Chin. Ann. Math. Ser. B
43(3), 2022, 417–424
DOI: 10.1007/s11401-022-0332-7

Chinese Annals of Mathematics, Series B

On a Supercongruence Conjecture of Z.-W. Sun*

Guo-shuai MAO[1]

Abstract In this paper, the author partly proves a supercongruence conjectured by Z.-W. Sun in 2013. Let p be an odd prime and let $a \in \mathbb{Z}^+$. Then, if $p \equiv 1 \pmod 3$,

$$\sum_{k=0}^{\lfloor \frac{5}{6}p^a \rfloor} \frac{\binom{2k}{k}}{16^k} \equiv \left(\frac{3}{p^a}\right) \pmod{p^2}$$

is obtained, where $\left(\frac{\cdot}{\cdot}\right)$ is the Jacobi symbol.

Keywords Supercongruences, Binomial coefficients, Fermat quotient, Jacobi symbol
2000 MR Subject Classification 11A07, 05A10, 11B65

1 Introduction

In the past years, congruences for sums of binomial coefficients have attracted the attention of many researchers (see, for instance, [1, 3–4, 6, 10, 12, 16–17, 19]). In 2011, Sun [17] proved that for any odd prime p and $a \in \mathbb{Z}^+$,

$$\sum_{k=0}^{p^a-1} \binom{2k}{k} \equiv \left(\frac{p^a}{3}\right) \pmod{p^2}.$$

Recently, Liu and Petrov [7] showed some congruences on sums of q-binomial coefficients.

Pan and Sun [13] proved that for any prime $p \equiv 1 \pmod 4$ or $1 < a \in \mathbb{Z}^+$,

$$\sum_{k=0}^{\lfloor \frac{3}{4}p^a \rfloor} \frac{\binom{2k}{k}}{(-4)^k} \equiv \left(\frac{2}{p^a}\right) \pmod{p^2}.$$

In 2017, Mao and Sun [11] showed that for any prime $p \equiv 1 \pmod 4$ or $1 < a \in \mathbb{Z}^+$,

$$\sum_{k=0}^{\lfloor \frac{3}{4}p^a \rfloor} \frac{\binom{2k}{k}^2}{(16)^k} \equiv \left(\frac{-1}{p^a}\right) \pmod{p^3}.$$

Sun [15] proved that for any odd prime p and $a \in \mathbb{Z}^+$,

$$\sum_{k=0}^{\frac{p^a-1}{2}} \frac{\binom{2k}{k}}{16^k} \equiv \left(\frac{3}{p^a}\right) \pmod{p^2}. \tag{1.1}$$

Manuscript received January 28, 2021. Revised October 15, 2021.
[1]Department of Mathematics, Nanjing University of Information Science and Technology, Nanjing 210044, China. E-mail: maogsmath@163.com
*This work was supported by the National Natural Science Foundation of China (Nos. 12001288, 12071208).

In this paper, we partly prove Sun's conjecture (see [15, Conjecture 1.2(i)]).

Theorem 1.1 *Let p be an odd prime and let $a \in \mathbb{Z}^+$. If $p \equiv 1 \pmod 3$, then*

$$\sum_{k=0}^{\lfloor \frac{5}{6}p^a \rfloor} \frac{\binom{2k}{k}}{16^k} \equiv \left(\frac{3}{p^a}\right) \pmod{p^2}.$$

We shall prove Theorem 1.1 in Section 2. Our result is much interesting because of much rarer are the examples where the upper limit of the sum is strictly between $\frac{p-1}{2}$ and $p-1$, and these congruences are much more difficult to handle.

2 Proof of Theorem 1.1

Lemma 2.1 (see [5]) *For any prime $p > 3$, we have the following congruences modulo p*

$$H_{\lfloor \frac{p}{2} \rfloor} \equiv -2q_p(2), \quad H_{\lfloor \frac{p}{3} \rfloor} \equiv -\frac{3}{2}q_p(3), \quad H_{\lfloor \frac{p}{6} \rfloor} \equiv -2q_p(2) - \frac{3}{2}q_p(3).$$

Proof of Theorem 1.1 In view of (1.1), we just need to verify that

$$\sum_{k=\frac{p^a+1}{2}}^{\lfloor \frac{5}{6}p^a \rfloor} \frac{\binom{2k}{k}}{16^k} \equiv 0 \pmod{p^2}. \tag{2.1}$$

Let k and l be positive integers with $k + l = p^a$ and $0 < l < \frac{p^a}{2}$. In view of [13], we have

$$\frac{l}{2}\binom{2l}{l} = \frac{(2l-1)!}{(l-1)!^2} \not\equiv 0 \pmod{p^a} \tag{2.2}$$

and

$$\binom{2k}{k} \equiv -p^a \frac{(l-1)!^2}{(2l-1)!} = -\frac{2p^a}{l\binom{2l}{l}} \pmod{p^2}. \tag{2.3}$$

So we have

$$\sum_{k=\frac{p^a+1}{2}}^{\lfloor \frac{5}{6}p^a \rfloor} \frac{\binom{2k}{k}}{16^k} \equiv \sum_{k=\frac{p^a+1}{2}}^{\lfloor \frac{5}{6}p^a \rfloor} \frac{-2p^a}{(p^a-k)\binom{2p^a-2k}{p^a-k}16^k} = \frac{-2p^a}{16^{p^a}} \sum_{k=\lfloor \frac{p^a}{6} \rfloor+1}^{\frac{p^a-1}{2}} \frac{16^k}{k\binom{2k}{k}} \pmod{p^2}.$$

It is easy to see that for $k = 1, 2, \cdots, \frac{p^a-1}{2}$,

$$\frac{\binom{\frac{p^a-1}{2}}{k}}{\frac{\binom{2k}{k}}{(-4)^k}} = \frac{\binom{\frac{p^a-1}{2}}{k}}{\binom{-\frac{1}{2}}{k}} = \prod_{j=0}^{k-1} \frac{\frac{p^a-1}{2} - j}{-\frac{1}{2} - j} = \prod_{j=0}^{k-1} \left(1 - \frac{p^a}{2j+1}\right) \equiv 1 \pmod{p}. \tag{2.4}$$

This, with Fermat's little theorem yields that

$$\sum_{k=\frac{p^a+1}{2}}^{\lfloor \frac{5}{6}p^a \rfloor} \frac{\binom{2k}{k}}{16^k} \equiv -\frac{p^a}{8} \sum_{k=\lfloor \frac{p^a}{6} \rfloor+1}^{\frac{p^a-1}{2}} \frac{(-4)^k}{k\binom{\frac{p^a-1}{2}}{k}} \equiv -p^a \sum_{k=\lfloor \frac{p^a}{6} \rfloor}^{\frac{p^a-3}{2}} \frac{(-4)^k}{\binom{\frac{p^a-3}{2}}{k}} \pmod{p^2}.$$

Thus, by (2.1) we only need to show that

$$p^{a-1}\sum_{k=\lfloor \frac{p^a}{6}\rfloor}^{\frac{p^a-3}{2}} \frac{(-4)^k}{\binom{\frac{p^a-3}{2}}{k}} \equiv 0 \pmod p. \tag{2.5}$$

Now we set

$$n=\frac{p^a-1}{2}, \quad m=\left\lfloor \frac{p^a}{6}\right\rfloor,\ \lambda=-4,$$

then we only need to prove that

$$p^{a-1}\sum_{k=m}^{n-1}\frac{\lambda^k}{\binom{n-1}{k}} \equiv 0 \pmod p. \tag{2.6}$$

Setting $n=n-1$ in the last equation of page 3 in [18], we have

$$\begin{aligned}\sum_{k=m}^{n-1}\frac{\lambda^k}{\binom{n-1}{k}} &= n\sum_{k=0}^{n-1-m}\frac{\lambda^{m+k}}{(\lambda+1)^{k+1}}\sum_{i=0}^{n-1-m-k}\frac{(-1)^i\binom{n-1-m-k}{i}}{m+i+1}\\ &\quad+\frac{n\lambda^n}{(\lambda+1)^{n+1}}\sum_{k=m}^{n-1}\frac{(\lambda+1)^{k+1}}{k+1}.\end{aligned}$$

It is easy to check that for each $0\le k\le n-1-m$,

$$\begin{aligned}&\sum_{i=0}^{n-1-m-k}\binom{n-1-m-k}{i}\frac{(-1)^i}{m+i+1}\\ &=\int_0^1\sum_{i=0}^{n-1-m-k}\binom{n-1-m-k}{i}(-x)^i x^m \mathrm{d}x\\ &=\int_0^1 x^m(1-x)^{n-1-m-k}\mathrm{d}x = B(m+1,n-m-k),\end{aligned}$$

where $B(P,Q)$ stands for the beta function. It is well known that the beta function relates to gamma function:

$$B(P,Q)=\frac{\Gamma(P)\Gamma(Q)}{\Gamma(P+Q)}.$$

So

$$\begin{aligned}B(m+1,n-m-k)&=\frac{\Gamma(m+1)\Gamma(n-m-k)}{\Gamma(n-k+1)}\\ &=\frac{m!(n-m-k-1)!}{(n-k)!}\\ &=\frac{1}{(m+1)\binom{n-k}{m+1}}.\end{aligned}$$

Therefore

$$\sum_{k=m}^{n-1}\frac{\lambda^k}{\binom{n-1}{k}}=\frac{n}{m+1}\sum_{k=0}^{n-1-m}\frac{\lambda^{m+k}}{(\lambda+1)^{k+1}\binom{n-k}{m+1}}+\frac{n\lambda^n}{(\lambda+1)^{n+1}}\sum_{k=m}^{n-1}\frac{(\lambda+1)^{k+1}}{k+1}$$
$$=\frac{n}{m+1}\sum_{k=m+1}^{n}\frac{\lambda^{m+n-k}}{(\lambda+1)^{n-k+1}\binom{k}{m+1}}+\frac{n\lambda^n}{(\lambda+1)^{n+1}}\sum_{k=m+1}^{n}\frac{(\lambda+1)^k}{k}$$
$$=\frac{n\lambda^n}{(\lambda+1)^{n+1}}\Big(\frac{\lambda^m}{m+1}\sum_{k=m+1}^{n}\frac{(\lambda+1)^k}{\lambda^k\binom{k}{m+1}}+\sum_{k=m+1}^{n}\frac{(\lambda+1)^k}{k}\Big).$$

Hence, by (2.6), we just need to show that

$$p^{a-1}\frac{\lambda^m}{m+1}\sum_{k=m+1}^{n}\frac{(\lambda+1)^k}{\lambda^k\binom{k}{m+1}}\equiv -p^{a-1}\sum_{k=m+1}^{n}\frac{(\lambda+1)^k}{k}\ (\mathrm{mod}\,p). \tag{2.7}$$

It is obvious that

$$\sum_{k=m+1}^{n}\frac{(\lambda+1)^k}{\lambda^k\binom{k}{m+1}}=\sum_{k=m+1}^{n}\frac{1}{\binom{k}{m+1}}\Big(\frac{3}{4}\Big)^k$$
$$=\sum_{k=m+1}^{n}\frac{1}{\binom{k}{m+1}}\sum_{j=0}^{k}\frac{\binom{k}{j}}{(-4)^j}$$
$$=\mathfrak{B}+\mathfrak{C},$$

where

$$\mathfrak{B}=\sum_{j=m+1}^{n}\frac{1}{(-4)^j}\sum_{k=j}^{n}\frac{\binom{k}{j}}{\binom{k}{m+1}},$$
$$\mathfrak{C}=\sum_{j=0}^{m}\frac{1}{(-4)^j}\sum_{k=m+1}^{n}\frac{\binom{k}{j}}{\binom{k}{m+1}}.$$

By the following transformation

$$\frac{\binom{k}{j}}{\binom{k}{m+1}}=\frac{k!(m+1)!(k-m-1)!}{j!(k-j)!k!}=\frac{(m+1)!(k-m-1)!(j-m-1)!}{j!(k-j)!(j-m-1)!}=\frac{\binom{k-m-1}{j-m-1}}{\binom{j}{m+1}},$$

we have

$$\mathfrak{B}=\sum_{j=m+1}^{n}\frac{1}{(-4)^j}\sum_{k=j}^{n}\frac{\binom{k-m-1}{j-m-1}}{\binom{j}{m+1}}=\sum_{j=m+1}^{n}\frac{1}{(-4)^j\binom{j}{m+1}}\sum_{k=0}^{n-j}\binom{k+j-m-1}{j-m-1}.$$

In view of [2, (1.48)], we have

$$\mathfrak{B}=\sum_{j=m+1}^{n}\frac{1}{(-4)^j\binom{j}{m+1}}\binom{n-m}{j-m},$$

and it is easy to check that

$$\frac{\binom{n-m}{j-m}}{\binom{j}{m+1}}=\frac{(n-m)!(m+1)!(j-m-1)!}{j!(n-j)!(j-m)!}=\frac{n+1}{j-m}\frac{\binom{n}{j}}{\binom{n+1}{m+1}}.$$

Thus

$$\mathfrak{B} = \frac{n+1}{\binom{n+1}{m+1}} \sum_{j=m+1}^{n} \frac{\binom{n}{j}}{(j-m)(-4)^j}.$$

Now we calculate $\mathfrak{C}$. First we have the following transformation

$$\frac{\binom{k}{j}}{\binom{k}{m+1}} = \frac{k!(m+1)!(k-m-1)!}{j!(k-j)!k!} = \frac{(m+1)!(k-m-1)!(m-j+1)!}{j!(k-j)!(m-j+1)!} = \frac{\binom{m+1}{j}}{\binom{k-j}{m-j+1}}.$$

Hence,

$$\mathfrak{C} = \sum_{j=0}^{m} \binom{m+1}{j} \frac{1}{(-4)^j} \sum_{k=m+1}^{n} \frac{1}{\binom{k-j}{m-j+1}} = \sum_{j=0}^{m} \binom{m+1}{j} \frac{1}{(-4)^j} \sum_{k=0}^{n-m-1} \frac{1}{\binom{k+m+1-j}{m-j+1}}.$$

With the help of package Sigma (see [14]), we find the following identity:

$$\sum_{k=0}^{N} \frac{1}{\binom{k+i}{i}} = \frac{i}{i-1} - \frac{N+1}{(i-1)\binom{N+i}{N}},$$

which can be easily proved by induction on N.

Substituting $N = n-m-1, i = m+1-j$ into the above identity, we have

$$\mathfrak{C} = \sum_{j=0}^{m-1} \binom{m+1}{j} \frac{1}{(-4)^j} \Big(\frac{m+1-j}{m-j} - \frac{n-m}{(m-j)\binom{n-j}{n-m-1}} \Big) + (m+1)\Big(-\frac{1}{4}\Big)^m \sum_{k=1}^{n-m} \frac{1}{k}.$$

It is easy to check that

$$\frac{(n-m)\binom{m+1}{j}}{\binom{n-j}{n-m-1}} = \frac{(m+1)!(n-m)!(m+1-j)!}{j!(n-j)!(m+1-j)!} = \frac{(m+1)!(n-m)!}{j!(n-j)!} = \frac{(n+1)\binom{n}{j}}{\binom{n+1}{m+1}}.$$

Therefore

$$\mathfrak{C} = (m+1) \sum_{j=0}^{m-1} \frac{\binom{m}{j}}{(m-j)(-4)^j} - \frac{n+1}{\binom{n+1}{m+1}} \sum_{j=0}^{m-1} \frac{\binom{n}{j}}{(m-j)(-4)^j} + (m+1)\Big(-\frac{1}{4}\Big)^m \sum_{k=1}^{n-m} \frac{1}{k}.$$

Hence

$$\mathfrak{B} + \mathfrak{C} = (m+1) \sum_{j=0}^{m-1} \frac{\binom{m}{j}}{(m-j)(-4)^j} + \frac{n+1}{\binom{n+1}{m+1}} \sum_{\substack{j=0 \\ j\neq m}}^{n} \frac{\binom{n}{j}}{(j-m)(-4)^j} + (m+1)\Big(-\frac{1}{4}\Big)^m \sum_{k=1}^{n-m} \frac{1}{k}.$$

That is

$$\frac{\lambda^m}{m+1}(\mathfrak{B} + \mathfrak{C}) = \lambda^m \sum_{j=0}^{m-1} \frac{\binom{m}{j}}{(m-j)(-4)^j} + \frac{\lambda^m}{\binom{n}{m}} \sum_{\substack{j=0 \\ j\neq m}}^{n} \frac{\binom{n}{j}}{(j-m)(-4)^j} + H_{n-m}. \tag{2.8}$$

One can easily check that

$$\begin{aligned}\sum_{k=1}^{n}\frac{(-3)^k}{k}&=\int_0^1\sum_{k=1}^{n}(-3)^k x^{k-1}\mathrm{d}x\\ &=-3\int_0^1\sum_{k=0}^{n-1}(-3x)^k\mathrm{d}x=-3\int_0^1\frac{1-(-3x)^n}{1+3x}\mathrm{d}x\\ &=3\int_0^1\sum_{k=1}^{n}\binom{n}{k}(-1)^k(1+3x)^{k-1}\mathrm{d}x=\int_1^4\sum_{k=1}^{n}\binom{n}{k}(-1)^k y^{k-1}\mathrm{d}y\\ &=\sum_{k=1}^{n}\binom{n}{k}(-1)^k\frac{4^k-1}{k}\end{aligned}$$

and (the following identity can be found in [2])

$$\begin{aligned}\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^k}{k}&=\int_0^1\sum_{k=1}^{n}\binom{n}{k}(-1)^k x^{k-1}\mathrm{d}x=\int_0^1\frac{(1-x)^n-1}{x}\mathrm{d}x=\int_0^1\frac{y^n-1}{1-y}\mathrm{d}y\\ &=-\int_0^1\sum_{k=0}^{n-1}y^k\mathrm{d}y=-\sum_{k=0}^{n-1}\frac{1}{k+1}=-\sum_{k=1}^{n}\frac{1}{k}.\end{aligned}$$

These yield that

$$\sum_{k=1}^{n}\frac{(\lambda+1)^k}{k}=\sum_{k=1}^{n}\frac{(-3)^k}{k}=\sum_{k=1}^{n}\binom{n}{k}\frac{(-4)^k}{k}+H_n.$$

Replacing n by m in the above equation, we have

$$\sum_{k=1}^{m}\frac{(\lambda+1)^k-1}{k}=\sum_{j=1}^{m}\binom{m}{j}\frac{(-4)^j}{j}=(-4)^m\sum_{j=0}^{m-1}\binom{m}{j}\frac{1}{(m-j)(-4)^j}.$$

Hence

$$\sum_{k=1}^{m}\frac{(\lambda+1)^k}{k}=(-4)^m\sum_{j=0}^{m-1}\frac{\binom{m}{j}}{(m-j)(-4)^j}+H_m.$$

So

$$\sum_{k=m+1}^{n}\frac{(\lambda+1)^k}{k}=\sum_{k=1}^{n}\binom{n}{k}\frac{(-4)^k}{k}+H_n-\lambda^m\sum_{j=0}^{m-1}\frac{\binom{m}{j}}{(m-j)(-4)^j}-H_m. \tag{2.9}$$

In view of [16, (1.20)], and by (2.2)–(2.4) we have

$$p^{a-1}\sum_{k=1}^{n}\binom{n}{k}\frac{(-4)^k}{k}\equiv p^{a-1}\sum_{k=1}^{n}\frac{\binom{2k}{k}}{k}\equiv p^{a-1}\sum_{k=1}^{p^a-1}\frac{\binom{2k}{k}}{k}\equiv 0 \pmod p. \tag{2.10}$$

It is obvious that

$$p^{a-1}H_n=p^{a-1}\sum_{k=1}^{n}\frac{1}{k}\equiv p^{a-1}\sum_{j=1}^{\frac{p-1}{2}}\frac{1}{jp^{a-1}}=H_{\frac{p-1}{2}} \pmod p$$

and $p^{a-1}H_m\equiv H_{\lfloor\frac{p}{6}\rfloor} \pmod p$, $p^{a-1}H_{n-m}\equiv H_{\lfloor\frac{p}{3}\rfloor} \pmod p$.

Since $p \equiv 1 \pmod 3$, by [8, Lemma 17(2)], we have

$$\binom{n}{m} \not\equiv 0 \pmod p.$$

These, with (2.7)–(2.10) yield that we only need to prove the following congruence:

$$p^{a-1} \sum_{\substack{j=0 \\ j \neq m}}^{n} \frac{\binom{n}{j}}{(j-m)(-4)^j} \equiv 0 \pmod p. \tag{2.11}$$

Now $n = \frac{p^a-1}{2}, m = \frac{p^a-1}{6}$, so, by Fermat's little theorem we have

$$p^{a-1} \sum_{\substack{j=0 \\ j \neq m}}^{n} \frac{\binom{n}{j}}{(j-m)(-4)^j} \equiv -3(-1)^{\frac{p^a-1}{2}} p^{a-1} \sum_{\substack{j=0 \\ j \neq n-m}}^{n} \frac{\binom{n}{j}(-4)^j}{3j+1} \pmod p.$$

There are only the items $3j+1 = p^{a-1}(3k+1)$ with $k = 0, 1, \cdots, \frac{p-1}{2}$ and $k \neq \frac{p-1}{3}$, so, by [9, Theorem 1.2] and Lucas congruence, we have

$$\begin{aligned} p^{a-1} \sum_{\substack{j=0 \\ j \neq m}}^{n} \frac{\binom{n}{j}}{(j-m)(-4)^j} &\equiv -3(-1)^{\frac{p^a-1}{2}} \sum_{\substack{k=0 \\ k \neq \frac{p-1}{3}}}^{\frac{p-1}{2}} \frac{\binom{n}{kp^{a-1}+\frac{p^{a-1}-1}{3}}(-4)^{kp^{a-1}+\frac{p^{a-1}-1}{3}}}{3k+1} \\ &\equiv -3(-1)^{\frac{p^a-1}{2}} (-4)^{\frac{p^{a-1}-1}{3}} \binom{\frac{p-1}{2}}{k} \binom{\frac{p^{a-1}-1}{2}}{\frac{p^{a-1}-1}{3}} \sum_{\substack{k=0 \\ k \neq \frac{p-1}{3}}}^{\frac{p-1}{2}} \frac{\binom{\frac{p-1}{2}}{k}(-4)^k}{3k+1} \equiv 0 \pmod p. \end{aligned}$$

Therefore the proof of Theorem 1.1 is complete.

Acknowledgement The author would like to thank the anonymous referees for helpful comments.

References

[1] Apagodu, M. and Zeilberger, D., Using the "freshman's dream" to prove combinatorial congruences, *Amer. Math. Monthly,* **124**, 2017, 597–608.

[2] Gould, H. W., Combinatorial Identities, Morgantown Printing and Binding Co., Morgantown, 1972.

[3] Guo, V. J. W., Proof of a supercongruence conjectured by Z.-H. Sun, *Integral Transforms Spec. Funct.,* **25**, 2014, 1009–1015.

[4] Guo, V. J. W. and Liu, J.-C., Some congruences related to a congruence of Van Hamme, *Integral Transforms Spec. Funct.,* **31**, 2020, 221–231.

[5] Lehmer, E., On congruences involving Bernoulli numbers and the quotients of Fermat and Wilson, *Ann. Math.,* **39**, 1938, 350–360.

[6] Liu, J.-C., Congruences for truncated hypergeometric series ${}_2F_1$, *Bull. Aust. Math. Soc.,* **96**, 2017, 14–23.

[7] Liu, J.-C. and Petrov, F., Congruences on sums of q-binomial coefficients, *Adv. Appl. Math.,* **116**, 2020, 102003.

[8] Long, L. and Ramakrishna, R., Some supercongruences occurring in truncated hypergeometric series, *Adv. Math.,* **290**, 2016, 773–808.

[9] Mao, G.-S., Proof of a conjecture of Adamchuk, *J. Combin. Theory, Ser. A,* **182**, 2021, Art. 105478.

[10] Mao, G.-S. and Cao, Z. J., On two congruence conjectures, *C. R. Acad. Sci. Paris, Ser. I,* **357**, 2019, 815–822.

[11] Mao, G.-S. and Sun, Z.-W., New congruences involving products of two binomial coefficients, *Ramanujan J.,* **49**(2), 2019, 237–256.

[12] Mao, G.-S. and Wang, J., On some congruences invloving Domb numbers and harmonic numbers, *Int. J. Number Theory,* **15**, 2019, 2179–2200.

[13] Pan, H. and Sun, Z.-W., Proof of three conjectures of congruences, *Sci. China Math.,* **57**(10), 2014, 2091–2102.

[14] Schneider, C., Symbolic summation assists combinatorics, *Sém. Lothar. Combin.,* **56**, 2007, B56b, 36pp.

[15] Sun, Z.-W., Fibonacci numbers modulo cubes of primes, *Taiwanese J. Math.,* **17**(5), 2013, 1523–1543.

[16] Sun, Z.-W. and Tauraso, R., New congruences for central binomial coefficients, *Adv. Appl. Math.,* **45**, 2010, 125–148.

[17] Sun, Z.-W. and Tauraso, R., On some new congruences for binomial coefficients, *Int. J. Number Theory,* **7**, 2011, 645–662.

[18] Sury, B., Wang, T.-M. and Zhao, F.-Z., Identities involving reciprocals of binomial coefficients, *J. Integer Seq.,* **7**, 2004, Art. 04.2.8.

[19] Wang, C. and Sun, Z.-W., Divisibility results on Franel numbers and related polynomials, *Int. J. Number Theory,* **15**(2), 2019, 433–444.