_0^2}{4}$. *Then there exist a constant* $c_\mu \in [\Lambda_\mu, \sup \mathcal{I}(Q)]$ *and a sequence* $\{u_n\} \subset E$ *satisfying*

$$\mathcal{I}_\mu(u_n) \to c, \quad \|\mathcal{I}'_\mu(u_n)\|(1 + \|z_n\|) \to 0,$$

where Q *is defined in* (3.3).

Proof Combining with Lemmas 3.1–3.2, 3.4 and Corollary 3.3, it is easy to verify Lemma 3.6. The proof will be omitted.

Lemma 3.7 *Assume that* (V), (F1)–(F4) *are satisfied and* $0 \leq \underline{\mu} \leq \overline{\mu} < \frac{(N-2)^2 l_0^2}{4}$. *Then there exist a constant* $c_\mu \in [\Lambda_\mu, m_\mu]$ *and a sequence* $\{u_n\} \subset E$ *satisfying*

$$\mathcal{I}_\mu(u_n) \to c_\mu, \quad \|\mathcal{I}'_\mu(u_n)\|(1 + \|u_n\|) \to 0. \tag{3.4}$$

Proof This is a standard result which can be found in [32–33], for the convenience of readers, we give the detailed proof process here. Choose $\xi_k \in \mathcal{N}_\mu$ such that

$$m_\mu \leq \mathcal{I}_\mu(\xi_k) < m_\mu + \frac{1}{k}, \quad k \in \mathbb{N}. \tag{3.5}$$

Using Lemma 3.4, we can derive $\|\xi_k^+\| \geq \sqrt{2m_\mu} > 0$. Let $e_k = \frac{\xi_k}{\|\xi_k\|}$. Then $e_k \in E^+$ with $\|e_k\| = 1$. Applying Corollary 3.3, there exists a constant $r_k > \max\{\rho, \|\xi_k\|\}$ satisfying $\sup \mathcal{I}_\mu(\partial Q_k) \leq 0$, where

$$Q_k = \{\zeta + se_k : \zeta \in E^-, s \geq 0, \|\zeta + se_k\| \leq r_k\}, \quad k \in \mathbb{N}. \tag{3.6}$$

Then, by Lemma 3.6, there exist a constant $c_k \in [\Lambda_\mu, \sup \mathcal{I}_\mu(Q_k)]$ and a sequence $\{u_{k,n}\}_{n \in \mathbb{N}} \subset E$,

$$\mathcal{I}_\mu(u_{k,n}) \to c_k, \quad \|\mathcal{I}'_\mu(u_{k,n})\|(1 + \|u_{k,n}\|) \to 0, \quad k \in \mathbb{N}. \tag{3.7}$$

In virtue of Corollary 3.1, we get

$$\mathcal{I}_\mu(\xi_k) \geq \mathcal{I}_\mu(\eta\xi_k + \zeta), \quad \forall \eta \geq 0,\ \zeta \in E^-. \tag{3.8}$$

Since $\xi_k \in Q_k$, by (3.6) and (3.8) we have $\mathcal{I}_\mu(\xi_k) = \sup \mathcal{I}_\mu(Q_k)$. Furthermore, by (3.5) and (3.7), we have

$$\mathcal{I}_\mu(u_{k,n}) \to c_k < m + \frac{1}{k}, \quad \|\mathcal{I}'_\mu(u_{k,n})\|(1 + \|u_{k,n}\|) \to 0, \quad k \in \mathbb{N}.$$

We can choose $\{n_k\} \subset \mathbb{N}$ such that

$$\mathcal{I}_\mu(u_{k,n_k}) < m_\mu + \frac{1}{k}, \quad \|\mathcal{I}'_\mu(u_{k,n_k})\|(1 + \|u_{k,n_k}\|) < \frac{1}{k}, \quad k \in \mathbb{N}.$$

Set $u_k = u_{k,n_k}, k \in \mathbb{N}$. Then we have

$$\mathcal{I}_\mu(u_n) \to c_* \in [\kappa, m_\mu], \quad \|\mathcal{I}'_\mu(u_n)\|(1 + \|u_n\|) \to 0.$$

Lemma 3.8 *Assume that* (V1'), (F1)–(F4) *are satisfied. Then for any* $u \in E \setminus E^-$, $\mathcal{N}^- \cap (E^- \oplus \mathbb{R}^+ z) \neq \emptyset$, *there exist* $\eta(u) > 0, \zeta(u) \in E^-$ *such that* $\eta(u)u + \zeta(u) \in \mathcal{N}_\mu$.

Proof Note that $E^- \oplus \mathbb{R}^+ u = E^- \oplus \mathbb{R}^+ u^+$, then we may assume that $u \in E^+$. It follows from Lemma 3.5 that there exists a constant $R > 0$ such that $\mathcal{I}_\mu(u) \leq 0$ for any $u \in (E^- \oplus \mathbb{R}^+ z) \setminus B_R(0)$. For sufficiently small $s \geq 0$, we have $\mathcal{I}_\mu(su) > 0$. Thus, $0 < \sup \mathcal{I}_\mu(E^- \oplus \mathbb{R}^+ u) < \infty$. It is easy to show that $\mathcal{I}_\mu$ is weakly continue on $E^- \oplus \mathbb{R}^+ u$, then for some $u_0 \in E^- \oplus \mathbb{R}^+ u$, we have $\mathcal{I}_\mu(u_0) = \sup \mathcal{I}_\mu(E^- \oplus \mathbb{R}^+ u)$. So u_0 is a critical point of $\mathcal{I}_\mu|_{E^- \oplus \mathbb{R}u}$. Moreover, $\langle \mathcal{I}'_\mu(u_0), u_0 \rangle = \langle \mathcal{I}'_\mu(u_0), \zeta \rangle$, $\forall \zeta \in E^- \oplus \mathbb{R}u$. From the above discussion, we can derive that $u_0 \in \mathcal{N}_\mu \cap (E^- \oplus \mathbb{R}^+ z)$.

4 Periodic Case

In this section, we assume that V and f are 1-periodic in each of $x_1, x_2, \cdots, x_N$, i.e., (V1) and (F1) are satisfied. In this case, $V_i = U_i$ and $W_i = 0$.

Lemma 4.1 *Assume that* (V1), (F1)–(F4) *are satisfied. Then for any* $\{u_n\} \subset E$ *such that*

$$0 < \overline{\mu^{(n)}} \leq \overline{\mu^{(0)}} < \frac{(N-2)^2 l_0^2}{4}$$

and

$$\sup_n |\mathcal{I}_{\mu^{(n)}}(u_n)| < \infty, \quad \|\mathcal{I}'_{\mu^{(n)}}(u_n)\|(1 + \|u_n\|) \to 0 \tag{4.1}$$

is bounded in E, *where* $\overline{\mu^{(n)}} := \max\{\mu_1^{(n)}, \mu_2^{(n)}, \cdots, \mu_K^{(n)}\}$ *and* $\overline{\mu^{(0)}} := \max\{\mu_1^{(0)}, \mu_2^{(0)}, \cdots, \mu_K^{(0)}\}$.

Proof Choose $M > 0$ such that $|\mathcal{I}_{\mu^{(n)}}(u_n)| \leq M$. We prove the boundedness of $\{u_n\}$ by negation, if the assertion would not hold, then $\|u_n\| \to \infty$. Denote $v_n = \frac{u_n}{\|u_n\|}$, we have $\|v_n\| = 1$. Taking into account Sobolev embedding theorem, there exists a constant $C_1 > 0$ such that $\|v_n\|_2 \leq C_1$. If

$$\delta := \limsup_{n\to\infty} \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |v_n^+|^2 \mathrm{d}x = 0,$$

it is easy to verify that $v_n^+ \to 0$ in L^p ($p \in (2, 2^*)$) by using Lions' concentration compactness principle. Fix $R > (N-2)\sqrt{\frac{2(1+M)l_0^2}{(N-2)^2 l_0^2 - 4\overline{\mu^{(0)}}}}$, combining (F1) with (F2), we see that there exists a constant $C_\varepsilon > 0$ such that

$$F(x,u) \le \varepsilon |u|^2 + C_\varepsilon |u|^p$$

for $\varepsilon = \frac{1}{4(RC_1)^2} > 0$, where $(x,u) \in \mathbb{R}^N \times \mathbb{R}^K$. Hence, we have

$$\begin{aligned}
&\limsup_{n\to\infty} \int_{\mathbb{R}^N} F\Big(x, \frac{Ru_n^+}{\|u_n\|}\Big)\mathrm{d}x \\
&= \limsup_{n\to\infty} \int_{\mathbb{R}^N} F(x, Rv_n^+)\mathrm{d}x \\
&\le \limsup_{n\to\infty} R^2\varepsilon \int_{\mathbb{R}^N} |v_n^+|^2 \mathrm{d}x + \limsup_{n\to\infty} R^p C_\varepsilon \int_{\mathbb{R}^N} |v_n^+|^2 \mathrm{d}x \\
&\le \varepsilon (RC_1)^2 = \frac{1}{4}.
\end{aligned} \tag{4.2}$$

Set $\eta_n = \frac{R}{\|u_n\|}$, by virtue of Corollary 3.2, we have, in light of (4.2),

$$\begin{aligned}
M &\ge \mathcal{I}_{\mu^{(n)}}(u_n) \\
&\ge \frac{\eta_n^2}{2}\|u_n\|^2 - \int_{\mathbb{R}^N} F(x, \eta_n u_n^+)\mathrm{d}x + \frac{1-\eta_n^2}{2}\langle \mathcal{I}'_{\mu_n}(u_n), u_n\rangle \\
&\quad + \eta_n^2 \langle \mathcal{I}'_{\mu_n}(u_n), u_n^-\rangle - \frac{\overline{\mu^{(n)}}\eta_n^2}{2}\int_{\mathbb{R}^N} \frac{|u_n^+|^2 - |u_n^-|^2}{|x|^2}\mathrm{d}x \\
&= \frac{R^2}{2}\|v_n\|^2 - \int_{\mathbb{R}^N} F(x, Rv_n^+)\mathrm{d}x + \Big(\frac{1}{2} - \frac{R^2}{2\|u_n\|^2}\Big)\langle \mathcal{I}'_{\mu_n}(u_n), u_n\rangle \\
&\quad + \frac{R^2}{\|u_n\|^2}\langle \mathcal{I}'_{\mu_n}(u_n), u_n^-\rangle - \frac{\overline{\mu^{(n)}}R^2}{2}\int_{\mathbb{R}^N} \frac{|v_n^+|^2 - |v_n^-|^2}{|x|^2}\mathrm{d}x \\
&\ge \frac{R^2}{2} - \int_{\mathbb{R}^N} F(x, Rv_n^+)\mathrm{d}x - \frac{2\overline{\mu^{(n)}}R^2}{(N-2)^2}\int_{\mathbb{R}^N} |\nabla v_n^+|^2 \mathrm{d}x + o(1) \\
&= \frac{R^2}{2} - \int_{\mathbb{R}^N} F(x, Rv_n^+)\mathrm{d}x - \frac{2\overline{\mu^{(n)}}R^2}{(N-2)^2 l_0^2} + o(1) \\
&\ge \frac{R^2}{2}\Big[1 - \frac{4\overline{\mu^{(0)}}}{(N-2)^2 l_0^2}\Big] - \frac{1}{4} + o(1) > \frac{3}{4} + M + o(1).
\end{aligned}$$

This leads to a contradiction, so $\delta > 0$.

Without loss of generality, we suppose the existence of $k_n \in \mathbb{Z}^N$ such that

$$\int_{B_{1+\sqrt{N}}(k_n)} |v_n^+|^2 \mathrm{d}x > \frac{\delta}{2}.$$

Denote $\zeta_n(x) = v_n(x + k_n)$, then

$$\int_{B_{1+\sqrt{N}}(0)} |\zeta_n^+|^2 \mathrm{d}x > \frac{\delta}{2}. \tag{4.3}$$

Set $\widetilde{u}_n(x) = u_n(x + k_n)$, $\frac{\widetilde{u}_n}{\|u_n\|} = \zeta_n$, then $\|\zeta_n\| = 1$. Passing to a subsequence, we may assume that $\zeta_n \rightharpoonup \zeta$ on E, and $\zeta_n \to \zeta$, $\zeta_n \to \zeta$ on L^2_{loc} a.e. on $\mathbb{R}^N$. It is evident that (4.3) implies that $\zeta \neq 0$. Thus, by virtue of (2.1), (F3) and Fatou' lemma, we see that

$$\begin{aligned}
0 &= \lim_{n\to\infty} \frac{\mathcal{I}_{\mu^{(n)}}(u_n)}{\|u_n\|^2} \\
&= \lim_{n\to\infty} \Big[\frac{1}{2}(\|v_n^+\|^2 - \|v_n^-\|^2) - \frac{1}{2\|u_n\|^2}\sum_{i=1}^{K}\int_{\mathbb{R}^N} \frac{\mu_i^{(n)}(u_n)_i^2}{|x|^2}\mathrm{d}x - \int_{\mathbb{R}^N} \frac{F(x,u_n)}{\|u_n\|^2}\mathrm{d}x\Big] \\
&\leq \lim_{n\to\infty} \Big[\frac{1}{2}(\|v_n^+\|^2 - \|v_n^-\|^2) - \frac{1}{2\|u_n\|^2}\sum_{i=1}^{K}\int_{\mathbb{R}^N} \frac{\mu_i^{(n)}(u_n)_i^2}{|x|^2}\mathrm{d}x - \int_{\mathbb{R}^N} \frac{F(x,u_n)}{|\widetilde{u}_n|^2}|\zeta_n|^2\mathrm{d}x\Big] \\
&\leq \frac{1}{2} - \liminf_{n\to\infty} \int_{\mathbb{R}^N} \frac{F(x,u_n)}{|\widetilde{u}_n|^2}|\zeta_n|^2\mathrm{d}x \\
&= -\infty,
\end{aligned}$$

which is a contradiction. Hence the statement of Lemma 4.1 is proved.

The following fact is very useful to deal with the Hardy type term and plays a very important role in the proof of the decomposition result. Their proofs are similar to those in [10], which we omit here.

Lemma 4.2 *If $|x_n| \to \infty$, then for any $u \in E$,*

$$\int_{\mathbb{R}^N} \frac{|u(\cdot - x_n)|}{|x|^2}\mathrm{d}x \to 0 \quad \text{as } n \to \infty.$$

Lemma 4.3 *Assume that $0 \leq \underline{\mu} \leq \overline{\mu} < \frac{(N-2)^2 l_0^2}{4}$ and let $\{u_n\}$ be a bounded $(C)_{c_\mu}$-sequence of $\mathcal{I}_\mu$ at level $c_\mu \geq 0$. Then there exists $u_\mu \in E$ such that $\mathcal{I}'_\mu(u_\mu) = 0$, and there exist a number $k \in \mathbb{N}\cup\{0\}$, nontrivial critical points $u_1, \cdots, u_k$ of $\mathcal{I}_0$ and k sequences of points $x_n^i \subset \mathbb{Z}^N, 1 \leq i \leq k$, such that*

$$\begin{aligned}
&|x_n^i| \to +\infty, \quad |x_n^i - x_n^j| \to +\infty, \quad i \neq j,\ i,j = 1,2,\cdots,k, \\
&\|u_n - u_\mu - \sum_{i=1}^{k} u_i(\cdot - x_n^i)\| \to 0, \\
&c_\mu = \mathcal{I}_\mu(u_\mu) + \sum_{i=1}^{k} \mathcal{I}_0(u_i).
\end{aligned}$$

Lemma 4.4 *Assume that $Q \in C(\mathbb{R}^N \times \mathbb{R}^K, \mathbb{R})$ and there exist $a_0, b_1, b_2 > 0, p \in (2, 2^*)$ and $1 < q_2 \leq q_1 < 2$ such that*

$$Q(x,u) \leq a_0(|u|^2 + |u|^p), \quad \forall (x,u) \in \mathbb{R}^N \times \mathbb{R}^K,$$

$$Q(x,u) \geq \begin{cases} b_1|u|^{q_1} & \text{for } |u| \leq 1, \\ b_2|u|^{q_2} & \text{for } |u| > 1. \end{cases}$$

If $u_n \rightharpoonup u$ in E, and

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} Q(x,u_n)\mathrm{d}x = \lim_{n\to\infty} \int_{\mathbb{R}^N} Q(x,u)\mathrm{d}x,$$

then $u_n \to u$ in $L^{q_1}(\mathbb{R}^N)$.

Proof of Theorem 1.1 In light of Lemma 4.1, there exists a bounded sequence $\{u_n\} \subset E$ satisfying Lemma 3.6. Hence, there exists a constant $C_2 > 0$ such that $\|u_n\|_2 \le C_2$. If $\delta := \limsup\limits_{n\to\infty} \sup\limits_{y\in\mathbb{R}^N} \int_{B_1(y)} |u_n|^2 = 0$, then $u_n \to 0$ in L^p, where $p \in (2, 2^*)$. On the other hand, by virtue of (F1) and (F2), for $\varepsilon = \frac{c_\mu}{4C_2^2} > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$F(x,u) \le \varepsilon|u|^2 + C_\varepsilon |u|^p, \quad \forall (x,u) \in \mathbb{R}^N \times \mathbb{R}^K.$$

Based on the above discussion, we have

$$\begin{aligned}
&\limsup_{n\to\infty} \int_{\mathbb{R}^N} \Big[\frac{1}{2}F_u(x,u_n)\cdot u_n - F(x,u_n)\Big]\mathrm{d}x \\
\le\ & \frac{3\varepsilon}{2}C_2^2 + \frac{3\varepsilon}{2}C_\varepsilon \lim_{n\to\infty} |u_n|^p \\
=\ & \frac{3c_\mu}{8}.
\end{aligned}$$

Thus,

$$\begin{aligned}
c_\mu &= \mathcal{I}_\mu(u_n) - \frac{1}{2}\langle \mathcal{I}'_\mu(u_n), u_n\rangle + o(1) \\
&= \int_{\mathbb{R}^N} \Big[\frac{1}{2}F_u(x,u_n)\cdot u_n - F(x,u_n)\Big]\mathrm{d}x + o(1) \\
&\le \frac{3c_\mu}{8} + o(1),
\end{aligned}$$

which is a contraction. Then $\delta > 0$.

Passing to the subsequence, we may assume that there exists $k_n \in \mathbb{Z}^N$ such that

$$\int_{B_{1+\sqrt{N}}(0)} |u_n^+|^2 > \frac{\delta}{2}.$$

Set $\zeta_n(x) = u_n(x + k_n)$, then

$$\int_{B_{1+\sqrt{N}}(0)} |\zeta_n^+|^2 \mathrm{d}x > \frac{\delta}{2}. \tag{4.4}$$

Due to the periodic assumption of $V(x)$ and $F(x,u)$, it follows that $\|\zeta_n\| = \|u_n\|$ and

$$\mathcal{I}_\mu(\zeta_n) \to c_\mu, \quad \|\mathcal{I}'_\mu(\zeta_n)\|(1 + \|\zeta_n\|) \to 0. \tag{4.5}$$

Thus, passing to the subsequence, suppose that $\zeta_n \rightharpoonup \zeta$ in E, $\zeta_n \to \zeta$ in L^2_{loc}, $\zeta_n(x) \to \zeta(x)$ a.e. on $\mathbb{R}^N$. In light of (4.4), we see that $\zeta \ne 0$. For every $\phi \in C_0^\infty(\mathbb{R}^N)$, by (2.2), we have $\langle \mathcal{I}'_\mu(\zeta), \phi\rangle = \lim\limits_{n\to\infty} \langle \mathcal{I}'_\mu(\zeta_n), \phi\rangle = 0$. Hence, $\mathcal{I}'_\mu(\zeta) = 0$, which implies that $\zeta \in \mathcal{N}_\mu$. Then, $\mathcal{I}_\mu(\zeta) \ge m_\mu$. On the other way, it follows from (F2), (F3), (F4), Lemmas 3.4, 3.7 and Fatou's lemma that

$$\begin{aligned}
m_\mu \ge c_\mu &= \lim_{n\to\infty} \Big[\mathcal{I}_\mu(u_n) - \frac{1}{2}\langle \mathcal{I}'_\mu(u_n), u_n\rangle\Big] \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^N} \Big[\frac{1}{2}F_u(x,u_n)\cdot u_n - F(x,u_n)\Big]\mathrm{d}x \\
&\ge \int_{\mathbb{R}^N} \lim_{n\to\infty} \Big[\frac{1}{2}F_u(x,u_n)\cdot u_n - F(x,u_n)\Big]\mathrm{d}x
\end{aligned}$$

$$= \int_{\mathbb{R}^N} \Big[\frac{1}{2} F_{u_\mu}(x, u_\mu) \cdot u_\mu - F(x, u_\mu)\Big] \mathrm{d}x$$
$$= \mathcal{I}_\mu(u_\mu) - \frac{1}{2} \langle \mathcal{I}'_\mu(u_\mu), u_\mu \rangle = \mathcal{I}_\mu(u_\mu),$$

which implies $\mathcal{I}_\mu(u_\mu) \le m_\mu$. So $\mathcal{I}_\mu(u_\mu) = m_\mu = \inf_{\mathcal{N}_\mu} \mathcal{I}_\mu > 0$. The proof is completed.

Next we claim that $u_\mu \ne 0$. Indeed, for $\mu = 0$, by Lemma 3.7 and the concentration compactness arguments, it is easy to prove that $\mathcal{I}_0$ has a nontrivial ground state solution $u_0 \in \mathcal{N}_0$ such that $\mathcal{I}_0(u_0) = m_0 = \inf_{u \in \mathcal{N}_0} \mathcal{I}_0$. Now let us assume that $0 < \underline{\mu} \le \overline{\mu} < \frac{(N-2)^2 l_0^2}{4}$ and consider

$$Q(u_0) := \{u = tu_0 + w : w \in E^-, t \ge 0, \|tu_0 + w\| \le R\}.$$

Observe that, let $t_n u_0 + w_n \in Q(u_0)$, then passing to a subsequence we may assume that $t_n \to t_0, w_n \rightharpoonup w_0$ in E^- and $L^2\big(\mathbb{R}^N, \frac{1}{|x|^2}\big)$, and $w_n(x) \to w_0(x)$ a.e. on $\mathbb{R}^N$. Hence $t_n u_0 + w_n \rightharpoonup t_0 u_0 + w_0 \in Q(u_0)$ by the weak lower semi-continuous of norm, which implies that $Q(z_0)$ is weakly sequentially closed. It follows from Fatou's lemma that

$$\limsup_{n\to\infty} \mathcal{I}_\mu(t_n u_0 + w_n) \le \mathcal{I}_\mu(t_0 u_0 + w_0),$$

this shows that $\mathcal{I}_\mu$ is weakly sequentially upper semi-continuous. Then $\mathcal{I}_\mu$ attains its maximum in $Q(u_0)$. Assume that $t_0 u_0 + w_0 \in Q(u_0)$ such that

$$\mathcal{I}_\mu(t_0 u_0 + w_0) = \max_{u \in Q(u_0)} \mathcal{I}_\mu(u) > 0,$$

then $t_0 u_0 + w_0 \in \mathcal{N}_\mu$. Therefore by Corollary 3.1, we have

$$m_0 = \mathcal{I}_0(u_0) \ge \mathcal{I}_0(t_0 u_0 + w_0) > \mathcal{I}_\mu(t_0 u_0 + w_0) \ge m_\mu \ge c_\mu,$$

similar to the Lemma 4.3, we get $u_n \to u_\mu$ in E, and so $u_\mu \ne 0$. The proof is completed.

Proof of Theorem 1.2 Let $u_\mu \in \mathcal{N}_\mu$ be a ground state solution of $\mathcal{I}_\mu$ and $0 \le \underline{\mu} \le \overline{\mu} < \frac{(N-2)^2 l_0^2}{4}$. In view of Lemma 3.8, there exist $t_\mu > 0$ and $w_\mu \in E^-$ such that $t_\mu u_\mu + w_\mu \in \mathcal{N}_0$. Then, by Corollary 3.1 we have

$$\begin{aligned} m_\mu &= \mathcal{I}_\mu(u_\mu) \ge \mathcal{I}_\mu(t_\mu u_\mu + w_\mu) \\ &= \mathcal{I}_0(t_\mu z_\mu + w_\mu) - \frac{1}{2} \sum_{i=1}^K \int_{\mathbb{R}^N} \frac{\mu_i |t_\mu (u_\mu)_i + (w_\mu)_i|^2}{|x|^2} \mathrm{d}x \\ &\ge m_0 - \frac{\overline{\mu}}{2} \int_{\mathbb{R}^N} \frac{|t_\mu u_\mu + w_\mu|^2}{|x|^2} \mathrm{d}x, \end{aligned} \tag{4.6}$$

this shows that conclusion (i) holds. Similarly, let $u_0 \in \mathcal{N}_0$ be a ground state solution of $\mathcal{I}_\mu$. By Lemma 3.8, there exist $t_0 > 0$ and $w_0 \in E^-$ such that $t_0 u_0 + w_0 \in \mathcal{N}_\mu$. Then, by Corollary 3.1 we have

$$\begin{aligned} m_0 &= \mathcal{I}_0(u_0) \ge \mathcal{I}_\mu(t_0 u_0 + w_0) \\ &= \mathcal{I}_\mu(t_0 u_0 + w_0) + \frac{1}{2} \sum_{i=1}^K \int_{\mathbb{R}^N} \frac{\mu_i |t_0 (u_0)_i + (w_0)_i|^2}{|x|^2} \mathrm{d}x \end{aligned}$$

$$\geq m_\mu + \frac{\mu}{2}\int_{\mathbb{R}^N} \frac{|t_0 u_0 + w_0|^2}{|x|^2}\mathrm{d}x, \tag{4.7}$$

which implies that conclusion (ii) holds.

Proof of Theorem 1.3 Since $\mu_i \geq 0$, we get by (4.7),

$$m_0 \geq m_\mu = \mathcal{I}_\mu(u_\mu), \tag{4.8}$$

and by Lemma 4.1 we have $\{u_\mu\}$ is bounded if $\mu \to 0^+$. We take a sequence $\mu^{(n)} \to 0^+$ and denote $u_n := u_{\mu^{(n)}}$. If

$$\limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B(y,1)} |u_n^+|^2 \mathrm{d}x = 0,$$

then by Lions' concentration compactness principle, we get $u_n^+ \to 0$ in L^p for $2 < p < 2^*$. Thus, from $u_n \in \mathcal{N}_{\mu^{(n)}}$, it follows that

$$\|u_n^+\|^2 = \sum_{i=1}^K \int_{\mathbb{R}^N} \frac{\mu_i^{(n)}(u_n)_i(u_n)_i^+}{|x|^2}\mathrm{d}x + \int_{\mathbb{R}^N} F_u(x,u_n)\cdot u_n^+\mathrm{d}x \to 0,$$

which shows that $\limsup\limits_{n\to\infty}\mathcal{I}_{\mu^{(n)}}(u_n) \leq 0$, this implies a contradiction with Lemma 3.4(i). Therefore, there exist $\varrho > 1$ and $\{y_n\} \subset \mathbb{Z}^N$ such that

$$\liminf_{n\to\infty} \int_{B(y_n,\varrho)} |u_n^+|^2\mathrm{d}x \geq 0,$$

then passing to a subsequence, we find $u \in E$ such that $u_n^+(\cdot + y_n) \to u^+$ in L^2_{loc} and $u^+ \neq 0$. Moreover, we may assume that $u_n(\cdot+y_n) \rightharpoonup u$ in E, $u_n(x+y_n) \to u(x)$, $u_n^+(x+y_n) \to u^+(x)$ a.e. on $\mathbb{R}^N$. Let $t_n u_n + w_n \in \mathcal{N}_0$ and $t_n > 0, w_n \in E^-$. By (F6), we have $f(x,u)\cdot u - 2F(x,u) \geq 0$. Then

$$\begin{aligned}\|u_n^+\|^2 &= \Big\|u_n^- + \frac{w_n}{t_n}\Big\|^2 + \frac{1}{t_n^2}\int_{\mathbb{R}^N} f(x,t_n u_n + w_n)\cdot(t_n u_n + w_n)\mathrm{d}x \\ &\geq \Big\|u_n^- + \frac{w_n}{t_n}\Big\|^2 + 2\int_{\mathbb{R}^N} \frac{F\big(x,t_n\big(u_n + \frac{w_n}{t_n}\big)\big)}{t_n^2}\mathrm{d}x,\end{aligned} \tag{4.9}$$

which implies that $\big\|u_n^- + \frac{w_n}{t_n}\big\|$ is bounded. Hence we may assume that $u_n^- + \frac{w_n}{t_n} \to w(x)$ a.e. on $\mathbb{R}^N$ for some $w \in E^-$. Now we claim that t_n is bounded. If not, then $|t_n u_n(x) + w_n(x)| = t\big|u_n^- + \frac{w_n}{t_n}\big| \to \infty$ provided that $u^+ + w(x) \neq 0$. It follows from (F4) and Fatou's lemma that

$$\int_{\mathbb{R}^N} \frac{F\big(x,t_n\big(u_n + \frac{w_n}{t_n}\big)\big)}{t_n^2}\mathrm{d}x \to \infty,$$

which contradicts (4.9), thus t_n is bounded. Then $\|t_n u_n^+\|$ and $\|t_n u_n^- + w_n\|$ are bounded, by the Hölder's inequality and (2.4) we get

$$\frac{1}{2}\sum_{i=1}^K \int_{\mathbb{R}^N} \frac{\mu_i|t_n u_n + w_n|^2}{|x|^2}\mathrm{d}x \to 0. \tag{4.10}$$

Therefore, (4.6), (4.8) and (4.10) imply that conclusion (i) holds.

Next, we will verify that (ii) holds. Let $\{u_{\mu^{(n)}}\}$ be a sequence of ground state solutions of $\mathcal{I}_{\mu^{(n)}}$, and we take a sequence $\mu^{(n)} \to 0^+$ and denote $u_n := u_{\mu^{(n)}}$. It follows from Lemma 4.1 that $\{u_n\}$ is bounded, then passing to a subsequence, we may assume that $u_n \rightharpoonup u_0$ in E, $u_n \to u_0$ in L^p_{loc} for $2 \le p < 2^*$ and $u_n(x) \to u_0(x)$ a.e. on $\mathbb{R}^N$.

Define $\widetilde{u}_n(x) = u_n(x + x_n)$, then $\|\widetilde{u}_n\| = \|u_n\|$. Passing to a subsequence, $\widetilde{u}_n \rightharpoonup u$ in E, $\widetilde{u}_n \to u_0$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$, $\forall s \in [2, 2^*)$, $\widetilde{u}_n \to u_0$ a.e. on $\mathbb{R}^N$. For each $\phi \in C_0^\infty(\mathbb{R}^N)$, set $\phi_n = \phi(x - y_n)$, In view of the Hölder's inequality and Lemma 4.2, we have

$$\lim_{n\to\infty} \sum_{i=1}^{K} \int_{\mathbb{R}^N} \frac{(u_n)_i(\phi_n)_i}{|x|^2} \mathrm{d}x = 0.$$

Noting that $V_i(x)$ and $f(x,u)$ are periodic in x, it follows that

$$\begin{aligned}
\langle \mathcal{I}_0'(\widetilde{u}_n), \phi\rangle &= \sum_{i=1}^{K} \int_{\mathbb{R}^N} [\nabla(\widetilde{u}_n)_i \nabla\phi_i + V_i(x)(\widetilde{u}_n)_i\phi_i]\mathrm{d}x - \int_{\mathbb{R}^N} f(x, \widetilde{u}_n)\cdot \phi \mathrm{d}x \\
&= \sum_{i=1}^{K} \int_{\mathbb{R}^N} [\nabla(\widetilde{u}_n)_i \nabla(\phi_n)_i + V_i(x)(\widetilde{u}_n)_i(\phi_n)_i]\mathrm{d}x - \int_{\mathbb{R}^N} f(x, \widetilde{u}_n)\cdot \phi_n \mathrm{d}x \\
&= \langle \mathcal{I}_{\mu_n}'(\widetilde{u}_n), \phi_n\rangle + \sum_{i=1}^{K} \int_{\mathbb{R}^N} \frac{\mu_i^{(n)}(u_n)_i(\phi_n)_n}{|x|^2}\mathrm{d}x \\
&= o(1).
\end{aligned}$$

Thus, we have $\mathcal{I}_0'(u_0) = 0$, which implies that u_0 is a nontrivial critical point of $\mathcal{I}_0$. We will claim that u_0 is a ground state solution of $\mathcal{I}_0$. Since $\mu \ge 0$, it is to show that $m_\mu = \mathcal{I}_\mu(u)$ is non-increasing on $\overline{\mu} \in \big[0, \frac{(N-2)^2}{4} l_0^2\big)$. Then we obtain

$$\begin{aligned}
m_0 &\ge \limsup_{n\to\infty} m_{\mu^{(n)}} = \limsup_{n\to\infty} \mathcal{I}_{\mu^{(n)}}(u_n) \\
&= \limsup_{n\to\infty} \Big[\mathcal{I}_{\mu_n}(u_n) - \frac{1}{2}\langle \mathcal{I}_{\mu_n}'(u_n), u_n\rangle\Big] \\
&= \limsup_{n\to\infty} \int_{\mathbb{R}^N} \hat{F}(x, u_n)\mathrm{d}x = \limsup_{n\to\infty} \int_{\mathbb{R}^N} \hat{F}(x, \widetilde{u}_n)\mathrm{d}x \\
&\ge \liminf_{n\to\infty} \int_{\mathbb{R}^N} \hat{F}(x, \widetilde{u}_n)\mathrm{d}x \\
&\ge \int_{\mathbb{R}^N} \hat{F}(x, u_0)\mathrm{d}x = \mathcal{I}_0(u_0) \ge m_0.
\end{aligned}$$

This implies

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \hat{F}(x, \widetilde{u}_n)\mathrm{d}x = \int_{\mathbb{R}^N} \hat{F}(x, u_0)\mathrm{d}x.$$

Applying Lemma 4.4 to $\hat{F}(x,u)$, we have $\|\widetilde{u}_n - u_0\|_2 = 0$. Since $|f(x,u)| \le c_1|u|$ for some $c_1 > 0$, we deduce that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} f(x, \widetilde{u}_n)\cdot(\widetilde{u}_n^\pm - u_0^\pm)\mathrm{d}x = 0.$$

Thus

$$\|\widetilde{u}_n^+ - u_0^+\| = \langle \mathcal{I}_{\mu_n}'(\widetilde{u}_n), \widetilde{u}_n^+ - u_0^+\rangle - (u^+, \widetilde{u}_n^+ - u_0^+)$$

$$
\begin{aligned}
&+\sum_{i=1}^{K}\int_{\mathbb{R}^N}\frac{\mu_i\widetilde{u}_n(\widetilde{u}_n^+-u_0^+)}{|x|^2}\mathrm{d}x+\int_{\mathbb{R}^N}f(x,\widetilde{u}_n)\cdot(\widetilde{u}_n^+-u_0^+)\mathrm{d}x\\
&=o(1)
\end{aligned}
\tag{4.11}
$$

and

$$
\begin{aligned}
\|\widetilde{u}_n^- - u_0^-\| &= \langle \mathcal{I}'_{\mu_n}(\widetilde{u}_n),\widetilde{u}_n^- - u_0^-\rangle - (u_0^+,\widetilde{u}_n^- - u_0^-)\\
&\quad+\sum_{i=1}^{K}\int_{\mathbb{R}^N}\frac{\mu_i\widetilde{u}_n\cdot(\widetilde{u}_n^- - u_0^-)}{|x|^2}\mathrm{d}x+\int_{\mathbb{R}^N}f(x,\widetilde{u}_n)\cdot(\widetilde{u}_n^- - u_0^-)\mathrm{d}x\\
&=o(1).
\end{aligned}
\tag{4.12}
$$

It follows from (4.11)–(4.12) that $\widetilde{u}_n \to u_0$ in E, which implies that (ii) holds. The proof is completed.

5 Asymptotically Periodic Case

In this section, we always assume that $V(x)$ satisfies (V1'). We define functional $\mathcal{J}_\mu$ as follows

$$
\begin{aligned}
\mathcal{J}_\mu(u) &= \frac{1}{2}\sum_{i=1}^{K}\int_{\mathbb{R}^N}(|\nabla u_i|^2+U_i(x)|u_i|^2)\mathrm{d}x-\frac{1}{2}\sum_{i=1}^{K}\int_{\mathbb{R}^N}\frac{\mu_i u_i^2}{|x|^2}\mathrm{d}x-\int_{\mathbb{R}^N}F(x,u)\mathrm{d}x\\
&=\frac{1}{2}(\|u^+\|^2-\|u^-\|^2)-\frac{1}{2}\sum_{i=1}^{K}\int_{\mathbb{R}^N}\frac{\mu_i u_i^2}{|x|^2}\mathrm{d}x-\int_{\mathbb{R}^N}F(x,u)\mathrm{d}x.
\end{aligned}
$$

Then (V1'), (F1'), (F2)–(F5) imply that $\mathcal{J}_\mu \in C^1(E,\mathbb{R})$ and

$$
\langle \mathcal{J}'_\mu(u),v\rangle = \sum_{i=1}^{K}\int_{\mathbb{R}^N}(\nabla u_i\cdot\nabla v_i+U_i(x)u_iv_i)\mathrm{d}x-\sum_{i=1}^{K}\int_{\mathbb{R}^N}\frac{\mu_i u_i v_i}{|x|^2}\mathrm{d}x-\int_{\mathbb{R}^N}f(x,u)\cdot v\mathrm{d}x.
$$

Similar to Lemma 3.3, we have the following lemma.

Lemma 5.1 *Assume that* (V1'), (F1'), (F2)–(F5) *are satisfied. Then for all* $\kappa \geq 0, u \in E, \zeta = (\zeta_1,\zeta_2,\cdots,\zeta_K) \in E^-$,

$$
\begin{aligned}
\mathcal{J}_\mu(u) &\geq \mathcal{J}_\mu(\kappa u+\zeta)+\frac{1}{2}\|\zeta\|^2-\frac{1}{2}\sum_{i=1}^{K}\int_{\mathbb{R}^N}W_i(x)\zeta_i^2\mathrm{d}x\\
&\quad+\frac{1}{2}\sum_{i=1}^{K}\int_{\mathbb{R}^N}\frac{\mu_i\zeta_i^2}{|x|^2}\mathrm{d}x+\frac{1-\kappa^2}{2}\langle \mathcal{J}'_\mu(u),u\rangle-\kappa\langle \mathcal{J}'_\mu(u),\zeta\rangle.
\end{aligned}
\tag{5.1}
$$

Lemma 5.2 *Assume that* (V1'), (F1'), (F2)–(F5) *are satisfied. Then any sequence* $\{u_n\} \subset E$ *satisfying* (4.1) *is bounded in* E.

Proof To prove the boundedness of $\{u_n\}$, arguing by contradiction, we suppose that $\|u_n\| \to \infty$. Let $v_n = \frac{u_n}{\|u_n\|}$. Then $\|v_n\| = 1$. By Sobolev imbedding theorem, there exists a constant $C_4 > 0$ such that $\|v_n\|_2 \leq C_4$. Passing to a subsequence, we have $v_n \rightharpoonup v$ in E. There are two possible cases: (i) $v = 0$ and (ii) $v \neq 0$.

Case (i) $v = 0$, i.e., $v_n \rightharpoonup 0$ in E. Then $v_n^+ \to 0$ and $v_n^- \to 0$ in $L^s_{\text{loc}}(\mathbb{R}^N)$. By (V1'), it is easy to show that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} W_i(x)(v_i^+)^2 \mathrm{d}x = \lim_{n\to\infty} \int_{\mathbb{R}^N} W_i(x)(v_i^-)^2 \mathrm{d}x = 0. \tag{5.2}$$

If

$$\delta := \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |v_n^+|^2 \mathrm{d}x = 0,$$

then by Lions' concentration compactness principle, $v_n^+ \to 0$ in $L^s(\mathbb{R}^N)$ for $2 < s < 2^*$. Fix $R > [2(1+c_\mu)]^{\frac{1}{2}}$. By virtue of (F0) and (F1), for $\varepsilon = \frac{1}{4}(RC_4)^2 > 0$, there exists $C_\varepsilon > 0$ such that

$$\begin{aligned}\limsup_{n\to\infty} \int_{\mathbb{R}^N} F(x, Rv_n^+)\mathrm{d}x &\le \limsup_{n\to\infty} [\varepsilon R^2 \|v_n^+\|_2^2 + C_\varepsilon R^p \|v_n^+\|_p^p] \\ &\le \varepsilon (RC_4)^2 = \frac{1}{4}.\end{aligned} \tag{5.3}$$

Let $\eta_n = \frac{R}{\|u_n\|}$. Hence, by virtue of (4.1), (5.2)–(5.3) and Corollary 3.2, one can get that

$$\begin{aligned}c_\mu + o(1) &= \mathcal{I}_\mu(u_n) \\ &\ge \frac{\eta_n^2}{2}\|u_n\|^2 - \int_{\mathbb{R}^N} F(x, \eta_n u_n^+)\mathrm{d}x + \frac{1-\eta_n^2}{2}\langle \mathcal{I}'_\mu(u_n), u_n\rangle \\ &\quad + \eta_n^2 \langle \mathcal{I}'_\mu(u_n), u_n^-\rangle + \frac{\eta_n^2}{2}\sum_{i=1}^K \int_{\mathbb{R}^N} W_i(x)[((u_n)_i^+)^2 - ((u_n)_i^-)^2]\mathrm{d}x \\ &= \frac{R^2}{2} - \int_{\mathbb{R}^N} F\Big(x, \frac{Ru_n^+}{\|u_n\|}\Big)\mathrm{d}x + \Big(\frac{1}{2} - \frac{R^2}{2\|u_n\|^2}\Big)\langle \mathcal{I}'_\mu(u_n), u_n\rangle \\ &\quad + \frac{R^2}{\|u_n\|^2}\langle \mathcal{I}'_\mu(u_n), u_n^-\rangle + \frac{R^2}{2}\sum_{i=1}^K \int_{\mathbb{R}^N} W_i(x)[((v_n)_i^+)^2 - ((v_n)_i^-)^2]\mathrm{d}x \\ &= \frac{R^2}{2} - \int_{\mathbb{R}^N} F\Big(x, \frac{Ru_n^+}{\|u_n\|}\Big)\mathrm{d}x + o(1) \\ &\ge \frac{R^2}{2} - \frac{1}{4} + o(1) > \frac{3}{4} + c_\mu + o(1).\end{aligned}$$

This leads to a contradiction, so $\delta > 0$. Without loss of generality, we suppose the existence of $k_n \in \mathbb{Z}^N$ such that $\int_{B_{1+\sqrt{N}}(0)} |\omega_n^+|^2 \mathrm{d}x > \frac{\delta}{2}$. Denote $\omega_n(x) = v_n(x+k_n)$, then

$$\int_{B_{1+\sqrt{N}}(0)} |\omega_n^+|^2 \mathrm{d}x > \frac{\delta}{2}. \tag{5.4}$$

Put $\widetilde{u}_n(x) = u_n(x+k_n)$, $\frac{\widetilde{u}_n}{\|u_n\|} = \omega_n$, then $\|\omega_n\| = 1$. Passing to a subsequence, we may assume that $\omega_n \rightharpoonup \omega$ on E, and $\omega_n \to \omega$ on L^2_{loc} a.e. on $\mathbb{R}^N$. It is evident that (5.4) implies that $\omega \neq 0$. Thus, by virtue of (F4) and Fatou's lemma, we see that

$$\begin{aligned}0 &= \lim_{n\to\infty} \frac{c_\mu + o(1)}{\|u_n\|^2} \\ &= \lim_{n\to\infty} \frac{\mathcal{I}_\mu(u_n)}{\|u_n\|^2}\end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty}\Big[\frac{1}{2}(\|\omega_n^+\|^2-\|\omega_n^-\|^2)+\frac{1}{2}\sum_{i=1}^K\int_{\mathbb{R}^N}W_i(x)((v_n)_i)^2\mathrm{d}x\\
&\quad -\sum_{i=1}^K\frac{1}{2\|u_n\|^2}\int_{\mathbb{R}^N}\frac{\mu_i|(u_n)_i|^2}{|x|^2}\mathrm{d}x-\int_{\mathbb{R}^N}\frac{F(x,u_n)}{\|u_n\|^2}\mathrm{d}x\Big]\\
&= \lim_{n\to\infty}\Big[\frac{1}{2}(\|\omega_n^+\|^2-\|\omega_n^-\|^2)+\frac{1}{2}\sum_{i=1}^K\int_{\mathbb{R}^N}W_i(x)((v_n)_i)^2\mathrm{d}x-\int_{\mathbb{R}^N}\frac{F(x,u_n)}{|\widetilde{u}_n|^2}|\omega_n|^2\mathrm{d}x\Big]\\
&\le \frac{1}{2}-\liminf_{n\to\infty}\int_{\mathbb{R}^N}\frac{F(x,u_n)}{|\widetilde{u}_n|^2}|\omega_n|^2\mathrm{d}x\\
&= -\infty,
\end{aligned}
$$

which is a contradiction. Hence the statement of Lemma 5.2 is proved.

Case (ii) $v \neq 0$. In this case, we can also deduce a contradiction by a standard argument. Cases (i) and (ii) show that $\{u_n\}$ is bounded in E.

Proof of Theorem 1.4 Applying Lemmas 3.7 and 4.1, we deduce that there exists a bounded sequence $\{u_n\}\subset E$ satisfying (3.4). Passing to a subsequence, we have $u_n \rightharpoonup u$ in E. Next, we prove $u\neq 0$.

Arguing by contradiction, suppose that $u=0$, i.e., $u_n\rightharpoonup 0$ in E, and so $u_n\rightharpoonup 0$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$, $2\le s<2^*$ and $u_n\to 0$ a.e. on $\mathbb{R}^N$. By (V1'), (F1') and (F5'), it is easy to show that

$$
\lim_{n\to\infty}\int_{\mathbb{R}^N}W_i(x)u_n^2\mathrm{d}x=0,\quad \lim_{n\to\infty}\int_{\mathbb{R}^N}W_i(x)u_nv\mathrm{d}x=0 \tag{5.5}
$$

and

$$
\lim_{n\to\infty}\int_{\mathbb{R}^N}H(x,u_n)\mathrm{d}x=0,\quad \lim_{n\to\infty}\int_{\mathbb{R}^N}\partial_u H(x,u_n)\cdot v\mathrm{d}x=0,\quad \forall v\in E. \tag{5.6}
$$

Note that

$$
\mathcal{J}_\mu(u)=\mathcal{I}_\mu(u)-\frac{1}{2}\sum_{i=1}^K\int_{\mathbb{R}^N}W_i(x)u_i^2\mathrm{d}x+\int_{\mathbb{R}^N}H(x,u)\mathrm{d}x,\quad \forall u\in E \tag{5.7}
$$

and

$$
\langle\mathcal{J}_\mu'(u),v\rangle=\langle\mathcal{I}_\mu'(u),v\rangle-\sum_{i=1}^K\int_{\mathbb{R}^N}W_i(x)u_iv_i\mathrm{d}x+\int_{\mathbb{R}^N}\partial_uH(x,u)\cdot v\mathrm{d}x,\quad \forall u,v\in E. \tag{5.8}
$$

From (5.5)–(5.8), one can get that

$$
\mathcal{I}_\mu(u_n)\to c_\mu,\quad \|\mathcal{I}_\mu'(u_n)\|(1+\|u_n\|)\to 0. \tag{5.9}
$$

Analogous to the proof of Theorem 1.2, we can prove that there exists $k_n\in\mathbb{Z}^N$ such that

$$
\int_{B_{1+\sqrt{N}}(k_n)}|u_n^+|^2\mathrm{d}x>\frac{\delta}{2}.
$$

Denote $v_n(x)=v_n(x+k_n)$, then

$$
\int_{B_{1+\sqrt{N}}(0)}|v_n^+|^2\mathrm{d}x>\frac{\delta}{2}. \tag{5.10}
$$

Passing to a subsequence, we have $v_n \rightharpoonup v$ in E, $v_n \to v$ in $L^s_{\text{loc}}(\mathbb{R}^N)$, $2 \le s < 2^*$ and $v_n \to v$ a.e. on $\mathbb{R}^N$. Obviously, (5.10) implies that $v \neq 0$. Since $U_i(x)$ and $g_i(x,u)$ are periodic in x, by (5.9), we have

$$\mathcal{J}_\mu(v_n) \to c_\mu, \quad \|\mathcal{J}'_\mu(v_n)\|(1+\|v_n\|) \to 0. \tag{5.11}$$

In the same way as the last part of the proof of Theorem 1.2, we can prove that $\mathcal{J}'_\mu(v) = 0$ and $\mathcal{J}_\mu(v) \le c_\mu$.

It is easy to show that $v^+ \neq 0$. By Lemma 3.8, there exist $\kappa_0 = \kappa(v) > 0$ and $w_0 = w(v) \in E^-$ such that $\kappa_0 v + w_0 \in \mathcal{N}_\mu$, and so $\mathcal{I}_\mu(\kappa_0 v + w_0) \ge m$.

Hence, from the fact that $H(x,u) - \sum\limits_{i=1}^{K} W_i(x)u_i^2 > 0$, for $(x,u) \in B_{1+\sqrt{N}}(0) \times \mathbb{R}^K \setminus \{0\}$, we have

$$\begin{aligned}
m_\mu \ge c_\mu \ge \mathcal{J}_\mu(v)
&= \mathcal{J}_\mu(\kappa_0 v + w_0) + \frac{1}{2}\|w_0\|^2 + \frac{1-\kappa_0^2}{2}\langle \mathcal{J}'_\mu(v), v\rangle - \kappa_0 \langle \mathcal{J}'_\mu(v), w_0\rangle \\
&\quad + \int_{\mathbb{R}^N} [G(x, \kappa_0 v + w_0) - G(x,v)]\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N} \Big[\frac{1-\kappa_0^2}{2} G_u(x,u)\cdot u - \kappa G_u(x,u)\cdot \zeta\Big]\mathrm{d}x \\
&\ge \mathcal{J}_\mu(\kappa_0 u + w_0) + \frac{1}{2}\|w_0\|^2 \\
&= \frac{1}{2}\|w_0\|^2 + \mathcal{I}_\mu(\kappa_0 V + w_0) \\
&\quad + \int_{\mathbb{R}^N} H(x, \kappa_0 v + w_0)\mathrm{d}x - \frac{1}{2}\sum_{i=1}^{K}\int_{\mathbb{R}^N} W_i(x)((\kappa_0 v + w_0)_i)^2 \mathrm{d}x \\
&> \mathcal{I}_\mu(\kappa_0 v + w_0) \ge m_\mu,
\end{aligned}$$

since $v(x) \not\equiv 0$ for $x \in B_{1+\sqrt{N}}(0)$. This contradiction implies that $u \neq 0$. In the same way as the last part of the proof of Theorem 1.2, we can certify that $\mathcal{I}'_\mu(u) = 0$ and $\mathcal{I}_\mu(u) = m = \inf\limits_{\mathcal{N}_\mu} \mathcal{I}_\mu$. This shows that $u \in E$ is a solution to (1.2) with $\mathcal{I}_\mu(u) = \inf\limits_{\mathcal{N}_\mu} \mathcal{I}_\mu > 0$. The proof is completed.

Similar to the proofs of Theorems 1.2–1.3, we can prove Theorems 1.5–1.6, we omit the proof process.

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References

[1] Cao, D. M. and Han, P. G., Solutions for semilinear elliptic equations with critical exponents and Hardy potential, *J. Differential Equations*, **205**, 2004, 521–537.

[2] Cao, D. M. and Peng, S. J., A global compactness result for singular elliptic problems involving critical Sobolev exponent, *Proc. Amer. Math. Soc.*, **131**, 2003, 1857–1866.

[3] Cao, D. M. and Peng, S. J., A note on the sign-changing solutions to elliptic problems with critical Sobolev and Hardy terms, *J. Differential Equations*, **193**, 2003, 424–434.

[4] Chen, P., Tang, X. H. and Zhang, L. M., Non-nehari manifold method for hamiltonian elliptic system with hardy potential: Existence and asymptotic properties of ground state solution, *J. Geom. Anal.*, **32**(2), 2022, 46.

[5] Chen, Z. J. and Zou, W. M, On an elliptic problem with critical exponent and Hardy potential, *J. Differential Equations*, **252**, 2012, 969–987.

[6] Deng, Y. B., Jin, L. and Peng, S. J., Solutions of Schrödinger equations with inverse square potential and critical nonlinearity, *J. Differential Equations*, **253**, 2012, 1376–1398.

[7] Felli, V., On the existence of ground state solutions to nonlinear Schrödinger equations with multisingular inverse-square anisotropic potentials, *J. Anal. Math.*, **108**, 2009, 189–217.

[8] Felli, V., Marchini, E. and Terracini, S., On Schrödinger operators with multipolar inverse-square potentials, *J. Funct. Anal.*, **250**, 2007, 265–316.

[9] Felli, V. and Terracini, S., Elliptic equations with multi-singular inverse-square potentials and critical nonlinearity, *Comm. Partial Differential Equations*, **31**, 2006, 469–495.

[10] Guo, Q. Q. and Mederski, J., Ground states of nonlinear Schrödinger equations with sum of periodic and inverse-square potentials, *J. Differential Equations*, **260**, 2016, 4180–4202.

[11] Guo, Y. J., Li, S. and Wei, J. C., Ground states of two-component attractive Bose-Einstein condensates I: Existence and uniqueness, *J. Funct. Anal.*, **276**(1), 2019, 183–230.

[12] Guo, Y. J., Li, S., Wei, J. C. and Zeng, X. Y., Ground States of two-component attractive Bose-Einstein condensates II: Semi-trivial limit behavior, *T. Am. Math. Soc.*, **371**(10), 2019, 6903–6948.

[13] Li, G. B. and Szulkin, A., An asymptotically periodic Schrödinger equation with indefinite linear part, *Commun. Contemp. Math.*, **4**, 2002, 763–776.

[14] Lin, T. C. and Wei, J. C., Spikes in two-component systems of nonlinear Schrödinger equations with trapping potentials, *J. Differential Equations*, **229**(2), 2006, 538–569.

[15] Lin, T. and Wei, J. C., Ground state of N coupled nonlinear Schrödinger equations in $\mathbb{R}^n, n \leq 3$, *Commu. Math. Phys.*, **277**(2), 2008, 573–576.

[16] Lin, X. Y., He, Y. B. and Tang, X. H., Existence and asymptotic behavior of ground state solutions for asymptotically linear Schrödinger equation with inverse square potential, *Commun. Pure. Appl. Anal.*, **18**(3), 2019, 1547–1565.

[17] Liu S. B., On superlinear Schrödinger equations with periodic potential, *Calc. Var. Partial Differential Equations*, **45**, 2012, 1–9.

[18] Maia, L. A., Montefusco, E. and Pellacci, B., Positive solutions for a weakly coupled nonlinear Schrödinger system, *J. Differential Equations*, **229**(2), 2006, 743–767.

[19] Malomed, B., Multi-component Bose-Einstein condensates: Theory, In: Emergent Nonlinear Phenomena in Bose-Einstein Condensation, P. G. Kevrekidis et al. (eds.), Atomic, Optical, and Plasma Physics, **45**, Springer-Verlag, Berlin, 2008, 287–305.

[20] Mederski, J., Ground states of a system of nonlinear Schrödinger equations with periodic potentials, *Comm. Partial Differential Equations*, **41**(9), 2016, 1426–1440.

[21] Montefusco, E., Pellacc, B. and Squassina, M., Semiclassical states for weakly coupled nonlinear Schrödinger systems, *J. Eur. Math. Soc.*, **10**(1), 2008, 47–71.

[22] Pankov, A., Periodic nonlinear schrödinger equation with application to photonic crystals, *Milan J. Math.*, **73**, 2005, 259–287.

[23] Pankov, A., On decay of solutions to nonlinear Schrödinger equations, *Proc. Amer. Math. Soc.,* **136**, 2008, 2565–2570.

[24] Peng, S. J. and Wang, Z. Q., Segregated and synchronized vector solutions for nonlinear Schrödinger systems, *Arch. Ration. Mech. Anal.*, **208**(1), 2013, 305–339.

[25] Reed, M. and Simon, B., Methods of Modern Mathematical Physics, Analysis of Operators, Vol. IV, Academic Press, New York, 1978.

[26] Ruegg, Ch. et al., Bose-Einstein condensation of the triplet states in the magnetic insulator TICuCI3, *Nature*, **423**, 2003, 62–65.

[27] Ruiz, D. and Willem, M., Elliptic problems with critical exponents and Hardy potentials, *J. Differential Equations*, **190**, 2003, 524–538.

[28] Simon, B., Schrödinger semigroups, *Bull. Amer. Math. Soc.*, **7**(3), 1982, 447–526.

[29] Sirakov, B., Least-energy solitary waves for a system of nonlinear Schrödinger equations in $\mathbb{R}^n$, *Comm. Math. Phys.*, **271**, 2007, 199–221.

[30] Smets, D., Nonlinear Schrödinger equations with Hardy potential and critical nonlinearities, *Trans. Amer. Math. Soc.*, **357**, 2005, 2909–2938.

[31] Szulkin, A. and Weth, T., Ground state solutions for some indefinite variational problems, *J. Funct. Anal.*, **257**(12), 2009, 3802–3822.

[32] Tang, X. H., Non-Nehari manifold method for superlinear Schrödinger equation, *Taiwanese J. Math.*, **18**, 2014, 1957–1979.

[33] Tang, X. H. and Chen, S. T., Ground state solutions of Nehari-Pohozaev type for Kirchhoff-type problems with general potentials, *Calc. Var. Partial Differential Equations*, **55**, 2017, 110.

[34] Tang, X. H, Chen, S. T., Lin, X. Y. and Yu, J. S., Ground state solutions of Nehari-Pankov type for Schrödinger equations with local super-quadratic conditions, *J. Differential Equations*, **268**, 2020, 4663–4690.

[35] Wei, J. C. and Wu, Y. Z., Ground states of nonlinear Schrödinger systems with mixed couplings, *J. Math. Pure. Appl.*, **141**, 2020, 50–88.

[36] Wu, Y. Z., Ground states of a K-component critical system with linear and nonlinear couplings: The attractive case, *Adv. Nonlinear Stud.*, **19**(3), 2019, 595–623.

[37] Yang, M. B., Chen, W. X. and Ding, Y. H., Solutions of a class of Hamiltonian elliptic systems in $\mathbb{R}^N$, *J. Math. Anal. Appl.*, **362**, 2010, 338–349.

[38] Zhang, J., Tang, X. H. and Zhang, W., Ground-state solutions for superquadratic Hamiltonian elliptic systems with gradient terms, *Nonlinear Anal.*, **95**, 2014, 1–10.

[39] Zhang, J., Zhang, W. and Xie, X. L., Existence and concentration of semiclassical solutions for Hamiltonian elliptic system, *Comm. Pure Appl. Anal.*, **15**, 2016, 599–622.

[40] Zhao, F. K. and Ding, Y. H., On Hamiltonian elliptic systems with periodic or non-periodic potentials, *J. Differential Equations*, **249**, 2010, 2964–2985.