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A Note on the Indirect Controls for a Coupled System of Wave Equations*

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Abstract By a procedure of successive projections, the authors decompose a coupled system of wave equations into a sequence of sub-systems. Then, they can clarify the indirect controls and the total number of controls. Moreover, the authors give a uniqueness theorem of solution to the system of wave equations under Kalman's rank condition.

Keywords Indirect controls, Approximate controllability, Coupled system of wave equations
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1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary Γ. Consider the following system for the variable $U = (u^{(1)}, \cdots, u^{(N)})^{\mathrm{T}}$:

$$\begin{cases} U'' - \Delta U + AU = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ U = DH & \text{on } \mathbb{R}^+ \times \Gamma, \end{cases} \tag{1.1}$$

where A is a matrix of order N and D is a matrix of order $N \times M$.

Obviously, system (1.1) is controlled by the controls H directly acted on the boundary, and also implicitly influenced by the interaction between the equations. It is well-known that when $\mathrm{rank}(D) < N$, because of the compactness of the coupling term AU, system (1.1) is never exactly controllable in the space $(L^2(\Omega) \times H^{-1}(\Omega))^N$ (see [6]). However, the following Kalman's rank condition

$$\mathrm{rank}(D, AD, \cdots, A^{N-1}D) = N \tag{1.2}$$

is necessary (and even sufficient in some special situations) for the approximate boundary controllability of system (1.1) (see [7]). This shows that the coupling term AU plays an important role for the approximate boundary controllability. It seems that $\mathrm{rank}(D, AD, \cdots, A^{N-1}D)$, called the total number of controls in [8], is a good indicator for the action of the coupling matrix A with the boundary control matrix D. Since $\mathrm{rank}(D)$ is the number of boundary controls

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H, it is natural to name $\text{rank}(D, AD, \cdots, A^{N-1}D) - \text{rank}(D)$ as the number of indirect controls, but we did not see what are these (internal or boundary) controls, nor how they intervene into the system.

In this paper, we try to explain the meaning of indirect controls and the mechanism of their roles. The basic idea is to project system (1.1) to $\text{Ker}(D^{\text{T}})$ for getting a system with a homogeneous boundary condition. We first show the idea by a simple example, and present the general procedure later.

Example 1 Consider the following system

$$\begin{cases} u_1'' - \Delta u_1 + u_2 = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ u_2'' - \Delta u_2 + u_3 = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ u_3'' - \Delta u_3 = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ u_1 = u_2 = 0, \quad u_3 = h & \text{on } \mathbb{R}^+ \times \Gamma. \end{cases} \tag{1.3}$$

First let

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

We have

$$\text{Ker}(D^{\text{T}}) = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Then, applying the row-vectors $(1, 0, 0)$ and $(0, 1, 0)$ in $\text{Ker}(D^{\text{T}})$ to system (1.3), we get

$$\begin{cases} u_1'' - \Delta u_1 + u_2 = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ u_2'' - \Delta u_2 = -u_3 & \text{in } \mathbb{R}^+ \times \Omega, \\ u_1 = u_2 = 0 & \text{on } \mathbb{R}^+ \times \Gamma. \end{cases} \tag{1.4}$$

The reduced system (1.4) is for the variables u_1 and u_2, so at the first step, the variable $h^{(1)} = -u_3$ can be formally regarded as an internal control appearing in system (1.4). However, the value of $h^{(1)}$ can not be freely chosen, then we call it as an indirect internal control.

Next let

$$A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

We have

$$\text{Ker}(D_1^{\text{T}}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Then, applying the row-vector $(1, 0)$ in $\text{Ker}(D_1^{\text{T}})$ to the reduced system (1.4), at the second step we get

$$\begin{cases} u_1'' - \Delta u_1 = -u_2 & \text{in } \mathbb{R}^+ \times \Omega, \\ u_1 = 0 & \text{on } \mathbb{R}^+ \times \Gamma. \end{cases} \tag{1.5}$$

This is a system for the variable u_1, in which the variable $h^{(2)} = -u_2$ can be regarded as an indirect internal control.

Finally, let

$$A_2 = (0), \quad D_2 = (-1). \tag{1.6}$$

Since $\mathrm{Ker}(D_2^{\mathrm{T}}) = (0)$, we stop the projection.

By this way, we decompose the original system (1.3) into two sub-systems (1.4) and (1.5). Consequently, besides the direct boundary control h acting on the boundary and appearing in the original system (1.3), we find two indirect internal controls $h^{(1)}$ and $h^{(2)}$, which are hidden in the sub-systems (1.4) and (1.5), respectively.

Related to the indirect controllability, the notion of the indirect stabilization was introduced by Russell [16] in the early 1990's. It concerns if the dissipation induced by one of the equations can be sufficiently transmitted to the other ones in order to realize the stability of the overall system (see [2–3] for wave equations and [14–15] for wave/heat equations). Moreover, as shown in [4, 13], the situation is more complicated for partially damped systems. The effectiveness of the indirect damping depends in a very complex way on all of the involved factors such as the nature of the coupling, the order of boundary dissipation, the hidden regularity, the accordance of boundary conditions and many others.

The paper is organized as follows. In §2, we will give a general procedure of projection, which decomposes a system of wave equations into a sequence of sub-systems. In §3, we establish the relation between the ranks of the matrices appearing in the procedure of projection. In §4, we identify the indirect internal controls in the reduced systems and explain its role in the systems. In §5, we establish a uniqueness theorem under Kalman's rank condition without any algebraic condition on the coupling matrix, neither any geometrical condition on the controlled domain. This result will be served as a base for the approximate controllability by locally distributed controls later. §6 is devoted to some questions to be developed in the forthcoming work.

2 An Algebraic Procedure of Reduction

Now we describe the general procedure of projection. Let

$$N_0 = N, \quad A_0 = A, \quad D_0 = D,$$

where A_0 is a matrix of order N_0, D_0 is a matrix of order $N_0 \times M$ with

$$N_1 = N_0 - \mathrm{rank}(D_0).$$

D_0 is not necessarily a full column-rank matrix.

Let

$$\mathrm{Ker}(D_0^{\mathrm{T}}) = \mathrm{Span}\{d_1, \cdots, d_{N_1}\}. \tag{2.1}$$

We choose

$$K_0 = (d_1, \cdots, d_{N_1}). \tag{2.2}$$

In particular, we have

$$D_0^{\mathrm{T}} K_0 = 0. \tag{2.3}$$

Noting

$$\mathrm{Im}(K_0) \oplus \mathrm{Im}(D_0) = \mathrm{Ker}(D_0^{\mathrm{T}}) \oplus \mathrm{Im}(D_0) = \mathbb{R}^N, \tag{2.4}$$

there exist a matrix A_1 of order N_1 and a matrix D_1 of order $N_1 \times M$, such that

$$A_0^{\mathrm{T}} K_0 = K_0 A_1^{\mathrm{T}} - D_0 D_1^{\mathrm{T}}. \tag{2.5}$$

Since K_0 is of full column-rank, A_1 is uniquely determined. While, since D_0 may be not of full column-rank, for guaranteeing the uniqueness of D_1, we require

$$\mathrm{Im}(D_1^{\mathrm{T}}) \cap \mathrm{Ker}(D_0) = \{0\}. \tag{2.6}$$

Then, noting (2.3), by applying K_0^{T} to system (1.1) and setting

$$U^{(1)} = K_0^{\mathrm{T}} U, \quad H^{(1)} = D_0^{\mathrm{T}} U, \tag{2.7}$$

we get

$$\begin{cases} (U^{(1)})'' - \Delta U^{(1)} + A_1 U^{(1)} = D_1 H^{(1)} & \text{in } \mathbb{R}^+ \times \Omega, \\ U^{(1)} = 0 & \text{on } \mathbb{R}^+ \times \Gamma. \end{cases} \tag{2.8}$$

The projected system (2.8) is not self-closed in general. It can be regarded as a system for the reduced variable $U^{(1)}$, associated with the internal control $H^{(1)}$.

Similarly, by the successive projections, for $l = 2, 3, \cdots$, we get

$$\begin{cases} U^{(l-1)''} - \Delta U^{(l-1)} + A_{l-1} U^{(l-1)} = D_{l-1} H^{(l-1)} & \text{in } \mathbb{R}^+ \times \Omega, \\ U^{(l-1)} = 0 & \text{on } \mathbb{R}^+ \times \Gamma. \end{cases} \tag{2.9}$$

For $l = 1, 2, \cdots$, let

$$N_l = N_{l-1} - \mathrm{rank}(D_{l-1}). \tag{2.10}$$

Define

$$\mathrm{Ker}(D_{l-1}^{\mathrm{T}}) = \mathrm{Span}\{d_1, \cdots, d_{N_l}\}, \quad K_{l-1} = (d_1, \cdots, d_{N_l}). \tag{2.11}$$

In particular, we have

$$D_{l-1}^{\mathrm{T}} K_{l-1} = 0. \tag{2.12}$$

Noting that

$$\mathrm{Im}(K_{l-1}) \oplus \mathrm{Im}(D_{l-1}) = \mathrm{Ker}(D_{l-1}^{\mathrm{T}}) \oplus \mathrm{Im}(D_{l-1}) = \mathbb{R}^{N_{l-1}}, \tag{2.13}$$

there exist a matrix A_l of order N_l and a matrix D_l of order $N_l \times M$, such that

$$K_{l-1}^{\mathrm{T}} A_{l-1} = A_l K_{l-1}^{\mathrm{T}} - D_l D_{l-1}^{\mathrm{T}}. \tag{2.14}$$

Then, noting (2.12), we have

$$K_{l-1}^{\mathrm{T}} K_{l-1} A_l^{\mathrm{T}} = K_{l-1}^{\mathrm{T}} A_{l-1}^{\mathrm{T}} K_{l-1} \tag{2.15}$$

and

$$D_{l-1}^{\mathrm{T}} D_{l-1} D_l^{\mathrm{T}} = -D_{l-1}^{\mathrm{T}} A_{l-1}^{\mathrm{T}} K_{l-1}. \tag{2.16}$$

Since K_{l-1} is of full column-rank, we have $\mathrm{Ker}(K_{l-1}) = \{0\}$. It follows from (2.15) that

$$A_l^{\mathrm{T}} = (K_{l-1}^{\mathrm{T}} K_{l-1})^{-1} K_{l-1}^{\mathrm{T}} A_{l-1}^{\mathrm{T}} K_{l-1}. \tag{2.17}$$

While, since D_{l-1} may be not of full column-rank, $\mathrm{Ker}(D_{l-1}) \neq \{0\}$ in general. In order to uniquely determine the matrix D_l by the relation (2.16), similarly to (2.6), we require

$$\mathrm{Im}(D_l^{\mathrm{T}}) \cap \mathrm{Ker}(D_{l-1}) = \{0\}. \tag{2.18}$$

Then, applying K_{l-1}^{T} to system (2.9) and setting

$$U^{(l)} = K_{l-1}^{\mathrm{T}} U^{(l-1)}, \quad H^{(l)} = D_{l-1}^{\mathrm{T}} U^{(l-1)}, \tag{2.19}$$

we get

$$\begin{cases} (U^{(l)})'' - \Delta U^{(l)} + A_l U^{(l)} = D_l H^{(l)} & \text{in } \mathbb{R}^+ \times \Omega, \\ U^{(l)} = 0 & \text{on } \mathbb{R}^+ \times \Gamma. \end{cases} \tag{2.20}$$

We continue the procedure of projection until

(i) either $D_L = 0$, then we get a self-closed conservative system

$$\begin{cases} (U^{(L)})'' - \Delta U^{(L)} + A_L U^{(L)} = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ U^{(L)} = 0 & \text{on } \mathbb{R}^+ \times \Gamma, \end{cases} \tag{2.21}$$

which is not approximately controllable, so is the original system (1.1);

(ii) or $\mathrm{Ker}(D_L^{\mathrm{T}}) = \{0\}$, then we get a non self-closed system

$$\begin{cases} (U^{(L)})'' - \Delta U^{(L)} + A_L U^{(L)} = D_L H^{(L)} & \text{in } \mathbb{R}^+ \times \Omega, \\ U^{(L)} = 0 & \text{on } \mathbb{R}^+ \times \Gamma. \end{cases} \tag{2.22}$$

Since the control matrix D_L is of full row-rank, this case is favorite for the approximate controllability of system (2.22), however, we don't know whether the original system (1.1) is actually approximately controllable or not.

The above procedure is purely algebraic. In order to clarify the leading idea, we do not take other type boundary conditions into account.

3 Mathematical Analysis

Let us recall the following fundamental result (see [7, Lemma 2.5]).

Lemma 3.1 *Let $d \geq 0$ be an integer. The rank condition*

$$\mathrm{rank}(D, AD, \cdots, A^{N-1}D) = N - d \tag{3.1}$$

holds if and only if d is the largest dimension of the subspaces, which are contained in $\mathrm{Ker}(D^{\mathrm{T}})$ *and invariant for A^{T}.*

Proposition 3.1 *Let l be an integer with $1 \leq l \leq L$. For any subspace V contained in* $\mathrm{Ker}(D_l^{\mathrm{T}})$ *and invariant for A_l^{T}, there exists a subspace W contained in* $\mathrm{Ker}(D_{l-1}^{\mathrm{T}})$ *and invariant for A_{l-1}^{T}, such that* $\dim(W) = \dim(V)$*, and vice versa.*

Proof First, let $V \subseteq \text{Ker}(D_l^{\text{T}})$ be an invariant subspace of A_l^{T}. Let $W = K_{l-1}(V)$ denote the direct image of V by K_{l-1}.

For any given $y \in W$, by the definition of W, there exists $x \in V$, such that $y = K_{l-1}x$. Applying x^{T} to (2.14) leads that

$$x^{\text{T}} K_{l-1}^{\text{T}} A_{l-1} = x^{\text{T}} A_l K_{l-1}^{\text{T}} - x^{\text{T}} D_l D_{l-1}^{\text{T}}.$$

Since $x \in V \subseteq \text{Ker}(D_l^{\text{T}})$, we have $x^{\text{T}} D_l D_{l-1}^{\text{T}} = 0$, then

$$A_{l-1}^{\text{T}} K_{l-1} x = K_{l-1} A_l^{\text{T}} x. \tag{3.2}$$

Moreover, since V is invariant for A_l^{T}, we have $A_l^{\text{T}} x \in V$, then it follows from (3.2) that

$$A_{l-1}^{\text{T}} y = A_{l-1}^{\text{T}} K_{l-1} x = K_{l-1} A_l^{\text{T}} x \subseteq W.$$

By (2.12), we have

$$D_{l-1}^{\text{T}} y = D_{l-1}^{\text{T}} K_{l-1} x = 0.$$

Thus W is contained in $\text{Ker}(D_{l-1}^{\text{T}})$ and invariant for A_{l-1}^{T}.

Inversely, let $W \subseteq \text{Ker}(D_{l-1}^{\text{T}})$ be an invariant subspace of A_{l-1}^{T}. Let

$$V = K_{l-1}^{-1}(W) = \{x : \ K_{l-1} x \in W\}$$

denote the inverse image of W by K_{l-1}. For any given $x \in V$, there exists $y \in W$, such that $K_{l-1}x = y$. Applying x^{T} to (2.14), we get

$$x^{\text{T}} K_{l-1}^{\text{T}} A_{l-1} = x^{\text{T}} A_l K_{l-1}^{\text{T}} - x^{\text{T}} D_l D_{l-1}^{\text{T}}. \tag{3.3}$$

Applying K_{l-1} from the right to the above relation, it follows that

$$x^{\text{T}} K_{l-1}^{\text{T}} A_{l-1} K_{l-1} = x^{\text{T}} A_l K_{l-1}^{\text{T}} K_{l-1} - x^{\text{T}} D_l D_{l-1}^{\text{T}} K_{l-1}.$$

By (2.12), $D_{l-1}^{\text{T}} K_{l-1} = 0$, then

$$x^{\text{T}} K_{l-1}^{\text{T}} A_{l-1} K_{l-1} = x^{\text{T}} A_l K_{l-1}^{\text{T}} K_{l-1}. \tag{3.4}$$

Since W is invariant for A_{l-1}^{T}, we have $A_{l-1}^{\text{T}} y \in W$. By the definition of V, there exists $\widetilde{x} \in V$, such that $K_{l-1}\widetilde{x} = A_{l-1}^{\text{T}} y$. Then, it follows from (3.4) that

$$\widetilde{x}^{\text{T}} K_{l-1}^{\text{T}} K_{l-1} = y^{\text{T}} A_{l-1} K_{l-1} = x^{\text{T}} K_{l-1}^{\text{T}} A_{l-1} K_{l-1} = x^{\text{T}} A_l K_{l-1}^{\text{T}} K_{l-1}.$$

Since $K_{l-1}^{\text{T}} K_{l-1}$ is invertible, we have

$$A_l^{\text{T}} x = \widetilde{x} \in V,$$

namely, V is invariant for A_l^{T}.

Finally, inserting $K_{l-1}x = y$ and $A_l^{\text{T}} x = \widetilde{x}$ into (3.3), and noting $K_{l-1}\widetilde{x} = A_{l-1}^{\text{T}} y$, we get

$$x^{\text{T}} D_l D_{l-1}^{\text{T}} = x^{\text{T}} A_l K_{l-1}^{\text{T}} - x^{\text{T}} K_{l-1}^{\text{T}} A_{l-1} = \widetilde{x}^{\text{T}} K_{l-1}^{\text{T}} - y^{\text{T}} A_{l-1} = 0.$$

Then $D_{l-1} D_l^{\text{T}} x = 0$. By (2.18), we get $D_l^{\text{T}} x = 0$. So, $V \subseteq \text{Ker}(D_l^{\text{T}})$. Moreover, since K_{l-1} is of full column-rank, we have $\dim(V) = \dim(W)$.

Proposition 3.2 *We have*

$$\operatorname{rank}(D_0, A_0D_0, \cdots, A_0^{N_0-1}D_0) = \sum_{l=0}^{L}\operatorname{rank}(D_l). \tag{3.5}$$

Proof We first show that for $1 \le l \le L$, we have

$$\begin{aligned}&\operatorname{rank}(D_l, A_lD_l, \cdots, A_l^{N_l-1}D_l)\\ =\ &\operatorname{rank}(D_{l-1}, A_{l-1}D_{l-1}, \cdots, A_{l-1}^{N_{l-1}-1}D_{l-1}) - \operatorname{rank}(D_{l-1}).\end{aligned} \tag{3.6}$$

In fact, let

$$\operatorname{rank}(D_{l-1}, A_{l-1}D_{l-1}, \cdots, A_{l-1}^{N_{l-1}-1}D_{l-1}) = N_{l-1} - p_{l-1}. \tag{3.7}$$

By Lemma 3.1, p_{l-1} is the dimension of the largest subspace which is contained in $\operatorname{Ker}(D_{l-1}^{\mathrm{T}})$ and invariant for A_{l-1}^{T}. By Proposition 3.1, the largest subspace which is contained in $\operatorname{Ker}(D_l^{\mathrm{T}})$ and invariant for A_l^{T} has also the dimension p_{l-1}. Then we have

$$\operatorname{rank}(D_l, A_lD_l, \cdots, A_l^{N_l-1}D_l) = N_l - p_{l-1}. \tag{3.8}$$

Noting (2.10) and combining (3.7)–(3.8), we get (3.6).

Then, the summation of (3.6) for l from 1 to L gives

$$\begin{aligned}&\operatorname{rank}(D_0, A_0D_0, \cdots, A_0^{N-1}D_0)\\ =&\sum_{l=0}^{L-1}\operatorname{rank}(D_l) + \operatorname{rank}(D_L, A_LD_L, \cdots, A_L^{N_L-1}D_L).\end{aligned} \tag{3.9}$$

At the L-th step of reduction, we have either $D_L = 0$, then

$$\operatorname{rank}(D_L, A_LD_L, \cdots, A_L^{N_L-1}D_L) = \operatorname{rank}(D_L) = 0; \tag{3.10}$$

or $\operatorname{Ker}(D_L^{\mathrm{T}}) = 0$, then

$$\operatorname{rank}(D_L, A_LD_L, \cdots, A_L^{N_L-1}D_L) = \operatorname{rank}(D_L). \tag{3.11}$$

Then, using (3.10) and (3.11) in (3.9), we get (3.5).

Proposition 3.3 $\operatorname{rank}(D_l, A_lD_l, \cdots, A_l^{N_l-1}D_l) - N_l$ *is a constant with respect to* l *with* $0 \le l \le L$. *Consequently, Kaman's rank condition*

$$\operatorname{rank}(D_l, A_lD_l, \cdots, A_l^{N_l-1}D_l) = N_l \tag{3.12}$$

holds for all l *with* $0 \le l \le L$ *if and only if* $\operatorname{Ker}(D_L^{\mathrm{T}}) = \{0\}$.

Proof First, using (2.10) and (3.6), we deduce

$$\begin{aligned}&\operatorname{rank}(D_l, A_lD_l, \cdots, A_l^{N_l-1}D_l) - N_l\\ =\ &\operatorname{rank}(D_{l-1}, A_{l-1}D_{l-1}, \cdots, A_{l-1}^{N_{l-1}-1}D_{l-1}) - \operatorname{rank}(D_{l-1}) - N_l\\ =\ &\operatorname{rank}(D_{l-1}, A_{l-1}D_{l-1}, \cdots, A_{l-1}^{N_{l-1}-1}D_{l-1}) - N_{l-1}, \quad 1 \le l \le L.\end{aligned} \tag{3.13}$$

Next, assume that condition (3.12) holds for all l with $1 \le l \le L$. In particular, we have

$$\operatorname{rank}(D_L, A_L D_L, \cdots, A_L^{N_L-1} D_L) = N_L. \tag{3.14}$$

Since $N_L > 0$, we have $D_L \neq 0$. By the alternative of reduction, we get $\operatorname{Ker}(D_L^{\mathrm{T}}) = \{0\}$.

Inversely, by Lemma 3.1, condition $\operatorname{Ker}(D_L^{\mathrm{T}}) = \{0\}$ implies condition (3.14). Then, it follows from relation (3.13) that condition (3.12) holds for all l with $1 \le l \le L$.

Proposition 3.4 *Let A be a cascade matrix and Ω satisfy the geometrical control condition. Then system* (1.1) *is approximately controllable if and only if* $\operatorname{Ker}(D_L^{\mathrm{T}}) = \{0\}$.

Proof By [7] (see also [1]), system (1.1) is approximately controllable if and only if the pair (A, D) satisfies Kalman rank condition (1.2), or equivalently, by Proposition 3.3, if and only if $\operatorname{Ker}(D_L^{\mathrm{T}}) = \{0\}$.

At the end of the section, we give two others examples for further illustrating the reduction procedure.

Example 2 Consider the following system

$$\begin{cases} u_1'' - \Delta u_1 + u_2 = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ u_2'' - \Delta u_2 = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ v_1'' - \Delta v_1 + v_2 = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ v_2'' - \Delta v_2 = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ u_1 = v_1 = 0, \quad u_2 = v_2 = h & \text{on } \mathbb{R}^+ \times \Gamma. \end{cases} \tag{3.15}$$

Let

$$A_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad D_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

Noting (2.1)–(2.2), we may take

$$K_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Then, using (2.16)–(2.17) with $l = 1$, a straightforward computation gives

$$D_1^{\mathrm{T}} = -(D_0^{\mathrm{T}} D_0)^{-1} D_0^{\mathrm{T}} A_0^{\mathrm{T}} K_0 = -\frac{1}{2}(1, 1, 0)$$

and

$$A_1^{\mathrm{T}} = (K_0^{\mathrm{T}} K_0)^{-1} K_0^{\mathrm{T}} A_0^{\mathrm{T}} K_0 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix}.$$

Applying K_0^{T} to system (3.15), we get

$$\begin{cases} u_1'' - \Delta u_1 + \dfrac{\eta_1}{2} = -\dfrac{h_1}{2} & \text{in } \mathbb{R}^+ \times \Omega, \\ v_1'' - \Delta v_1 - \dfrac{\eta_1}{2} = -\dfrac{h_1}{2} & \text{in } \mathbb{R}^+ \times \Omega, \\ \eta_1'' - \Delta \eta_1 = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ u_1 = v_1 = \eta_1 = 0 & \text{on } \mathbb{R}^+ \times \Gamma, \end{cases} \tag{3.16}$$

where

$$U^{(1)} = K_0^{\mathrm{T}} U^{(0)} = \begin{pmatrix} u_1 \\ v_1 \\ u_2 - v_2 =: \eta_1 \end{pmatrix}$$

and

$$H^{(1)} = D_0^{\mathrm{T}} U^{(0)} = u_2 + v_2 =: h_1.$$

This is a system for the variables u_1, v_1 and η_1. The variable h_1 can be regarded as an internal control in system (3.16).

Next, applying (2.16)–(2.17) with $l = 2$ to

$$A_1 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad D_1 = -\frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad K_1 = \begin{pmatrix} 0 & 1 \\ 0 & -1 \\ 1 & 0 \end{pmatrix},$$

we get

$$A_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Applying K_1^{T} to system (3.16), we get

$$\begin{cases} \eta_1'' - \Delta\eta_1 = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ \eta_2'' - \Delta\eta_2 + \eta_1 = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ \eta_1 = \eta_2 = 0 & \text{on } \mathbb{R}^+ \times \Gamma, \end{cases} \tag{3.17}$$

where

$$U^{(2)} = K_1^{\mathrm{T}} U^{(1)} = \begin{pmatrix} \eta_1 \\ u_1 - v_1 =: \eta_2 \end{pmatrix}.$$

Since $D_2 = 0$, we stop the projection with $N_2 = 2$. By Proposition 3.3, none of the pairs $(A_0, D_0), (A_1, D_1)$ or (A_2, D_2) satisfies Kalman's rank condition (3.12). More precisely, we have

$$\operatorname{rank}(D_0, A_0D_0, A_0^2D_0, A_0^3D_0) - 4 = \operatorname{rank}(D_1, A_1D_1, A_1^2D_1) - 3 = \operatorname{rank}(D_2, A_2D_2) - 2.$$

Noting $\operatorname{rank}(D_2, A_2D_2) = 0$, it follows that

$$\operatorname{rank}(D_0, A_0D_0, A_0^2D_0, A_0^3D_0) = 4 - 2 = 2, \quad \operatorname{rank}(D_1, A_1D_1, A_1^2D_1) = 3 - 2 = 1.$$

Example 3 Consider the following system.

$$\begin{cases} u_1'' - \Delta u_1 + u_1 + u_2 + u_3 = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ u_2'' - \Delta u_2 + u_1 + 2u_2 + 3u_3 = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ u_3'' - \Delta u_3 + 3u_1 + 2u_2 + u_3 = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ u_1 = u_2 = u_3 = h & \text{on } \mathbb{R}^+ \times \Gamma. \end{cases} \tag{3.18}$$

Let

$$A_0 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \quad D_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Noting (2.1)–(2.2), we may take

$$K_0 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{pmatrix}.$$

Using (2.16)–(2.17) with $l = 1$, a straightforward computation gives

$$D_1^{\mathrm{T}} = -(D_0^{\mathrm{T}} D_0)^{-1} D_0^{\mathrm{T}} A_0^{\mathrm{T}} K_0 = (1, 0)$$

and

$$A_1^{\mathrm{T}} = (K_0^{\mathrm{T}} K_0)^{-1} K_0^{\mathrm{T}} A_0^{\mathrm{T}} K_0 = \begin{pmatrix} 1 & -2 \\ 1 & -2 \end{pmatrix}.$$

Then, applying K_0^{T} to system (3.18), we get

$$\begin{cases} v_1'' - \Delta v_1 + v_1 + v_2 = h_1 & \text{in } \mathbb{R}^+ \times \Omega, \\ v_2'' - \Delta v_2 - 2v_1 - 2v_2 = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ v_1 = v_2 = 0 & \text{on } \mathbb{R}^+ \times \Gamma, \end{cases} \tag{3.19}$$

where

$$U^{(1)} = K_0^{\mathrm{T}} U = \begin{pmatrix} u_1 - u_2 =: v_1 \\ u_2 - u_3 =: v_2 \end{pmatrix},$$
$$H^{(1)} = D_0^{\mathrm{T}} U = u_1 + u_2 + u_3 =: h_1.$$

This is a system for the variables v_1, v_2 with an internal control h_1.

Next, applying (2.16)–(2.17) with $l = 2$ to

$$A_1 = \begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad K_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

we get

$$A_2 = (-2), \quad D_2 = (2).$$

Then, applying K_1^{T} to system (3.18), we get

$$\begin{cases} v_2'' - \Delta v_2 - 2v_2 = 2h_2 & \text{in } \mathbb{R}^+ \times \Omega, \\ v_2 = 0 & \text{on } \mathbb{R}^+ \times \Gamma, \end{cases} \tag{3.20}$$

where

$$U^{(2)} = K_1^{\mathrm{T}} U^{(1)} =: v_2, \quad H^{(2)} = D_1^{\mathrm{T}} U^{(1)} = v_1 =: h_2.$$

This is a system for the variable v_2 with an internal control h_2. Since $\mathrm{Ker}(D_2^{\mathrm{T}}) = \{0\}$, we stop the projection.

Since $\mathrm{Ker}(D_2^{\mathrm{T}}) = \{0\}$, we stop the projection with $N_2 = 2$. By Proposition 3.3, the pairs (A_0, D_0), (A_1, D_1) and (A_2, D_2) satisfy Kalman's rank condition (3.12).

4 Notion of Indirect Controls

For $1 \le l \le L$, the term $H^{(l)}$ can be formally regarded as internal controls in the sub-system (2.20). But the value of $H^{(l)}$ is given by (2.19), therefore, it can not be freely chosen. So, $H^{(l)} (1 \le l \le L)$ will be called indirect internal controls, and accordingly, $\mathrm{rank}(D_l)$ denotes its number. Thus, the original system (1.1) is directly controlled by the boundary control $H^{(0)}$, and indirectly controlled by the internal controls $H^{(1)}, \cdots, H^{(L)}$ which are hidden in the sub-system (2.20) and intervene into the systems at different steps of the reduction. Moreover, the formula (3.5) justifies well the notion of the total number of (direct and indirect) controls previously introduced in [8]. This gives a pretty good explanation to the indirect controls.

The term "direct controls" or "indirect controls" is related to the sub-system (2.20). For $1 \le l \le L$, $H^{(l)}$ can be regarded as direct internal controls in (2.20) at the l-th step or as indirect controls for the original system (1.1). In any case, this is simply a terminology that we can use as we want.

Proposition 4.1 *Assume that system* (1.1) *is approximately controllable. Then for all l with $1 \le l \le L$, the rank condition* (3.12) *holds and the sub-system* (2.20) *is approximately controllable by the internal indirect control $H^{(l)}$.*

Proof First by Proposition 3.3 and noting (1.2), we have

$$\operatorname{rank}(D_l, A_l D_l, \cdots, A_l^{N_l-1} D_l) - N_l = \operatorname{rank}(D_0, A_0 D_0, \cdots, A_0^{N_0-1} D_0) - N_0 = 0.$$

On the other hand, by (2.19), we have

$$U^{(l)} = (K_0 \cdots K_{l-1})^{\mathrm{T}} U^{(0)}, \quad 1 \le l \le L.$$

Then, the approximate controllability of system (1.1) implies that of the sub-system (2.20) for all l with $1 \le l \le L$.

We know few about the structure of indirect controls $H^{(l)}$ with $1 \le l \le L$, however, the following result shows that the indirect controls $H^{(l)}$ should be so smooth that its action on the sub-system (2.20) will be very weak, especially as the step l increases.

Proposition 4.2 *For any given l with $1 \le l \le L$, let*

$$(K_0 \cdots K_{l-1})^{\mathrm{T}} \widehat{U}_0 \in (H_0^l(\Omega))^{N_l}, \quad (K_0 \cdots K_{l-1})^{\mathrm{T}} \widehat{U}_1 \in (H_0^{l-1}(\Omega))^{N_l}. \tag{4.1}$$

Then, we have

$$U^{(l)} \in (C_{\mathrm{loc}}^{l-k}(\mathbb{R}^+; H_0^k(\Omega)))^{N_l}, \quad 0 \le k \le l. \tag{4.2}$$

Proof For any given $(\widehat{U}_0, \widehat{U}_1) \in (L^2(\Omega) \times H^{-1}(\Omega))^N$ and any given $H \in (L^2_{\mathrm{loc}}(\mathbb{R}^+; L^2(\Gamma)))^M$, the solution to problem (1.1)–(1.2) has the regularity (see [7–8, 10]):

$$U^{(0)} \in (C^0_{\mathrm{loc}}(\mathbb{R}^+; L^2(\Omega)) \cap C^1_{\mathrm{loc}}(\mathbb{R}^+; H^{-1}(\Omega)))^N.$$

For $l = 1$, consider the reduced system

$$\begin{cases} (U^{(1)})'' - \Delta U^{(1)} + A_1 U^{(1)} = D_1 H^{(1)} & \text{in } \mathbb{R}^+ \times \Omega, \\ U^{(1)} = 0 & \text{on } \mathbb{R}^+ \times \Gamma \end{cases} \tag{4.3}$$

with the initial data:

$$t = 0: \ U^{(1)} = K_0^{\mathrm{T}} \widehat{U}_0 \in (H_0^1(\Omega))^{N_1}, \quad (U^{(1)})' = K_0^{\mathrm{T}} \widehat{U}_1 \in (L^2(\Omega))^{N_1}. \tag{4.4}$$

Since the right-hand side

$$H^{(1)} = D_0 U^{(0)} \in (C^0_{\mathrm{loc}}(\mathbb{R}^+; L^2(\Omega)))^{N_1},$$

the solution to problem (4.3)–(4.4) has the regularity (see [11] or [12])

$$U^{(1)} \in (C^0_{\mathrm{loc}}(\mathbb{R}^+; H_0^1(\Omega)) \cap C^1_{\mathrm{loc}}(\mathbb{R}^+; L^2(\Omega)))^{N_1}.$$

The general case can be easily completed by a bootstrap argument.

5 Approximate Controllability by Locally Distributed Controls

This section gives only a brief abstract on the internal controllability. It will be carefully completed in a forthcoming work.

Now we consider the system for the variable $U = (u^{(1)}, \cdots, u^{(N)})^{\mathrm{T}}$:

$$\begin{cases} U'' - \Delta U + AU = \chi_\omega DH & \text{in } \mathbb{R}^+ \times \Omega, \\ U = 0 & \text{on } \mathbb{R}^+ \times \Gamma \end{cases} \tag{5.1}$$

with the initial data

$$t = 0: \ U = \widehat{U}_0, \quad U' = \widehat{U}_1 \quad \text{in } \Omega, \tag{5.2}$$

where $H \in (L^2_{\mathrm{loc}}(\mathbb{R}^+; L^2(\Omega)))^M$ and χ_ω is the characteristic function of a subset ω of Ω.

Remark 5.1 The global case $\omega = \Omega$ is trivial, so less interesting. For the exact controllability, ω is often assumed to be a neighbour of Γ in the literature, while for the approximate controllability, it seems that no restriction on ω will be necessary.

Definition 5.1 *System* (5.1) *is approximately controllable at the time* $T > 0$ *if for any given initial data* $(\widehat{U}_0, \widehat{U}_1) \in (L^2(\Omega) \times H^{-1}(\Omega))^N$, *there exists a sequence* $\{H_n\}$ *of controls in* $(L^2_{\mathrm{loc}}(\mathbb{R}^+; L^2(\Omega)))^M$ *with support in* $[0, T]$, *such that the corresponding sequence* $\{U_n\}$ *of solutions satisfies*

$$U_n \to 0 \quad \textit{in } (C^0_{\mathrm{loc}}(\mathbb{R}^+; H^1_0(\Omega)) \cap C^1_{\mathrm{loc}}(\mathbb{R}^+; L^2(\Omega)))^N \quad \textit{as } n \to +\infty. \tag{5.3}$$

Similarly to the approximate boundary controllability in [9], we can show the equivalence between the approximate controllability of system (5.1) and the uniqueness of solution to the following adjoint system for the variable $\Phi = (\phi^{(1)}, \cdots, \phi^{(N)})^{\mathrm{T}}$:

$$\begin{cases} \Phi'' - \Delta\Phi + A^{\mathrm{T}}\phi = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ \Phi = 0 & \text{on } \mathbb{R}^+ \times \Gamma \end{cases} \tag{5.4}$$

associated with the internal observation

$$D^{\mathrm{T}}\Phi = 0 \quad \text{in } (0, T) \times \omega. \tag{5.5}$$

Moreover, condition (1.2) is still necessary for the uniqueness of solution to the overdetermined system (5.4)–(5.5).

Theorem 5.1 *If A satisfies Kalman's rank condition* (1.2), *then system* (5.4)–(5.5) *has only the trivial solution. Consequently, system* (5.1) *is approximately controllable by locally distributed controls.*

Proof To be clear, let $\Phi \in (C^0(\mathbb{R}; H^1_0(\Omega)) \cap C^1(\mathbb{R}; L^2(\Omega)))^N$ be a solution to system (5.4)–(5.5). First, applying D^{T} to the equations in (5.4) and noting (5.5), we get

$$D^{\mathrm{T}}A^{\mathrm{T}}\Phi = 0 \quad \text{in } (0, T) \times \omega. \tag{5.6}$$

Then, successively applying $D^{\mathrm{T}}A^{\mathrm{T}}, D^{\mathrm{T}}(A^2)^{\mathrm{T}} \cdots$ to the equations in (5.4), we get

$$D^{\mathrm{T}}\Phi = D^{\mathrm{T}}A^{\mathrm{T}}\Phi = D^{\mathrm{T}}(A^2)^{\mathrm{T}}\Phi = \cdots = 0 \quad \text{in } (0, T) \times \omega, \tag{5.7}$$

therefore,

$$\Phi^{\mathrm{T}}(D, AD, \cdots, A^{N-1}D) = 0 \quad \text{in } (0,T) \times \omega. \tag{5.8}$$

By (1.2), the matrix $(D, AD, \cdots, A^{N-1}D)$ is of full row-rank, then

$$\Phi = 0 \quad \text{in } (0,T) \times \omega. \tag{5.9}$$

Thus, applying Holmgren's uniqueness theorem, we get $\Phi \equiv 0$ in $(0,T) \times \Omega$, provided that

$$T > 2d(\Omega), \tag{5.10}$$

where $d(\Omega)$ denotes the geodesic diameter of Ω (see [10, Theorem 8.2]).

Remark 5.2 Since the differential operator Δ commutes with the internal D-observation (5.5):

$$D^T \chi_\omega \Delta\Phi = \Delta D^T \chi_\omega \Phi \quad \text{in } \mathcal{D}'(\omega), \tag{5.11}$$

the situation is almost the same as for ordinary differential equations (see [5]). This is why the uniqueness in Theorem 5.1 holds without any restriction on the coupling matrix A, nor on the damped sub-domain ω.

Remark 5.3 Recall that the controllability time (optimal) for system (1.1) is given by

$$T > 2(N - \text{rank}(D) + 1)d(\Omega). \tag{5.12}$$

It should be sufficiently large, especially as N is large (see [7, 17]). However, the controllability time given by (5.10) is independent of the number of equations and of the number of applied controls. It is exactly the same as for a sole equation in [10].

6 Comments

After having discussed the notion of indirect controls, further work would be needed to develop new results. For example, some interesting problems could be considered as follows.

Question 1 Since the value of $H^{(l)}$ can not be freely chosen, the indirect internal controls $H^{(l)}$ in the sub-system (2.20) has not the same meaning as the direct internal controls H in (5.1). Any initiative for further clarifying their relations would be interesting to pursue.

Question 2 The adaptation of the procedure to the coupled system of wave equations with coupled Robin controls (with two coupling matrices) might be an interesting direction to be investigated.

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