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Products of Toeplitz and Hankel Operators on Fock-Sobolev Spaces*

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Abstract In this paper, the authors investigate the boundedness of Toeplitz product T_fT_g and Hankel product $H_f^*H_g$ on Fock-Sobolev space for $f, g \in \mathcal{P}$. As a result, the boundedness of Toeplitz operator T_f and Hankel operator H_f with $f \in \mathcal{P}$ is characterized.

Keywords Toeplitz product, Hankel product, Fock-Sobolev space
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1 Introduction

Let $\mathbb{C}^n$ be the Euclidean space of complex dimension n and $\mathrm{d}v$ be the Lebesgue measure on $\mathbb{C}^n$. For $z = (z_1, \cdots, z_n)$ and $w = (w_1, \cdots, w_n)$ in $\mathbb{C}^n$, we denote

$$z \cdot \overline{w} = \sum_{j=1}^{n} z_j \overline{w}_j, \quad |z| = (z \cdot \overline{z})^{\frac{1}{2}}.$$

The Fock space F^2 consists of all entire functions f on $\mathbb{C}^n$ such that

$$\|f\|_2 = \Big(\frac{1}{\pi^n}\int_{\mathbb{C}^n} |f(z)|^2 \mathrm{e}^{-|z|^2} \mathrm{d}v(z)\Big)^{\frac{1}{2}} < \infty.$$

Let $\mathbb{N}$ be the set of nonnegative integers. For any multi-index $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$ and $z \in \mathbb{C}^n$, we write

$$|\alpha| = \alpha_1 + \cdots + \alpha_n, \quad \alpha! = \alpha_1! \cdots \alpha_n!, \quad \partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}, \quad z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n},$$

where ∂_j denotes the partial derivative with respect to the z_j.

For any $m \in \mathbb{N}$, the Fock-Sobolev space $F^{2,m}$ consists of all entire functions f on $\mathbb{C}^n$ such that

$$\|f\|_{2,m} := \sum_{|\alpha| \le m} \|\partial^\alpha f\|_2 < \infty.$$

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The Fock-Sobolev space was introduced by Cho and Zhu in [4], where they proved that $f \in F^{2,m}$ if and only if the function $z^\alpha f(z)$ is in F^2 for all multi-indexes α with $|\alpha| = m$, which allows us to introduce the equivalent norm on $F^{2,m}$:

$$\|f\|_{2,m} = \Big(\omega_{n,m} \int_{\mathbb{C}^n} |f(z)|^2 |z|^{2m} \mathrm{e}^{-|z|^2} \mathrm{d}v(z)\Big)^{\frac{1}{2}},$$

where

$$\omega_{n,m} = \frac{(n-1)!}{\pi^n \Gamma(m+n)}$$

is a normalizing constant such that the constant function 1 has norm 1 in $F^{2,m}$.

For any $z \in \mathbb{C}^n$, Let

$$\mathrm{d}V_m(z) := \omega_{n,m} |z|^{2m} \mathrm{e}^{-|z|^2} \mathrm{d}v(z).$$

Denote L^2_m by the space of Lebesgue measurable functions f on $\mathbb{C}^n$ so that the function $f(z) \in L^2(\mathbb{C}^n, \mathrm{d}V_m)$. It is well-known that the space L^2_m is a Hilbert space with the inner product

$$\langle f, g\rangle_m = \int_{\mathbb{C}^n} f(z)\overline{g(z)} \mathrm{d}V_m(z).$$

It is clear that the Fock-Sobolev space $F^{2,m}$ is a closed subspace of L^2_m. Let P_m be the orthogonal projection from L^2_m to $F^{2,m}$, that is

$$P_m f(z) = \int_{\mathbb{C}^n} f(w) K_m(z, w) \mathrm{d}V_m(z),$$

where $K_m(z, w)$ is the reproducing kernel of $F^{2,m}$.

For a Lebesgue measurable function f on $\mathbb{C}^n$ such that $fK_m(z, \cdot)$ are in $L^2(\mathbb{C}^n, \mathrm{d}V_m)$ for all $z \in \mathbb{C}^n$, the Toeplitz operator with symbol f on $F^{2,m}$ is defined by

$$T_f g = P_m(fg),$$

and the Hankel operator H_f with symbol f is given by

$$H_f g = (I - P_m)(fg),$$

where I is the identity operator on L^2_m.

The original Toeplitz product problem was raised by Sarason in [8], to ask whether one can give a characterization for the pairs of outer functions g, h in the Hardy space H^2 such that the operator $T_g T_{\overline{h}}$ is bounded on H^2. The famous Sarason's conjecture on this problem has attracted the attention of some mathematical researchers in operator theory. This problem was partially solved on the Hardy space of the unit circle in [13], on the Bergman space of the unit disk in [9], on the Bergman space of the polydisk in [10] and on the Bergman space of the unit ball in [7, 11]. Unfortunately, Sarason's conjecture was eventually proved to be false, both on the Hardy space and the Bergman space, see [1, 6] for counterexamples. However, in [2–3], the Sarason's conjecture was proved to be true on the Fock space, and in this setting, the explicit forms of the symbols f and g were given. Although the boundedness of a single Toeplitz operator on Fock space is still an open problem, some progress has been made in Toeplitz products and Hankel products. Ma, Yan, Zheng and Zhu [5] gave a sufficient but not necessary condition on bounded Hankel product $H_{\overline{f}}^*H_{\overline{g}}$ for f, g in the Fock space. Yan and Zheng [12] characterized bounded Toeplitz product T_fT_g and Hankel product $H_f^*H_g$ on Fock space for two polynomials f and g in z, $\overline{z} \in \mathbb{C}$. Inspired by these work, we study the boundedness of Toeplitz product T_fT_g and Hankel product $H_f^*H_g$ on $F^{2,m}$ for two polynomials $f, g \in \mathcal{P}$, where

$$\mathcal{P} := \Big\{ \prod_{s=1}^{n} \Big(\sum_{\beta_s \le k_s} \sum_{\gamma_s \le l_s} a_{\beta_s\gamma_s,s} z_s^{\beta_s} \overline{z}_s^{\gamma_s} \Big) : k_s, l_s \in \mathbb{N},\ z_s \in \mathbb{C} \text{ and } a_{\beta_s\gamma_s,s} \text{ are constants} \Big\}.$$

Our main results can be stated as follows.

Theorem 1.1 *Let $f, g \in \mathcal{P}$. Then the Toeplitz product T_fT_g is bounded on $F^{2,m}$ if and only if both f and g are constants.*

Theorem 1.2 *Let $f, g \in \mathcal{P}$. Then the Hankel product $H_f^*H_g$ is bounded on $F^{2,m}$ if and only if at least one of the following statements holds*:

(1) *f is holomorphic.*

(2) *g is holomorphic.*

(3) *$n = 1$ and there exist two holomorphic polynomials f_1 and g_1 such that*

$$f = f_1 + a\overline{z}, \quad g = g_1 + b\overline{z},$$

where a, b are constants and $z, \overline{z} \in \mathbb{C}$.

We would like to mention that all the conclusions for the Fock-Sobolev space $F^{2,m}$ in this paper are consistent with the results in [12] when $m = 0$ and $n = 1$, but the boundedness characterization of Hankel product for $n \ge 2$ is essentially different from $n = 1$ and all the results for $m \ge 1$ are new.

The layout of the paper is as follows. In Section 2 we give the proof of characterizations of bounded Toeplitz product T_fT_g on $F^{2,m}$. In Section 3 we give the proof of characterizations of bounded Hankel product $H_f^*H_g$.

In what follows, denote by χ_E the characteristic function of a measurable set E. We say a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$ tends to ∞ if each component α_i tends to ∞. For two arbitrary sequences A_α and B_α depending on multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$, we use the notation $A_\alpha \sim B_\alpha$ to denote the relationship:

$$\lim_{\alpha\to\infty} \frac{A_\alpha}{B_\alpha} = C,$$

where C is a positive constant independent of α.

Recall that the Stirling's formula is stated as

$$k! \sim \sqrt{2\pi k}\Big(\frac{k}{\mathrm{e}}\Big)^k,$$

where k is a positive integer and "$\sim$" can be understood in the sense that the ratio of the two sides tends to 1 as k goes to ∞.

2 Toeplitz Products

In this section, we are going to characterize bounded Toeplitz product T_fT_g with $f,g\in\mathcal{P}$. For $\alpha\in\mathbb{N}^n$ and $z\in\mathbb{C}^n$, the functions

$$e_\alpha(z)=\sqrt{\frac{(m+n-1)!(n-1+|\alpha|)!}{\alpha!(n-1)!(m+n-1+|\alpha|)!}}z^\alpha$$

form an orthonormal basis for $F^{2,m}$, see [4] for more details.

Given $\alpha=(\alpha_1,\cdots,\alpha_n)$, $\beta=(\beta_1,\cdots,\beta_n)\in\mathbb{N}^n$, the addition and the subtraction of α and β are defined by

$$\alpha\pm\beta:=(\alpha_1\pm\beta_1,\cdots,\alpha_n\pm\beta_n).$$

We call $\alpha\geq\beta$ (resp. $\alpha>\beta$, $\alpha\leq\beta$, $\alpha<\beta$) if $\alpha_i\geq\beta_i$ (resp. $\alpha_i>\beta_i$, $\alpha_i\leq\beta_i$, $\alpha_i<\beta_i$) for each $i=1,\cdots,n$.

We now give a technical result that will be frequently used in the following.

Lemma 2.1 *Let $\{e_\alpha:\alpha\in\mathbb{N}^n\}$ be any orthonormal basis of $F^{2,m}$. Then for any β, $\gamma\in\mathbb{N}^n$ and $z\in\mathbb{C}^n$, we have*

$$T_{z^\beta\overline{z}^\gamma}e_\alpha=\begin{cases}\sqrt{\frac{\alpha!(n-1+|\alpha|)!(n-1+|\alpha+\beta-\gamma|)!}{(\alpha+\beta-\gamma)!(m+n-1+|\alpha|)!(m+n-1+|\alpha+\beta-\gamma|)!}}\frac{(\alpha+\beta)!(m+n-1+|\alpha+\beta|)!}{\alpha!(n-1+|\alpha+\beta|)!}e_{\alpha+\beta-\gamma}, & \alpha+\beta-\gamma\geq0,\\ 0, & \textit{otherwise.}\end{cases}$$

Proof Direct verifications give

$$T_{z^\beta\overline{z}^\gamma}e_\alpha=\sqrt{\frac{(m+n-1)!(n-1+|\alpha|)!}{\alpha!(n-1)!(m+n-1+|\alpha|)!}}P_m(z^{\alpha+\beta}\overline{z}^\gamma)\tag{2.1}$$

and

$$\begin{aligned}P_m(z^{\alpha+\beta}\overline{z}^\gamma)&=\sum_{\eta\in\mathbb{N}^n}\langle z^{\alpha+\beta}\overline{z}^\gamma,e_\eta\rangle_m e_\eta\\&=\sum_{\eta\in\mathbb{N}^n}\sqrt{\frac{(m+n-1)!(n-1+|\eta|)!}{\eta!(n-1)!(m+n-1+|\eta|)!}}\langle z^{\alpha+\beta},z^{\eta+\gamma}\rangle_m e_\eta.\end{aligned}\tag{2.2}$$

For $\eta\neq\alpha+\beta-\gamma$, it is easy to see that

$$\langle z^{\alpha+\beta},z^{\eta+\gamma}\rangle_m=0.\tag{2.3}$$

For $\eta=\alpha+\beta-\gamma$, applying integration in polar coordinates and using [14, Lemma 1.11], we obtain

$$\langle z^{\alpha+\beta},z^{\eta+\gamma}\rangle_m=\frac{(\alpha+\beta)!(n-1)!(m+n-1+|\alpha+\beta|)!}{(m+n-1)!(n-1+|\alpha+\beta|)!}.$$

Notice that if $\alpha+\beta-\gamma \geq 0$, then there exists a unique η in (2.2) such that $\eta = \alpha+\beta-\gamma$. Thus

$$\begin{aligned} & P_m(z^{\alpha+\beta}\overline{z}^{\gamma}) \\ &= \sqrt{\frac{(m+n-1)!(n-1+|\alpha+\beta-\gamma|)!}{(\alpha+\beta-\gamma)!(n-1)!(m+n-1+|\alpha+\beta-\gamma|)!}}\langle z^{\alpha+\beta}, z^{\alpha+\beta}\rangle_m e_{\alpha+\beta-\gamma} \\ &= \sqrt{\frac{(n-1)!(n-1+|\alpha+\beta-\gamma|)!}{(\alpha+\beta-\gamma)!(m+n-1)!(m+n-1+|\alpha+\beta-\gamma|)!}}\frac{(\alpha+\beta)!(m+n-1+|\alpha+\beta|)!}{(n-1+|\alpha+\beta|)!}e_{\alpha+\beta-\gamma}. \end{aligned}$$

This together with (2.1) gives

$$\begin{aligned} & T_{z^{\beta}\overline{z}^{\gamma}}e_\alpha \\ &= \sqrt{\frac{\alpha!(n-1+|\alpha|)!(n-1+|\alpha+\beta-\gamma|)!}{(\alpha+\beta-\gamma)!(m+n-1+|\alpha|)!(m+n-1+|\alpha+\beta-\gamma|)!}}\frac{(\alpha+\beta)!(m+n-1+|\alpha+\beta|)!}{\alpha!(n-1+|\alpha+\beta|)!}e_{\alpha+\beta-\gamma}. \end{aligned}$$

If $\alpha+\beta-\gamma$ is less than 0, then $\eta \neq \alpha+\beta-\gamma$ for all η in (2.2), it follows from (2.1)–(2.3) that $T_{z^{\beta}\overline{z}^{\gamma}}e_\alpha = 0$. This completes the proof.

In order to state the following lemma effectively, for any function f, we define

$$f^{(j)} := \begin{cases} f, & j=0, \\ \overline{f}, & j=1. \end{cases} \tag{2.4}$$

Lemma 2.2 *Suppose $\beta = (\beta_1,\cdots,\beta_n)$, $\gamma = (\gamma_1,\cdots,\gamma_n)$, $k=(k_1,\cdots,k_n)$ and $l = (l_1,\cdots,l_n)$ are in $\mathbb{N}^n$. For any $z=(z_1,\cdots,z_n)\in\mathbb{C}^n$, let*

$$f_{\beta_i}(z_i) = \sum_{\mu_i\leq k_i} a_{\mu_i} z_i^{\mu_i+\beta_i}\overline{z}_i^{\mu_i}, \quad g_{\gamma_i}(z_i) = \sum_{\nu_i\leq l_i} b_{\nu_i} z_i^{\nu_i+\gamma_i}\overline{z}_i^{\nu_i},$$

where a_{μ_i}, b_{ν_i} are constants with a_{k_i}, b_{l_i} nonzero for each $i=1,\cdots,n$. For $i_1,\cdots,i_n$, $j_1,\cdots,j_n\in\{0,1\}$, let

$$f_\beta(z) = f_{\beta_1}^{(i_1)}(z_1)\cdots f_{\beta_n}^{(i_n)}(z_n), \quad g_\gamma(z) = g_{\gamma_1}^{(j_1)}(z_1)\cdots g_{\gamma_n}^{(j_n)}(z_n).$$

Then each of the Toeplitz products $T_{f_\beta}T_{g_\gamma}$ is bounded on $F^{2,m}$ if and only if $\beta=\gamma=k=l=(0,\cdots,0)$.

Proof For simplicity, we set $i=(i_1,\cdots,i_n)$, $j=(j_1,\cdots,j_n)$ and denote

$$\begin{aligned} \theta &:= \theta_{\mu,\beta,i} = (\mu_1+\chi_{\{0\}}(i_1)\beta_1,\cdots,\mu_n+\chi_{\{0\}}(i_n)\beta_n), \\ \vartheta &:= \vartheta_{\mu,\beta,i} = (\mu_1+\chi_{\{1\}}(i_1)\beta_1,\cdots,\mu_n+\chi_{\{1\}}(i_n)\beta_n), \\ \varphi &:= \varphi_{\nu,\gamma,j} = (\nu_1+\chi_{\{0\}}(j_1)\gamma_1,\cdots,\nu_n+\chi_{\{0\}}(j_n)\gamma_n), \\ \psi &:= \psi_{\nu,\gamma,j} = (\nu_1+\chi_{\{1\}}(j_1)\gamma_1,\cdots,\nu_n+\chi_{\{1\}}(j_n)\gamma_n). \end{aligned}$$

For $\alpha\in\mathbb{N}^n$ satisfying $\alpha_s \geq \chi_{\{1\}}(j_s)\gamma_s + \chi_{\{1\}}(i_s)\beta_s$ $(s=1,\cdots,n)$, we apply Lemma 2.1 twice to obtain

$$\begin{aligned} & T_{f_\beta}T_{g_\gamma}e_\alpha \\ &= \sum_{\mu_1\leq k_1}\cdots\sum_{\mu_n\leq k_n}\sum_{\nu_1\leq l_1}\cdots\sum_{\nu_n\leq l_n} a_{\mu_1}^{(i_1)}\cdots a_{\mu_n}^{(i_n)} b_{\nu_1}^{(j_1)}\cdots b_{\nu_n}^{(j_n)} T_{z^\theta\overline{z}^\vartheta}T_{z^\varphi\overline{z}^\psi}e_\alpha \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\mu_1\le k_1}\cdots\sum_{\mu_n\le k_n}\sum_{\nu_1\le l_1}\cdots\sum_{\nu_n\le l_n} a_{\mu_1}^{(i_1)}\cdots a_{\mu_n}^{(i_n)} b_{\nu_1}^{(j_1)}\cdots b_{\nu_n}^{(j_n)} T_{z^\theta\overline{z}^\vartheta}\\
&\quad\times\Big(\sqrt{\frac{\alpha!(n-1+|\alpha|)!(n-1+|\alpha+\varphi-\psi|)!}{(\alpha+\varphi-\psi)!(m+n-1+|\alpha|)!(m+n-1+|\alpha+\varphi-\psi|)!}}\\
&\quad\times\frac{(\alpha+\varphi)!(m+n-1+|\alpha+\varphi|)!}{\alpha!(n-1+|\alpha+\varphi|)!}\Big)e_{\alpha+\varphi-\psi}\\
&= \sum_{\mu_1\le k_1}\cdots\sum_{\mu_n\le k_n}\sum_{\nu_1\le l_1}\cdots\sum_{\nu_n\le l_n} a_{\mu_1}^{(i_1)}\cdots a_{\mu_n}^{(i_n)} b_{\nu_1}^{(j_1)}\cdots b_{\nu_n}^{(j_n)}\\
&\quad\times\Big(\sqrt{\frac{\alpha!(n-1+|\alpha|)!(n-1+|\alpha+\varphi-\psi|)!}{(\alpha+\varphi-\psi)!(m+n-1+|\alpha|)!(m+n-1+|\alpha+\varphi-\psi|)!}}\\
&\quad\cdot\frac{(\alpha+\varphi)!(m+n-1+|\alpha+\varphi|)!}{\alpha!(n-1+|\alpha+\varphi|)!}\\
&\quad\times\sqrt{\frac{(\alpha+\varphi-\psi)!(n-1+|\alpha+\varphi-\psi|)!(n-1+|\alpha+\varphi-\psi+\theta-\vartheta|)!}{(\alpha+\varphi-\psi+\theta-\vartheta)!(m+n-1+|\alpha+\varphi-\psi|)!(m+n-1+|\alpha+\varphi-\psi+\theta-\vartheta|)!}}\\
&\quad\times\frac{(\alpha+\varphi-\psi+\theta)!(m+n-1+|\alpha+\varphi-\psi+\theta|)!}{(\alpha+\varphi-\psi)!(n-1+|\alpha+\varphi-\psi+\theta|)!}\Big)e_{\alpha+\varphi-\psi+\theta-\vartheta}\\
&= \sum_{\mu_1\le k_1}\cdots\sum_{\mu_n\le k_n}\sum_{\nu_1\le l_1}\cdots\sum_{\nu_n\le l_n} a_{\mu_1}^{(i_1)}\cdots a_{\mu_n}^{(i_n)} b_{\nu_1}^{(j_1)}\cdots b_{\nu_n}^{(j_n)} A_\alpha^{\theta\vartheta\varphi\psi} e_{\alpha+\varphi-\psi+\theta-\vartheta}, \qquad (2.5)
\end{aligned}
$$

where

$$
\begin{aligned}
A_\alpha^{\theta\vartheta\varphi\psi} := &\sqrt{\frac{\alpha!(n-1+|\alpha|)!(n-1+|\alpha+\varphi-\psi+\theta-\vartheta|)!}{(\alpha+\varphi-\psi+\theta-\vartheta)!(m+n-1+|\alpha|)!(m+n-1+|\alpha+\varphi-\psi+\theta-\vartheta|)!}}\\
&\times\frac{(\alpha+\varphi)!(\alpha+\varphi-\psi+\theta)!(m+n-1+|\alpha+\varphi|)!(n-1+|\alpha+\varphi-\psi|)!}{\alpha!(\alpha+\varphi-\psi)!(n-1+|\alpha+\varphi|)!(m+n-1+|\alpha+\varphi-\psi|)!}\\
&\times\frac{(m+n-1+|\alpha+\varphi-\psi+\theta|)!}{(n-1+|\alpha+\varphi-\psi+\theta|)!}.
\end{aligned}
$$

An application of Stirling's formula implies that

$$
A_\alpha^{\theta\vartheta\varphi\psi}\sim\alpha^{\frac12(\varphi+\psi+\theta+\vartheta)}=\alpha^{\frac12(\beta+\gamma)+\mu+\nu},\quad \mu\le k,\ \nu\le l. \qquad (2.6)
$$

Since a_{k_i}, b_{l_i} are nonzero constants for each $i=1,\cdots,n$, it follows from (2.5) and (2.6) that

$$
\begin{aligned}
\|T_{f_\beta}T_{g_\gamma}e_\alpha\|_{2,m} &= \Big|\sum_{\mu_1\le k_1}\cdots\sum_{\mu_n\le k_n}\sum_{\nu_1\le l_1}\cdots\sum_{\nu_n\le l_n} a_{\mu_1}^{(i_1)}\cdots a_{\mu_n}^{(i_n)} b_{\nu_1}^{(j_1)}\cdots b_{\nu_n}^{(j_n)} A_\alpha^{\theta\vartheta\varphi\psi}\Big|\\
&\sim |a_{k_1}^{(i_1)}\cdots a_{k_n}^{(i_n)} b_{l_1}^{(j_1)}\cdots b_{l_n}^{(j_n)}\alpha^{\frac12(\beta+\gamma)+k+l}|.
\end{aligned}
$$

Therefore, if we denote

$$
\mathcal{A}=\{\alpha\in\mathbb{N}^n:\alpha_s\ge\chi_{\{1\}}(j_s)\gamma_s+\chi_{\{1\}}(i_s)\beta_s \text{ for any } s=1,\cdots,n\},
$$

then the Toeplitz product $T_{f_\beta}T_{g_\gamma}$ is bounded if and only if

$$
\{\|T_{f_\beta}T_{g_\gamma}e_\alpha\|_{2,m}\}_{\alpha\in\mathcal{A}}
$$

is uniformly bounded on $F^{2,m}$, which is equivalent to $\beta = \gamma = k = l = (0, \cdots, 0)$. This completes the proof of Lemma 2.2.

Next, we will use Lemma 2.2 to prove the main theorem in this section. To this end, we first recall that, if $f \in \mathcal{P}$, then there exist $k = (k_1, \cdots, k_n)$ and $l = (l_1, \cdots, l_n) \in \mathbb{N}^n$ such that

$$f(z, \overline{z}) = \prod_{s=1}^{n} \Big(\sum_{\beta_s \le k_s} \sum_{\gamma_s \le l_s} a_{\beta_s \gamma_s, s} z_s^{\beta_s} \overline{z}_s^{\gamma_s} \Big). \tag{2.7}$$

For any $s = 1, \cdots, n$, let

$$i_{0,s} = \min\{\beta_s - \gamma_s : a_{\beta_s \gamma_s} \neq 0, \beta_s \le k_s, \gamma_s \le l_s\}$$

and

$$i_{1,s} = \max\{\beta_s - \gamma_s : a_{\beta_s \gamma_s} \neq 0, \beta_s \le k_s, \gamma_s \le l_s\}.$$

For each integer θ_s satisfying $i_{0,s} \le \theta_s \le i_{1,s}$ $(s = 1, \cdots, n)$, let $F_{\theta_s}(z_s, \overline{z_s})$ be the sum of all those terms $a_{\beta_s \gamma_s} z^{\beta_s} \overline{z}^{\gamma_s}$ in the polynomial formula (2.7) of f such that $\beta_s - \gamma_s = \theta_s$. If there is no such kind of term, we set $F_{\theta_s} = 0$. Then F_{θ_s} is of the same form as the function f_{β_s} (if $\theta_s \ge 0$) or the complex conjugate of f_{β_s} (if $\theta_s < 0$) in Lemma 2.2. Thus, with this new notation, the expression in (2.7) may be rewritten as

$$f(z, \overline{z}) = \prod_{s=1}^{n} \Big(\sum_{\theta_s = i_{0,s}}^{i_{1,s}} F_{\theta_s}(z_s, \overline{z_s}) \Big).$$

Now, we give the proof of the first main result.

Proof of Theorem 1.1 If both f and g are constants, then it is easy to check that Toeplitz operators T_f and T_g are both bounded on $F^{2,m}$. Hence the Toeplitz product $T_f T_g$ is bounded on $F^{2,m}$.

Conversely, suppose that the Toeplitz product $T_f T_g$ is bounded. Since $f, g \in \mathcal{P}$, from the above discussion, f and g admit expansions:

$$f(z, \overline{z}) = \prod_{s=1}^{n} \Big(\sum_{\theta_s = i_{0,s}}^{i_{1,s}} F_{\theta_s}(z_s, \overline{z_s}) \Big), \quad g(z, \overline{z}) = \prod_{t=1}^{n} \Big(\sum_{\tau_t = j_{0,t}}^{j_{1,t}} G_{\tau_t}(z_t, \overline{z_t}) \Big),$$

where $F_{i_{0,s}}(z_s, \overline{z_s})$, $F_{i_{1,s}}(z_s, \overline{z_s})$, $G_{j_{0,t}}(z_t, \overline{z_t})$ and $G_{j_{1,t}}(z_t, \overline{z_t})$ are nonzero for all $s, t = 1, \cdots, n$. In what follows, we write

$$F_{\theta_s} := F_{\theta_s}(z_s, \overline{z_s}), \quad G_{\tau_t} := G_{\tau_t}(z_t, \overline{z_t})$$

for simplicity. Therefore

$$\begin{aligned} T_f T_g e_\alpha &= \sum_{\theta_1 = i_{0,1}}^{i_{1,1}} \cdots \sum_{\theta_n = i_{0,n}}^{i_{1,n}} \sum_{\tau_1 = j_{0,1}}^{j_{1,1}} \cdots \sum_{\tau_n = j_{0,n}}^{j_{1,n}} T_{F_{\theta_1} \cdots F_{\theta_n}} T_{G_{\tau_1} \cdots G_{\tau_n}} e_\alpha \\ &= T_{F_{i_{1,1}} \cdots F_{i_{1,n}}} T_{G_{j_{1,1}} \cdots G_{j_{1,n}}} e_\alpha + \sum_{\substack{(\theta_1, \cdots, \theta_n, \tau_1, \cdots, \tau_n) \neq \\ (i_{1,1}, \cdots, i_{1,n}, j_{1,1}, \cdots, j_{1,n})}} T_{F_{\theta_1} \cdots F_{\theta_n}} T_{G_{\tau_1} \cdots G_{\tau_n}} e_\alpha. \end{aligned} \tag{2.8}$$

Set multi-index

$$\kappa = (\max\{|i_{0,1}|, |i_{1,1}|\} + \max\{|j_{0,1}|, |j_{1,1}|\}, \cdots, \max\{|i_{0,n}|, |i_{1,n}|\} + \max\{|j_{0,n}|, |j_{1,n}|\}).$$

It follows from the definitions of F_{θ_s}, G_{τ_t} and the proof of Lemma 2.2 that for any $\alpha \geq \kappa$, $\beta = (\theta_1, \cdots, \theta_n)$, $\gamma = (\tau_1, \cdots, \tau_n)$ with $i_{0,s} \leq \theta_s \leq i_{1,s}$ and $j_{0,t} \leq \tau_t \leq j_{1,t}$ $(s, t = 1, \cdots, n)$ such that $(\theta_1, \cdots, \theta_n, \tau_1, \cdots, \tau_n) \neq (i_{1,1}, \cdots, i_{1,n}, j_{1,1}, \cdots, j_{1,n})$, we have

$$T_{F_{\theta_1} \cdots F_{\theta_n}} T_{G_{\tau_1} \cdots G_{\tau_n}} e_\alpha \in \mathrm{Span}\{e_{\alpha+\beta+\gamma}\}.$$

Notice that the first term of (2.8),

$$T_{F_{i_{1,1}} \cdots F_{i_{1,n}}} T_{G_{j_{1,1}} \cdots G_{j_{1,n}}} e_\alpha \in \mathrm{Span}\{e_{\alpha+\beta'+\gamma'}\},$$

where $\beta' = (i_{1,1}, \cdots, i_{1,n})$ and $\gamma' = (j_{1,1}, \cdots, j_{1,n})$, we see that $T_{F_{i_{1,1}} \cdots F_{i_{1,n}}} T_{G_{j_{1,1}} \cdots G_{j_{1,n}}} e_\alpha$ is orthogonal to the second term of (2.8) for $\alpha \geq \kappa$. It follows that

$$\|T_f T_g e_\alpha\|_{2,m} \geq \|T_{F_{i_{1,1}} \cdots F_{i_{1,n}}} T_{G_{j_{1,1}} \cdots G_{j_{1,n}}} e_\alpha\|_{2,m}.$$

Obviously, the boundedness of $T_f T_g$ implies the boundedness of $T_{F_{i_{1,1}} \cdots F_{i_{1,n}}} T_{G_{j_{1,1}} \cdots G_{j_{1,n}}}$. This along with Lemma 2.2 implies that $F_{i_{1,1}}, \cdots, F_{i_{1,n}}$ and $G_{j_{1,1}}, \cdots, G_{j_{1,n}}$ must be constants. Similarly, we can also conclude that $T_{F_{i_{0,1}} \cdots F_{i_{0,n}}} T_{G_{j_{0,1}} \cdots G_{j_{0,n}}}$ is bounded if (2.8) is replaced by

$$T_f T_g e_\alpha = T_{F_{i_{0,1}} \cdots F_{i_{0,n}}} T_{G_{j_{0,1}} \cdots G_{j_{0,n}}} e_\alpha + \sum_{\substack{(\theta_1, \cdots, \theta_n, \tau_1, \cdots, \tau_n) \neq \\ (i_{0,1}, \cdots, i_{0,n}, j_{0,1}, \cdots, j_{0,n})}} T_{F_{\theta_1} \cdots F_{\theta_n}} T_{G_{\tau_1} \cdots G_{\tau_n}} e_\alpha, \tag{2.9}$$

where the summation is taken over all $i_{0,1} \leq \theta_1 \leq i_{1,1}, \cdots, i_{0,n} \leq \theta_n \leq i_{1,n}$, $j_{0,1} \leq \tau_1 \leq j_{1,1}, \cdots, j_{0,n} \leq \tau_n \leq j_{1,n}$ such that $(\theta_1, \cdots, \theta_n, \tau_1, \cdots, \tau_n) \neq (i_{0,1}, \cdots, i_{0,n}, j_{0,1}, \cdots, j_{0,n})$. By Lemma 2.2 again, $F_{i_{0,1}}, \cdots, F_{i_{0,n}}$ and $G_{j_{0,1}}, \cdots, G_{j_{0,n}}$ must be constants. Thus f and g are constants. This completes the proof of Theorem 1.1.

Corollary 2.1 *Assume that $f \in \mathcal{P}$. Then the Toeplitz operator T_f is bounded on $F^{2,m}$ if and only if f is a constant.*

Proof It follows from Theorem 1.1 by setting $g = 1$ or $g = f$.

3 Hankel Products

In this section, we are to characterize bounded Hankel products $H_f^* H_g$ with $f, g \in \mathcal{P}$. For technical reasons, we require the following lemma.

Lemma 3.1 *Assume $\beta = (\beta_1, \cdots, \beta_n)$, $\gamma = (\gamma_1, \cdots, \gamma_n)$, $\mu = (\mu_1, \cdots, \mu_n)$ and $\nu = (\nu_1, \cdots, \nu_n)$ are all in $\mathbb{N}^n$. Let $f = z^\beta \overline{z}^\gamma$ and $g = z^\mu \overline{z}^\nu$ for $z, \overline{z} \in \mathbb{C}^n$. Then for any $\alpha \geq (|\gamma_1 - \beta_1| + |\mu_1 - \nu_1|, \cdots, |\gamma_n - \beta_n| + |\mu_n - \nu_n|)$, we have*

$$H_f^* H_g e_\alpha = A_\alpha e_{\alpha+\gamma+\mu-\beta-\nu},$$

where

$$A_\alpha = \Big(\frac{(\alpha+\gamma+\mu)!(m+n-1+|\alpha+\gamma+\mu|)!}{\alpha!(n-1+|\alpha+\gamma+\mu|)!} - \frac{(\alpha+\mu)!(\alpha+\gamma+\mu-\nu)!}{\alpha!(\alpha+\mu-\nu)!} \times \frac{(m+n-1+|\alpha+\mu|)!(n-1+|\alpha+\mu-\nu|)!(m+n-1+|\alpha+\gamma+\mu-\nu|)!}{(n-1+|\alpha+\mu|)!(m+n-1+|\alpha+\mu-\nu|)!(n-1+|\alpha+\gamma+\mu-\nu|)!}\Big) \times \sqrt{\frac{\alpha!(n-1+|\alpha|)!(n-1+|\alpha+\gamma+\mu-\beta-\nu|)!}{(\alpha+\gamma+\mu-\beta-\nu)!(m+n-1+|\alpha|)!(m+n-1+|\alpha+\gamma+\mu-\beta-\nu|)!}}. \tag{3.1}$$

Furthermore, $A_\alpha = 0$ *if and only if* $\gamma = 0$ *or* $\nu = 0$*. And if* $A_\alpha \neq 0$*, then*

$$A_\alpha \sim \Big(\sum_{j=1}^{n} \gamma_j \nu_j \alpha_j^{-1}\Big) \alpha^{\frac{\beta+\nu+\gamma+\mu}{2}}. \tag{3.2}$$

Proof We only give the proof for the case of $m \neq 0$, since the case of $m = 0$ is much simpler. It is easy to verify that

$$H_f^* H_g = T_{\overline{f}g} - T_{\overline{f}} T_g = T_{z^{\gamma+\mu}\overline{z}^{\beta+\nu}} - T_{z^\gamma \overline{z}^\beta} T_{z^\mu \overline{z}^\nu}. \tag{3.3}$$

It follows from Lemma 2.1 that for any $\alpha \geq (|\gamma_1 - \beta_1| + |\mu_1 - \nu_1|, \cdots, |\gamma_n - \beta_n| + |\mu_n - \nu_n|)$, we have

$$\begin{aligned} &T_{z^{\gamma+\mu}\overline{z}^{\beta+\nu}} e_\alpha \\ &= \sqrt{\frac{\alpha!(n-1+|\alpha|)!(n-1+|\alpha+\gamma+\mu-\beta-\nu|)!}{(\alpha+\gamma+\mu-\beta-\nu)!(m+n-1+|\alpha|)!(m+n-1+|\alpha+\gamma+\mu-\beta-\nu|)!}} \\ &\quad\times \frac{(\alpha+\gamma+\mu)!(m+n-1+|\alpha+\gamma+\mu|)!}{\alpha!(n-1+|\alpha+\gamma+\mu|)!} e_{\alpha+\gamma+\mu-\beta-\nu}. \end{aligned} \tag{3.4}$$

Applying Lemma 2.1 again, we obtain

$$\begin{aligned} &T_{z^\gamma \overline{z}^\beta} T_{z^\mu \overline{z}^\nu} e_\alpha \\ &= T_{z^\gamma \overline{z}^\beta} \sqrt{\frac{\alpha!(n-1+|\alpha|)!(n-1+|\alpha+\mu-\nu|)!}{(\alpha+\mu-\nu)!(m+n-1+|\alpha|)!(m+n-1+|\alpha+\mu-\nu|)!}} \\ &\quad\times \frac{(\alpha+\mu)!(m+n-1+|\alpha+\mu|)!}{\alpha!(n-1+|\alpha+\mu|)!} e_{\alpha+\mu-\nu} \\ &= \sqrt{\frac{(\alpha+\mu-\nu)!(n-1+|\alpha+\mu-\nu|)!(n-1+|\alpha+\gamma+\mu-\beta-\nu|)!}{(\alpha+\gamma+\mu-\beta-\nu)!(m+n-1+|\alpha+\mu-\nu|)!(m+n-1+|\alpha+\gamma+\mu-\beta-\nu|)!}} \\ &\quad\times \frac{(\alpha+\mu)!(\alpha+\gamma+\mu-\nu)!(m+n-1+|\alpha+\mu|)!(m+n-1+|\alpha+\gamma+\mu-\nu|)!}{\alpha!(\alpha+\mu-\nu)!(n-1+|\alpha+\mu|)!(n-1+|\alpha+\gamma+\mu-\nu|)!} \\ &= \sqrt{\frac{\alpha!(n-1+|\alpha|)!(n-1+|\alpha+\gamma+\mu-\beta-\nu|)!}{(\alpha+\gamma+\mu-\beta-\nu)!(m+n-1+|\alpha|)!(m+n-1+|\alpha+\gamma+\mu-\beta-\nu|)!}} \\ &\quad\times \frac{(m+n-1+|\alpha+\mu|)!(n-1+|\alpha+\mu-\nu|)!(m+n-1+|\alpha+\gamma+\mu-\nu|)!}{(n-1+|\alpha+\mu|)!(m+n-1+|\alpha+\mu-\nu|)!(n-1+|\alpha+\gamma+\mu-\nu|)!} \\ &\quad\times \frac{(\alpha+\mu)!(\alpha+\gamma+\mu-\nu)!}{\alpha!(\alpha+\mu-\nu)!} e_{\alpha+\gamma+\mu-\beta-\nu}. \end{aligned} \tag{3.5}$$

Combining (3.3)–(3.5), we deduce the explicit formula for A_α in (3.1). From this formula, it is not hard to see that $A_\alpha = 0$ is equivalent to $\gamma = 0$ or $\nu = 0$.

If $A_\alpha \neq 0$, then by Stirling's formula, we have

$$\sqrt{\frac{\alpha!(n-1+|\alpha|)!(n-1+|\alpha+\gamma+\mu-\beta-\nu|)!}{(\alpha+\gamma+\mu-\beta-\nu)!(n-1+m+|\alpha|)!(n-1+m+|\alpha+\gamma+\mu-\beta-\nu|)!}} \sim \alpha^{\frac{\beta+\nu-\gamma-\mu}{2}}|\alpha|^{-m}. \tag{3.6}$$

Denote

$$B_\alpha := \frac{(\alpha+\gamma+\mu)!(m+n-1+|\alpha+\gamma+\mu|)!}{\alpha!(n-1+|\alpha+\gamma+\mu|)!} - \frac{(\alpha+\mu)!(\alpha+\gamma+\mu-\nu)!}{\alpha!(\alpha+\mu-\nu)!} \times \frac{(m+n-1+|\alpha+\mu|)!(n-1+|\alpha+\mu-\nu|)!(m+n-1+|\alpha+\gamma+\mu-\nu|)!}{(n-1+|\alpha+\mu|)!(m+n-1+|\alpha+\mu-\nu|)!(n-1+|\alpha+\gamma+\mu-\nu|)!} \tag{3.7}$$

for simplicity. Next, we study the asymptotic behavior of B_α as each component α_j tends to infinity. Firstly, we estimate the first term of B_α.

$$\begin{aligned}
&\frac{(\alpha+\gamma+\mu)!(m+n-1+|\alpha+\gamma+\mu|)!}{\alpha!(n-1+|\alpha+\gamma+\mu|)!} \\
&= \Big(\prod_{j=1}^{n}\prod_{i=1}^{\gamma_j+\mu_j}(\alpha_j+i)\Big)\prod_{i=1}^{m}(n-1+|\alpha+\gamma+\mu|+i) \\
&= \Big(\prod_{j=1}^{n}\Big(\alpha_j^{\gamma_j+\mu_j}+\Big(\sum_{i=1}^{\gamma_j+\mu_j} i\Big)\alpha_j^{\gamma_j+\mu_j-1}+O(\alpha_j^{\gamma_j+\mu_j-2})\Big)\Big) \\
&\quad \times \Big(|\alpha|^m+\Big(\sum_{i=1}^{m}(n-1+|\gamma|+|\mu|+i)\Big)|\alpha|^{m-1}+O(|\alpha|^{m-2})\Big) \\
&= \Big(\alpha^{\gamma+\mu}+\sum_{j=1}^{n}\Big(\sum_{i=1}^{\gamma_j+\mu_j} i\Big)\alpha_j^{-1}\alpha^{\gamma+\mu}+\sum_{j,k=1}^{n}O(\alpha^{\gamma+\mu}\alpha_j^{-1}\alpha_k^{-1})\Big) \\
&\quad \times \Big(|\alpha|^m+\Big(\sum_{i=1}^{m}(n-1+|\gamma|+|\mu|+i)\Big)|\alpha|^{m-1}+O(|\alpha|^{m-2})\Big) \\
&= \alpha^{\gamma+\mu}|\alpha|^m+\sum_{j=1}^{n}\Big(\sum_{i=1}^{\gamma_j+\mu_j} i\Big)\alpha_j^{-1}\alpha^{\gamma+\mu}|\alpha|^m+\Big(\sum_{i=1}^{m}(n-1+|\gamma|+|\mu|+i)\Big)\alpha^{\gamma+\mu}|\alpha|^{m-1} \\
&\quad +O(\alpha^{\gamma+\mu}|\alpha|^{m-2})+\sum_{j=1}^{n}O(\alpha^{\gamma+\mu}\alpha_j^{-1}|\alpha|^{m-1})+\sum_{j,k=1}^{n}O(\alpha^{\gamma+\mu}\alpha_j^{-1}\alpha_k^{-1}|\alpha|^m).
\end{aligned} \tag{3.8}$$

Besides,

$$\begin{aligned}
&\frac{(\alpha+\mu)!(\alpha+\gamma+\mu-\nu)!}{\alpha!(\alpha+\mu-\nu)!} \\
&= \Big(\prod_{j=1}^{n}\prod_{i=1}^{\mu_j}(\alpha_j+i)\Big)\Big(\prod_{j=1}^{n}\prod_{i=1}^{\gamma_j}(\alpha_j+\mu_j-\nu_j+i)\Big) \\
&= \Big(\prod_{j=1}^{n}\Big(\alpha_j^{\mu_j}+\Big(\sum_{i=1}^{\mu_j} i\Big)\alpha_j^{\mu_j-1}+O(\alpha_j^{\mu_j-2})\Big)\Big)
\end{aligned}$$

$$\times\Big(\prod_{j=1}^{n}\Big(\alpha_j^{\gamma_j}+\Big(\sum_{i=1}^{\gamma_j}(\mu_j-\nu_j+i)\Big)\alpha_j^{\gamma_j-1}+O(\alpha_j^{\gamma_j-2})\Big)\Big)$$

$$=\Big(\alpha^{\mu}+\sum_{j=1}^{n}\Big(\sum_{i=1}^{\mu_j}i\Big)\alpha_j^{-1}\alpha^{\mu}+\sum_{j,k=1}^{n}O(\alpha^{\mu}\alpha_j^{-1}\alpha_k^{-1})\Big)$$

$$\times\Big(\alpha^{\gamma}+\sum_{j=1}^{n}\Big(\sum_{i=1}^{\gamma_j}(\mu_j-\nu_j+i)\Big)\alpha_j^{-1}\alpha^{\gamma}+\sum_{j,k=1}^{n}O(\alpha^{\gamma}\alpha_j^{-1}\alpha_k^{-1})\Big)$$

$$=\alpha^{\mu+\gamma}+\sum_{j=1}^{n}\Big(\Big(\sum_{i=1}^{\mu_j}i\Big)+\Big(\sum_{i=1}^{\gamma_j}(\mu_j-\nu_j+i)\Big)\Big)\alpha_j^{-1}\alpha^{\mu+\gamma}+\sum_{j,k=1}^{n}O(\alpha^{\mu+\gamma}\alpha_j^{-1}\alpha_k^{-1}) \tag{3.9}$$

and

$$\frac{(m+n-1+|\alpha+\mu|)!(n-1+|\alpha+\mu-\nu|)!(m+n-1+|\alpha+\gamma+\mu-\nu|)!}{(n-1+|\alpha+\mu|)!(m+n-1+|\alpha+\mu-\nu|)!(n-1+|\alpha+\gamma+\mu-\nu|)!}$$

$$=\prod_{i=1}^{m}\frac{(n-1+|\alpha+\mu|+i)(n-1+|\alpha+\gamma+\mu-\nu|+i)}{(n-1+|\alpha+\mu-\nu|+i)}$$

$$=\frac{|\alpha|^m+\big(\sum_{i=1}^{m}(n-1+|\mu|+i)\big)|\alpha|^{m-1}+O(|\alpha|^{m-2})}{|\alpha|^m+\big(\sum_{i=1}^{m}(n-1+|\mu|-|\nu|+i)\big)|\alpha|^{m-1}+O(|\alpha|^{m-2})}$$

$$\times\Big(|\alpha|^m+\Big(\sum_{i=1}^{m}(n-1+|\mu|+|\gamma|-|\nu|+i)\Big)|\alpha|^{m-1}+O(|\alpha|^{m-2})\Big)$$

$$=|\alpha|^m+\Big(\sum_{i=1}^{m}(n-1+|\mu|+|\gamma|+i)\Big)|\alpha|^{m-1}+O(|\alpha|^{m-2}), \tag{3.10}$$

which implies that

$$\frac{(\alpha+\mu)!(\alpha+\gamma+\mu-\nu)!}{\alpha!(\alpha+\mu-\nu)!}\frac{(m+n-1+|\alpha+\mu|)!(n-1+|\alpha+\mu-\nu|)!(m+n-1+|\alpha+\gamma+\mu-\nu|)!}{(n-1+|\alpha+\mu|)!(m+n-1+|\alpha+\mu-\nu|)!(n-1+|\alpha+\gamma+\mu-\nu|)!}$$

$$=\alpha^{\gamma+\mu}|\alpha|^m+\sum_{j=1}^{n}\Big(\Big(\sum_{i=1}^{\mu_j}i\Big)+\Big(\sum_{i=1}^{\gamma_j}(\mu_j-\nu_j+i)\Big)\Big)\alpha_j^{-1}\alpha^{\gamma+\mu}|\alpha|^m$$

$$+\Big(\sum_{i=1}^{m}(n-1+|\mu|+|\gamma|+i)\Big)\alpha^{\gamma+\mu}|\alpha|^{m-1}$$

$$+O(\alpha^{\gamma+\mu}|\alpha|^{m-2})+\sum_{j=1}^{n}O(\alpha^{\gamma+\mu}\alpha_j^{-1}|\alpha|^{m-1})+\sum_{j,k=1}^{n}O(\alpha^{\gamma+\mu}\alpha_j^{-1}\alpha_k^{-1}|\alpha|^m). \tag{3.11}$$

(3.11) Subtracted from (3.8), we obtain

$$B_\alpha=\Big(\sum_{j=1}^{n}\gamma_j\nu_j\alpha_j^{-1}\Big)\alpha^{\gamma+\mu}|\alpha|^m+O(\alpha^{\gamma+\mu}|\alpha|^{m-2})+\sum_{j=1}^{n}O(\alpha^{\gamma+\mu}\alpha_j^{-1}|\alpha|^{m-1})$$

$$+\sum_{j,k=1}^{n}O(\alpha^{\gamma+\mu}\alpha_j^{-1}\alpha_k^{-1}|\alpha|^m).$$

This along with (3.6) gives

$$A_\alpha\sim\Big(\sum_{j=1}^{n}\gamma_j\nu_j\alpha_j^{-1}\Big)\alpha^{\frac{\beta+\nu+\gamma+\mu}{2}}.$$

This completes the proof of Lemma 3.1.

Lemma 3.2 *Suppose* $\beta = (\beta_1, \cdots, \beta_n)$, $\gamma = (\gamma_1, \cdots, \gamma_n)$, $k = (k_1, \cdots, k_n)$ *and* $l = (l_1, \cdots, l_n)$ *are in* $\mathbb{N}^n$. *For any* $z = (z_1, \cdots, z_n) \in \mathbb{C}^n$, *let*

$$f_{\beta_i}(z_i) = \sum_{\mu_i \le k_i} a_{\mu_i} z_i^{\mu_i+\beta_i} \overline{z}_i^{\mu_i}, \quad g_{\gamma_i}(z_i) = \sum_{\nu_i \le l_i} b_{\nu_i} z_i^{\nu_i+\gamma_i} \overline{z}_i^{\nu_i},$$

where a_{μ_i}, b_{ν_i} *are constants with* a_{k_i}, b_{l_i} *nonzero for each* $i = 1, \cdots, n$. *For* $i_1, \cdots, i_n$, $j_1, \cdots, j_n \in \{0, 1\}$, *let*

$$f_\beta(z) = f_{\beta_1}^{(i_1)}(z_1) \cdots f_{\beta_n}^{(i_n)}(z_n), \quad g_\gamma(z) = g_{\gamma_1}^{(j_1)}(z_1) \cdots g_{\gamma_n}^{(j_n)}(z_n).$$

Then the Hankel product $H_{f_\beta}^* H_{g_\gamma}$ *is bounded on* $F^{2,m}$ *if and only if at least one of the following conditions holds*:

(1) $k = (0, \cdots, 0)$ *and* $\beta_s = 0$ *for any* $1 \le s \le n$ *such that* $i_s = 1$.

(2) $l = (0, \cdots, 0)$ *and* $\gamma_t = 0$ *for any* $1 \le t \le n$ *such that* $j_t = 1$.

(3) $n = \beta_1 = \gamma_1 = i_1 = j_1 = 1$ *and* $k_1 = l_1 = 0$.

Proof To begin with, we use the same notations θ, ϑ, φ and ψ as in Lemma 2.2. Then by Lemma 3.1, for any $\alpha \in \mathbb{N}^n$ satisfying $\alpha \ge \beta + \gamma$,

$$\begin{aligned} & H_{f_\beta}^* H_{g_\gamma} e_\alpha \\ &= \sum_{\mu_1 \le k_1} \cdots \sum_{\mu_n \le k_n} \sum_{\nu_1 \le l_1} \cdots \sum_{\nu_n \le l_n} a_{\mu_1}^{(i_1)} \cdots a_{\mu_n}^{(i_n)} b_{\nu_1}^{(j_1)} \cdots b_{\nu_n}^{(j_n)} H_{z^\theta \overline{z}^\vartheta}^* H_{z^\varphi \overline{z}^\psi} e_\alpha \\ &= \sum_{\mu_1 \le k_1} \cdots \sum_{\mu_n \le k_n} \sum_{\nu_1 \le l_1} \cdots \sum_{\nu_n \le l_n} a_{\mu_1}^{(i_1)} \cdots a_{\mu_n}^{(i_n)} b_{\nu_1}^{(j_1)} \cdots b_{\nu_n}^{(j_n)} B_\alpha^{\theta\vartheta\varphi\psi} e_{\alpha+\vartheta+\varphi-\theta-\psi}, \end{aligned}$$

where

$$\begin{aligned} B_\alpha^{\theta\vartheta\varphi\psi} := \Big(& \frac{(\alpha+\vartheta+\varphi)!(m+n-1+|\alpha+\vartheta+\varphi|)!}{\alpha!(n-1+|\alpha+\vartheta+\varphi|)!} - \frac{(\alpha+\varphi)!(\alpha+\vartheta+\varphi-\psi)!}{\alpha!(\alpha+\varphi-\psi)!} \\ & \times \frac{(m+n-1+|\alpha+\varphi|)!(n-1+|\alpha+\varphi-\psi|)!(m+n-1+|\alpha+\vartheta+\varphi-\psi|)!}{(n-1+|\alpha+\varphi|)!(m+n-1+|\alpha+\varphi-\psi|)!(n-1+|\alpha+\vartheta+\varphi-\psi|)!}\Big) \\ & \times \sqrt{\frac{\alpha!(n-1+|\alpha|)!(n-1+|\alpha+\vartheta+\varphi-\theta-\psi|)!}{(\alpha+\vartheta+\varphi-\theta-\psi)!(m+n-1+|\alpha|)!(m+n-1+|\alpha+\vartheta+\varphi-\theta-\psi|)!}}. \end{aligned}$$

If $\vartheta \ne 0$ and $\psi \ne 0$, then by Lemma 3.1 again, we have $B_\alpha^{\theta\vartheta\varphi\psi} \ne 0$ and

$$\begin{aligned} B_\alpha^{\theta\vartheta\varphi\psi} &\sim \sum_{s=1}^n \vartheta_s \psi_s \alpha_s^{-1} \alpha^{\frac{\theta+\vartheta+\varphi+\psi}{2}} \\ &= \sum_{s=1}^n (\mu_s + \chi_{\{1\}}(i_s)\beta_s)(\nu_s + \chi_{\{1\}}(j_s)\gamma_s) \alpha_s^{-1} \alpha^{\frac{\beta+\gamma}{2}+\mu+\nu}. \end{aligned}$$

Since a_{k_i}, b_{l_i} are nonzero constants for each $i = 1, \cdots, n$, we have

$$\begin{aligned}&\|H^*_{f_\beta} H_{g_\gamma} e_\alpha\|_{2,m}\\ &= \Big| \sum_{\mu_1 \le k_1} \cdots \sum_{\mu_n \le k_n} \sum_{\nu_1 \le l_1} \cdots \sum_{\nu_n \le l_n} a^{(i_1)}_{\mu_1} \cdots a^{(i_n)}_{\mu_n} b^{(j_1)}_{\nu_1} \cdots b^{(j_n)}_{\nu_n} B^{\theta\vartheta\varphi\psi}_\alpha \Big|\\ &\sim |a^{(i_1)}_{k_1} \cdots a^{(i_n)}_{k_n} b^{(j_1)}_{l_1} \cdots b^{(j_n)}_{l_n}| \sum_{s=1}^n (k_s + \chi_{\{1\}}(i_s)\beta_s)(l_s + \chi_{\{1\}}(j_s)\gamma_s)\alpha_s^{-1} \alpha^{\frac{\beta+\gamma}{2}+k+l}\end{aligned}$$

for $\alpha \ge \beta + \gamma$. Consequently, the Hankel product $H^*_{f_\beta} H_{g_\gamma}$ is bounded on $F^{2,m}$ if and only if the following expression

$$(k_s + \chi_{\{1\}}(i_s)\beta_s)(l_s + \chi_{\{1\}}(j_s)\gamma_s)\alpha_s^{-1} \alpha^{\frac{\beta+\gamma}{2}+k+l}$$

is independent of α for each $s = 1, \cdots, n$, which is equivalent to that at least one of the following statements holds:

(a) $(k_s + \chi_{\{1\}}(i_s)\beta_s)(l_s + \chi_{\{1\}}(j_s)\gamma_s) = 0$ for each $s = 1, \cdots, n$.

(b) $n = \beta_1 = \gamma_1 = i_1 = j_1 = 1$ and $k_1 = l_1 = 0$.

Since (a) is equivalent to condition (1) or (2), the desired result is then obtained.

We proceed to prove the main theorem in this section.

Proof of Theorem 1.2 If the statement (1) or (2) is true, then $H^*_f = 0$ or $H_g = 0$, it follows that $H^*_f H_g$ is bounded on $F^{2,m}$. If the statement (3) is true, then we have

$$\begin{aligned}H^*_f H_g e_\alpha &= \overline{a} b H^*_{\overline{z}} H_{\overline{z}} e_\alpha\\ &= \overline{a} b e_\alpha\end{aligned}$$

by Lemma 3.2, which implies that the Hankel product $H^*_f H_g$ is bounded on $F^{2,m}$.

Conversely, assume the Hankel product $H^*_f H_g$ is bounded on $F^{2,m}$. If neither f nor g is holomorphic, we are to show that the statement (3) must be true. Since $f \in \mathcal{P}$, there exist $k = (k_1, \cdots, k_n)$ and $l = (l_1, \cdots, l_n) \in \mathbb{N}^n$ such that

$$f(z, \overline{z}) = \prod_{s=1}^n \Big(\sum_{\beta_s \le k_s} \sum_{\gamma_s \le l_s} a_{\beta_s\gamma_s,s} z_s^{\beta_s} \overline{z}_s^{\gamma_s} \Big). \tag{3.12}$$

Let

$$f_1(z, \overline{z}) = \prod_{s=1}^n \Big(\sum_{\beta_s \le k_s} a_{\beta_s 0,s} z_s^{\beta_s} \Big).$$

Then f_1 is said to be the pure holomorphic part of f. Similarly, denote g_1 by the pure holomorphic part of g. Let $f_2 = f - f_1$ and $g_2 = g - g_1$. Then by our assumption, we see that neither f_2 nor g_2 is 0. Moreover, from the discussion before Theorem 1.1, f_2 and g_2 admit expansions

$$f_2 = \prod_{s=1}^n \Big(\sum_{\theta_s = i_{0,s}}^{i_{1,s}} F_{\theta_s} \Big), \quad g_2 = \prod_{t=1}^n \Big(\sum_{\tau_t = j_{0,t}}^{j_{1,t}} G_{\tau_t} \Big),$$

where $F_{i_{0,s}}$, $F_{i_{1,s}}$, $G_{j_{0,s}}$ and $G_{j_{1,s}}$ are nonzero. Therefore,

$$\begin{aligned} H_f^* H_g e_\alpha &= H_{f_2}^* H_{g_2} e_\alpha \\ &= \sum_{\theta_1=i_{0,1}}^{i_{1,1}} \cdots \sum_{\theta_n=i_{0,n}}^{i_{1,n}} \sum_{\tau_1=j_{0,1}}^{j_{1,1}} \cdots \sum_{\tau_n=j_{0,n}}^{j_{1,n}} H_{F_{\theta_1}\cdots F_{\theta_n}}^* H_{G_{\tau_1}\cdots G_{\tau_n}} e_\alpha \\ &= H_{F_{i_{0,1}}\cdots F_{i_{0,n}}}^* H_{G_{j_{1,1}}\cdots G_{j_{1,n}}} e_\alpha + \sum_{\substack{(\theta_1,\cdots,\theta_n,\tau_1,\cdots,\tau_n)\neq \\ (i_{0,1},\cdots,i_{0,n},j_{1,1},\cdots,j_{1,n})}} H_{F_{\theta_1}\cdots F_{\theta_n}}^* H_{G_{\tau_1}\cdots G_{\tau_n}} e_\alpha \end{aligned} \tag{3.13}$$

for any $\alpha \in \mathbb{N}^n$. Set multi-index

$$\kappa = (\max\{|i_{0,1}|, |i_{1,1}|\} + \max\{|j_{0,1}|, |j_{1,1}|\}, \cdots, \max\{|i_{0,n}|, |i_{1,n}|\} + \max\{|j_{0,n}|, |j_{1,n}|\}).$$

It follows from the definitions of F_{θ_s}, G_{τ_t} and the proof of Lemma 3.2 that for any $\alpha \geq \kappa$, $\beta = (\theta_1, \cdots, \theta_n)$, $\gamma = (\tau_1, \cdots, \tau_n)$ with $i_{0,s} \leq \theta_s \leq i_{1,s}$ and $j_{0,t} \leq \tau_t \leq j_{1,t}$ $(s, t = 1, \cdots, n)$ satisfying $(\theta_1, \cdots, \theta_n, \tau_1, \cdots, \tau_n) \neq (i_{1,1}, \cdots, i_{1,n}, j_{1,1}, \cdots, j_{1,n})$, we have

$$H_{F_{\theta_1}\cdots F_{\theta_n}}^* H_{G_{\tau_1}\cdots G_{\tau_n}} e_\alpha \in \mathrm{Span}\{e_{\alpha+\gamma-\beta}\}.$$

But the first term of (3.13),

$$H_{F_{i_{0,1}}\cdots F_{i_{0,n}}}^* H_{G_{j_{1,1}}\cdots G_{j_{1,n}}} e_\alpha \in \mathrm{Span}\{e_{\alpha+\gamma'-\beta'}\},$$

where $\gamma' = (i_{0,1}, \cdots, i_{0,n})$ and $\beta' = (j_{1,1}, \cdots, j_{1,n})$. Therefore, we conclude that

$$H_{F_{i_{0,1}}\cdots F_{i_{0,n}}}^* H_{G_{j_{1,1}}\cdots G_{j_{1,n}}} e_\alpha$$

is orthogonal to the second term of (3.13) for $\alpha \geq \kappa$. This makes

$$\|H_f^* H_g e_\alpha\|_{2,m} \geq \|H_{F_{i_{0,1}}\cdots F_{i_{0,n}}}^* H_{G_{j_{1,1}}\cdots G_{j_{1,n}}} e_\alpha\|_{2,m}$$

for $\alpha \geq \kappa$. Carefully examining the proof of Lemma 3.2, we see that $H_{F_{i_{1,1}}\cdots F_{i_{1,n}}}^* H_{G_{j_{1,1}}\cdots G_{j_{1,n}}} e_\alpha$ is bounded on $F^{2,m}$ if and only if the sequence

$$\{\|H_{F_{i_{0,1}}\cdots F_{i_{0,n}}}^* H_{G_{j_{1,1}}\cdots G_{j_{1,n}}} e_\alpha\|_{2,m}\}_{\alpha\geq\kappa}$$

is bounded on $F^{2,m}$.

Notice from the definitions of f_2 and g_2 that, for $\theta_s \geq 0$ (resp. $\tau_t \geq 0$), F_{θ_s} (resp. G_{τ_t}) does not contain any term as $a_{\theta_s} z_s^{\theta_s}$ (resp. $b_{\tau_t} z_t^{\tau_t}$), where a_{θ_s} (resp. b_{τ_t}) denotes the coefficient. In other words, for $\theta_s \geq 0$ (resp. $\tau_t \geq 0$), the term F_{θ_s} (resp. G_{τ_t}) is of the following form:

$$\sum_{1\leq\mu_s\leq k_s} a_{\mu_s,s} z_s^{\mu_s+\theta_s} \overline{z}_s^{\mu_s} \quad \Big(\text{resp.} \sum_{1\leq\nu_t\leq l_t} b_{\nu_t,t} z_t^{\nu_t+\tau_t} \overline{z}_t^{\nu_t}\Big), \tag{3.14}$$

where k_s and l_t are positive integers greater than or equal to 1.

For $\theta_s < 0$ (resp. $\tau_t < 0$), the term F_{θ_s} (resp. G_{τ_t}) is of the following form:

$$\sum_{\mu_s\leq k_s} a_{\mu_s,s} z_s^{\mu_s} \overline{z}_s^{\mu_s+|\theta_s|} \quad \Big(\text{resp.} \sum_{\nu_t\leq l_t} b_{\nu_t,t} z_t^{\nu_t} \overline{z}_t^{\nu_t+|\gamma_t|}\Big). \tag{3.15}$$

If $i_{0,s} \geq 0$ or $j_{1,t} \geq 0$ for all $s, t = 1, \cdots, n$, then it follows from (3.14) and Lemma 3.2 that $H^*_{F_{i_{0,1}} \cdots F_{i_{0,n}}} H_{G_{j_{1,1}} \cdots G_{j_{1,n}}}$ is unbounded. Thus, the boundedness of $H^*_{F_{i_{0,1}} \cdots F_{i_{0,n}}} H_{G_{j_{1,1}} \cdots G_{j_{1,n}}}$ implies that $i_{0,s} < 0$ and $j_{1,t} < 0$ for all $s, t = 1, \cdots, n$. Then F_{θ_s} is the form of (3.15). It follows from (3) of Lemma 3.2, we have $n = 1$ and $F_{i_{0,1}} = a_0 \overline{z}$, $G_{j_{1,1}} = b_0 \overline{z}$, where a_0, b_0 are nonzero constants and $\overline{z} \in \mathbb{C}$.

As discussed above, we can also conclude that the Hankel product is bounded if (3.13) is replaced by

$$H^*_f H_g e_\alpha = H^*_{F_{i_{1,1}} \cdots F_{i_{1,n}}} H_{G_{j_{0,1}} \cdots G_{j_{0,n}}} e_\alpha + \sum_{\substack{(\theta_1, \cdots, \theta_n, \tau_1, \cdots, \tau_n) \neq \\ (i_{1,1}, \cdots, i_{1,n}, j_{0,1}, \cdots, j_{0,n})}} H^*_{F_{\theta_1} \cdots F_{\theta_n}} H_{G_{\tau_1} \cdots G_{\tau_n}} e_\alpha.$$

Similar to the discussion of $H^*_{F_{i_{0,1}} \cdots F_{i_{0,n}}} H_{G_{j_{1,1}} \cdots G_{j_{1,n}}} e_\alpha$, we can also conclude that $n = 1$ and $F_{i_{1,1}} = a_0' \overline{z}$, $G_{j_{0,1}} = b_0' \overline{z}$, where a_0', b_0' are nonzero constants and $\overline{z} \in \mathbb{C}$. Therefore, $f_2(z) = a\overline{z}$ and $g_2(z) = b\overline{z}$, where a and b are nonzero constants and $\overline{z} \in \mathbb{C}$, hence the statement (3) is true. This completes the proof of Theorem 1.2.

Corollary 3.1 *Assume that $f \in \mathcal{P}$. Then the Hankel operator H_f is bounded on $F^{2,m}$ if and only if one of the following statements is true*:

(1) *f is holomorphic.*

(2) *$n = 1$ and there exists a holomorphic polynomial f_1 such that*

$$f = f_1 + a\overline{z},$$

where a is a constant and $z \in \mathbb{C}$.

Proof It is a direct consequence of Theorem 1.2 by setting $g = f$.

Corollary 3.2 *Assume that $f \in \mathcal{P}$. Then the Hankel operator H_f is compact on $F^{2,m}$ if and only if f is holomorphic.*

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