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# A Generalization of Lappan's Theorem to Higher Dimensional Complex Projective Space*

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**Abstract** In this paper, the authors discuss a generalization of Lappan's theorem to higher dimensional complex projective space and get the following result: Let $f$ be a holomorphic mapping of $\Delta$ into $\mathbb{P}^n(\mathbb{C})$, and let $H_1, \cdots, H_q$ be hyperplanes in general position in $\mathbb{P}^n(\mathbb{C})$. Assume that

$$\sup\Big\{(1-|z|^2)f^\sharp(z) : z \in \bigcup_{j=1}^{q} f^{-1}(H_j)\Big\} < \infty,$$

if $q \geq 2n^2+3$, then $f$ is normal.

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## 1 Introduction

In the theory of normal families, perhaps the following criterion of Montel [1] is the most celebrated theorem.

**Theorem 1.1** *Let $\mathcal{F}$ be a family of meromorphic functions in a domain $D \subset \mathbb{C}$, and let $a, b, c$ be three distinct points in $\overline{\mathbb{C}}$. Assume that all functions in $\mathcal{F}$ omit three points $a, b, c$ in $D$. Then $\mathcal{F}$ is a normal family in $D$.*

In 1957, Lehto and Virtanen [2] proved the following well-known result, which says that a function $f(z)$ meromorphic in the unit disc $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ is normal if and only if

$$\sup_{z\in\Delta}\{(1-|z|^2)f^\sharp(z)\} < +\infty,$$

where $f^\sharp := \frac{|f'|}{1+|f|^2}$ is the spherical derivative of $f$.

In 1972, Pommerenke [3] posed an open question: For a given positive number $M > 0$, does there exist a finite subset $E \subset \overline{\mathbb{C}}$ such that if $f$ is a meromorphic function in $\Delta$, then the condition that $(1-|z|^2)f^\sharp(z) \leq M$ for each $z \in f^{-1}(E)$ implies that $f$ is a normal function?

Latter, Lappan [4] answered the above question and proved the following well-known result named Lappan's theorem.

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**Theorem 1.2** *Let $E$ be any set consisting of five complex numbers, finite or infinite. If $f$ is a meromorphic function in $\Delta$ such that*

$$\sup\{(1-|z|^2)f^\sharp(z) : z \in f^{-1}(E)\} < \infty,$$

*then $f$ is a normal function.*

In 2020, Tan [5] generalized the above theorem to the $n$ dimensional complex projective space, and proved the following theorem.

**Theorem 1.3** *Let $f$ be a holomorphic mapping of $\Delta$ into $\mathbb{P}^n(\mathbb{C})$, and let $H_1, \cdots, H_q$ be hyperplanes in general position in $\mathbb{P}^n(\mathbb{C})$. Assume that*

$$\sup\left\{(1-|z|^2)f^\sharp(z) : z \in \bigcup_{j=1}^{q} f^{-1}(H_j)\right\} < \infty,$$

*if $q \geq n(2n+1)+2$, then $f$ is normal.*

Inspried by the method of the proof of the main theorem in Chen and Yan [6], we reduce the number of hyperplanes in Theorem 1.3 and obtain the following main result.

**Theorem 1.4** *Let $f$ be a holomorphic mapping of $\Delta$ into $\mathbb{P}^n(\mathbb{C})$, and let $H_1, \cdots, H_q$ be hyperplanes in general position in $\mathbb{P}^n(\mathbb{C})$. Assume that*

$$\sup\left\{(1-|z|^2)f^\sharp(z) : z \in \bigcup_{j=1}^{q} f^{-1}(H_j)\right\} < \infty,$$

*if $q \geq 2n^2+3$, then $f$ is normal.*

## 2 Notations and Preliminaries

In this section, we introduce some notations and preliminaries related to this paper. For more details see [7].

Let $f = [f_0 : \cdots : f_n]$ be a holomorphic mapping from a domain in $\mathbb{C}$ to $\mathbb{P}^n(\mathbb{C})$ given by homogeneous coordinate function $f_j$ $(j = 0, 1, \cdots, n)$ which are holomorphic without common zeros. In this paper, we also need the following formula named Fubini-Study derivative $f^\sharp$ of $f$ (for details, see [8]),

$$(f^\sharp)^2 := \frac{\partial^2}{\partial z \partial \overline{z}} \log \sum_{i=0}^{n} |f_i|^2 = \frac{\sum\limits_{0\leq s<t\leq n} |W(f_s, f_t)|^2}{\|f\|^4}.$$

**Definition 2.1** (see [7]) *Let $\nu$ be an effective divisor on $\mathbb{C}$. For each positive integer (or $+\infty$) $p$, we define the counting function of $\nu$ (where multiplicities are truncated by $p$) by*

$$N^{[p]}(r,\nu) := \int_1^r \frac{n_\nu^{[p]}}{t} \mathrm{d}t, \quad 1 < r < \infty,$$

*where* $n_\nu^{[p]}(t) = \sum_{|z|\le t} \min\{\nu(z), p\}$. *For brevity, we will omit the character* $[p]$ *in the counting function if* $p = +\infty$.

For a meromorphic function $\varphi$ on $\mathbb{C}$ ($\varphi \not\equiv 0, \varphi \not\equiv \infty$), we denote by $(\varphi)_0$ the divisor of zeros of $\varphi$. We have the following Jensen's formula for the counting function:

$$N(r,(\varphi)_0) - N\Big(r, \Big(\frac{1}{\varphi}\Big)_0\Big) = \frac{1}{2\pi}\int_0^{2\pi} \log|\varphi(re^{i\theta})|d\theta + O(1).$$

**Definition 2.2** (see [7]) *We define the proximity function of* $\varphi$ *by*

$$m(r,\varphi) = \frac{1}{2\pi}\int_0^{2\pi} \log^+|\varphi(re^{i\theta})|d\theta,$$

*where* $\log^+ x = \max\{0, \log x\}$ *for* $x > 0$.

If $\varphi$ is nonconstant, then $m\big(r, \frac{\varphi'}{\varphi}\big) = o(T_\varphi(r))$ as $r \to \infty$, outside a set of finite Lebesgue measure (Nevanlinna's lemma on the logarithmic derivative).

Nevanlinna's first main theorem for $\varphi$ states that, for any $a \in \overline{\mathbb{C}}$,

$$T_\varphi(r) = N\Big(r, \Big(\frac{1}{\varphi - a}\Big)_0\Big) + \Big(r, \frac{1}{\varphi - a}\Big) + O(1).$$

**Definition 2.3** (see [7]) *Let* $f$ *be a holomorphic mapping of* $\mathbb{C}$ *into* $\mathbb{P}^n(\mathbb{C})$ *with a reduced representation* $(f_0, \cdots, f_n)$. *The characteristic function* $T_f(r)$ *of* $f$ *is defined by*

$$T_f(r) := \frac{1}{2\pi}\int_0^{2\pi} \log\|f(re^{i\theta})\|d\theta - \frac{1}{2\pi}\int_0^{2\pi} \log\|f(e^{i\theta})\|d\theta, \quad r > 1,$$

*where* $\|f\| = \max_{0\le i\le n} |f_i|$.

Let $H = \big\{(\omega_0 : \cdots : \omega_n) \in \mathbb{P}^n(\mathbb{C}) : \sum_{i=0}^{n} a_i\omega_i = 0\big\}$ be a hyperplane in $\mathbb{P}^n(\mathbb{C})$ such that $f(\mathbb{C}) \not\subset H$. Denote by $(H(f))_0$ the divisor of zeros of $\sum_{i=0}^{n} a_i f_i(z)$, and put $N_f^{[p]}(r, H) := N^{[p]}(r, (H(f))_0)$.

**Definition 2.4** *Let* $q$, $k$ *be two positive integers, satisfying* $q \ge k \ge n$ *and let* $H_1, \cdots, H_q$ *be* $q$ *hyperplanes in* $\mathbb{P}^n(\mathbb{C})$. *These hyperplanes are said to be in* $k$*-subgenerral position if* $\bigcap_{i=0}^{k} H_{j_i} = \emptyset$, *for all* $1 \le j_0 < \cdots < j_k \le q$.

**Definition 2.5** (see [7]) *Let* $f$ *be a holomorphic mapping of* $\mathbb{C}$ *into* $\mathbb{P}^n(\mathbb{C})$. *If there exists a hyperplance* $H$ *in* $\mathbb{P}^n(\mathbb{C})$, *such that* $f(\mathbb{C}) \subset H$, *then we call that* $f$ *is a linearly degenerate holomorphic mapping, otherwise* $f$ *is linearly non-degenerate.*

**Definition 2.6** (see [3]) *Let* $f : \Delta \to \overline{\mathbb{C}}$ *be a meromorphic function, and let* $\mathcal{F} = \{f \circ \varphi \mid \varphi : \Delta \to \Delta$ *be a conformal mapping*$\}$. *If* $\mathcal{F}$ *is normal in* $\Delta$, *then* $f$ *is called a normal function.*

Similarly, we can give the following definition for the normal curve.

**Definition 2.7** *Let $f : \Delta \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic mapping, and let $\mathcal{F} = \{f \circ \varphi \mid \varphi : \Delta \to \Delta$ be a conformal mapping$\}$. If $\mathcal{F}$ is normal in $\Delta$, then $f$ is called a normal curve.*

**Nochka's Second Main Theorem** (see [9]) Let $f$ be a linearly non-degenerate holomorphic mapping of $\mathbb{C}$ into $\mathbb{P}^n(\mathbb{C})$, and let $H_1, \cdots, H_q$ be $q$ hyperplanes in $k$-subgeneral position in $\mathbb{P}^n(\mathbb{C})$ ($k \geq n$ and $q \geq 2k - n + 1$). Then

$$||(q - 2k + n - 1)T_f(r) \leq \sum_{j=1}^{q} N_f^{[n]}(r, H_j) + o(T_f(r)),$$

where "$\|$" means the estimate holds for all large $r$ outside a set of finite Lebesgue measure.

## 3 Preliminary Lemmas

Before we give the proof of our main theorem, we need the following version of Zalcman's lemma for holomorphic mappings from the domain $\Omega \subseteq \mathbb{C}$ to $\mathbb{P}^n(\mathbb{C})$.

**Lemma 3.1** (see [10]) *Let $\mathcal{F}$ be a family of holomorphic mappings of a hyperbolic domain $\Omega$ in $\mathbb{C}$ into $\mathbb{P}^n(\mathbb{C})$. The family $\mathcal{F}$ is not normal on $\Omega$ if and only if there exist sequences $\{f_k\} \subset \mathcal{F}$, $\{z_k\} \subset \Omega$ with $z_k \to z_0 \in \Omega$, $\{r_k\}$ with $r_k > 0$ and $r_k \to 0$, such that*

$$g_k(\xi) := f_k(z_k + r_k\xi)$$

*converges uniformly on compact subsets of $\mathbb{C}$ to a nonconstant holomorphic map $g$ of $\mathbb{C}$ into $\mathbb{P}^n(\mathbb{C})$.*

**Lemma 3.2** (see [7]) *Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic mapping, and $H_1, \cdots, H_q$ be ($q \geq 2n + 1$) hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position. If for each $j = 1, \cdots, q$, either $f(\mathbb{C})$ is contained in $H_j$, or $f(\mathbb{C})$ omits $H_j$, then $f$ must be a constant.*

The following lemma plays an important role in the proof of Theorem 1.3.

**Lemma 3.3** (see [5]) *Let $f$ be a linearly non-degenerate holomorphic mapping of $\mathbb{C}$ into $\mathbb{P}^n(\mathbb{C})$, and let $H_1, \cdots, H_q$ be $q$ hyperplanes in $k$-subgeneral position in $\mathbb{P}^n(\mathbb{C})$, where $k \geq n$ and $q \geq 2k - n + 1$. Assume that $f(z) \in H_j \Rightarrow f^\sharp(z) = 0$, $j = 1, \cdots, q$. Then $q \leq 2k(n+1) - n + 1$.*

In this paper, inspired by the method of Chen and Yan [6], we improve the above lemma and get the following lemma, which plays a key role in the proof of our main theorem.

**Lemma 3.4** *Let $f$ be a linearly non-degenerate holomorphic mapping of $\mathbb{C}$ into $\mathbb{P}^n(\mathbb{C})$, and let $H_1, \cdots, H_q$ be $q$ hyperplanes in $k$-subgeneral position in $\mathbb{P}^n(\mathbb{C})$, where $k \geq n$ and $q \geq 2k - n + 1$. Assume that $f(z) \in H_j \Rightarrow f^\sharp(z) = 0$, $j = 1, \cdots, q$. Then $q < 2k(n+1) - 2n + 3$.*

**Proof** We pick up a reduced presentation $(f_0, \cdots, f_n)$ of $f$. We also write it as $f =$

$(f_0, \cdots, f_n)$ and let $f' = (f'_0, \cdots, f'_n)$. For each $z \in \bigcup_{j=1}^{q} f^{-1}(H_j)$, we define

$$\mathcal{A}_z := \{(a_0, \cdots, a_n) \in \mathbb{C}^{n+1} : a_0 f_0(z) + \cdots + a_n f_n(z) = 0\}.$$

Since $\mathcal{A}_z$ is a vector subspace of dimension $n$ of $\mathbb{C}^{n+1}$ and since $\bigcup_{j=1}^{q} f^{-1}(H_j)$ is at most countable, it follows that there exists a vector

$$\ell = (l_0, \cdots, l_n) \in \mathbb{C}^{n+1} \setminus \Big( \bigcup_{j=1}^{q} \bigcup_{z \in f^{-1}(H_j)} \mathcal{A}_z \Big).$$

Let $L$ be a hyperplane in $\mathbb{P}^n(\mathbb{C})$, where $L(\omega)$ is defined by the equation

$$\langle \omega, \ell \rangle = l_0\omega_0 + \cdots + l_n\omega_n = 0.$$

By our choice, we have

$$f^{-1}(L) \bigcap \Big( \bigcup_{j=1}^{q} f^{-1}(H_j) \Big) = \emptyset.$$

Moreover, we can also choose such $L(\omega)$ to satisfy that for all $i \in \{1, 2, \cdots, q\}$, $\frac{H_i}{L}(\omega) \not\equiv$ constant.

Set $\Phi_i = \frac{H_i(f)}{L(f)}$, where $i = 1, 2, \cdots, q$.

If there exists some $i_0 \in \{1, 2, \cdots, q\}$, such that $\Phi_{i_0} \equiv C$, then $H_{i_0}(f) \equiv CL(f)$, and we have $\langle f, \alpha_{i_0} \rangle \equiv \langle f, C\ell \rangle$, which implies that $\langle f, \alpha_{i_0} - C\ell \rangle \equiv 0$. Since $\frac{H_{i_0}}{L}(\omega) \not\equiv C$, we have $\alpha_{i_0} - C\ell \neq 0$, which means that $f$ is linearly degenerate, a contradiction.

Then, for every $i \in \{1, 2, \cdots, q\}$, $\Phi_i \not\equiv$ constant. and then

$$\Phi'_i = \Big( \frac{H_i(f)}{L(f)} \Big)' = \frac{(H_i(f))' L(f) - H_i(f)(L(f))'}{(L(f))^2} \not\equiv 0.$$

For any $z_0 \in \mathbb{C}$, $\Phi'_i(z_0) = 0$, we divided into two cases.

**Case 1** $f(z_0) \in H_i$, then $\langle f(z_0), \alpha_i \rangle = 0$. And if $z_0 \in \{z \mid \nu_{\langle f, H_i \rangle}(z) \geq 2\}$, we have $\langle f'(z_0), \alpha_i \rangle = 0$. Then, $z_0$ is a zero of $\Phi'_i$, and

$$\nu_{\Phi'_i}(z_0) \geq \min\{\nu_{\langle f, H_i \rangle}(z_0), \nu_{\langle f', H_i \rangle}(z_0)\} = \nu_{\langle f', H_i \rangle}(z_0) \geq 1.$$

We note that

$$\{z \mid \nu_{\langle f, H_i \rangle}(z) \geq 2\} = \{z \mid \nu_{\langle f, H_i \rangle}(z) = 2\} \cup \{z \mid \nu_{\langle f, H_i \rangle}(z) = 3\} \\ \cup \cdots \cup \{z \mid \nu_{\langle f, H_i \rangle}(z) = n\} \cup \{z \mid \nu_{\langle f, H_i \rangle}(z) > n\}.$$

If $z_0 \in \{z \mid \nu_{\langle f, H_i \rangle}(z) = l\} (= \{z \mid \nu_{\langle f', H_i \rangle}(z) = l - 1\})$, $2 \leq l \leq n$, then

$$\nu_{\Phi'_i}(z_0) \geq \min\{\nu_{\langle f, H_i \rangle}(z_0), \nu_{\langle f', H_i \rangle}(z_0)\} = \nu_{\langle f', H_i \rangle}(z_0) = l - 1$$

$$= \min\{\nu_{\langle f,H_i\rangle}(z_0), n\} - \min\{\nu_{\langle f,H_i\rangle}(z_0), 1\}.$$

If $z_0 \in \{z \mid \nu_{\langle f,H_i\rangle}(z) > n\}$, then $z_0 \in \{z \mid \nu_{\langle f',H_i\rangle}(z) \geq n\}$,

$$\nu_{\Phi_i'}(z_0) \geq \min\{\nu_{\langle f,H_i\rangle}(z_0), \nu_{\langle f',H_i\rangle}(z_0)\} = \nu_{\langle f',H_i\rangle}(z_0) \geq n \geq \min\{\nu_{\langle f,H_i\rangle}(z_0), n\}.$$

**Case 2** $f(z_0) \in H_j$, where $j \in \{1, \cdots, q\}$ and $j \neq i$. By the condition of this lemma, we have

$$f^\sharp(z_0) = \frac{|f \wedge f'|}{\|f\|^2}(z_0) = 0$$

and

$$\big[f_0 : \cdots : f_n\big](z_0) = \big[f_0' : \cdots : f_n'\big](z_0).$$

Then $H_i(f)(z_0) = \lambda(z_0)H_i(f')(z_0) = \lambda(z_0)(H_i(f))'(z_0)$ and $L(f)(z_0) = \lambda(z_0)L(f')(z_0) = \lambda(z_0)(L(f)')(z_0)$, where $\lambda(z_0)$ is a constant. Thus, $\Phi_i'(z_0) = 0$. So, we have any zero of $\langle f(z_0), \alpha_j\rangle$ is also a zero of $\Phi_i'$.

Since $H_1, \cdots, H_q$ are in $k$-subgeneral position in $\mathbb{P}^n(\mathbb{C})$, combining with the discussion above, we have

$$\begin{aligned}\sum_{j=1,j\neq i}^{q} N^{[1]}_{\langle f,H_j\rangle}(r) + N^{[n]}_{\langle f,H_i\rangle}(r) - N^{[1]}_{\langle f,H_i\rangle}(r) &\leq kN_{\Phi_i'}(r)\\ &\leq kT_{\Phi_i'}(r) + O(1).\end{aligned} \tag{3.1}$$

By the first main theorem and the logarithmic derivative lemma of Nevanlinna theory for meromorphic function, we can easily get

$$T_{\Phi_i'}(r) \leq 2T_{\Phi_i}(r) + o(T_{\Phi_i}(r)). \tag{3.2}$$

By (3.1)–(3.2) and [7, p.162], we have

$$\begin{aligned}\sum_{j=1,j\neq i}^{q} N^{[1]}_{\langle f,H_j\rangle}(r) + N^{[n]}_{\langle f,H_i\rangle}(r) - N^{[1]}_{\langle f,H_i\rangle}(r) &\leq 2kT_{\Phi_i}(r) + o(T_{\Phi_i}(r))\\ &= 2kT_f(r) + o(T_f(r)).\end{aligned} \tag{3.3}$$

Take summation of (3.3) over $1 \leq i \leq q$, we have

$$\begin{aligned}(q-1)\sum_{j=1}^{q} N^{[1]}_{\langle f,H_j\rangle}(r) + \sum_{i=1}^{q} N^{[n]}_{\langle f,H_i\rangle}(r) - \sum_{i=1}^{q} N^{[1]}_{\langle f,H_i\rangle}(r) &\leq 2kqT_f(r) + o(T_f(r))\\ (q-2)\sum_{j=1}^{q} N^{[1]}_{\langle f,H_j\rangle}(r) + \sum_{i=1}^{q} N^{[n]}_{\langle f,H_i\rangle}(r) &\leq 2kqT_f(r) + o(T_f(r))\\ \frac{(q-2)}{n}\sum_{j=1}^{q} N^{[n]}_{\langle f,H_j\rangle}(r) + \sum_{i=1}^{q} N^{[n]}_{\langle f,H_i\rangle}(r) &\leq 2kqT_f(r) + o(T_f(r))\end{aligned}$$

$$\frac{(q-2+n)}{n}\sum_{j=1}^{q} N^{[n]}_{\langle f,H_j\rangle}(r) \le 2kqT_f(r)+o(T_f(r)).$$

By Nochka's second main theorem, it follows that

$$\Big\| \frac{(q-2+n)(q-2k+n-1)}{n}T_f(r)-o(T_f(r)) \le 2kqT_f(r)+o(T_f(r)).$$

Comparing the coefficients of $T_f(r)$ in the both sides of above inequality, we have

$$\frac{(q-2+n)(q-2k+n-1)}{n} \le 2kq,$$

then

$$q^2+(2n-2k-3-2kn)q+(n-2)(n-1-2k) \le 0.$$

Since,

$$\begin{aligned}\Delta &= (2n-2k-3-2kn)^2-4(n-2)(n-1-2k)\\ &= 4k^2n^2+8k^2n-8kn^2+12kn+4k^2-4k+1\\ &< (2kn+2k-2n+3)^2.\end{aligned}$$

By calculation,

$$q \le \frac{-(2n-2k-3-2kn)+\sqrt{\Delta}}{2} < 2k(n+1)-2n+3.$$

Thus, this lemma is proved.

## 4 Proof of Theorem 1.4

If not the case, we may assume that $\sup\limits_{z\in\Delta}\{(1-|z|^2)f^\sharp(z)\}=+\infty$. Then there exist a sequence $z_k$, $|z_k|<1$, such that

$$\lim_{k\to\infty}(1-|z_k|^2)f^\sharp(z_k)=+\infty.$$

Let

$$f_k(z)=f(z_k+(1-|z_k|)z), \quad z\in\Delta.$$

Since $|z_k+(1-|z_k|)z| \le |z_k|+(1-|z_k|)|z| < |z_k|+1-|z_k|=1$, we have $f_k$ is well-defined. By calculation,

$$f_k^\sharp(z)=(1-|z_k|)f^\sharp(z_k+(1-|z_k|)z),$$

then

$$f_k^\sharp(0)=(1-|z_k|)f^\sharp(z_k)=\frac{(1-|z_k|^2)f^\sharp(z_k)}{1+|z_k|} > \frac{(1-|z_k|^2)f^\sharp(z_k)}{2}.$$

Thus, $\lim_{k\to\infty} f_k^\sharp(0) = +\infty$ and $\{f_k(z)\}$ is not normal at $z = 0$.

By Lemma 3.1, there exist points $z_k^* \in \Delta$, positive numbers $\rho_k$ with $\rho_k \to 0^+$ such that

$$g_k(\xi) := f_k(z_k^* + \rho_k\xi) \Rightarrow g(\xi),$$

where $g$ is a nonconstant holomorphic mapping of $\mathbb{C}$ into $\mathbb{P}^n(\mathbb{C})$.

If for each $j \in \{1, \cdots, q\}$, $g(\mathbb{C})$ is contained in $H_j$, or $g(\mathbb{C})$ omits $H_j$. By Lemma 3.2, $g$ is a constant, a contradiction. So there exist some $j \in \{1, \cdots, q\}$ and $\xi_0 \in \mathbb{C}$, such that $\langle g(\xi_0), \alpha_j\rangle = 0$ but $\langle g(\xi), \alpha_j\rangle \not\equiv 0$.

We now prove that $g^\sharp(\xi_0) = 0$. By Hurwitz's theorem, there exist points $\xi_k$ with $\xi_k \to \xi_0$ (as $k \to \infty$), such that $g_k(\xi_k) \in H_j$, and hence $f_k(z_k^* + \rho_k\xi_k) \in H_j$. Then $f(z_k + (1 - |z_k|)(z_k^* + \rho_k\xi_k)) \in H_j$.

Denoting

$$a_k = z_k + (1 - |z_k|)(z_k^* + \rho_k\xi_k),$$

we have, by the condition of this theorem, there is a positive constant $M$ such that

$$(1 - |a_k|^2)f^\sharp(a_k) \le M$$

for all $k$ sufficiently large.

By calculation,

$$\begin{aligned}
g_k^\sharp(\xi_k) &= \rho_k f_k^\sharp(z_k^* + \rho_k\xi_k) \\
&= \rho_k(1 - |z_k|)f^\sharp(z_k + (1 - |z_k|)(z_k^* + \rho_k\xi_k)) \\
&= \frac{\rho_k(1 - |z_k|)}{1 - |z_k + (1 - |z_k|)(z_k^* + \rho_k\xi_k)|^2}(1 - |a_k|^2)f^\sharp(a_k) \\
&= \frac{\rho_k\left(\frac{1}{1-|z_k|} - \left|\frac{z_k}{1-|z_k|} + (z_k^* + \rho_k\xi_k)\right|\right)^{-1}}{1 + |z_k + (1 - |z_k|)(z_k^* + \rho_k\xi_k)|}(1 - |a_k|^2)f^\sharp(a_k) \\
&\le \frac{\rho_k(1 - |z_k^* + \rho_k\xi_k|)^{-1}}{1 + |z_k + (1 - |z_k|)(z_k^* + \rho_k\xi_k)|}M \\
&= \frac{\rho_k\left(\frac{1}{1-|z_k^*|} - \left|\frac{z_k^*}{1-|z_k^*|} + \frac{\rho_k}{1-|z_k^*|}\xi_k\right|\right)^{-1}}{(1 - |z_k^*|)(1 + |z_k + (1 - |z_k|)(z_k^* + \rho_k\xi_k)|)}M.
\end{aligned}$$

By the proof of Lemma 3.1, we have $\lim_{k\to\infty} \frac{\rho_k}{1-|z_k^*|} = 0$, then

$$\begin{aligned}
\frac{1}{1 - |z_k^*|} - \left|\frac{z_k^*}{1 - |z_k^*|} + \frac{\rho_k}{1 - |z_k^*|}\xi_k\right| &\ge \left(\frac{1}{1 - |z_k^*|} - \frac{|z_k^*|}{1 - |z_k^*|}\right) - \frac{\rho_k}{1 - |z_k^*|}|\xi_k| \\
&= 1 - \frac{\rho_k}{1 - |z_k^*|}|\xi_k| > \frac{1}{2}
\end{aligned}$$

as $k \to \infty$.

So, we have

$$g_k^\sharp(\xi_k) \le 2M \cdot \frac{\rho_k}{1 - |z_k^*|}.$$

Let $k \to \infty$, $g^\sharp(\xi_0) = 0$.

Without loss of the generality, we may assume that there exists some integer $q_0$ with $1 \le q_0 \le q$, such that for any $j \in \{1, \cdots, q_0\}$, $g(\mathbb{C}) \not\subset H_j$, and $j \in \{q_0+1, \cdots, q\}$, $g(\mathbb{C}) \subset H_j$. Denote the smallest subspace of $\mathbb{P}^n(\mathbb{C})$ containing $g(\mathbb{C})$ by $\mathbb{P}$. Then $p := \dim \mathbb{P} \ge 1$, and $g$ is a linearly non-degenerate entire curve in $\mathbb{P}$. Since $H_1, \cdots, H_q$ are in general position, we have $q - q_0 + p \le n$, furthermore, $\widetilde{H_1} := H_1 \cap \mathbb{P}, \cdots, \widetilde{H_{q_0}} := H_{q_0} \cap \mathbb{P}$ are hyperplanes in $n - (q - q_0)$ subgeneral position in $\mathbb{P}$.

For each $j = 1, \cdots, q_0$ and for all $\xi_0 \in \mathbb{C}$, such that $g(\xi_0) \in H_j$, we have $g(\xi_0) \in \widetilde{H_j}$ and $g^\sharp(\xi_0) = 0$.

Since $q \ge 2n^2+3 > 2n+1$, we have $q_0 > q_0-(q-q_0)-(q-2n-1)-p = 2[n-(q-q_0)]-p+1$.

Applying Lemma 3.4, we have

$$q_0 < 2(n - (q - q_0))(p + 1) - 2p + 3.$$

Then

$$q_0 + 2(p + 1)(q - q_0) < 2n(p + 1) - 2p + 3.$$

Therefore,

$$\begin{aligned} q &= q_0 + (q - q_0) \\ &\le q_0 + 2(p + 1)(q - q_0) \\ &< 2n(p + 1) - 2p + 3 \\ &= 2n + (2n - 2)p + 3 \\ &\le 2n + (2n - 2)n + 3 \\ &= 2n^2 + 3. \end{aligned}$$

This contradicts to the assumption that $q \ge 2n^2 + 3$. Thus, we have $\sup\limits_{z\in\Delta}\{(1 - |z|^2)f^\sharp(z)\} < \infty$.

The proof of Theorem 1.4 is finished.

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