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Cartan's Second Main Theorem and Mason's Theorem for Jackson Difference Operator*

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Abstract Let $f : \mathbb{C} \to \mathbb{P}^n$ be a holomorphic curve of order zero. The authors establish a Jackson difference analogue of Cartan's second main theorem for the Jackson q-Casorati determinant and introduce a truncated second main theorem of Jackson difference operator for holomorphic curves. In addition, a Jackson difference Mason's theorem is proved by using a Jackson difference radical of a polynomial. Furthermore, they extend the Mason's theorem for $m+1$ polynomials. Some examples are constructed to show that their results are accurate.

Keywords Jackson difference operator, Nevanlinna theory, Holomorphic curve, Cartan's second main theorem, Mason's theorem, Polynomial
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1 Introduction

In the 1980s, an intensive and somewhat surprising interest in the subject reappeared in many areas of mathematics and applications including mainly new difference calculus, orthogonal polynomials and variational q-calculus (see [10]). In 1908, Jackson investigated the Jackson difference operator (or called q-derivative) and studied the q-difference equations. These polynomials, the Jackson difference operator and related topics have found numerous applications, such as q-hypergeometric series (see [10]), boundary value problems of q-difference equations (see [24]).

Recently, there has been a renewed interest in the difference analogues and Jackson difference operator. In 2006, Halburd and Korhonen [12] introduced the c-difference operator: $\Delta_c f = f(z+c) - f(z)$, and got the c-difference analogue of the second main theorem for meromorphic functions in the complex plane. In 2014, Korhonen and Tohge [13] extended the results of c-difference analogue to holomorphic curves intersecting hyperplanes in general position. Meanwhile, many scholars investigated the value distribution theory of q-difference analogue in several complex variables (see [5–7]). For the Jackson difference operator, Li and

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Song [18] considered the functional and dynamical properties. Cao, Dai and Wang [4] studied the Nevanlinna theory of Jackson difference operator and the solutions of q-difference equations. In this paper, we study the Cartan's second main theorem of Jackson difference operator and a Jackson difference analogue of Mason's theorem (see [14, 21]).

This paper is organised as follows. In Section 2 we will give some basic notations of Cartan characteristic function and Jackson difference operator theory. Section 3 contains a Jackson difference analogue of the second main theorem for holomorphic curves. Here we introduce a Jackson difference counterpart of the radical, and use it to prove a truncated version of the Cartan's second main theorem for Jackson difference operator. In Section 4, we state a Jackson difference analogue of Mason's theorem and extend it for $m+1$ polynomials. In addition, the sharpness of the obtained results is discussed with the help of examples.

2 Preliminaries

2.1 First we recall some known properties of the Cartan characteristic function from [8, 11, 13, 17, 20]. The order of growth of a holomorphic curve $g : \mathbb{C} \to \mathbb{P}^n$ with homogeneous coordinates $g = [g_0 : \cdots : g_n]$ is defined by

$$\sigma(g) = \limsup_{r\to\infty} \frac{\log^+ T_g(r)}{\log r}, \tag{2.1}$$

where $\log^+ x = \max\{0, \log x\}$ for all $x \geq 0$. If the functions g_j $(j = 0, \cdots, n)$ are entire functions without common zeros, then $g = [g_0 : \cdots : g_n]$ with $n \geq 1$ is called the reduced representation of g. The Cartan characteristic function of g is

$$T_g(r) := \int_0^{2\pi} u(re^{i\theta})\frac{d\theta}{2\pi} - u(0), \quad u(z) = \sup_{j\in\{0,1,\cdots,n\}} \log|g_j(z)|.$$

Moreover, if $f_0, \cdots, f_q$ are linear combinations of $g_0, \cdots, g_n$, and satisfy that any $n+1$ of the functions $f_0, \cdots, f_q$ are linearly independent over $\mathbb{C}$, $q > n$, then

$$T\Big(r, \frac{f_\mu}{f_\nu}\Big) \leq T_g(r) + O(1), \quad r \to \infty, \tag{2.2}$$

where μ and ν are distinct integers in the set $\{0, \cdots, q\}$. Especially, if $n = 1$, then (2.2) becomes an asymptotic identity. For more detailed concepts of Nevanlinna theory and Cartan's value distribution theory, we can refer to [9, 13, 17, 20, 23].

2.2 The Jackson difference operator

$$D_q f(z) = \frac{f(qz) - f(z)}{qz - z}, \quad z \in \mathbb{C},\ 0 < |q| < 1 \tag{2.3}$$

was initially investigated by Jackson [15–16] in 1908. In [4], Cao, Dai and Wang studied the Nevanlinna theory of Jackson difference operator on the complex plane. For a positive integer $k \in \mathbb{N}$, the Jackson kth-order difference operator is denoted by

$$D_q^0 f(z) := f(z), \quad D_q^k f(z) := D_q(D_q^{k-1} f(z)),$$

it follows from the equality [1, page 13],

$$D_q^k f(z) = (q-1)^{-k} z^{-k} q^{\frac{-k(k-1)}{2}} \sum_{j=0}^{k} (-1)^j \begin{bmatrix} k \\ j \end{bmatrix}_q q^{\frac{j(j-1)}{2}} f(q^{k-j} z), \tag{2.4}$$

where

$$\begin{bmatrix} k \\ j \end{bmatrix}_q = \frac{(q;q)_k}{(q;q)_j (q;q)_{k-j}}, \quad (a;q)_0 = 1$$

and

$$(a;q)_k = (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{k-1}), \quad k \in \mathbb{N},\ a \in \mathbb{C}.$$

In addition, we can find more notations about Jackson difference operator in [4].

Let $f(z)$ be a non-constant meromorphic function of zero order and $q \in \mathbb{C}\backslash\{0,1\}$. The q-difference operator $\Delta_q f(z) := f(qz) - f(z)$ is defined by Barnett [2]. Thus by the definition of Jackson difference operator, we can easily show that

$$D_q f(z) = \frac{\Delta_q f(z)}{qz - z}, \quad z \in \mathbb{C}.$$

The Wronskian determinant is indispensable in the proof of the Cartan's generalization of the main second theorem. In [13], a q-difference analogue of Cartan's result where the ramification term has been replaced by a quantity was expressed in terms of the q-Casorati determinant of functions which are linearly independent over a field. The q-Casorati determinant (see [13]) is defined by

$$C_q(g_0, \cdots, g_n) = \begin{vmatrix} g_0(z) & g_1(z) & \cdots & g_n(z) \\ g_0(qz) & g_1(qz) & \cdots & g_n(qz) \\ \vdots & \vdots & \ddots & \vdots \\ g_0(q^n z) & g_1(q^n z) & \cdots & g_n(q^n z) \end{vmatrix}.$$

Similarly, we denote by

$$C_J(g_0, \cdots, g_n) = \begin{vmatrix} g_0 & g_1 & \cdots & g_n \\ D_q g_0 & D_q g_1 & \cdots & D_q g_n \\ \vdots & \vdots & \ddots & \vdots \\ D_q^n g_0 & D_q^n g_1 & \cdots & D_q^n g_n \end{vmatrix}$$

the Jackson q-Casorati determinant of $g_0, \cdots, g_n$.

In fact, we can get the relationship between $\mathbb{C}_q(g_0, g_1, \cdots, g_n)$ and $\mathbb{C}_J(g_0, g_1, \cdots, g_n)$:

$$C_q(f_0, \cdots, f_n) = C_J(f_0, \cdots, f_n) \cdot B,$$

where

$$B = z^{\frac{n(n+1)}{2}} \cdot (q-1)^{\frac{n(n+1)}{2}} \cdot q^{\frac{n(n-1)(n+1)}{6}}.$$

We will give it a concrete proof in Lemma 3.1.

2.3 In order to obtain a truncated version of the second main theorem for Jackson difference operator, we introduce the following definitions.

As in [9, 14], denote by $\mathrm{ord}_\zeta(f)=\mu\in\mathbb{Z}$ the order of a meromorphic function f at $\zeta\in\mathbb{C}$ if

$$\lim_{z\to\zeta}\frac{f(z)}{(z-\zeta)^\mu}\in\mathbb{C}\backslash\{0\}.$$

As we all know, $\mathrm{ord}_\zeta(f)>0$ implies that f has a zero of order $\mathrm{ord}_\zeta(f)>0$ at ζ, and $\mathrm{ord}_\zeta(f)<0$ implies that f has a pole of order $-\mathrm{ord}_\zeta(f)$ at ζ. We also adopt the notation $\mathrm{ord}_\zeta^+(f)=\max\{0,\mathrm{ord}_\zeta(f)\}$ and $\mathrm{ord}_\zeta^-(f)=\max\{0,-\mathrm{ord}_\zeta(f)\}$. For the closed disc $\overline{D}(z_0,s)=\{z\in\mathbb{C}:|z-z_0|\le s\}$, we define

$$\widetilde{n}_J^{[n]}\Big(r,\frac{1}{f}\Big)=\sum_{\omega\in\overline{D}(0,r)}\Big(\mathrm{ord}_\omega^+(f)-\min\{\mathrm{ord}_\omega^+(f),\mathrm{ord}_\omega^+(D_q(f)),\cdots,\mathrm{ord}_\omega^+(D_q^n(f))\}\Big)$$

as a Jackson difference analogue of the truncated counting function for the zeros of f. The corresponding integrated counting function is defined as

$$\widetilde{N}_J^{[n]}\big(r,\frac{1}{f}\big)=\int_0^r\frac{\widetilde{n}_J^{[n]}\big(t,\frac{1}{f}\big)-\widetilde{n}_J^{[n]}\big(0,\frac{1}{f}\big)}{t}\mathrm{d}t+\widetilde{n}_J^{[n]}\Big(0,\frac{1}{f}\Big)\log r. \tag{2.5}$$

where $n\in\mathbb{N}$, $q\in\mathbb{C}\backslash\{0,1\}$.

3 Jackson Difference Analogue of Cartan's Second Main Theorem

Before describing Theorem 3.1, we give a few other remarks. Let f be a non-constant zero-order meromorphic function, and $q\in\mathbb{C}\backslash\{0,1\}$. We denote by $\mathcal{P}_q^0$ the field of meromorphic functions which have forward invariant preimages under f with respect to the rescaling $\tau(z)=qz$, that is $f(qz)\equiv f(z)$. In particular, if $|q|<1$, then for any $f\in\mathcal{P}_q^0$, f is a constant. Besides, we set meromorphic function

$$L=\frac{f_0\cdot D_qf_1\cdot D_q^2f_2\cdots D_q^nf_n\cdot f_{n+1}\cdots f_p}{C_J(g_0,g_1,\cdots,g_n)},\quad q\in\mathbb{C}\backslash\{0,1\}. \tag{3.1}$$

We are now ready to state the Jackson difference analogue of Cartan's second main theorem.

Theorem 3.1 *Let $g=[g_0:\cdots:g_n]:\mathbb{C}\to\mathbb{P}^n$ be a holomorphic curve with $\sigma(g)=0$, $g_0,\cdots,g_n$ $(n\ge1)$ are entire functions without common zeros, linearly independent over $\mathcal{P}_q^0$, and $z\in\mathbb{C}\backslash\{0\}$. Suppose $f_0,\cdots,f_p$ are linear combinations of the functions $g_0,\cdots,g_n$ $(p>n)$, such that any $n+1$ of the functions $f_0,\cdots,f_p$ are linearly independent. L satisfies* (3.1), *then*

$$(p-n)T_g(r)\le N\Big(r,\frac{1}{L}\Big)-N(r,L)-\frac{n(n+1)}{2}\log r+o(T_g(r))+S(r,f),$$

where $S(r,f)=o(T(r,f_j))$, $j=0,\cdots,p$ and r approaches infinity on a set of logarithmic density 1.

To prove Theorem 3.1, it suffices to introduce some lemmas. The first lemma is that the relationship between $\mathbb{C}_q(g_0,g_1,\cdots,g_n)$ and $\mathbb{C}_J(g_0,g_1,\cdots,g_n)$.

Lemma 3.1 *Let $g_0, \cdots, g_n$ be entire functions, $n > 1$, $q \in \mathbb{C}\backslash\{0,1\}$. Then*

$$C_q(g_0, \cdots, g_n) = C_J(g_0, \cdots, g_n) \cdot B,$$

where

$$B = (qz - z)^{\frac{n(n+1)}{2}} \cdot q^{\frac{n(n-1)(n+1)}{6}}.$$

Proof It follows from the definition of $\mathbb{C}_J(g_0, g_1, \cdots, g_n)$ and $D_q^k g_k$ that

$$\begin{aligned}
&C_J(g_0, \cdots, g_n)\\
&= \begin{vmatrix} g_0(z) & g_1(z) & \cdots & g_n(z) \\ D_q g_0(z) & D_q g_1(z) & \cdots & D_q g_n(z) \\ \vdots & \vdots & \ddots & \vdots \\ D_q^n g_0(z) & D_q^n g_1(z) & \cdots & D_q^n g_n(z) \end{vmatrix}\\
&= B_1 \cdot \begin{vmatrix} g_0(z) & \cdots & g_n(z) \\ \sum\limits_{j=0}^{1} (-1)^j q^{\frac{j(j-1)}{2}} \begin{bmatrix} 1 \\ j \end{bmatrix}_q g_0(q^{1-j}z) & \cdots & \sum\limits_{j=0}^{1} (-1)^j q^{\frac{j(j-1)}{2}} \begin{bmatrix} 1 \\ j \end{bmatrix}_q g_n(q^{1-j}z) \\ \vdots & \ddots & \vdots \\ \sum\limits_{j=0}^{n} (-1)^j q^{\frac{j(j-1)}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_q g_0(q^{n-j}z) & \cdots & \sum\limits_{j=0}^{n} (-1)^j q^{\frac{j(j-1)}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_q g_n(q^{n-j}z) \end{vmatrix}\\
&= B_1 \cdot \begin{vmatrix} g_0(z) & g_1(z) & \cdots & g_n(z) \\ g_0(qz) & g_1(qz) & \cdots & g_n(qz) \\ \vdots & \vdots & \ddots & \vdots \\ g_0(q^n z) & g_1(q^n z) & \cdots & g_n(q^n z) \end{vmatrix},
\end{aligned}$$

where

$$\begin{aligned}
B_1 = \prod_{k=0}^{n} \frac{1}{(q-1)^k z^k q^{\frac{k(k-1)}{2}}} &= \frac{1}{(qz-z)^{\frac{n(n+1)}{2}} \cdot q^{\sum_{k=0}^{n} \frac{k(k-1)}{2}}}\\
&= \frac{1}{(qz-z)^{\frac{n(n+1)}{2}} \cdot q^{\frac{n(n-1)(n+1)}{6}}},
\end{aligned}$$

set $B = \frac{1}{B_1}$, we obtain $C_q(g_0, \cdots, g_n) = C_J(g_0, \cdots, g_n) \cdot B$.

For the classical Nevanlinna theory, we have the fact that entire functions $f_0, \cdots, f_n$ are linearly dependent over $\mathbb{C}$ if and only if the Wronskian $W(f_0, \cdots, f_n)$ vanishes identically. Let $c \in \mathbb{C}$, and let $\mathcal{P}_c^1$ be the field of period c meromorphic functions defined in $\mathbb{C}$ of hyper-order strictly less than one. In general, we know (see [13]) that entire functions $f_0, \cdots, f_n$ are linearly dependent over the field $\mathcal{P}_c^1$ if and only if the Casorati determinant of $C(f_0, \cdots, f_n)$ vanishes identically. Here we introduce the analogue of these results for the case of Jackson difference operator.

Lemma 3.2 *Let the holomorphic curve $g = [g_0 : \cdots : g_n] : \mathbb{C} \to P^n$ satisfy $\sigma(g) = 0$ and $q \in \mathbb{C}\backslash\{0,1\}$. Then $C_J(g_0, \cdots, g_n) \equiv 0$ if and only if the entire functions $g_0, \cdots, g_n$ are linearly dependent over the field $\mathcal{P}_q^0$.*

Proof Suppose that $g_0, \cdots, g_n$ are linearly dependent over $\mathcal{P}_q^0$. Then there exist $A_0, \cdots, A_n \in \mathcal{P}_q^0$ which are not all identically zero and satisfy $A_0g_0 + \cdots + A_ng_n = 0$. This clearly implies that

$$\begin{cases} A_0g_0 + \cdots + A_ng_n = 0, \\ A_0D_qg_0 + \cdots + A_nD_qg_n = 0, \\ \qquad\qquad \vdots \\ A_0D_q^ng_0 + \cdots + A_nD_q^ng_n = 0. \end{cases} \tag{3.2}$$

The determinant of the coefficient matrix corresponding to system (3.2) is the Jackson q-Casoratian determinant $C_J(g_0, \cdots, g_n)$. We notice that (3.2) has a nontrivial solution, hence $C_J(g_0, \cdots, g_n) \equiv 0$.

On the other hand, we will continue to prove it by induction. For $n = 1$, we assume that $C_J(g_0, g_1) \equiv 0$, and consider

$$\begin{cases} A_0g_0 + A_1g_1 = 0, \\ A_0D_qg_0 + A_1D_qg_1 = 0, \end{cases} \tag{3.3}$$

where A_0, A_1 are meromorphic functions. Furthermore, (3.3) is equivalent to

$$\begin{cases} A_0g_0 + A_1g_1 = 0, \\ A_0\Delta_qg_0 + A_1\Delta_qg_1 = 0, \end{cases} \tag{3.4}$$

and we have

$$\begin{cases} A_0g_0 + A_1g_1 = 0, \\ A_1C_q(g_0, g_1) = 0. \end{cases}$$

Since $C_J(g_0, g_1) = 0$, by Lemma 3.1 we get $C_q(g_0, g_1) = 0$, it follows that $A_0 = \frac{g_1}{g_0}$ and $A_1 = -1$ is a nontrivial solution of (3.3). Moreover, the usual order of A_0 satisfies $\sigma(A_0) = \sigma(\frac{g_1}{g_0}) \leq \sigma(g) = 0$ by (2.2). It is clear that $A_1 \in \mathcal{P}_q^0$. To complete the proof in the case $n = 1$, we just need to show that $A_0 \in \mathcal{P}_q^0$. From (3.4),

$$g_0(qz) \cdot \triangle_qA_0 = 0,$$

which implies that $\triangle_qA_0 = 0$. Then $A_0 \in \mathcal{P}_q^0$.

For all $j \in \{1, \cdots, k-1\}$ and $k \leq n$, suppose $C_J(g_0, \cdots, g_j) \equiv 0$, then $g_0, \cdots, g_j$ are linearly dependent over $\mathcal{P}_q^0$. We will prove that $g_0, \cdots, g_k$ are linearly dependent when $C_J(g_0, \cdots, g_k) \equiv 0$ over $\mathcal{P}_q^0$. We consider the linear system

$$\begin{cases} A_0g_0 + \cdots + A_kg_k = 0, \\ A_0D_qg_0 + \cdots + A_kD_qg_k = 0, \\ \qquad\qquad \vdots \\ A_0D_q^kg_0 + \cdots + A_kD_q^kg_k = 0, \end{cases} \tag{3.5}$$

where $A_0, \cdots, A_k$ are meromorphic functions. According to (2.3) and (2.4) we have

$$D_q^kg_i(z) = (q-1)^{-k}z^{-k}q^{-\frac{k(k-1)}{2}}\sum_{j=0}^{k}(-1)^j \begin{bmatrix} k \\ j \end{bmatrix}_q q^{\frac{j(j-1)}{2}}g_i(q^{k-j}z),$$

where $i = 0, \cdots, k$. Therefore

$$A_0 D_q^m g_0(z) + A_1 D_q^m g_1(z) + \cdots + A_k D_q^m g_k(z) = 0, \quad m = 1, \cdots, k$$

implies

$$A_0 g_0(q^m z) + A_1 g_1(q^m z) + \cdots + A_k g_k(q^m z) = 0, \quad m = 1, \cdots, k.$$

So the linear system (3.5) is equivalent to

$$\begin{cases} A_0 g_0 + \cdots + A_k g_k = 0, \\ A_0 g_0(qz) + \cdots + A_k g_k(qz) = 0, \\ \qquad\qquad \vdots \\ A_0 g_0(q^k z) + \cdots + A_k g_k(q^k z) = 0. \end{cases}$$

Then following the same method in [13, Lemma 3.2], we can get $g_0, \cdots, g_k$ are linearly dependent over $\mathcal{P}_q^0$.

Lemma 3.3 (see [2]) *Let f be a non-constant zero-order meromorphic function, and $q \in \mathbb{C}\backslash\{0\}$. Then*

$$m\Big(r, \frac{f(qz)}{f(z)}\Big) = o(T(r, f))$$

on a set of logarithmic density 1.

Lemma 3.4 (see [4]) *Let f be a nonconstant meromorphic function with zero order and $q \in \mathbb{C}\backslash\{0\}$. Then*

$$m\Big(r, \frac{D_q^k f(z)}{f(z)}\Big) = o(T(r, f))$$

on a set of logarithmic density 1.

Lemma 3.5 (see [8]) *Let $n \geq 1$, $z \in \mathbb{C}$ and let $g_0, \cdots, g_n$ be linearly independent entire functions such that* $\max\{|g_0(z)|, \cdots, |g_n(z)|\} > 0$ *for each $z \in \mathbb{C}$. If $f_0, \cdots, f_p$ are $p+1$ linear combinations of the $n+1$ functions $g_0, \cdots, g_n$, where $p > n$, such that any $n+1$ of the $p+1$ functions $f_0, \cdots, f_p$ are linearly independent, then there exists a positive constant A that does not depend on z, such that*

$$|g_j(z)| \leq A|f_{m_v}(z)|,$$

where $0 \leq j \leq n$, $0 \leq \nu \leq p-n$ and the integers $m_0, \cdots, m_p$ are chosen so that

$$|f_{m_0}(z)| \geq |f_{m_1}(z)| \geq \cdots \geq |f_{m_p}(z)|.$$

In particular, there exist at least $p-n+1$ functions f_j that do not vanish at z.

Proof of Theorem 3.1 We follow the reasoning behind the original Cartan's second main theorem (see [11]) and the difference analogue of Cantan's second main theorem (see [13]). Because the functions $g_0, \cdots, g_n$ are linearly independent over $\mathcal{P}_q^0$, by Lemma 3.2 we have

$C_J(g_0, \cdots, g_n) \not\equiv 0$ and the function L given by (3.1) is well defined. The functions $g_0, \cdots, g_n$ are also linearly independent over $\mathbb{C}$ (since $\mathbb{C} \subset \mathcal{P}_q^0$), thus by Lemma 3.5 the auxiliary function

$$\upsilon(z) = \max_{\{a_j\}_{j=0}^{p-n-1} \subset \{0,\cdots,p\}} \log |f_{a_0}(z) \cdots f_{a_{p-n-1}}(z)| \tag{3.6}$$

gives a finite real number for all $z \in \mathbb{C}$. Let $\{a_0, \cdots, a_{p-n-1}\} \subset \{0, \cdots, p\}$, and $\{b_0, \cdots, b_n\} = \{0, \cdots, p\} \backslash \{a_0, \cdots, a_{p-n-1}\}$. Similarly, for $n+1$ linearly independent functions $f_{b_0}, \cdots, f_{b_n}$, we get $C_J(f_{b_0}, \cdots, f_{b_n}) \not\equiv 0$ and

$$\begin{pmatrix} f_{b_0} & \cdots & f_{b_n} \\ D_q f_{b_0} & \cdots & D_q f_{b_n} \\ \vdots & \ddots & \vdots \\ D_q^n f_{b_0} & \cdots & D_q^n f_{b_n} \end{pmatrix} = \begin{pmatrix} g_0 & \cdots & g_n \\ D_q g_0 & \cdots & D_q g_n \\ \vdots & \ddots & \vdots \\ D_q^n g_0 & \cdots & D_q^n g_n \end{pmatrix} \begin{pmatrix} \pi_{00} & \cdots & \pi_{0n} \\ \pi_{10} & \cdots & \pi_{1n} \\ \vdots & \ddots & \vdots \\ \pi_{n0} & \cdots & \pi_{nn} \end{pmatrix},$$

where $\pi_{jm} \in \mathbb{C}$ for all $j, m = 0, \cdots, n$. Therefore,

$$C_J(g_0, \cdots, g_n) = A(b_0, \cdots, b_n) C_J(f_{b0}, \cdots, f_{bn}), \tag{3.7}$$

where $A(b_0, \cdots, b_n) =: A_b \in \mathbb{C}\backslash\{0\}$. By Lemma 3.1 and (3.1), we have

$$\begin{aligned} L &= \frac{f_0 \cdot D_q f_1 \cdot D_q^2 f_2 \cdots D_q^n f_n \cdot f_{n+1} \cdots f_p}{C_J(g_0, \cdots, g_n)} \\ &= \frac{f_0 \cdot D_q f_1 \cdot D_q^2 f_2 \cdots D_q^n f_n \cdot f_{n+1} \cdots f_p}{A_b C_J(f_{b_0}, f_{b_1}, \cdots, f_{b_n})} \\ &= \frac{f_0 f_1 \cdots f_p \big(\frac{D_q f_1}{f_1}\big)\big(\frac{D_q^2 f_2}{f_2}\big) \cdots \big(\frac{D_q^n f_n}{f_n}\big)}{A_b C_J(f_{b_0}, \cdots, f_{b_n})} \\ &= \frac{M}{A_b C_J(f_{b_0}, \cdots, f_{b_n})} \\ &= \frac{M \cdot B}{A_b C_q(f_{b_0}, \cdots, f_{b_n})} \\ &= \frac{f_{a_0} \cdots f_{a_{p-n-1}} \big(\frac{D_q f_1}{f_1}\big) \cdot \big(\frac{f_{b_1}}{D_q f_{b_1}}\big) \cdots \big(\frac{D_q^n f_n}{f_n}\big) \cdot \big(\frac{f_{b_n}}{D_q^n f_{b_n}}\big) \cdot B}{\Big(\frac{A_b f_0(z) \cdot f_0(qz) \cdots f_0(q^n z) C_q\big(\frac{f_{b_0}}{f_0}, \frac{f_{b_1}}{f_0}, \cdots, \frac{f_{b_n}}{f_0}\big)}{f_{b_0}(D_q f_{b_1}) \cdots (D_q^n f_{b_n})}\Big)} \\ &= \frac{f_{a_0} \cdots f_{a_{q-n-1}} \big(\frac{D_q f_1}{f_1}\big)\big(\frac{f_{b_1}}{D_q f_{b_1}}\big) \cdots \big(\frac{D_q^n f_n}{f_n}\big)\big(\frac{f_{b_n}}{D_q^n f_{b_n}}\big) \cdot B}{\Big(\frac{A_b C_q\big(\frac{f_{b_0}}{f_0}, \frac{f_{b_1}}{f_0}, \cdots, \frac{f_{b_n}}{f_0}\big)}{(f_{b_0}/f_0) \cdot \big(\frac{D_q f_{b_1}}{f_0(qz)}\big) \cdots \big(\frac{D_q^n f_{b_n}}{f_0(q^n z)}\big)}\Big)}, \end{aligned}$$

where

$$M = f_{b_0}(D_q f_{b_1}) \cdots (D_q^n f_{b_n}) \cdot f_{a_0} \cdots f_{a_{p-n-1}} \Big(\frac{D_q f_1}{f_1}\Big) \cdots \Big(\frac{D_q^n f_n}{f_n}\Big) \cdot \Big(\frac{f_{b_1}}{D_q f_{b_1}}\Big) \cdots \Big(\frac{f_{b_n}}{D_q^n f_{b_n}}\Big).$$

Therefore

$$L = \frac{f_{a_0} \cdots f_{a_{p-n-1}}}{A_b G},$$

where

$$G = \frac{G_1}{B},$$

$$G_1=\frac{\Big(\frac{C_q\left(\frac{f_{b_0}}{f_0},\frac{f_{b_1}}{f_0},\cdots,\frac{f_{b_n}}{f_0}\right)}{(f_{b_0}/f_0)\cdot\left(D_q\frac{f_{b_1}}{f_0}(qz)\right)\cdots\left(D_q^n\frac{f_{b_n}}{f_0}(q^nz)\right)}\Big)\left(\frac{D_qf_{b_1}}{f_{b_1}}\right)\cdots\left(\frac{D_qf_{b_n}}{f_{b_n}}\right)}{\left(\frac{D_qf_1}{f_1}\right)\cdots\left(\frac{D_qf_n}{f_n}\right)}. \tag{3.8}$$

Set

$$\omega(z)=\max_{\{b_j\}_{j=0}^n\subset\{0,\cdots,p\}}\log|A_bG(z)|,$$

it follows that

$$\int_0^{2\pi}\nu(re^{i\theta})d\theta=\int_0^{2\pi}\log|L(re^{i\theta})|d\theta+\int_0^{2\pi}\omega(re^{i\theta})d\theta. \tag{3.9}$$

Suppose $\{c_0,\cdots,c_{p-n-1}\}$ is the set of indexes that make (3.6) have the maximum value for a specific $z\in\mathbb{C}$. Then by Lemma 3.5, it is not difficult to see the inequality

$$\log|g_j(z)|\le\log|f_{c_\upsilon}(z)|+\log A$$

holds for all $0\le j\le n$ and $0\le\upsilon\le p-n-1$, and

$$(p-n)T_g(r)\le\frac{1}{2\pi}\int_0^{2\pi}\upsilon(re^{i\theta})d\theta+O(1),\quad r\to\infty. \tag{3.10}$$

Since the function G_1 in (3.8) consists purely of sums, products and quotients of fractions of the form $\frac{\left(\frac{f_j(q^lz)}{f_k(q^lz)}\right)}{\frac{f_j}{f_k}}$, $D_q^n\frac{f_j}{f_j}$ and $\frac{f_j(q^lz)}{f_j(z)}$, where $l\in\{1,\cdots,n\}$, $j,k\in\{0,\cdots,p\}$. We deduce from Lemmas 3.3–3.4 that

$$\omega=\max_{\{b_j\}_{j=0}^n\subset\{0,\cdots,p\}}\Big\{\log|G_1|-\frac{n(n+1)}{2}\log r+O(1)\Big\},$$

$$\frac{1}{2\pi}\int_0^{2\pi}\omega(re^{i\theta})d\theta\le\sum_{j=0}^p\sum_{k=0}^p o\Big(T\Big(r,\frac{f_j}{f_k}\Big)\Big)-\frac{n(n+1)}{2}\log r+o(T(r,f_j)) \tag{3.11}$$

as r approaches infinity on a set of logarithmic density 1. It yields from (3.11) and (2.2) that

$$\frac{1}{2\pi}\int_0^{2\pi}\omega(re^{i\theta})d\theta=o(T_g(r))-\frac{n(n+1)}{2}\log r+o(T(r,f_j)) \tag{3.12}$$

as r tends to infinity on a set of logarithmic density 1.

Finally, by Jensen's formula

$$\frac{1}{2\pi}\int_0^{2\pi}\log|L(re^{i\theta})|d\theta=N\Big(r,\frac{1}{L}\Big)-N(r,L)+O(1) \tag{3.13}$$

as $r\to\infty$. Combining (3.9), (3.10), (3.12) and (3.13), it yields out

$$(p-n)T_g(r)\le N\Big(r,\frac{1}{L}\Big)-N(r,L)+o(T_g(r))-\frac{n(n+1)}{2}\log r+S(r,f),$$

where $S(r,f)=o(T(r,f_j))$, $j=0,\cdots,p$.

Furthermore, we obtain the following result about a truncated second main theorem of Jackson difference operator for holomorphic curve.

Theorem 3.2 *Let $g_0, \cdots, g_n$ $(n \geq 1)$ be entire functions with no common zeros, linearly independent over $\mathcal{P}_q^0$, and $g_{n+1} = g_0 + \cdots + g_n$. If the holomorphic curve $g = [g_0 : \cdots : g_n] : \mathbb{C} \to P^n$ satisfies $\sigma(g) = 0$, then*

$$T_g(r) \leq \sum_{j=0}^{n+1} \widetilde{N}_J^{[n]}\Big(r, \frac{1}{g_j}\Big) + o(T_g(r)), \quad q \in \mathbb{C}\backslash\{0,1\} \tag{3.14}$$

as $r \to \infty$ on a set of logarithmic density 1.

Proof We define

$$L = \frac{g_0 \cdot D_q g_1 \cdots D_q^n g_n \cdot g_{n+1}}{C_J(g_0 \cdots g_n)},$$

where $C_J(g_0, \cdots, g_n)$ is the Jackson q-Casoratian determinant of $g_0, \cdots, g_n$. Suppose ω is a zero of L, we assert that

$$\mathrm{ord}_\omega^+(L) \leq \sum_{j=0}^{n+1} (\mathrm{ord}_\omega^+(g_j) - \min_{i \in \{0,\cdots,n\}} \{\mathrm{ord}_\omega^+(D_q^{\mathrm{i}} g_j)\}). \tag{3.15}$$

Since $n \geq 1$, $g_0, \cdots, g_n$ are linearly independent over $\mathcal{P}_q^0$ with no common zeros, and $g_{n+1} = g_0 + \cdots + g_n$. We have $g_{n+1}(z) \neq 0$ for all $z \in \mathbb{C}$. To illustrate that (3.15) is correct, we write

$$\frac{1}{L} = \frac{1}{g_{n+1}} \cdot \begin{vmatrix} 1 & \frac{g_1}{D_q g_1} & \cdots & \frac{g_n}{D_q^n g_n} \\ \frac{D_q g_0}{g_0} & 1 & \cdots & \frac{D_q g_n}{D_q^n g_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{D_q^n g_0}{g_0} & \frac{D_q^n g_1}{D_q g_1} & \cdots & 1 \end{vmatrix}.$$

Note that the zeros of L are the poles of some of $\frac{D_q^{\mathrm{i}} g_0(z)}{g_0(z)}, \frac{D_q^{\mathrm{i}} g_1(z)}{D_q g_1(z)}, \cdots, \frac{D_q^{\mathrm{i}} g_n(z)}{D_q^n g_n(z)}$, $1 \leq i \leq n$. The maximal order of poles among the first column in the determinant above is given by

$$\max_{i \in \{0,\cdots,n\}} \{\mathrm{ord}_\omega^+(g_0) - \{\mathrm{ord}_\omega^+(D_q^{\mathrm{i}}(g_0))\}\} = \mathrm{ord}_\omega^+(g_0) - \min_{i \in \{0,\cdots,n\}} \{\mathrm{ord}_\omega^+(D_q^{\mathrm{i}}(g_0))\}.$$

It is worth noting that $\mathrm{ord}_\omega^+(D_q^j(g_p)) \leq \mathrm{ord}_\omega^+(g_p)$ $(j = 2, \cdots, n; p = 1, \cdots, n)$, thus the maximal order of poles among the j column in the determinant above is given by

$$\max_{i \in \{0,\cdots,n\}} \{\mathrm{ord}_\omega^+(D_q^j g_p) - \{\mathrm{ord}_\omega^+(D_q^{\mathrm{i}}(g_p))\}\} \leq \mathrm{ord}_\omega^+(g_p) - \min_{i \in \{0,\cdots,n\}} \{\mathrm{ord}_\omega^+(D_q^{\mathrm{i}}(g_p))\}.$$

Hence for all $z \in \mathbb{C}$, the inequality (3.15) holds and we get

$$n\Big(r, \frac{1}{L}\Big) \leq \sum_{j=0}^{n+1} \widetilde{n}_J^{[n]}\Big(r, \frac{1}{g_j}\Big). \tag{3.16}$$

Then by applying Theorem 3.1 and integrating (3.16), we obtain (3.14).

4 Mason's Theorem with a Jackson Difference Radical

Mason's theorem is a counterpart of the *abc* conjecture in number theory. We state it here as follows.

Mason's Theorem (see [14, 19, 24]) If relatively prime polynomials a, b and c, not all identically zero, satisfy

$$a + b = c,$$

then $\deg c \leq \deg \operatorname{rad}(abc) - 1$, where the radical $\operatorname{rad}(abc)$ is the product of distinct linear factors of abc.

In [14], we know that the Fermat's last theorem for polynomials is an elementary application of Mason's theorem. For any polynomial $p(z) \in \mathbb{C}[z]$, the usual radical $\operatorname{rad} p(z)$ (see [3, 14]) is defined by

$$\operatorname{rad} p(z) = \prod_{\omega \in \mathbb{C}} (z - \omega)^{d(\omega)},$$

where

$$d(\omega) = \operatorname{ord}_\omega(p) - \min\{\operatorname{ord}_\omega(p), \operatorname{ord}_\omega(p')\} \in \{0, 1\}.$$

Similarly, we define the q-difference radical $\operatorname{rad}_q(p)$ and the Jackson difference radical $\operatorname{rad}_J(p)$ of the polynomial $p(z)$.

Definition 4.1 *Let $q \in \mathbb{C}\backslash\{0, 1\}$, we define the q-difference radical $\operatorname{rad}_q(p)$ and the Jackson difference radical* $\operatorname{rad}_J(p)$ *of the polynomial $p(z)$ as*

$$\operatorname{rad}_q(p(z)) = \prod_{\omega \in \mathbb{C}} (z - \omega)^{d_q(\omega)},$$
$$\operatorname{rad}_J(p(z)) = \prod_{\omega \in \mathbb{C}} (z - \omega)^{d_J(\omega)},$$

where

$$d_q(\omega) = \operatorname{ord}_\omega(p) - \min\{\operatorname{ord}_\omega(p), \operatorname{ord}_\omega(p(qz))\},$$
$$d_J(\omega) = \operatorname{ord}_\omega(p) - \min\{\operatorname{ord}_\omega(p), \operatorname{ord}_\omega(D_q(p))\}$$

with $\operatorname{ord}_\omega(p) \geq 0$ *being the order of zero of the polynomial $p(z)$ at $\omega \in \mathbb{C}$.*

Definition 4.2 *We denote $\widetilde{n}_q(p) = \deg \operatorname{rad}_q(p)$ and $\widetilde{n}_J(p) = \deg \operatorname{rad}_J(p)$, more precisely,*

$$\widetilde{n}_q(p) = \sum_{\omega \in \mathbb{C}} (\operatorname{ord}_\omega(p) - \min\{\operatorname{ord}_\omega(p), \operatorname{ord}_\omega(p(qz))\}),$$
$$\widetilde{n}_J(p) = \sum_{\omega \in \mathbb{C}} (\operatorname{ord}_\omega(p) - \min\{\operatorname{ord}_\omega(p), \operatorname{ord}_\omega(D_q(p))\}). \tag{4.1}$$

Remark 4.1 In fact, we obtain the following properties for $\widetilde{n}_J(p)$:

(1) $\widetilde{n}_J(p) \leq \deg p$ for any $p(z) \in \mathbb{C}[z]$;

(2) $\widetilde{n}_J(p^m) = m \cdot \widetilde{n}_J(p)$ for any $p(z) \in \mathbb{C}[z]$ and $m \in \mathbb{N}$;

(3)

$$\widetilde{n}_J(ph) \leq \begin{cases} \widetilde{n}_J(p) + \widetilde{n}_J(h), & 0 \notin (ph)^{-1}(0), \\ \widetilde{n}_J(p) + \widetilde{n}_J(h) + 1, & 0 \in (ph)^{-1}(0) \end{cases}$$

for any $p(z), h(z) \in C[z]$, where the equality holds exactly when both $p(z)$ and $p(qz)$, $p(z)$ and $h(qz)$, $h(z)$ and $p(qz)$, as well as $h(z)$ and $h(qz)$ are relatively prime.

As we all know, for any $a_i(z) \in \mathbb{C}[z]$, the differential operator $\overline{D} = \frac{\mathrm{d}}{\mathrm{d}z}$ satisfies

$$\deg(\overline{D}^j a_i(z)) = \deg a_i(z) - j, \quad 0 < j < \deg a_i(z).$$

To better interpret the proof of Theorem 4.2, we need to introduce some notations.

Definition 4.3 *The corresponding q-difference radical and Jackson difference radical of truncation level m is defined by*

$$\begin{aligned} \mathrm{rad}_q^{[m]}(a_i(z)) &= \frac{a_i(z)}{\gcd\{a_i(z), \Delta_q a_i(z), \Delta_q^2 a_i(z), \cdots, \Delta_q^m a_i(z)\}}, \\ \mathrm{rad}_J^{[m]}(a_i(z)) &= \frac{a_i(z)}{\gcd\{a_i(z), D_q a_i(z), D_q^2 a_i(z), \cdots, D_q^m a_i(z)\}}. \end{aligned}$$

Remark 4.2 When $0 \notin \{a_i(z)^{-1}(0)\}$, we deduce that

$$\begin{aligned} \mathrm{rad}_J^{[m]}(a_i(z)) &= \frac{a_i(z)}{\gcd\{a_i(z), \Delta_q a_i(z), \Delta_q^2 a_i(z), \cdots, \Delta_q^m a_i(z)\}} \\ &= \mathrm{rad}_q^{[m]}(a_i(z)). \\ \widetilde{n}_J^{[m-1]}(a_i(z)) &= \deg \mathrm{rad}_J^{[m-1]}(a_i(z)) \\ &= \sum_{\omega \in \mathbb{C}\setminus\{0\}} (\mathrm{ord}_\omega a_i(z) - \min_{0 \leq j \leq m-1} \{\mathrm{ord}_\omega (D_q^j a_i(z))\}) \\ &= \sum_{\omega \in \mathbb{C}\setminus\{0\}} (\mathrm{ord}_\omega a_i(z) - \min_{0 \leq j \leq m-1} \{\mathrm{ord}_\omega (a_i(q^j z))\}) \\ &= \widetilde{n}_q^{[m-1]}(a_i(z)). \end{aligned}$$

Here, we will use the Jackson difference radical $\mathrm{rad}_J(p)$ to prove the Jackson difference analogue of Mason's theorem. Furthermore, we extend the Jackson difference analogue of Mason's theorem for $m+1$ polynomials. We get the following theorems similarly as [14].

Theorem 4.1 *Let a_1, a_2, a_3 be relatively prime polynomials in $\mathbb{C}[z]$ such that*

$$a_1 + a_2 = a_3, \tag{4.2}$$

a_1, a_2, a_3 are not all constant and $0 \notin \{a_i^{-1}(0)\}$, $i = 1, 2, 3$. Then

$$\max\{\deg a_1, \deg a_2, \deg a_3\} \leq \widetilde{n}_J(a_1) + \widetilde{n}_J(a_2) + \widetilde{n}_J(a_3) - 1, \tag{4.3}$$

where $q \in \mathbb{C}\setminus\{0, 1\}$.

Theorem 4.2 *Let $a_1, \cdots, a_{m+1}$ be pairwise relatively prime polynomials in $\mathbb{C}[z]$ such that*

$$a_1 + a_2 + \cdots + a_m = a_{m+1},$$

where $a_1, \cdots, a_m$ are not all constant and linearly independent over $\mathbb{C}$. In addition, $0 \notin a_i^{-1}(0)$, $i = 1, \cdots, m+1$. Then

$$\max_{1 \le i \le m+1} \{\deg a_i\} \le \sum_{i=1}^{m+1} \widetilde{n}_J^{[m-1]}(a_i),$$

where we denote

$$\begin{aligned} \widetilde{n}_J^{[m-1]}(a_i) &= \widetilde{n}_q^{[m-1]}(a_i) \\ &= \sum_{\omega \in \mathbb{C}\setminus\{0\}} (\mathrm{ord}_\omega(a_i) - \min_{0 \le j \le m-1} \{\mathrm{ord}_\omega(a_i(q^j z))\}) \end{aligned} \tag{4.4}$$

and $q \in \mathbb{C}\setminus\{0, 1\}$.

Lemma 4.1 *Suppose that $p(z) \not\equiv 0$ is a polynomial in $\mathbb{C}[z]$, $q \in \mathbb{C}\setminus\{0, 1\}$. Then*

$$\begin{aligned} p &= \gcd(p, D_q p) \cdot \mathrm{rad}_J(p), \\ \deg p &= \deg \gcd(p, D_q p) + \widetilde{n}_J(p). \end{aligned}$$

Proof Let $\alpha_i \not\equiv 0$ be the roots of $p(z)$ with multiplicities a_i respectively, and not all of a_i equal to zero, where $i = 1, 2, \cdots, m$, that is

$$p(z) = c(z - \alpha_1)^{a_1}(z - \alpha_2)^{a_2} \cdots (z - \alpha_m)^{a_m},$$

and thus

$$p(qz) = c(qz - \alpha_1)^{a_1}(qz - \alpha_2)^{a_2} \cdots (qz - \alpha_m)^{a_m}, \quad q \in \mathbb{C}\setminus\{0, 1\}.$$

We will prove the lemma by distinguishing two cases:

Case 1 If the origin is not the root of $p(z)$, we have

$$\begin{aligned} \gcd(p, D_q p(z)) &= \gcd(p, p(qz) - p(z)) = \gcd(p, p(qz)), \\ \gcd(p, p(qz)) &= c(z - \alpha_1)^{b_1} \cdots (z - \alpha_m)^{b_m}, \end{aligned}$$

where $b_i = \min\{\mathrm{ord}_{\alpha_i} p(z), \mathrm{ord}_{\alpha_i} p(qz)\}$, $i = 1, 2, \cdots, m$.

Since

$$\mathrm{rad}_J(p) = \prod_{\alpha_i \in \mathbb{C}} (z - \alpha_i)^{d_J(\alpha_i)},$$

it leads to

$$\begin{aligned} d_J(\alpha_i) = d_q(\alpha_i) &= \mathrm{ord}_{\alpha_i}(p) - \min\{\mathrm{ord}_{\alpha_i}(p), \mathrm{ord}_{\alpha_i}(p(qz))\} \\ &= a_i - b_i, \end{aligned} \tag{4.5}$$

hence

$$p(z) = \gcd(p(z), D_q p(z)) \cdot \mathrm{rad}_J(p(z)). \tag{4.6}$$

Case 2 Let the origin be the root of $p(z)$ and the multiple be m, then

$$\begin{aligned} p(z) &= cz^m(z-\alpha_1)^{a_1}(z-\alpha_2)^{a_2}\cdots(z-\alpha_m)^{a_m}, \\ p(qz) &= c(qz)^m(qz-\alpha_1)^{a_1}(qz-\alpha_2)^{a_2}\cdots(qz-\alpha_m)^{a_m}, \\ \gcd(p, D_q p(qz)) &= cz^{m-1}\cdot(z-\alpha_1)^{b_1}\cdots(z-\alpha_m)^{b_m}, \end{aligned}$$

where $b_i = \min\{\mathrm{ord}_{\alpha_i} p(z), \mathrm{ord}_{\alpha_i} p(qz)\}$, $i = 1, 2, \cdots, m$.

While

$$\mathrm{rad}_J(p) = z \cdot \prod_{\alpha_i \in \mathbb{C}\setminus\{0\}} (z-\alpha_i)^{d_J(\alpha_i)},$$

so combine with (4.5) in Case 2 we still have (4.6).

Therefore by Cases 1–2, it gives

$$\begin{aligned} \deg p(z) &= \deg\gcd(p(z), D_q p(z)) + \deg \mathrm{rad}_J(p(z)) \\ &= \deg\gcd(p(z), D_q p(z)) + \widetilde{n}_J(p). \end{aligned}$$

Proof of Theorem 4.1 Without loss of generality, we may assume that

$$\max\{\deg a_i\} = \deg a_3, \quad i = 1, 2, 3.$$

From (4.2) we see that

$$\triangle_q a_1 + \triangle_q a_2 = \triangle_q a_3,$$

Clearly,

$$a_1\triangle_q a_1 + a_1\triangle_q a_2 = a_1\triangle_q a_3, \tag{4.7}$$

$$a_1\triangle_q a_1 + a_2\triangle_q a_1 = a_3\triangle_q a_1. \tag{4.8}$$

By subtracting (4.8) from (4.7), we have

$$a_1\triangle_q a_2 - a_2\triangle_q a_1 = a_1\triangle_q a_3 - a_3\triangle_q a_1.$$

It is easy to see that $\gcd(a_i, \triangle_q a_i)$ are factors of $a_1\triangle_q a_2 - b\triangle_q a_1$.

Notice that a_1, a_2, a_3 are relatively prime, it means that $\gcd(a_i, \triangle_q a_i)$ are relatively prime. Therefore,

$$\gcd(a_1, \triangle_q a_1)\gcd(a_2, \triangle_q a_2)\gcd(a_3, \triangle_q a_3)$$

is a factor of $a_1\triangle_q a_2 - a_2\triangle_q a_1$. By calculation, we get

$$\deg\gcd(a_1, \triangle_q a_1) + \deg\gcd(a_2, \triangle_q a_2) + \deg\gcd(a_3, \triangle_q a_3) \le \deg a_1 + \deg a_2 - 1. \tag{4.9}$$

We claim that $a_1\triangle_q a_2 - a_2\triangle_q a_1 \neq 0$. Suppose for the contrary that

$$a_1\triangle_q a_2 - a_2\triangle_q a_1 = 0, \tag{4.10}$$

then $a_1\triangle_q a_2 = a_2\triangle_q a$ and a_1 is a factor of $b\triangle_q a_1$. Since a_1 and a_2 are relatively prime polynomials, we get a_1 is a factor of $\triangle_q a_1$. This is only possible if $\triangle_q a_1 = 0$. Similarly, it is easy to deduce $\triangle_q a_2 = 0$ and $\triangle_q a_3 = 0$. Thus, it contradicts the hypothesis of the theorem. Hence (4.9) is valid.

According to

$$\deg\gcd(a_i, \triangle_q a_i) = \deg\gcd(a_i, D_q a_i), \quad i = 1, 2, 3. \tag{4.11}$$

By adding $\deg a_3$ to both sides of (4.9) and reorganizing the term, we obtain

$$\deg a_3 \leq \deg a_1 - \deg\gcd(a_1, D_q a_1) + \deg a_2 - \deg\gcd(a_2, D_q a_2) + \deg a_3 - \deg\gcd(a_3, D_q a_3) - 1.$$

So (4.3) holds by Lemma 4.1. This finishes the proof of Theorem 4.1.

Example 4.1 Set $a_1(z) = \left(\frac{1}{q}z + \alpha\right)(z + \alpha)$, $a_2(z) = -\left(\frac{1}{q}z + \beta\right)(z + \beta)$, $a_3(z) = (\alpha - \beta)\left(\frac{1}{q}z + z + \alpha + \beta\right)$, where $\alpha, \beta \in \mathbb{C}\backslash\{0\}$, $\alpha \neq \beta$ and $\alpha + \beta \neq \alpha(1 + q)$. Clearly, a_1, a_2, a_3 are relatively prime polynomials in $\mathbb{C}[z]$ and satisfy $a_1 + a_2 = a_3$. Besides, none of the differences $\triangle_q a_1$, $\triangle_q a_2$, $\triangle_q a_3$ are identically zero. By calculation we get $\max\{\deg a_1, \deg a_2, \deg a_3\} = 2$, $\widetilde{n}_J(a_1) = 1$, $\widetilde{n}_J(a_2) = 1$, $\widetilde{n}_J(a_3) = 1$ and $\widetilde{n}_J(a_1) + \widetilde{n}_J(a_2) + \widetilde{n}_J(a_3) - 1 = 2$. The example shows that the assertion of Theorem 4.1 is sharp.

Example 4.2 Suppose $a_1(z) = \left(\frac{1}{q}z\right)z$, $a_2(z) = -\left(\frac{1}{q}z + \beta\right)(z + \beta)$, $a_3(z) = (-\beta)\left(\frac{1}{q}z + z + \beta\right)$. On account of that a_1, a_2, a_3 are relatively prime polynomials in $\mathbb{C}[z]$, we have $\beta \neq 0$. When $0 \in a_i^{-1}(0), i = 1, 2, 3$, we get $\deg\gcd(a_i, \triangle_q a_i) = \deg\gcd(a_i, D_q a_i) + 1$. Combining the inequality (4.9) yields

$$\deg\gcd(a_1, D_q a_1) + \deg\gcd(a_2, D_q a_2) + \deg\gcd(a_3, D_q a_3) \leq \deg a_1 + \deg a_2 - 2. \tag{4.12}$$

By adding $\deg a_3$ to both sides of (4.12) and reorganizing the term, we have

$$\max\{\deg a_1, \deg a_2, \deg a_3\} \leq \widetilde{n}_J(a_1) + \widetilde{n}_J(a_2) + \widetilde{n}_J(a_3) - 2. \tag{4.13}$$

However, by calculation we get $\max\{\deg a_1, \deg a_2, \deg a_3\} = 2$, $\widetilde{n}_J(a_1) = 1$, $\widetilde{n}_J(a_2) = 1$, $\widetilde{n}_J(a_3) = 1$ and $\widetilde{n}_J(a_1) + \widetilde{n}_J(a_2) + \widetilde{n}_J(a_3) - 2 = 1$, this contradicts (4.13). So the example shows that the condition of $0 \notin \{a_i^{-1}(0)\}$ $(i = 1, 2, 3)$ in Theorem 4.1 cannot be simply dropped.

Proof of Theorem 4.2 Through the introduction of the second part, we know that $C_q(z) \not\equiv 0$ is the q-Casorati determinant of $a_1(z), \cdots, a_m(z)$. Let $z_0 \neq 0$ be a zero of some $a_i(z)$ with $1 \leq i \leq m + 1$. Obviously, $z = z_0$ is also a zero of $C_q(z)$ with multiplicity not smaller than

$$\max_{0 \leq j \leq m-1}\{\mathrm{ord}_{z_0}(a_i(q^j z))\}.$$

Since $a_1, \cdots, a_{m+1}$ are pairwise relatively prime polynomials, we get

$$q(z) := \prod_{i=1}^{m+1} \gcd(a_i, a_i(qz), \cdots, a_i(q^{m-1}z))$$

is a factor of $C_q(z)$, and there exists a polynomial $p(z) \in \mathbb{C}[z]$ such that $C_q(z) = p(z)q(z)$. Combining with (4.4), we deduce that the degree of $q(z)$ is not less than

$$\sum_{i=1}^{m+1} \sum_{\omega \in \mathbb{C}\setminus\{0\}} \min_{0 \le j \le m-1} \{\mathrm{ord}_\omega(a_i(q^j z))\} = \sum_{i=1}^{m+1} \Big[\sum_{\omega \in \mathbb{C}\setminus\{0\}} \mathrm{ord}_\omega(a_i) - \widetilde{n}_q^{[m-1]}(a_i) \Big].$$

On the other hand, the degree of $C_q(z)$ is never beyond any sum of distinct m of the $\deg a_i(z)$ $(1 \le i \le m+1)$. That is

$$\begin{aligned} \min_{1 \le k \le m+1} \sum_{1 \le i \le m+1, i \ne k} \deg a_i &\ge \sum_{i=1}^{m+1} \Big[\sum_{\omega \in \mathbb{C}\setminus\{0\}} \mathrm{ord}_\omega(a_i) - \widetilde{n}_q^{[m-1]}(a_i) \Big] \\ &= \sum_{i=1}^{m+1} \Big[\sum_{\omega \in \mathbb{C}\setminus\{0\}} \mathrm{ord}_\omega(a_i) - \widetilde{n}_J^{[m-1]}(a_i) \Big]. \end{aligned}$$

Hence

$$\max_{1 \le i \le m+1} \{\deg a_i\} \le \sum_{i=1}^{m+1} \widetilde{n}_J^{[m-1]}(a_i).$$

The following example shows that Theorem 4.2 is accurate when $m = 3$.

Example 4.3 Take

$$\begin{aligned} a_1(z) &= \Big(\frac{q(z+1)}{2} - \alpha\Big)\Big((z+1) - \frac{\alpha}{2}\Big), \\ a_2(z) &= \Big(\frac{q(z+1)}{2} + \alpha\Big)\Big((z+1) + \frac{\alpha}{2}\Big), \\ a_3(z) &= -q(z+1)^2, \quad a_4(z) = \alpha^2, \end{aligned}$$

and $q \in \mathbb{C}\setminus\{0,1\}$, $\alpha \in \mathbb{C}\setminus\{0\}$. Thus a_1, a_2, a_3, a_4 satisfy the equation $a_1(z) + a_2(z) + a_3(z) = a_4(z)$ and the condition of Theorem 4.2. This example gives $\max\limits_{1 \le i \le 4}\{\deg a_i\} = 2$ and

$$\sum_{i=1}^{4} \widetilde{n}_J^{[2]}(a_i) = 2 + 2 + 2 = 6.$$

Remark 4.3 In Theorem 4.2, we assume that the origin is not a root of a_i, $i = 1, \cdots, m+1$. Otherwise, suppose that $0 \in a_i^{-1}(0)$, we can get the relationship between $\widetilde{n}_J^{[m-1]}(a_i)$ and $\widetilde{n}_q^{[m-1]}(a_i)$. Denote by $\mathrm{ord}_0(a_i)$ the order of polynomial a_i at the origin. Combining the definition of $D_q a_i$ and $\mathrm{rad}_J^{[m]}(a_i)$, we easily verify that if $\mathrm{ord}_0(a_i) > m-1$,

$$\widetilde{n}_J^{[m-1]}(a_i) = \sum_{\omega \in \mathbb{C}\setminus\{0\}} (\mathrm{ord}_\omega(a_i) - \min_{0 \le j \le m-1} \{\mathrm{ord}_\omega(D_q^j(a_i))\})$$

$$+ (\mathrm{ord}_0(a_i) - \min_{0\leq j\leq m-1}\{\mathrm{ord}_0(D_q^j(a_i))\})$$
$$= \widetilde{n}_q^{[m-1]}(a_i) + m - 1.$$

If $\mathrm{ord}_0(a_i) \leq m-1$, then $\mathrm{rad}_J^{[m]}(a_i) = \mathrm{rad}_q^{[m]}(a_i) + \mathrm{ord}_0(a_i)$. Unfortunately, this situation may not hold for Theorem 4.2.

Finally, we propose an interesting problem deserved to be further studied.

Problem 4.1 Factorial polynomial is defined as

$$t^{\overline{n}} = t(t+1)\cdots(t+n-1).$$

Ishizaki-Korhonen-Li-Tohge [14] introduce the notation for the factorial of a polynomial $p(z)$ in $\mathbb{C}[z]$ as

$$[p(z)]_\kappa^{\overline{n}} = p(z)p(z+\kappa)\cdots p(z+(n-1)\kappa),$$

where the shift $\kappa \in \mathbb{C}\backslash\{0\}$.

If we extend this notation for the factorial of a polynomial $p(z)$ in $\mathbb{C}[z]$ as

$$[p(z)]_J^{\overline{n}} = p(z)D_q p(z)\cdots D_q^{n-1}p(z),$$

where $q \in \mathbb{C}\backslash\{0,1\}$. What about the existence of polynomial solutions to a Jackson difference Fermat equations such as

$$[p_1(z)]_J^{\overline{n}} + [p_2(z)]_J^{\overline{n}} + \cdots + [p_m(z)]_J^{\overline{n}} = [p_{m+1}(z)]_J^{\overline{n}}?$$

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