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Critical Trace Trudinger-Moser Inequalities on a Compact Riemann Surface with Smooth Boundary*

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Abstract In this paper, the author concerns two trace Trudinger-Moser inequalities and obtains the corresponding extremal functions on a compact Riemann surface (Σ, g) with smooth boundary $\partial\Sigma$. Explicitly, let

$$\lambda_1(\partial\Sigma) = \inf_{u\in W^{1,2}(\Sigma,g),\int_{\partial\Sigma} u\mathrm{d}s_g=0,u\not\equiv 0} \frac{\int_\Sigma(|\nabla_g u|^2+u^2)\mathrm{d}v_g}{\int_{\partial\Sigma} u^2\,\mathrm{d}s_g}$$

and

$$\mathcal{H} = \Big\{u\in W^{1,2}(\Sigma,g): \int_\Sigma(|\nabla_g u|^2+u^2)\mathrm{d}v_g - \alpha\int_{\partial\Sigma} u^2\mathrm{d}s_g \le 1 \quad\text{and}\quad \int_{\partial\Sigma} u\,\mathrm{d}s_g = 0\Big\},$$

where $W^{1,2}(\Sigma, g)$ denotes the usual Sobolev space and ∇_g stands for the gradient operator. By the method of blow-up analysis, we obtain

$$\sup_{u\in\mathcal{H}}\int_{\partial\Sigma} \mathrm{e}^{\pi u^2}\mathrm{d}s_g \begin{cases} < +\infty, & 0\le\alpha<\lambda_1(\partial\Sigma),\\ = +\infty, & \alpha\ge\lambda_1(\partial\Sigma).\end{cases}$$

Moreover, the author proves the above supremum is attained by a function $u_\alpha \in \mathcal{H}\cap C^\infty(\overline{\Sigma})$ for any $0 \le \alpha < \lambda_1(\partial\Sigma)$. Further, he extends the result to the case of higher order eigenvalues. The results generalize those of [Li, Y. and Liu, P., Moser-Trudinger inequality on the boundary of compact Riemannian surface, *Math. Z.*, **250**, 2005, 363–386], [Yang, Y., Moser-Trudinger trace inequalities on a compact Riemannian surface with boundary, *Pacific J. Math.*, **227**, 2006, 177–200] and [Yang, Y., Extremal functions for Trudinger-Moser inequalities of Adimurthi-Druet type in dimension two, *J. Diff. Eq.*, **258**, 2015, 3161–3193].

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^2$ be a smooth bounded domain and $W_0^{1,2}(\Omega)$ be the completion of $C_0^\infty(\Omega)$ under the Sobolev norm $\|\nabla_{\mathbb{R}^2} u\|_2^2 = \int_\Omega |\nabla_{\mathbb{R}^2} u|^2\mathrm{d}x$, where $\nabla_{\mathbb{R}^2}$ is the gradient operator on $\mathbb{R}^2$ and $\|\cdot\|_2$

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denotes the standard L^2-norm. The classical Trudinger-Moser inequality (see [20, 24–25, 27, 32]), as the limit case of the Sobolev embedding, says

$$\sup_{u\in W_0^{1,2}(\Omega),\,\|\nabla_{\mathbb{R}^2}u\|_2\le 1}\int_\Omega \mathrm{e}^{\beta u^2}\mathrm{d}x<+\infty,\quad \forall\,\beta\le 4\pi. \tag{1.1}$$

Moreover, 4π is the best constant for this inequality in the sense that when $\beta>4\pi$, all integrals in (1.1) are finite and the supremum is infinite. It is interesting to know whether or not the supremum in (1.1) can be attained. For this topic, see Carleson-Chang [5], Flucher [11], Lin [17], Adimurthi-Struwe [2], Li [13–14], Zhu [38], Tintarev [26], Zhang [33–34, 37] and the references therein.

Trudinger-Moser inequalities were studied on Riemann manifolds by Aubin [3], Cherrier [6], Fontana [12] and others. In particular, let (Σ,g) be a compact Riemann surface with smooth boundary $\partial\Sigma$ and $W^{1,2}(\Sigma,g)$ be the completion of $C^\infty(\Sigma)$ under the norm

$$\|u\|^2_{W^{1,2}(\Sigma,g)}=\int_\Sigma(|\nabla_g u|^2+u^2)\,\mathrm{d}v_g,$$

where ∇_g and v_g stand for the gradient operator and the volume element on Σ with respect to the metric g, respectively. Liu [18] derived a trace Trudinger-Moser inequality in his doctoral thesis from the result of Osgood-Phillips-Sarnak [23]: For all functions $u\in W^{1,2}(\Sigma,g)$, there holds some constant C depending only on $(\Sigma,\ g)$ such that

$$\ln\int_{\partial\Sigma}\mathrm{e}^u\mathrm{d}s_g\le\frac{1}{4\pi}\int_\Sigma|\nabla_g u|^2\mathrm{d}v_g+\int_{\partial\Sigma}u\mathrm{d}s_g+C, \tag{1.2}$$

where s_g denotes the induced length element on $\partial\Sigma$ with respect to g. Later Li-Liu [15] obtained a strong version of (1.2), namely

$$\sup_{u\in W^{1,2}(\Sigma,g),\ \int_\Sigma|\nabla_g u|^2\mathrm{d}v_g=1,\int_{\partial\Sigma}u\mathrm{d}s_g=0}\int_{\partial\Sigma}\mathrm{e}^{\gamma u^2}\mathrm{d}s_g<+\infty \tag{1.3}$$

for any $\gamma\le\pi$. This inequality is sharp in the sense that all integrals in (1.3) are finite when $\gamma>\pi$ and the supremum is infinite. Moreover, for any $\gamma\le\pi$, the supremum is attained. After that, Yang [28] established the boundary estimate without direct boundary conditions, which is

$$\sup_{u\in W^{1,2}(\Sigma,g),\ \int_\Sigma(|\nabla_g u|^2+u^2)\mathrm{d}v_g=1}\int_{\partial\Sigma}\mathrm{e}^{\pi u^2}\mathrm{d}s_g<+\infty. \tag{1.4}$$

Moreover, the supremum in (1.4) can be attained.

A different form was also derived by Yang [30], namely

$$\sup_{u\in W^{1,2}(\Sigma,g),\ \int_\Sigma(|\nabla_g u|^2-\alpha u^2)\mathrm{d}v_g\le 1,\ \int_\Sigma u\mathrm{d}v_g=0}\int_\Sigma\mathrm{e}^{4\pi u^2}\mathrm{d}v_g<+\infty \tag{1.5}$$

for all $0\le\alpha<\lambda_1(\Sigma)$, where

$$\lambda_1(\Sigma)=\inf_{u\in W^{1,2}(\Sigma,g),\int_\Sigma u\mathrm{d}v_g=0,u\not\equiv 0}\frac{\int_\Sigma|\nabla_g u|^2\mathrm{d}v_g}{\int_\Sigma u^2\mathrm{d}v_g}$$

is the first eigenvalue of the Laplace-Beltrami operator Δ_g. Further, he extended (1.5) to the case of higher order eigenvalues. Precisely, let $\lambda_1(\Sigma)<\lambda_2(\Sigma)<\cdots$ be all distinct eigenvalues of the Laplace-Beltrami operator and $E_{\lambda_k(\Sigma)} = \{u \in W^{1,2}(\Sigma, g) : \Delta_g u = \lambda_k(\Sigma)u\},\ k = 1, 2, \cdots$ be associated eigenfunction spaces. For any positive integer k, we set

$$E_k(\Sigma) = E_{\lambda_1(\Sigma)} \oplus E_{\lambda_2(\Sigma)} \oplus \cdots \oplus E_{\lambda_k(\Sigma)}, \quad k = 1, 2, \cdots$$

and

$$E_k^{\perp}(\Sigma) = \Big\{u \in W^{1,2}(\Sigma, g) : \int_\Sigma uv\mathrm{d}v_g = 0,\ \forall v \in E_k(\Sigma)\Big\}, \quad k = 1, 2, \cdots .$$

Then we have

$$\sup_{u\in E_k^{\perp}(\Sigma),\ \int_\Sigma(|\nabla_g u|^2-\alpha u^2)\mathrm{d}v_g\le 1,\ \int_\Sigma u\mathrm{d}v_g=0} \int_\Sigma \mathrm{e}^{4\pi u^2}\mathrm{d}v_g < +\infty \tag{1.6}$$

for all $0 \le \alpha < \lambda_k(\Sigma)$; moreover the above supremum can be attained by some function $u_\alpha \in E_k^{\perp}(\Sigma)$.

In this paper, we will establish two new trace Trudinger-Moser inequalities, which are extensions of (1.5) and (1.6), respectively. Precisely we first have the following theorem.

Theorem 1.1 *Let (Σ, g) be a compact Riemann surface with smooth boundary $\partial\Sigma$, $\lambda_1(\partial\Sigma)$ be defined as*

$$\lambda_1(\partial\Sigma) = \inf_{u\in W^{1,2}(\Sigma,g),\int_{\partial\Sigma} u\mathrm{d}s_g=0,u\not\equiv 0} \frac{\int_\Sigma(|\nabla_g u|^2 + u^2)\mathrm{d}v_g}{\int_{\partial\Sigma} u^2\,\mathrm{d}s_g} \tag{1.7}$$

and

$$\mathcal{H} = \Big\{u \in W^{1,2}(\Sigma, g) : \int_\Sigma(|\nabla_g u|^2 + u^2)\mathrm{d}v_g - \alpha\int_{\partial\Sigma} u^2\mathrm{d}s_g \le 1 \ \text{ and } \int_{\partial\Sigma} u\,\mathrm{d}s_g = 0\Big\}.$$

Then we have

(i) *when $0 \le \alpha < \lambda_1(\partial\Sigma)$, we have*

$$\sup_{u\in\mathcal{H}} \int_{\partial\Sigma} \mathrm{e}^{\pi u^2}\mathrm{d}s_g < +\infty \tag{1.8}$$

and it can be attained by some function $u_\alpha \in \mathcal{H} \cap C^\infty(\overline{\Sigma})$;

(ii) *when $\alpha \ge \lambda_1(\partial\Sigma)$, the above supremum is infinite.*

An interesting consequence of Theorem 1.1 is the following weak form of (1.8).

Theorem 1.2 *Let (Σ, g) be a compact Riemann surface with smooth boundary $\partial\Sigma$, $\lambda_1(\partial\Sigma)$ be defined as in (1.7). Then for any $0 \le \alpha < \lambda_1(\partial\Sigma)$, $u \in W^{1,2}(\Sigma, g)$ and $\int_{\partial\Sigma} u\,\mathrm{d}s_g = 0$, there exists a constant $C > 0$ depending only on α and (Σ, g), such that*

$$4\pi \ln \int_{\partial\Sigma} \mathrm{e}^u\mathrm{d}s_g \le \int_\Sigma(|\nabla_g u|^2 + u^2)\mathrm{d}v_g - \alpha\int_{\partial\Sigma} u^2\mathrm{d}s_g + C.$$

Moreover, we extend Theorem 1.1 to the case of higher order eigenvalues. Let us introduce some notations. For any positive integer k, we set

$$E_{\lambda_k(\partial\Sigma)} = \Big\{u \in W^{1,2}(\Sigma, g) : \Delta_g u + u = 0 \text{ in } (\Sigma, g) \text{ and } \frac{\partial u}{\partial \mathbf{n}} = \lambda_k(\partial\Sigma)u \text{ on } \partial\Sigma\Big\},$$

where $\mathbf{n}$ denotes the outward unit normal vector on $\partial\Sigma$ and

$$E^{\perp}_{\lambda_k(\partial\Sigma)} = \Big\{u \in W^{1,2}(\Sigma, g) : \int_{\partial\Sigma} uv\mathrm{d}s_g = 0,\ \forall v \in E_{\lambda_k(\partial\Sigma)}\Big\}.$$

Then we set

$$\lambda_{k+1}(\partial\Sigma) = \inf_{u\in E_k^{\perp}(\partial\Sigma), \int_{\partial\Sigma} u\mathrm{d}s_g=0, u\not\equiv 0} \frac{\int_\Sigma(|\nabla_g u|^2 + u^2)\mathrm{d}v_g}{\int_{\partial\Sigma} u^2\,\mathrm{d}s_g}, \tag{1.9}$$

which is the $(k+1)$-th eigenvalue of Δ_g on $\partial\Sigma$, where

$$E_k(\partial\Sigma) = E_{\lambda_1(\partial\Sigma)} \oplus E_{\lambda_2(\partial\Sigma)} \oplus \cdots \oplus E_{\lambda_k(\partial\Sigma)} \tag{1.10}$$

and

$$E^{\perp}_k(\partial\Sigma) = \Big\{u \in W^{1,2}(\Sigma, g) : \int_{\partial\Sigma} uv\mathrm{d}s_g = 0,\ \forall v \in E_k(\partial\Sigma)\Big\}. \tag{1.11}$$

We note that $W^{1,2}(\Sigma, g) = E_k(\partial\Sigma) \oplus E^{\perp}_k(\partial\Sigma)$. Then a generalization of Theorem 1.1 can be stated as follows.

Theorem 1.3 *Let (Σ, g) be a compact Riemann surface with smooth boundary $\partial\Sigma$ and $\lambda_{k+1}(\partial\Sigma)$ be defined by* (1.9)*. For any $0 \le \alpha < \lambda_{k+1}(\partial\Sigma)$, let*

$$\mathcal{S} = \Big\{u \in E^{\perp}_k(\partial\Sigma) : \int_\Sigma (|\nabla_g u|^2 + u^2)\mathrm{d}v_g - \alpha \int_{\partial\Sigma} u^2\mathrm{d}s_g \le 1 \ \text{ and } \int_{\partial\Sigma} u\,\mathrm{d}s_g = 0\Big\}, \tag{1.12}$$

where $E^{\perp}_k(\partial\Sigma)$ is defined as in (1.11)*. Then the supremum*

$$\sup_{u\in\mathcal{S}} \int_{\partial\Sigma} \mathrm{e}^{\pi u^2}\mathrm{d}s_g \tag{1.13}$$

is attained by some function $u_\alpha \in \mathcal{S} \cap C^\infty(\overline{\Sigma})$.

Clearly Theorems 1.1 and 1.3 extend (1.5) and (1.6) to the trace Trudinger-Moser inequalities, respectively. For theirs proofs, we employ the method of blow-up analysis, which was originally used by Carleson-Chang [5], Ding-Jost-Li-Wang [8], Adimurthi-Struwe [2], Li [13], Liu [18], Li-Liu [15] and Yang [28–29]. This method is now standard. For related works, we refer Adimurthi-Druet [1], do Ó-de Souza [7, 9], Nguyen [21–22], Zhu [39], Fang-Zhang [10], Mancini-Martinazzi [19] and Zhang [35–36].

In the remaining part of this paper, we prove Theorem 1.1 in Section 2 and Theorem 1.3 in Section 3, respectively.

2 The First Eigenvalue Case

In this section, we will prove Theorem 1.1(ii) first, and then we will prove Theorem 1.1(i). Without loss of generality, we do not distinguish sequence and subsequence in the following.

2.1 The case of $\alpha \ge \lambda_1(\partial\Sigma)$

Let $\lambda_1(\partial\Sigma)$ be defined in (1.7). It is easy to know that $\lambda_1(\partial\Sigma)$ is attained by some function $u_0 \in W^{1,2}(\Sigma, g)$ satisfying $\int_{\partial\Sigma} u_0\mathrm{d}s_g = 0$ and $\int_{\partial\Sigma} u_0^2\,\mathrm{d}s_g = 1$. By a direct calculation, we derive that u_0 satisfies the Euler-Lagrange equation

$$\begin{cases} \Delta_g u_0 + u_0 = 0 & \text{in } \Sigma, \\ \dfrac{\partial u_0}{\partial \mathbf{n}} = \lambda_1(\partial\Sigma)u_0 & \text{on } \partial\Sigma, \end{cases} \tag{2.1}$$

where Δ_g denotes the Laplace-Beltrami operator, $\mathbf{n}$ denotes the outward unit normal vector on $\partial\Sigma$. Applying elliptic estimates to (2.1), we have $u_0 \in W^{1,2}(\Sigma, g) \cap C^0(\overline{\Sigma})$. Then we obtain that $\lambda_1(\partial\Sigma)$ can be attained by some function $tu_0 \in W^{1,2}(\Sigma, g) \cap C^0(\overline{\Sigma})$ for any positive integer t.

Since $\alpha \geq \lambda_1(\partial\Sigma)$, we have

$$\int_\Sigma (|\nabla_g(tu_0)|^2 + (tu_0)^2)\mathrm{d}v_g - \alpha \int_{\partial\Sigma} (tu_0)^2 \, \mathrm{d}s_g \leq 0.$$

Then there holds $tu_0 \in \mathcal{H} \cap C^0(\overline{\Sigma})$. In view of $u_0 \not\equiv 0$, $\int_{\partial\Sigma} u_0 \mathrm{d}s_g = 0$ and $u_0 \in C^0(\overline{\Sigma})$, we obtain that there is a point $x_0 \in \partial\Sigma$ with $u_0(x_0) > 0$. Moreover, there exists a neighborhood U of x_0 satisfying $u_0(x) \geq \frac{u_0(x_0)}{2} > 0$ in U. Then we get

$$\int_{\partial\Sigma} \mathrm{e}^{\pi(tu_0)^2} \mathrm{d}s_g \geq \int_{\partial\Sigma\cap \mathrm{U}} \mathrm{e}^{\pi(tu_0)^2} \mathrm{d}s_g \geq \mathrm{e}^{\frac{\pi}{4} u_0^2(x_0)\, t^2} \int_{\partial\Sigma\cap\mathrm{U}} 1 \mathrm{d}s_g.$$

Letting $t \to +\infty$, one has Theorem 1.1(ii).

2.2 The case of $0 \leq \alpha < \lambda_1(\partial\Sigma)$

In this subsection, we will prove Theorem 1.1(i) by four steps: Firstly, we consider the existence of maximizers for subcritical functionals and the corresponding Euler-Lagrange equation; secondly, we deal with the asymptotic behavior of the maximizers through blow-up analysis; thirdly, we deduce an upper bound of the supremum $\sup\limits_{u\in\mathcal{H}} \int_{\partial\Sigma} \mathrm{e}^{\pi u^2} \mathrm{d}s_g$ under the assumption that blow-up occurs; finally, we construct a sequence of functions to show that Theorem 1.1(i) holds.

Step 1 Existence of maximizers for subcritical functionals.

For any $0 \leq \alpha < \lambda_1(\partial\Sigma)$, we let

$$\|u\|_{1,\alpha}^2 = \int_\Sigma (|\nabla_g u|^2 + u^2)\mathrm{d}v_g - \alpha \int_{\partial\Sigma} u^2 \mathrm{d}s_g.$$

We have the following lemma.

Lemma 2.1 *For any* $0 < \varepsilon < \pi$, *the supremum* $\sup\limits_{u\in\mathcal{H}} \int_{\partial\Sigma} \mathrm{e}^{(\pi-\varepsilon)u^2} \mathrm{d}s_g$ *is attained by some function* $u_\varepsilon \in \mathcal{H} \cap C^\infty(\overline{\Sigma})$.

Proof Let $0 < \varepsilon < \pi$ be fixed. By the definition of supremum, we can choose a maximizing sequence $\{u_i\}_{i=1}^\infty$ in $\mathcal{H}$ such that

$$\lim_{i\to\infty} \int_{\partial\Sigma} \mathrm{e}^{(\pi-\varepsilon)u_i^2} \mathrm{d}s_g = \sup_{u\in\mathcal{H}} \int_{\partial\Sigma} \mathrm{e}^{(\pi-\varepsilon)u^2} \mathrm{d}s_g. \tag{2.2}$$

Moreover, u_i converges to some function u_ε weakly in $W^{1,2}(\Sigma, g)$ and strongly in $L^p(\partial\Sigma, g)$ for any $p > 1$. Then we have $\int_{\partial\Sigma} u_\varepsilon \, \mathrm{d}s_g = 0$. According to the definition of weak convergence and the Hölder's inequality, we get $\|\nabla_g u_\varepsilon\|_{L^2(\Sigma)} \leq \lim\limits_{i\to\infty} \|\nabla_g u_i\|_{L^2(\Sigma)}$, which gives $\|u_\varepsilon\|_{1,\alpha}^2 \leq 1$. From Lagrange's mean value theorem, the Hölder's inequality and (1.3), there holds

$$\lim_{i\to\infty} \Big| \int_{\partial\Sigma} \mathrm{e}^{(\pi-\varepsilon)u_i^2} \mathrm{d}s_g - \int_{\partial\Sigma} \mathrm{e}^{(\pi-\varepsilon)u_\varepsilon^2} \mathrm{d}s_g \Big| = 0.$$

In view of (2.2), we have

$$\int_{\partial\Sigma} \mathrm{e}^{(\pi-\varepsilon)u_\varepsilon^2}\mathrm{d}s_g = \sup_{u\in\mathcal{H}}\int_{\partial\Sigma} \mathrm{e}^{(\pi-\varepsilon)u^2}\mathrm{d}s_g. \tag{2.3}$$

Suppose $\|u_\varepsilon\|_{1,\alpha}^2 < 1$, then one gets

$$\int_{\partial\Sigma} \mathrm{e}^{(\pi-\varepsilon)u_\varepsilon^2}\mathrm{d}s_g < \int_{\partial\Sigma} \mathrm{e}^{(\pi-\varepsilon)(\frac{u_\varepsilon}{\|u_\varepsilon\|_{1,\alpha}})^2}\mathrm{d}s_g \le \sup_{u\in\mathcal{H}}\int_{\partial\Sigma} \mathrm{e}^{(\pi-\varepsilon)u^2}\mathrm{d}s_g.$$

This result contradicts with (2.3). Hence $\|u_\varepsilon\|_{1,\alpha}^2 = 1$ holds and $u_\varepsilon \in \mathcal{H}$.

By a direct calculation, we derive that u_ε satisfies the Euler-Lagrange equation

$$\begin{cases} \Delta_g u_\varepsilon + u_\varepsilon = 0 & \text{in } \Sigma, \\ \dfrac{\partial u_\varepsilon}{\partial \mathbf{n}} = \dfrac{1}{\lambda_\varepsilon} u_\varepsilon \mathrm{e}^{(\pi-\varepsilon)u_\varepsilon^2} + \alpha u_\varepsilon - \dfrac{\mu_\varepsilon}{\lambda_\varepsilon} & \text{on } \partial\Sigma, \\ \lambda_\varepsilon = \displaystyle\int_{\partial\Sigma} u_\varepsilon^2 \mathrm{e}^{(\pi-\varepsilon)u_\varepsilon^2}\mathrm{d}s_g, \\ \mu_\varepsilon = \dfrac{1}{\ell(\partial\Sigma)}\Big(\displaystyle\int_{\partial\Sigma} u_\varepsilon \mathrm{e}^{(\pi-\varepsilon)u_\varepsilon^2}\mathrm{d}s_g - \lambda_\varepsilon \int_\Sigma u_\varepsilon \,\mathrm{d}v_g\Big), \end{cases} \tag{2.4}$$

where $\ell(\partial\Sigma)$ denotes the length of $\partial\Sigma$. Applying elliptic estimates to (2.4), we have $u_\varepsilon \in \mathcal{H}\cap C^\infty(\overline{\Sigma})$. Then Lemma 2.1 follows.

Moreover, we have

$$\lim_{\varepsilon\to 0}\int_{\partial\Sigma} \mathrm{e}^{(\pi-\varepsilon)u_\varepsilon^2}\mathrm{d}s_g = \sup_{u\in\mathcal{H}}\int_{\partial\Sigma} \mathrm{e}^{\pi u^2}\mathrm{d}s_g \tag{2.5}$$

from Lebesgue's dominated convergence theorem. It follows from (2.5) and the fact of $\mathrm{e}^t \le 1+t\mathrm{e}^t$ for any $t \ge 0$ that

$$\liminf_{\varepsilon\to 0} \lambda_\varepsilon > 0. \tag{2.6}$$

From (2.6) and $u_\varepsilon \in \mathcal{H}$, one gets

$$\begin{aligned} \Big|\frac{\mu_\varepsilon}{\lambda_\varepsilon}\Big| &\le \frac{1}{\lambda_\varepsilon \ell(\partial\Sigma)}\Big(\int_{\{u\in\partial\Sigma:|u_\varepsilon|\ge 1\}} |u_\varepsilon|\mathrm{e}^{(\pi-\varepsilon)u_\varepsilon^2}\mathrm{d}s_g + \int_{\{u\in\partial\Sigma:|u_\varepsilon|<1\}} |u_\varepsilon|\mathrm{e}^{(\pi-\varepsilon)u_\varepsilon^2}\mathrm{d}s_g\Big) + C \\ &\le \frac{1}{\lambda_\varepsilon \ell(\partial\Sigma)}\Big(\int_{\{u\in\partial\Sigma:|u_\varepsilon|\ge 1\}} u_\varepsilon^2\mathrm{e}^{(\pi-\varepsilon)u_\varepsilon^2}\mathrm{d}s_g + \int_{\{u\in\partial\Sigma:|u_\varepsilon|<1\}} \mathrm{e}^{(\pi-\varepsilon)u_\varepsilon^2}\mathrm{d}s_g\Big) + C \\ &\le \frac{1}{\ell(\partial\Sigma)} + \frac{\mathrm{e}^\pi}{\lambda_\varepsilon} + C \\ &\le C. \end{aligned} \tag{2.7}$$

Step 2 Blow-up analysis.

Let us perform the blow-up analysis. Without loss of generality, we set $c_\varepsilon = |u_\varepsilon(x_\varepsilon)| = \max_{\overline{\Sigma}} |u_\varepsilon|$. If c_ε is bounded, by elliptic estimates, we complete the proof of Theorem 1.1(i). In the following, we assume $\lim_{\varepsilon\to 0} c_\varepsilon = \lim_{\varepsilon\to 0} u_\varepsilon(x_\varepsilon) = +\infty$ and $x_\varepsilon \to p$ as $\varepsilon \to 0$. Applying maximum principle to (2.4), we have $p \in \partial\Sigma$. Then we have the following lemma.

Lemma 2.2 *There hold $u_\varepsilon \rightharpoonup 0$ weakly in $W^{1,2}(\Sigma, g)$ and $u_\varepsilon \to 0$ strongly in $L^2(\partial\Sigma, g)$ as $\varepsilon \to 0$. Furthermore, $|\nabla_g u_\varepsilon|^2 \mathrm{d}v_g \rightharpoonup \delta_p$ in sense of measure, where δ_p is the usual Dirac measure centered at p.*

Proof Since u_ε is bounded in $W^{1,2}(\Sigma, g)$, there exists some function u_0 such that $u_\varepsilon \rightharpoonup u_0$ weakly in $W^{1,2}(\Sigma, g)$ and $u_\varepsilon \to u_0$ strongly in $L^2(\partial\Sigma, g)$ as $\varepsilon \to 0$. Then we have $\int_{\partial\Sigma} u_0 \mathrm{d}s_g = 0$ and $\|u_0\|_{1,\alpha}^2 \leq 1$.

Suppose $u_0 \not\equiv 0$, then one has $\int_\Sigma (|\nabla_g u_0|^2 + u_0^2)\mathrm{d}v_g > 0$ and

$$1 \geq \|u_0\|_{1,\alpha}^2 \geq \Big(1 - \frac{\alpha}{\lambda_1(\partial\Sigma)}\Big) \int_\Sigma (|\nabla_g u_0|^2 + u_0^2)\mathrm{d}v_g > 0.$$

Then we obtain $\|\nabla_g(u_\varepsilon - u_0)\|_2^2 \to 1 - \|u_0\|_{1,\alpha}^2$ as $\varepsilon \to 0$. Letting $\zeta = 1 - \|u_0\|_{1,\alpha}^2$, one has $0 \leq \zeta < 1$. For sufficiently small ε, there holds

$$\|\nabla_g(u_\varepsilon - u_0)\|_2^2 \leq \frac{\zeta + 1}{2} < 1. \tag{2.8}$$

From the Hölder's inequality, (1.3) and (2.8), we get $\mathrm{e}^{(\pi-\varepsilon)u_\varepsilon^2}$ is bounded in $L^q(\partial\Sigma, g)$ for sufficiently small ε. Applying the elliptic estimate to (2.4), one gets that u_ε is uniformly bounded, which contradicts $c_\varepsilon \to +\infty$. That is to say $u_0 \equiv 0$.

Suppose $|\nabla_g u_\varepsilon|^2 \mathrm{d}v_g \rightharpoonup \mu \neq \delta_p$ in sense of measure. Then there exists some positive number $r > 0$ such that $\lim\limits_{\varepsilon \to 0} \int_{B_r(p)} |\nabla_g u_\varepsilon|^2 \mathrm{d}v_g = \eta < 1$, where $B_r(p)$ is a geodesic ball centered at p with radius r. Moreover, we obtain $\int_{B_r(p)} |\nabla_g u_\varepsilon|^2 \mathrm{d}v_g \leq \frac{\eta+1}{2} < 1$ for sufficiently small ε. We choose a cut-off function $\rho \in C_0^1(B_r(p))$, which is equal to 1 in $\overline{B_{\frac{r}{2}}(p)}$ and $\int_{B_r(p)} |\nabla_g(\rho u_\varepsilon)|^2 \mathrm{d}v_g \leq \frac{\eta+3}{4} < 1$ for sufficiently small ε. Hence there holds

$$\begin{aligned}\int_{B_{\frac{r}{2}}(p)\cap\partial\Sigma} \mathrm{e}^{q(\pi-\varepsilon)u_\varepsilon^2} \mathrm{d}s_g &\leq \int_{B_r(p)\cap\partial\Sigma} \mathrm{e}^{q(\pi-\varepsilon)(\rho u_\varepsilon)^2} \mathrm{d}s_g \\ &\leq \int_{B_r(p)\cap\partial\Sigma} \mathrm{e}^{q(\pi-\varepsilon)\frac{\eta+3}{4}\frac{(\rho u_\varepsilon)^2}{\int_{B_r(p)}|\nabla_g(\rho u_\varepsilon)|^2 \mathrm{d}v_g}} \mathrm{d}s_g\end{aligned}$$

for some $q > 1$. In view of (1.3), we obtain $\mathrm{e}^{(\pi-\varepsilon)u_\varepsilon^2}$ is bounded in $L^q(B_{\frac{r}{2}}(p) \cap \partial\Sigma, g)$ for sufficiently small ε. Applying the elliptic estimate to (2.4), we get that u_ε is uniformly bounded in $B_{\frac{r}{4}}(p) \cap \partial\Sigma$, which contradicts $c_\varepsilon \to +\infty$. Therefore, Lemma 2.2 follows.

Now we analyse the asymptotic behavior of u_ε near the concentration point p. Let

$$r_\varepsilon = \frac{\lambda_\varepsilon}{c_\varepsilon^2 \mathrm{e}^{(\pi-\varepsilon)c_\varepsilon^2}}. \tag{2.9}$$

Following [31, Lemma 4], we can take an isothermal coordinate system (U, ϕ) near x_0, such that $\phi(x_0) = 0$, $\phi(U) = \mathbb{B}_r^+$ and $\phi(U \cap \partial\Sigma) = \partial\mathbb{R}_+^2 \cap \mathbb{B}_r$ for some fixed $r > 0$, where $\mathbb{B}_r^+ = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq r^2, x_2 > 0\}$ and $\mathbb{R}_+^2 = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$. In such coordinates, the metric g has the representation $g = \mathrm{e}^{2f}(\mathrm{d}x_1^2 + \mathrm{d}x_2^2)$ and f is a smooth function with $f(0) = 0$. Denote $\overline{u}_\varepsilon = u_\varepsilon \circ \phi^{-1}$, $\overline{x}_\varepsilon = \phi(x_\varepsilon)$ and $U_\varepsilon = \{x \in \mathbb{R}^2 : \overline{x}_\varepsilon + r_\varepsilon x \in \phi(U)\}$. Define two blowing up functions in U_ε,

$$\psi_\varepsilon(x) = \frac{\overline{u}_\varepsilon(\overline{x}_\varepsilon + r_\varepsilon x)}{c_\varepsilon} \tag{2.10}$$

and

$$\varphi_\varepsilon(x) = c_\varepsilon(\overline{u}_\varepsilon(\overline{x}_\varepsilon + r_\varepsilon x) - c_\varepsilon). \tag{2.11}$$

In view of (2.4) and (2.9)–(2.11), for any fixed $R>0$, we obtain

$$\begin{cases}-\Delta_{\mathbb{R}^2}\psi_\varepsilon=\mathrm{e}^{2f(\overline{x}_\varepsilon+r_\varepsilon x)}r_\varepsilon^2\psi_\varepsilon & \text{in } \mathbb{B}_R^+,\\ \dfrac{\partial\psi_\varepsilon}{\partial\mathbf{v}}=-\mathrm{e}^{f(\overline{x}_\varepsilon+r_\varepsilon x)}\Big(c_\varepsilon^{-2}\psi_\varepsilon\mathrm{e}^{(\pi-\varepsilon)(\psi_\varepsilon+1)\varphi_\varepsilon}+\alpha r_\varepsilon\psi_\varepsilon-\dfrac{r_\varepsilon\mu_\varepsilon}{c_\varepsilon\lambda_\varepsilon}\Big) & \text{on } \partial\mathbb{R}_+^2\cap\mathbb{B}_R\end{cases}\tag{2.12}$$

and

$$\begin{cases}-\Delta_{\mathbb{R}^2}\varphi_\varepsilon=\mathrm{e}^{2f(\overline{x}_\varepsilon+r_\varepsilon x)}r_\varepsilon^2(\varphi_\varepsilon+c_\varepsilon^2) & \text{in } \mathbb{B}_R^+,\\ \dfrac{\partial\varphi_\varepsilon}{\partial\mathbf{v}}=-\mathrm{e}^{f(\overline{x}_\varepsilon+r_\varepsilon x)}\Big(\psi_\varepsilon\mathrm{e}^{(\pi-\varepsilon)(\psi_\varepsilon+1)\varphi_\varepsilon}+\alpha c_\varepsilon^2r_\varepsilon\psi_\varepsilon-\dfrac{c_\varepsilon r_\varepsilon\mu_\varepsilon}{\lambda_\varepsilon}\Big) & \text{on } \partial\mathbb{R}_+^2\cap\mathbb{B}_R,\end{cases}\tag{2.13}$$

where $\Delta_{\mathbb{R}^2}$ denotes the Laplace operator on $\mathbb{R}^2$, $\mathbf{v}$ denotes the outward unit normal vector on $\partial\mathbb{R}_+^2$, $\mathbb{B}_r=\{x\in\mathbb{R}^2:\operatorname{dist}(x,0)\le r\}$ and $\mathbb{B}_r^+=\{x=(x_1,x_2)\in\mathbb{B}_r:\ x_2>0\}$ for any $r>0$. Applying elliptic estimates to (2.12), we have $\lim\limits_{\varepsilon\to0}\psi_\varepsilon=\psi$ in $C^1(\mathbb{B}^+_{\frac{R}{2}})$ for any fixed $R>0$ with $\psi(0)=1$. According to (2.4), (2.7) and (2.9), we get $\lim\limits_{\varepsilon\to0}\frac{\partial\psi_\varepsilon}{\partial\mathbf{v}}=0$ on $\partial\mathbb{R}_+^2\cap\mathbb{B}_{\frac{R}{2}}$. Then there holds

$$\lim_{\varepsilon\to0}\psi_\varepsilon=1\quad\text{in } C^1_{\mathrm{loc}}(\overline{\mathbb{R}_+^2}).\tag{2.14}$$

Using the same argument for (2.13) as above, we obtain

$$\lim_{\varepsilon\to0}\varphi_\varepsilon=\varphi\quad\text{in } C^1_{\mathrm{loc}}(\overline{\mathbb{R}_+^2}),\tag{2.15}$$

where φ satisfies

$$\begin{cases}\Delta_{\mathbb{R}^2}\varphi=0 & \text{in } \mathbb{B}_R^+,\\ \dfrac{\partial\varphi}{\partial\mathbf{v}}=-\mathrm{e}^{2\pi\varphi} & \text{on } \partial\mathbb{R}_+^2\cap\mathbb{B}_R,\\ \varphi(0)=\sup\varphi=0.\end{cases}$$

It is not difficult to check that

$$\int_{\partial\mathbb{R}_+^2\cap\mathbb{B}_R}\mathrm{e}^{2\pi\varphi}\mathrm{d}x_1\le\liminf_{\varepsilon\to0}\int_{\partial\Sigma\cap B_{Rr_\varepsilon}(x_\varepsilon)}\frac{1}{\lambda_\varepsilon}u_\varepsilon^2\mathrm{e}^{(\pi-\varepsilon)u_\varepsilon^2}\mathrm{d}s_g\le1$$

for any fixed $R>0$, that is to say $\int_{\partial\mathbb{R}_+^2}\mathrm{e}^{2\pi\varphi}\mathrm{d}x_1\le1$. By a result of Li-Zhu [16], we obtain

$$\varphi(x)=-\frac{1}{2\pi}\ln(\pi^2x_1^2+(1+\pi x_2)^2).\tag{2.16}$$

A direct calculation gives

$$\int_{\partial\mathbb{R}_+^2}\mathrm{e}^{2\pi\varphi}\mathrm{d}x_1=1.\tag{2.17}$$

Next we discuss the convergence behavior of u_ε away from p. Denote $u_{\varepsilon,\beta}=\min\{\beta c_\varepsilon,u_\varepsilon\}\in W^{1,2}(\Sigma,g)$ for any real number $0<\beta<1$. Following [29, Lemma 4.7], we can easily get

$$\lim_{\varepsilon\to0}\|\nabla_gu_{\varepsilon,\beta}\|_2^2=\beta.\tag{2.18}$$

Lemma 2.3 *Letting λ_ε be defined by (2.4), we obtain*

$$\limsup_{\varepsilon\to 0}\int_{\partial\Sigma} \mathrm{e}^{(\pi-\varepsilon)u_\varepsilon^2}\mathrm{d}s_g = \ell(\partial\Sigma) + \lim_{\varepsilon\to 0}\frac{\lambda_\varepsilon}{c_\varepsilon^2} \tag{2.19}$$

and

$$\lim_{\varepsilon\to 0}\frac{\lambda_\varepsilon}{c_\varepsilon^2} = \lim_{R\to+\infty}\lim_{\varepsilon\to 0}\int_{\phi^{-1}(\mathbb{B}_{Rr_\varepsilon}(\overline{x}_\varepsilon))\cap\partial\Sigma} \mathrm{e}^{(\pi-\varepsilon)u_\varepsilon^2}\,\mathrm{d}s_g. \tag{2.20}$$

Proof Recalling (2.4) and (2.18), one gets

$$\begin{aligned}
&\int_{\partial\Sigma}\mathrm{e}^{(\pi-\varepsilon)u_\varepsilon^2}\mathrm{d}s_g - \ell(\partial\Sigma)\\
&= \int_{\{x\in\partial\Sigma:\, u_\varepsilon\le\beta c_\varepsilon\}}(\mathrm{e}^{(\pi-\varepsilon)u_\varepsilon^2}-1)\,\mathrm{d}s_g + \int_{\{x\in\partial\Sigma:\, u_\varepsilon>\beta c_\varepsilon\}}(\mathrm{e}^{(\pi-\varepsilon)u_\varepsilon^2}-1)\,\mathrm{d}s_g\\
&\le \int_{\partial\Sigma}(\mathrm{e}^{(\pi-\varepsilon)u_{\varepsilon,\beta}^2}-1)\,\mathrm{d}s_g + \frac{1}{\beta^2c_\varepsilon^2}\int_{\{x\in\partial\Sigma:\, u_\varepsilon>\beta c_\varepsilon\}}u_\varepsilon^2\mathrm{e}^{(\pi-\varepsilon)u_\varepsilon^2}\,\mathrm{d}s_g\\
&\le \int_{\partial\Sigma}\mathrm{e}^{(\pi-\varepsilon)u_{\varepsilon,\beta}^2}(\pi-\varepsilon)u_\varepsilon^2\,\mathrm{d}s_g + \frac{\lambda_\varepsilon}{\beta^2c_\varepsilon^2}\\
&\le \Big(\int_{\partial\Sigma}\mathrm{e}^{r(\pi-\varepsilon)u_{\varepsilon,\beta}^2}\mathrm{d}s_g\Big)^{\frac1r}\Big(\int_{\partial\Sigma}(\pi-\varepsilon)^s u_\varepsilon^{2s}\,\mathrm{d}s_g\Big)^{\frac1s} + \frac{\lambda_\varepsilon}{\beta^2c_\varepsilon^2}
\end{aligned}$$

for any real number $0<\beta<1$ and some $r,\ s>1$ with $\frac1r+\frac1s=1$. From (1.3) and (2.18), $\mathrm{e}^{(\pi-\varepsilon)u_{\varepsilon,\beta}^2}$ is bounded in $L^r(\partial\Sigma,g)$. Letting $\varepsilon\to 0$ first and then $\beta\to 1$, we obtain

$$\limsup_{\varepsilon\to 0}\int_{\partial\Sigma}\mathrm{e}^{(\pi-\varepsilon)u_\varepsilon^2}\mathrm{d}s_g - \ell(\partial\Sigma) \le \lim_{\varepsilon\to 0}\frac{\lambda_\varepsilon}{c_\varepsilon^2}. \tag{2.21}$$

According to $c_\varepsilon = \max_{\overline{\Sigma}} u_\varepsilon$, (2.4) and Lemma 2.2, we have

$$\int_{\partial\Sigma}\mathrm{e}^{(\pi-\varepsilon)u_\varepsilon^2}\mathrm{d}s_g - \ell(\partial\Sigma) \ge \frac{\lambda_\varepsilon}{c_\varepsilon^2} - \int_{\partial\Sigma}\frac{u_\varepsilon^2}{c_\varepsilon^2}\,\mathrm{d}s_g,$$

that is to say

$$\limsup_{\varepsilon\to 0}\int_{\partial\Sigma}\mathrm{e}^{(\pi-\varepsilon)u_\varepsilon^2}\mathrm{d}s_g - \ell(\partial\Sigma) \ge \lim_{\varepsilon\to 0}\frac{\lambda_\varepsilon}{c_\varepsilon^2}. \tag{2.22}$$

Combining (2.21) with (2.22), one gets (2.19).

Applying (2.4) and (2.9)–(2.11), we have

$$\begin{aligned}
\int_{\phi^{-1}(\mathbb{B}_{Rr_\varepsilon}(\overline{x}_\varepsilon))\cap\partial\Sigma}\mathrm{e}^{(\pi-\varepsilon)u_\varepsilon^2}\mathrm{d}s_g &= \int_{\mathbb{B}_R\cap\partial\mathbb{R}^2_+} r_\varepsilon \mathrm{e}^{(\pi-\varepsilon)c_\varepsilon^2}\mathrm{e}^{(\pi-\varepsilon)(\psi_\varepsilon+1)\varphi_\varepsilon}\mathrm{e}^{f(\overline{x}_\varepsilon+r_\varepsilon x)}\mathrm{d}x_1\\
&= \int_{\mathbb{B}_R\cap\partial\mathbb{R}^2_+}\frac{\lambda_\varepsilon}{c_\varepsilon^2}\mathrm{e}^{(\pi-\varepsilon)(\psi_\varepsilon+1)\,\varphi_\varepsilon}\mathrm{e}^{f(\overline{x}_\varepsilon+r_\varepsilon x)}\mathrm{d}x_1.
\end{aligned}$$

From (2.14)–(2.17), (2.20) holds.

Next we consider the properties of $c_\varepsilon u_\varepsilon$. Combining Lemma 2.3 with [29, Lemma 4.9], we obtain

$$\frac{1}{\lambda_\varepsilon}c_\varepsilon u_\varepsilon \mathrm{e}^{(\pi-\varepsilon)u_\varepsilon^2}\mathrm{d}s_g \rightharpoonup \delta_p. \tag{2.23}$$

Furthermore, one has the follwing lemma.

Lemma 2.4 *There hold*

$$\begin{cases} c_\varepsilon u_\varepsilon \rightharpoonup G & \text{weakly in } W^{1,q}(\Sigma, g),\ \forall 1 < q < 2, \\ c_\varepsilon u_\varepsilon \to G & \text{strongly in } L^2(\partial\Sigma, g), \\ c_\varepsilon u_\varepsilon \to G & \text{in } C^1_{\text{loc}}(\overline{\Sigma}\backslash\{p\}), \end{cases}$$

where G is a Green function satisfying

$$\begin{cases} \Delta_g G + G = \delta_p & \text{in } \overline{\Sigma}, \\ \dfrac{\partial G}{\partial \mathbf{n}} = \alpha G - \dfrac{1}{\ell(\partial\Sigma)} & \text{on } \partial\Sigma\backslash\{p\}, \\ \int_{\partial\Sigma} G \mathrm{d}s_g = 0. \end{cases} \tag{2.24}$$

Proof From (2.4), there hold

$$\begin{cases} \Delta_g(c_\varepsilon u_\varepsilon) + c_\varepsilon u_\varepsilon = 0 & \text{in } \Sigma, \\ \dfrac{\partial(c_\varepsilon u_\varepsilon)}{\partial \mathbf{n}} = \dfrac{1}{\lambda_\varepsilon} c_\varepsilon u_\varepsilon \mathrm{e}^{(\pi-\varepsilon)u_\varepsilon^2} + \alpha c_\varepsilon u_\varepsilon - c_\varepsilon \dfrac{\mu_\varepsilon}{\lambda_\varepsilon} & \text{on } \partial\Sigma, \\ \displaystyle\int_{\partial\Sigma} c_\varepsilon u_\varepsilon \,\mathrm{d}s_g = 0. \end{cases} \tag{2.25}$$

Combining (2.4) with (2.23), we obtain

$$\begin{aligned} \Big|\frac{c_\varepsilon \mu_\varepsilon}{\lambda_\varepsilon}\Big| &= \frac{1}{\ell(\partial\Sigma)} \Big| \int_{\partial\Sigma} \frac{1}{\lambda_\varepsilon} c_\varepsilon u_\varepsilon \mathrm{e}^{(\pi-\varepsilon)u_\varepsilon^2} \mathrm{d}s_g - \int_\Sigma c_\varepsilon u_\varepsilon \mathrm{d}v_g \Big| \\ &\le C + \int_\Sigma |c_\varepsilon u_\varepsilon| \mathrm{d}v_g. \end{aligned} \tag{2.26}$$

Moreover, it follows from the Poincaré inequality that

$$\int_\Sigma |c_\varepsilon u_\varepsilon - \overline{c_\varepsilon u_\varepsilon}| \mathrm{d}v_g \le C||c_\varepsilon u_\varepsilon - \overline{c_\varepsilon u_\varepsilon}||_{L^q(\Sigma)} \le C||\nabla_g(c_\varepsilon u_\varepsilon)||_{L^q(\Sigma)},$$

where $\overline{c_\varepsilon u_\varepsilon} = \frac{\int_\Sigma c_\varepsilon u_\varepsilon \mathrm{d}v_g}{|\Sigma|}$, then we have

$$\int_\Sigma |c_\varepsilon u_\varepsilon| \mathrm{d}v_g \le C||\nabla_g(c_\varepsilon u_\varepsilon)||_{L^q(\Sigma)} + C. \tag{2.27}$$

From the Hölder's inequality and the Sobolev embedding theorem, one gets

$$\int_{\partial\Sigma} |c_\varepsilon u_\varepsilon| \mathrm{d}s_g \le C\|c_\varepsilon u_\varepsilon\|_{L^q(\partial\Sigma)} \le C\|\nabla_g(c_\varepsilon u_\varepsilon)\|_{L^q(\Sigma)} \tag{2.28}$$

for some $q > 1$. It is well known (see for example [15, Proposition 3.5]) that

$$\int_\Sigma |\nabla_g(c_\varepsilon u_\varepsilon)|^q \mathrm{d}v_g \le \sup_{\|\Phi\|_{W^{1,q'}(\Sigma,g)}=1} \int_\Sigma \nabla_g\Phi\nabla_g(c_\varepsilon u_\varepsilon) \mathrm{d}v_g, \tag{2.29}$$

where $\frac{1}{q} + \frac{1}{q'} = 1$. For any $1 < q < 2$, the Sobolev embedding theorem implies that $\|\Phi\|_{C^0(\overline{\Sigma})} \le C$, where C is a constant depending only on (Σ, g). Using (2.23), (2.25)–(2.29) and the divergence theorem, we have

$$\|\nabla_g(c_\varepsilon u_\varepsilon)\|^q_{L^q(\Sigma)} \le \int_\Sigma \nabla_g\Phi\nabla_g(c_\varepsilon u_\varepsilon) \mathrm{d}v_g$$

$$\leq \int_{\partial\Sigma}\Phi\frac{1}{\lambda_\varepsilon}c_\varepsilon u_\varepsilon \mathrm{e}^{(\pi-\varepsilon)u_\varepsilon^2}\mathrm{d}s_g+\alpha\int_{\partial\Sigma}\Phi c_\varepsilon u_\varepsilon \mathrm{d}s_g-c_\varepsilon\frac{\mu_\varepsilon}{\lambda_\varepsilon}\int_{\partial\Sigma}\Phi\mathrm{d}s_g-\int_\Sigma \Phi c_\varepsilon u_\varepsilon \mathrm{d}v_g$$
$$\leq \Phi(p)+C(\alpha+1)\|\nabla_g(c_\varepsilon u_\varepsilon)\|_{L^q(\Sigma)}+C.$$

That is to say $\|\nabla_g(c_\varepsilon u_\varepsilon)\|_{L^q(\Sigma)}\leq C$. The Poincaré inequality implies that $c_\varepsilon u_\varepsilon$ is bounded in $W^{1,q}(\Sigma,g)$ for any $1<q<2$. Hence there exists some function G such that $c_\varepsilon u_\varepsilon \rightharpoonup G$ weakly in $W^{1,q}(\Sigma,g)$ and $c_\varepsilon u_\varepsilon\to G$ strongly in $L^2(\partial\Sigma,g)$ as $\varepsilon\to 0$. By (2.25), we obtain (2.24).

For any fixed $\delta>0$, we choose a cut-off function $\eta\in C^\infty(\overline{\Sigma})$ such that $\eta\equiv 0$ on $\overline{B_\delta(p)}$ and $\eta\equiv 1$ on $\overline{\Sigma}/B_{2\delta}(p)$. Using Lemma 2.2, we have $\lim\limits_{\varepsilon\to 0}\|\nabla_g(\eta u_\varepsilon)\|_2=0$. Hence $\mathrm{e}^{(\pi-\varepsilon)u_\varepsilon^2}$ is bounded in $L^s(\overline{\Sigma}/B_{2\delta}(p))$ for any $s>1$. It follows from (2.25) that $\frac{\partial(c_\varepsilon u_\varepsilon)}{\partial\mathbf{n}}\in L^{s_0}(\overline{\Sigma}\backslash B_{2\delta}(p))$ for some $s_0>2$. Applying the elliptic estimate to (2.25), we get that $c_\varepsilon u_\varepsilon$ is bounded in $C^1(\overline{\Sigma}/B_{4\delta}(p))$. Then there holds $\lim\limits_{\varepsilon\to 0}c_\varepsilon u_\varepsilon=G$ in $C^1_{\mathrm{loc}}(\overline{\Sigma}/\{p\})$. This completes the proof of the lemma.

Applying the elliptic estimate to (2.24), we can decompose G near p,

$$G=-\frac{1}{\pi}\ln r+A_p+O(r), \tag{2.30}$$

where $r=\mathrm{dist}(x,p)$ and A_p is a constant depending only on $\alpha,\ p$ and (Σ,g).

Step 3 Upper bound estimate.

To derive an upper bound of $\sup\limits_{u\in\mathcal{H}}\int_{\partial\Sigma}\mathrm{e}^{\pi u^2}\mathrm{d}s_g$, we use the capacity estimate, which was first used by Li [13] in this topic and also used by Li-Liu [15].

Lemma 2.5 *Under the hypotheses $c_\varepsilon\to+\infty$ and $x_\varepsilon\to p\in\partial\Sigma$ as $\varepsilon\to 0$, there holds*

$$\sup_{u\in\mathcal{H}}\int_{\partial\Sigma}\mathrm{e}^{\pi u^2}\mathrm{d}s_g\leq \ell(\partial\Sigma)+2\pi\mathrm{e}^{\pi A_p}. \tag{2.31}$$

Proof We take an isothermal coordinate system (U,ϕ) near p such that $\phi(p)=0$, ϕ maps U to $\mathbb{R}^2_+$, and $\phi(U\cap\partial\Sigma)\subset\partial\mathbb{R}^2_+$. In such coordinates, the metric g has the representation $g=\mathrm{e}^{2f}(\mathrm{d}x_1^2+\mathrm{d}x_2^2)$ and f is a smooth function with $f(0)=0$. We claim that

$$\lim_{\varepsilon\to 0}\frac{\lambda_\varepsilon}{c_\varepsilon^2}\leq 2\pi\mathrm{e}^{\pi A_p}. \tag{2.32}$$

To confirm this claim, we set $a=\sup\limits_{\partial\mathbb{B}_\delta\cap\mathbb{R}^2_+}\overline{u}_\varepsilon$ and $b=\inf\limits_{\partial\mathbb{B}_{Rr_\varepsilon}\cap\mathbb{R}^2_+}\overline{u}_\varepsilon$ for sufficiently small $\delta>0$ and some fixed $R>0$, where $\overline{u}_\varepsilon=u_\varepsilon\circ\phi^{-1}$. It follows from (2.30) and Lemma 2.4 that on $\partial\mathbb{B}_\delta\cap\mathbb{R}^2_+$, $\overline{u}_\varepsilon=\frac{G+o_\varepsilon(1)}{c_\varepsilon}$, which leads to

$$a=\frac{1}{c_\varepsilon}\Big(\frac{1}{\pi}\ln\frac{1}{\delta}+A_p+o_\delta(1)+o_\varepsilon(1)\Big),$$

where $o_\delta(1)\to 0$, $o_\varepsilon(1)\to 0$ as $\varepsilon\to 0$. According to (2.15)–(2.16), we have on $\partial\mathbb{B}_{Rr_\varepsilon}\cap\mathbb{R}^2_+$,

$$\overline{u}_\varepsilon(x)=c_\varepsilon+\frac{\varphi(x)+o_\varepsilon(1)}{c_\varepsilon},$$

then there holds

$$b=c_\varepsilon+\frac{1}{c_\varepsilon}\Big(-\frac{1}{2\pi}\ln(1+\pi^2R^2)+o_\varepsilon(1)\Big).$$

From a direct computation, there holds

$$\pi(a-b)^2 = \pi c_\varepsilon^2 + 2\ln\delta - 2\pi A_p - \ln(1+\pi^2 R^2) + o_\delta(1) + o_\varepsilon(1). \tag{2.33}$$

Define

$$W_{a,b} = \Big\{ \overline{u} \in W^{1,2}(\mathbb{B}_\delta^+ \setminus \mathbb{B}_{Rr_\varepsilon}^+) :\ \overline{u}|_{\partial\mathbb{B}_\delta \cap \mathbb{R}_+^2} = a,\ \overline{u}|_{\partial\mathbb{B}_{Rr_\varepsilon} \cap \mathbb{R}_+^2} = b,\ \frac{\partial \overline{u}}{\partial \mathbf{v}}\Big|_{\partial\mathbb{R}_+^2 \cap (\mathbb{B}_\delta \setminus \mathbb{B}_{Rr_\varepsilon})} = 0 \Big\}.$$

Suppose that $\inf\limits_{u\in W_{a,b}} \int_{\mathbb{B}_\delta^+ \setminus \mathbb{B}_{Rr_\varepsilon}^+} |\nabla_{\mathbb{R}^2} u|^2 \mathrm{d}x$ can be attained by some function $m(x) \in W_{a,b}$ with $\Delta_{\mathbb{R}^2} m(x) = 0$. We can check that

$$m(x) = \frac{a(\ln|x| - \ln(Rr_\varepsilon)) + b(\ln\delta - \ln|x|)}{\ln\delta - \ln(Rr_\varepsilon)}$$

and

$$\int_{\mathbb{B}_\delta^+ \setminus \mathbb{B}_{Rr_\varepsilon}^+} |\nabla_{\mathbb{R}^2} m(x)|^2 \mathrm{d}x = \frac{\pi(a-b)^2}{\ln\delta - \ln(Rr_\varepsilon)}. \tag{2.34}$$

Recalling (2.4) and (2.9), we have

$$\ln\delta - \ln(Rr_\varepsilon) = \ln\delta - \ln R - \ln\frac{\lambda_\varepsilon}{c_\varepsilon^2} + (\pi - \varepsilon)c_\varepsilon^2. \tag{2.35}$$

Letting $\overline{u}_\varepsilon \in W_{a,b}$ and $u_\varepsilon^* = \max\{a,\ \min\{b,\ \overline{u}_\varepsilon\}\}$, one gets $|\nabla_{\mathbb{R}^2} u_\varepsilon^*| \le |\nabla_{\mathbb{R}^2} \overline{u}_\varepsilon|$ in $\mathbb{B}_\delta^+ \setminus \mathbb{B}_{Rr_\varepsilon}^+$ for sufficiently small ε. Further using $\|u_\varepsilon\|_{1,\alpha}^2 = 1$, we obtain

$$\begin{aligned}\int_{\mathbb{B}_\delta^+ \setminus \mathbb{B}_{Rr_\varepsilon}^+} |\nabla_{\mathbb{R}^2} m(x)|^2 \mathrm{d}x &\le \int_{\mathbb{B}_\delta^+ \setminus \mathbb{B}_{Rr_\varepsilon}^+} |\nabla_{\mathbb{R}^2} u_\varepsilon^*(x)|^2 \mathrm{d}x \\ &\le \Big(1 + \alpha\int_{\partial\Sigma} u_\varepsilon^2 \mathrm{d}s_g - \int_\Sigma u_\varepsilon^2 \mathrm{d}v_g\Big) - \int_{\Sigma\setminus\phi^{-1}(\mathbb{B}_\delta^+)} |\nabla_g u_\varepsilon|^2 \mathrm{d}v_g \\ &\quad - \int_{\phi^{-1}(\mathbb{B}_{Rr_\varepsilon}^+)} |\nabla_g u_\varepsilon|^2 \mathrm{d}v_g. \end{aligned} \tag{2.36}$$

Now we compute $\int_{\Sigma\setminus\phi^{-1}(\mathbb{B}_\delta^+)} |\nabla_g u_\varepsilon|^2 \mathrm{d}v_g$ and $\int_{\phi^{-1}(\mathbb{B}_{Rr_\varepsilon}^+)} |\nabla_g u_\varepsilon|^2 \mathrm{d}v_g$. In view of (2.30), we obtain

$$\int_{\Sigma\setminus\phi^{-1}(\mathbb{B}_\delta^+)} |\nabla_g G|^2 \mathrm{d}v_g = \frac{1}{\pi}\ln\frac{1}{\delta} + A_p + \alpha\|G\|_{L^2(\partial\Sigma)}^2 + o_\varepsilon(1) + o_\delta(1).$$

Hence we have by Lemma 2.4,

$$\int_{\Sigma\setminus\phi^{-1}(\mathbb{B}_\delta^+)} |\nabla_g u_\varepsilon|^2 \mathrm{d}v_g = \frac{1}{c_\varepsilon^2}\Big(\frac{1}{\pi}\ln\frac{1}{\delta} + A_p + \alpha\|G\|_{L^2(\partial\Sigma)}^2 + o_\varepsilon(1) + o_\delta(1)\Big). \tag{2.37}$$

According to (2.11), (2.15) and (2.16), one gets

$$\int_{\phi^{-1}(\mathbb{B}_{Rr_\varepsilon}^+)} |\nabla_g u_\varepsilon|^2 \mathrm{d}v_g = \frac{1}{c_\varepsilon^2}\Big(\frac{1}{\pi}\ln R + \frac{1}{\pi}\ln\frac{\pi}{2} + o_\varepsilon(1) + o_R(1)\Big), \tag{2.38}$$

where $o_R(1) \to 0$ as $R \to +\infty$. In view of (2.33)–(2.38), we obtain

$$\ln\frac{\lambda_\varepsilon}{c_\varepsilon^2} \le \ln(2\pi) + \pi A_p + o(1),$$

where $o(1) \to 0$ as $\varepsilon \to 0$ first, then $R \to +\infty$ and $\delta \to 0$. Hence (2.32) follows. Combining (2.5), (2.32) with Lemma 2.3, we finish the proof of the lemma.

Step 4 Existence result.

In this step, we always assume that $0 \le \alpha < \lambda_1(\partial\Sigma)$. We take an isothermal coordinate system (U, ϕ) near p such that $\phi(p) = 0$, ϕ maps U to $\mathbb{R}^2_+$, and $\phi(U \cap \partial\Sigma) \subset \partial\mathbb{R}^2_+$. In such coordinates, the metric g has the representation $g = \mathrm{e}^{2f}(\mathrm{d}x_1^2 + \mathrm{d}x_2^2)$ and f is a smooth function with $f(0) = 0$. Set a cut-off function $\xi \in C_0^\infty(\phi^{-1}(\mathbb{B}^+_{2R\varepsilon}))$ with $\xi = 1$ on $\phi^{-1}(\mathbb{B}^+_{R\varepsilon})$ and $\|\nabla_g \xi\|_{L^\infty} = O\big(\frac{1}{R\varepsilon}\big)$. Denote $\beta = G + \frac{1}{\pi}\ln r - A_p$ for any $r > 0$, where G is defined by (2.30). Let $R = \ln^2 \varepsilon$, then $R \to +\infty$ and $R\varepsilon \to 0$ as $\varepsilon \to 0$. We construct a blow-up sequence of functions

$$v_\varepsilon = \begin{cases} \Big(c - \dfrac{1}{2\pi c}\ln\dfrac{\pi^2 x_1^2 + (\pi x_2 + \varepsilon)^2}{\varepsilon^2} + \dfrac{B}{c}\Big)\circ\phi, & x \in \phi^{-1}(\mathbb{B}^+_{R\varepsilon}),\\ \dfrac{G - \xi\beta}{c}, & x \in \phi^{-1}(\mathbb{B}^+_{2R\varepsilon}\backslash\mathbb{B}^+_{R\varepsilon}),\\ \dfrac{G}{c}, & x \in \Sigma\backslash\phi^{-1}(\mathbb{B}^+_{2R\varepsilon}) \end{cases} \tag{2.39}$$

for some constants B, c to be determined later, such that

$$\int_\Sigma (|\nabla_g v_\varepsilon|^2 + (v_\varepsilon - \overline{v}_\varepsilon)^2)\mathrm{d}v_g - \alpha\int_{\partial\Sigma}(v_\varepsilon - \overline{v}_\varepsilon)^2 \mathrm{d}s_g = 1 \tag{2.40}$$

and $v_\varepsilon - \overline{v}_\varepsilon \in \mathcal{H}$, where $\overline{v}_\varepsilon = \frac{\int_{\partial\Sigma} v_\varepsilon \mathrm{d}s_g}{\ell(\partial\Sigma)}$.

Note that $\int_{\partial\Sigma} G\mathrm{d}s_g = 0$, one has $\overline{v}_\varepsilon = O(R\varepsilon\ln^2(R\varepsilon))$, and then

$$\begin{cases} \displaystyle\int_{\partial\Sigma} |v_\varepsilon - \overline{v}_\varepsilon|^2 \mathrm{d}s_g = \frac{\|G\|^2_{L^2(\partial\Sigma)}}{c^2} + O(R\varepsilon\ln^2(R\varepsilon)),\\ \displaystyle\int_\Sigma |v_\varepsilon - \overline{v}_\varepsilon|^2 \mathrm{d}v_g = \int_{\Sigma\backslash B_{R\varepsilon}(p)} \frac{G^2}{c^2}\mathrm{d}v_g + O(R\varepsilon\ln^2(R\varepsilon)). \end{cases} \tag{2.41}$$

A delicate calculation shows

$$\int_0^{\arcsin\frac{1}{\pi R}} \ln\sin\theta\mathrm{d}\theta = \int_0^{\frac{1}{\pi R}} \frac{\ln t}{\sqrt{1-t^2}}\mathrm{d}t \le C\int_0^{\frac{1}{\pi R}} \ln t\mathrm{d}t = O\Big(\frac{\ln R}{R}\Big) \tag{2.42}$$

and

$$\begin{aligned} &\int_{\phi^{-1}(\mathbb{B}^+_{R\varepsilon})} |\nabla_g v_\varepsilon|^2 \mathrm{d}v_g\\ &= \frac{1}{4\pi^2 c^2}\int_{Q(R)} |\nabla_{\mathbb{R}^2}\ln(\pi^2 x_1^2 + \pi^2 x_2^2)|^2 \mathrm{d}x_1\mathrm{d}x_2\\ &= \frac{1}{4\pi^2 c^2}\int_{\arcsin\frac{1}{\pi R}}^{\pi - \arcsin\frac{1}{\pi R}} \mathrm{d}\theta\int_{\frac{1}{\pi\sin\theta}}^{R} |\nabla_{\mathbb{R}^2}\ln(\pi^2 R^2)|^2 r\mathrm{d}r\\ &= \frac{1}{\pi^2 c^2}\Big(\pi\ln(\pi R) + \int_0^\pi \ln\sin\theta\ \mathrm{d}\theta - 2\int_0^{\arcsin\frac{1}{\pi R}} \ln\sin\theta\ \mathrm{d}\theta + O\Big(\frac{\ln R}{R}\Big)\Big)\\ &= \frac{1}{\pi c^2}\Big(\ln R + \ln\frac{\pi}{2} + O\Big(\frac{\ln R}{R}\Big)\Big), \end{aligned} \tag{2.43}$$

where $Q(R)=\{(x_1,x_2): x_1^2+x_2^2\le R^2,\ x_2\ge \frac{1}{\pi}\}$. According to (2.37) and (2.39), one has

$$\begin{aligned}&\int_{\Sigma\setminus\phi^{-1}(\mathbb{B}^+_{R\varepsilon})}|\nabla_g v_\varepsilon|^2\mathrm{d}v_g\\ &=\frac{1}{c^2}\Big(A_p+\alpha\|G\|^2_{L^2(\partial\Sigma)}-\frac{1}{\pi}\ln(R\varepsilon)-\int_{\Sigma\setminus B_{R\varepsilon}(p)}G^2\mathrm{d}v_g+O(R\varepsilon\ln^2(R\varepsilon))\Big).\end{aligned}\tag{2.44}$$

In view of (2.40)–(2.44), there holds

$$c^2=A_p+\frac{1}{\pi}\ln\Big(\frac{\pi}{2\varepsilon}\Big)+O\Big(\frac{\ln R}{R}\Big)+O(R\varepsilon\ln^2(R\varepsilon)).\tag{2.45}$$

Moreover, in order to assure that $v_\varepsilon\in W^{1,2}(\Sigma,g)$, we obtain

$$c^2-\frac{1}{2\pi}\ln(\pi^2R^2)+B+O\Big(\frac{1}{R}\Big)=-\frac{1}{\pi}\ln(R\varepsilon)+A_p,$$

which is equivalent to

$$c^2=\frac{1}{\pi}\ln\pi-\frac{1}{\pi}\ln\varepsilon-B+A_p+O\Big(\frac{1}{R}\Big).\tag{2.46}$$

According to (2.45)–(2.46), one gets

$$B=\frac{1}{\pi}\ln 2+O\Big(\frac{\ln R}{R}\Big)+O(R\varepsilon\ln^2(R\varepsilon)).$$

It follows that in $\partial\Sigma\cap B_{R\varepsilon}(p)$,

$$\pi(v_\varepsilon-\overline{v}_\varepsilon)^2\ge\ln(2\pi)+\pi A_p-\ln\Big(\frac{\varepsilon^2+\pi^2x_1^2}{\varepsilon}\Big)+O\Big(\frac{\ln R}{R}\Big)+O(R\varepsilon\ln^2(R\varepsilon)).$$

Hence

$$\int_{\partial\Sigma\cap\phi^{-1}(\mathbb{B}^+_{R\varepsilon})}\mathrm{e}^{\pi(v_\varepsilon-\overline{v}_\varepsilon)^2}\,\mathrm{d}s_g\ge 2\pi\mathrm{e}^{\pi A_p}+O\Big(\frac{\ln R}{R}\Big)+O(R\varepsilon\ln^2(R\varepsilon)).\tag{2.47}$$

On the other hand, from the fact $\mathrm{e}^t\ge t+1$ for any $t>0$ and (2.39), we get

$$\begin{aligned}\int_{\partial\Sigma\setminus\phi^{-1}(\mathbb{B}^+_{R\varepsilon})}\mathrm{e}^{\pi(v_\varepsilon-\overline{v}_\varepsilon)^2}\,\mathrm{d}s_g&\ge\int_{\partial\Sigma\setminus\phi^{-1}(\mathbb{B}^+_{R\varepsilon})}(1+\pi(v_\varepsilon-\overline{v}_\varepsilon)^2)\,\mathrm{d}s_g\\ &\ge\ell(\partial\Sigma)+\frac{\pi\|G\|^2_{L^2(\partial\Sigma)}}{c^2}+O(R\varepsilon\ln^2(R\varepsilon)).\end{aligned}\tag{2.48}$$

From (2.47)–(2.48) and $R=\ln^2\varepsilon$, there holds

$$\int_{\partial\Sigma}\mathrm{e}^{\pi(v_\varepsilon-\overline{v}_\varepsilon)^2}\,\mathrm{d}s_g>\ell(\partial\Sigma)+2\pi\mathrm{e}^{\pi A_p}\tag{2.49}$$

for sufficiently small $\varepsilon>0$. The contradiction between (2.31) and (2.49) indicates that c_ε must be bounded. Then Theorem 1.1 follows.

3 Higher Order Eigenvalue Cases

In this section, we will prove Theorem 1.3 involving higher order eigenvalues through blow-up analysis. Let k be a positive integer and $E_k(\partial\Sigma)$ be defined by (1.10). Denote the dimension of $E_k(\partial\Sigma)$ by s_k. From [4, Theorem 9.31], it is known that s_k is a finite constant only depending on k. Then we can find a set of normal orthogonal basis $\{e_i\}_{i=1}^{s_k} \in C^\infty(\overline{\Sigma})$ of $E_k(\partial\Sigma)$ satisfying

$$\begin{cases} \displaystyle\int_{\partial\Sigma} e_i \mathrm{d}s_g = 0, \\ \Delta_g e_i + e_i = 0 & \text{in } \Sigma, \\ \dfrac{\partial e_i}{\partial \mathbf{n}} = \lambda_{k_0}(\partial\Sigma) e_i & \text{on } \partial\Sigma, \end{cases} \tag{3.1}$$

where $k_0 \le k$ is a positive integer.

3.1 Blow-up analysis

Let $\lambda_{k+1}(\partial\Sigma)$ and $\mathcal{S}$ be defined by (1.9) and (1.12), respectively. In view of Lemma 2.1 and (3.1), we have the following lemma.

Lemma 3.1 *Let $0 \le \alpha < \lambda_{k+1}(\partial\Sigma)$ be fixed. For any $0 < \varepsilon < \pi$, the supremum*

$$\sup_{u \in \mathcal{S}} \int_{\partial\Sigma} \mathrm{e}^{(\pi-\varepsilon)u^2} \mathrm{d}s_g$$

is attained by some function $u_\varepsilon \in \mathcal{S} \cap C^\infty(\overline{\Sigma})$. Moreover, the Euler-Lagrange equation of u_ε is

$$\begin{cases} \Delta_g u_\varepsilon + u_\varepsilon = 0 & \text{in } \Sigma, \\ \dfrac{\partial u_\varepsilon}{\partial \mathbf{n}} = \dfrac{1}{\lambda_\varepsilon} u_\varepsilon \mathrm{e}^{(\pi-\varepsilon)u_\varepsilon^2} + \alpha u_\varepsilon - \dfrac{\mu_\varepsilon}{\lambda_\varepsilon} - \displaystyle\sum_{i=1}^{s_k} \dfrac{\beta_{\varepsilon,i}}{\lambda_\varepsilon} e_i & \text{on } \partial\Sigma, \\ \lambda_\varepsilon = \displaystyle\int_{\partial\Sigma} u_\varepsilon^2 \mathrm{e}^{(\pi-\varepsilon)u_\varepsilon^2} \mathrm{d}s_g, \\ \mu_\varepsilon = \dfrac{1}{\ell(\partial\Sigma)} \Big(\displaystyle\int_{\partial\Sigma} u_\varepsilon \mathrm{e}^{(\pi-\varepsilon)u_\varepsilon^2} \mathrm{d}s_g - \lambda_\varepsilon \int_\Sigma u_\varepsilon \mathrm{d}v_g \Big), \\ \beta_{\varepsilon,i} = \displaystyle\int_{\partial\Sigma} u_\varepsilon \mathrm{e}^{(\pi-\varepsilon)u_\varepsilon^2} e_i \mathrm{d}s_g. \end{cases} \tag{3.2}$$

Without loss of generality, we set $c_\varepsilon = |u_\varepsilon(x_\varepsilon)| = \max_{\overline{\Sigma}} |u_\varepsilon|$. We first assume that c_ε is bounded, which together with elliptic estimates completes the proof of Theorem 1.3. In the remainder of Section 3, we assume

$$\lim_{\varepsilon \to 0} c_\varepsilon = \lim_{\varepsilon \to 0} u_\varepsilon(x_\varepsilon) = +\infty$$

and $x_\varepsilon \to p$ as $\varepsilon \to 0$. Applying maximum principle to (3.2), we have $p \in \partial\Sigma$. Similar to Lemma 4, we get the following lemma.

Lemma 3.2 *There hold $c_\varepsilon u_\varepsilon \rightharpoonup G$ weakly in $W^{1,q}(\Sigma, g)$ $(\forall 1 < q < 2)$, $c_\varepsilon u_\varepsilon \to G$ strongly in $L^2(\partial\Sigma, g)$ and $c_\varepsilon u_\varepsilon \to G$ in $C^1_{\mathrm{loc}}(\overline{\Sigma} \backslash \{p\})$ as $\varepsilon \to 0$, where G is a Green function satisfying*

$$\begin{cases} \Delta_g G + G = \delta_p & \text{in } \overline{\Sigma}, \\ \dfrac{\partial G}{\partial \mathbf{n}} = \alpha G - \dfrac{1}{\ell(\partial\Sigma)} - \displaystyle\sum_{i=1}^{s_k} e_i e_i(p) & \text{on } \partial\Sigma \backslash \{p\}, \\ \displaystyle\int_{\partial\Sigma} G \mathrm{d}s_g = 0. \end{cases}$$

Moreover, G near p can be decomposed into

$$G = -\frac{1}{\pi}\ln r + A_p + O(r), \tag{3.3}$$

where $r = \mathrm{dist}(x,p)$ and A_p is a constant depending only on α, p and (Σ, g). Analogous to Lemma 2.5, using the capacity estimate, we derive an upper bound of the supremum (1.13):

$$\sup_{u\in\mathcal{S}}\int_{\partial\Sigma} \mathrm{e}^{\pi u^2}\mathrm{d}s_g \le \ell(\partial\Sigma) + 2\pi\mathrm{e}^{\pi A_p}. \tag{3.4}$$

3.2 Existence result

We always assume that $c_\varepsilon \to +\infty$ as $\varepsilon \to 0$. Take an isothermal coordinate system (U,ϕ) near p such that $\phi(p) = 0$, ϕ maps U to $\mathbb{R}^2_+$, and $\phi(U\cap\partial\Sigma) \subset \partial\mathbb{R}^2_+$. In such coordinates, the metric g has the representation $g = \mathrm{e}^{2f}(\mathrm{d}x_1^2 + \mathrm{d}x_2^2)$ and f is a smooth function with $f(0) = 0$. Set a cut-off function $\xi \in C_0^\infty(\phi^{-1}(\mathbb{B}^+_{2R\varepsilon}))$ with $\xi = 1$ on $\phi^{-1}(\mathbb{B}^+_{R\varepsilon})$ and $\|\nabla_g\xi\|_{L^\infty} = O\big(\frac{1}{R\varepsilon}\big)$. Denote $\beta = G + \frac{1}{\pi}\ln r - A_p$, where G is defined by (3.3). Let $R = \ln^2\varepsilon$, then $R \to +\infty$ and $R\varepsilon \to 0$ as $\varepsilon \to 0$. We construct a blow-up sequence of functions

$$v_\varepsilon = \begin{cases} \Big(c - \dfrac{1}{2\pi c}\ln\dfrac{\pi^2x_1^2 + (\pi x_2+\varepsilon)^2}{\varepsilon^2} + \dfrac{B}{c}\Big)\circ\phi, & x\in\phi^{-1}(\mathbb{B}^+_{R\varepsilon}),\\ \dfrac{G-\xi\beta}{c}, & x\in\phi^{-1}(\mathbb{B}^+_{2R\varepsilon}\backslash\mathbb{B}^+_{R\varepsilon}),\\ \dfrac{G}{c}, & x\in\Sigma\backslash\phi^{-1}(\mathbb{B}^+_{2R\varepsilon}) \end{cases}$$

for some constants B, c to be determined later, such that

$$\int_\Sigma(|\nabla_g v_\varepsilon|^2 + (v_\varepsilon - \overline{v}_\varepsilon)^2)\mathrm{d}v_g - \alpha\int_{\partial\Sigma}(v_\varepsilon - \overline{v}_\varepsilon)^2\mathrm{d}s_g = 1$$

and $v_\varepsilon - \overline{v}_\varepsilon \in \mathcal{S}$, where $\overline{v}_\varepsilon = \frac{\int_{\partial\Sigma}v_\varepsilon\mathrm{d}s_g}{\ell(\partial\Sigma)}$. Analogous to Subsection 2.4, we determine the constants

$$B = \frac{1}{\pi}\ln 2 + O\Big(\frac{\ln R}{R}\Big) + O(R\varepsilon\ln^2(R\varepsilon))$$

and

$$c^2 = A_p + \frac{1}{\pi}\ln\Big(\frac{\pi}{2\varepsilon}\Big) + O\Big(\frac{\ln R}{R}\Big) + O(R\varepsilon\ln^2(R\varepsilon)).$$

Then we get

$$\int_{\partial\Sigma}\mathrm{e}^{\pi(v_\varepsilon-\overline{v}_\varepsilon)^2}\mathrm{d}s_g \ge 2\pi\mathrm{e}^{\pi A_p} + \ell(\partial\Sigma) + \frac{\pi\|G\|^2_{L^2(\partial\Sigma)}}{c^2} + O(R\varepsilon\ln^2(R\varepsilon)) + O\Big(\frac{\ln R}{R}\Big). \tag{3.5}$$

Setting

$$v_\varepsilon^* = (v_\varepsilon - \overline{v}_\varepsilon) - \sum_{i=1}^{s_k} e_i\int_{\partial\Sigma}(v_\varepsilon - \overline{v}_\varepsilon)e_i\ \mathrm{d}s_g \in E_k^\perp,$$

one gets $\int_{\partial\Sigma} v_\varepsilon^*\,\mathrm{d}s_g = 0$ and

$$\int_\Sigma(|\nabla_g v_\varepsilon^*|^2 + (v_\varepsilon^*)^2)\mathrm{d}v_g - \alpha\int_{\partial\Sigma}(v_\varepsilon^*)^2\mathrm{d}s_g = 1 + O\Big(\frac{1}{R^2}\Big).$$

It is easy to verify $V_\varepsilon = v_\varepsilon^*/\|v_\varepsilon^*\|_{1,\alpha}^2 \in \mathcal{S}$. In view of (3.5) and $R = \ln^2 \varepsilon$, we have

$$\begin{aligned}\int_{\partial\Sigma} \mathrm{e}^{\pi V_\varepsilon^2}\, \mathrm{d}s_g &\geq \Big(1 + O\Big(\frac{1}{R^2}\Big)\Big) \int_{\partial\Sigma} \mathrm{e}^{\pi(v_\varepsilon - \overline{v}_\varepsilon)^2}\, \mathrm{d}s_g \\ &\geq 2\pi \mathrm{e}^{\pi A_p} + \ell(\partial\Sigma) + \frac{\pi\|G\|_{L^2(\partial\Sigma)}^2}{c^2} + O(R\varepsilon \ln^2(R\varepsilon)) + O\Big(\frac{\ln R}{R}\Big) \\ &> 2\pi \mathrm{e}^{\pi A_p} + \ell(\partial\Sigma) \end{aligned} \tag{3.6}$$

for sufficiently small $\varepsilon > 0$. The contradiction between (3.4) and (3.6) indicates that the assumption of $\lim_{\varepsilon\to 0} c_\varepsilon = \lim_{\varepsilon\to 0} u_\varepsilon(x_\varepsilon) = +\infty$ is not true. Then c_ε must be bounded and Theorem 1.3 follows from the elliptic estimate.

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