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Canonical Connections and Algebraic Ricci Solitons of Three-dimensional Lorentzian Lie Groups*

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Abstract In this paper, the author computes canonical connections and Kobayashi-Nomizu connections and their curvature on three-dimensional Lorentzian Lie groups with some product structure. He defines algebraic Ricci solitons associated to canonical connections and Kobayashi-Nomizu connections. He classifies algebraic Ricci solitons associated to canonical connections and Kobayashi-Nomizu connections on three-dimensional Lorentzian Lie groups with some product structure.

Keywords Canonical connections, Kobayashi-Nomizu connections, Algebraic Ricci solitons, Three-dimensional Lorentzian Lie groups
2000 MR Subject Classification 53C40, 53C42

1 Introduction

The concept of the algebraic Ricci soliton was first introduced by Lauret in Riemannian case in [6], where the author studied the relation between solvsolitons and Ricci solitons on Riemannian manifolds. More precisely, he proved that any Riemannian solvsoliton metric is a Ricci soliton. The concept of the algebraic Ricci soliton was extended to the pseudo-Riemannian case in [1], where Batat and Onda studied algebraic Ricci solitons of three-dimensional Lorentzian Lie groups. They got a complete classification of algebraic Ricci solitons of three-dimensional Lorentzian Lie groups and they proved that, contrary to the Riemannian case, Lorentzian Ricci solitons needed not be algebraic Ricci solitons. In [7], Onda provided a study of algebraic Ricci solitons in the pseudo-Riemannian case and obtained a steady algebraic Ricci soliton and a shrinking algebraic Ricci soliton in the Lorentzian setting. In [5], Etayo and Santamaria studied some affine connections on manifolds with the product structure or the complex structure. In particular, the canonical connection and the Kobayashi-Nomizu connection for a product structure were studied. In this paper, we introduce a particular product structure on three-dimensional Lorentzian Lie groups and we compute canonical connections and Kobayashi-Nomizu connections and their curvatures on three-dimensional Lorentzian Lie groups with this product structure. We define algebraic Ricci solitons associated to canonical connections and Kobayashi-Nomizu connections. We classify algebraic Ricci solitons associated to canonical

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connections and Kobayashi-Nomizu connections on three-dimensional Lorentzian Lie groups with this product structure.

In Section 2, we classify algebraic Ricci solitons associated to canonical connections and Kobayashi-Nomizu connections on three-dimensional unimodular Lorentzian Lie groups with the product structure. In Section 3, we classify algebraic Ricci solitons associated to canonical connections and Kobayashi-Nomizu connections on three-dimensional non-unimodular Lorentzian Lie groups with the product structure.

2 Algebraic Ricci Solitons Associated to Canonical Connections and Kobayashi-Nomizu Connections on Three-dimensional Lorentzian Lie Groups

Three-dimensional Lorentzian Lie groups have been classified in [2, 4] (see [1, Theorems 2.1–2.2]). Throughout this paper, we shall by $\{G_i\}_{i=1,\cdots,7}$, denote the connected, simply connected three-dimensional Lie group equipped with a left-invariant Lorentzian metric g and having Lie algebra $\{\mathfrak{g}\}_{i=1,\cdots,7}$. Let ∇ be the Levi-Civita connection of G_i and R be its curvature tensor, taken with the convention

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z. \tag{2.1}$$

The Ricci tensor of (G_i, g) is defined by

$$\rho(X,Y) = -g(R(X,e_1)Y,e_1) - g(R(X,e_2)Y,e_2) + g(R(X,e_3)Y,e_3), \tag{2.2}$$

where $\{e_1, e_2, e_3\}$ is a pseudo-orthonormal basis, with e_3 timelike and the Ricci operator Ric is given by

$$\rho(X,Y) = g(\mathrm{Ric}(X), Y). \tag{2.3}$$

We define a product structure J on G_i by

$$Je_1 = e_1, \quad Je_2 = e_2, \quad Je_3 = -e_3, \tag{2.4}$$

then $J^2 = \mathrm{Id}$ and $g(Je_j, Je_j) = g(e_j, e_j)$. By [5], we define the canonical connection and the Kobayashi-Nomizu connection as follows:

$$\nabla^0_X Y = \nabla_X Y - \frac{1}{2}(\nabla_X J)JY, \tag{2.5}$$

$$\nabla^1_X Y = \nabla^0_X Y - \frac{1}{4}[(\nabla_Y J)JX - (\nabla_{JY} J)X]. \tag{2.6}$$

We define

$$R^0(X,Y)Z = \nabla^0_X\nabla^0_Y Z - \nabla^0_Y\nabla^0_X Z - \nabla^0_{[X,Y]}Z, \tag{2.7}$$

$$R^1(X,Y)Z = \nabla^1_X\nabla^1_Y Z - \nabla^1_Y\nabla^1_X Z - \nabla^1_{[X,Y]}Z. \tag{2.8}$$

The Ricci tensors of (G_i, g) associated to the canonical connection and the Kobayashi-Nomizu connection are defined by

$$\rho^0(X,Y) = -g(R^0(X,e_1)Y,e_1) - g(R^0(X,e_2)Y,e_2) + g(R^0(X,e_3)Y,e_3), \tag{2.9}$$

$$\rho^1(X,Y) = -g(R^1(X,e_1)Y,e_1) - g(R^1(X,e_2)Y,e_2) + g(R^1(X,e_3)Y,e_3). \tag{2.10}$$

The Ricci operators Ric^0 and Ric^1 is given by

$$\rho^0(X,Y) = g(\mathrm{Ric}^0(X),Y), \quad \rho^1(X,Y) = g(\mathrm{Ric}^1(X),Y). \tag{2.11}$$

Let

$$\widetilde{\rho}^0(X,Y) = \frac{\rho^0(X,Y)+\rho^0(Y,X)}{2}, \quad \widetilde{\rho}^1(X,Y) = \frac{\rho^1(X,Y)+\rho^1(Y,X)}{2} \tag{2.12}$$

and

$$\widetilde{\rho}^0(X,Y) = g(\widetilde{\mathrm{Ric}}^0(X),Y), \quad \widetilde{\rho}^1(X,Y) = g(\widetilde{\mathrm{Ric}}^1(X),Y). \tag{2.13}$$

Definition 2.1 (G_i, g, J) *is called the first* (*resp. second*) *kind algebraic Ricci soliton associated to the connection* ∇^0 *if it satisfies*

$$\mathrm{Ric}^0 = c\mathrm{Id} + D \quad (\textit{resp.}\ \widetilde{\mathrm{Ric}}^0 = c\mathrm{Id} + D), \tag{2.14}$$

where c is a real number, and D is a derivation of $\mathfrak{g}$, that is

$$D[X,Y] = [DX,Y] + [X,DY] \quad for\ X,Y \in \mathfrak{g}. \tag{2.15}$$

(G_i, g, J) is called the first (*resp. second*) *kind algebraic Ricci soliton associated to the connection ∇^1 if it satisfies*

$$\mathrm{Ric}^1 = c\mathrm{Id} + D \quad (\textit{resp.}\ \widetilde{\mathrm{Ric}}^1 = c\mathrm{Id} + D). \tag{2.16}$$

By [1, Lemma 3.1], we have that for G_1, there exists a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$ with e_3 timelike such that the Lie algebra of G_1 satisfies

$$[e_1,e_2] = \alpha e_1 - \beta e_3, \quad [e_1,e_3] = -\alpha e_1 - \beta e_2, \quad [e_2,e_3] = \beta e_1 + \alpha e_2 + \alpha e_3, \quad \alpha \neq 0. \tag{2.17}$$

We recall (see [1, 3]) that the Levi-Civita connection ∇ of G_1 is given by

$$\begin{aligned}
&\nabla_{e_1}e_1 = -\alpha e_2 - \alpha e_3, \quad \nabla_{e_2}e_1 = \frac{\beta}{2}e_3, \quad \nabla_{e_3}e_1 = \frac{\beta}{2}e_2,\\
&\nabla_{e_1}e_2 = \alpha e_1 - \frac{\beta}{2}e_3, \quad \nabla_{e_2}e_2 = \alpha e_3, \quad \nabla_{e_3}e_2 = -\frac{\beta}{2}e_1 - \alpha e_3,\\
&\nabla_{e_1}e_3 = -\alpha e_1 - \frac{\beta}{2}e_2, \quad \nabla_{e_2}e_3 = \frac{\beta}{2}e_1 + \alpha e_2, \quad \nabla_{e_3}e_3 = -\alpha e_2.
\end{aligned} \tag{2.18}$$

By (2.4) and (2.18), we have that for G_1, the following equalities hold

$$\begin{aligned}&\nabla_{e_1}(J)e_1=-2\alpha e_3,\quad \nabla_{e_1}(J)e_2=-\beta e_3,\quad \nabla_{e_1}(J)e_3=2\alpha e_1+\beta e_2,\\&\nabla_{e_2}(J)e_1=\beta e_3,\quad \nabla_{e_2}(J)e_2=2\alpha e_3,\quad \nabla_{e_2}(J)e_3=-\beta e_1-2\alpha e_2,\\&\nabla_{e_3}(J)e_1=0,\quad \nabla_{e_3}(J)e_2=-2\alpha e_3,\quad \nabla_{e_3}(J)e_3=2\alpha e_2.\end{aligned}\tag{2.19}$$

By (2.4)–(2.5) and (2.18)–(2.19), we have the following.

The canonical connection ∇^0 of (G_1, J) is given by

$$\begin{aligned}&\nabla^0_{e_1}e_1=-\alpha e_2,\quad \nabla^0_{e_1}e_2=\alpha e_1,\quad \nabla^0_{e_1}e_3=0,\\&\nabla^0_{e_2}e_1=0,\quad \nabla^0_{e_2}e_2=0,\quad \nabla^0_{e_2}e_3=0,\\&\nabla^0_{e_3}e_1=\frac{\beta}{2}e_2,\quad \nabla^0_{e_3}e_2=-\frac{\beta}{2}e_1,\quad \nabla^0_{e_3}e_3=0.\end{aligned}\tag{2.20}$$

By (2.7) and (2.20), we have that the curvature R^0 of the canonical connection ∇^0 of (G_1, J) is given by

$$\begin{aligned}&R^0(e_1,e_2)e_1=\Big(\alpha^2+\frac{\beta^2}{2}\Big)e_2,\quad R^0(e_1,e_2)e_2=-\Big(\alpha^2+\frac{\beta^2}{2}\Big)e_1,\quad R^0(e_1,e_2)e_3=0,\\&R^0(e_1,e_3)e_1=-\alpha^2e_2,\quad R^0(e_1,e_3)e_2=\alpha^2e_1,\quad R^0(e_1,e_3)e_3=0,\\&R^0(e_2,e_3)e_1=\frac{\alpha\beta}{2}e_2,\quad R^0(e_2,e_3)e_2=-\frac{\alpha\beta}{2}e_1,\quad R^0(e_2,e_3)e_3=0.\end{aligned}\tag{2.21}$$

By (2.9), (2.11) and (2.21), we get

$$\mathrm{Ric}^0\begin{pmatrix}e_1\\e_2\\e_3\end{pmatrix}=\begin{pmatrix}-\big(\alpha^2+\frac{\beta^2}{2}\big)&0&0\\0&-\big(\alpha^2+\frac{\beta^2}{2}\big)&0\\\frac{\alpha\beta}{2}&\alpha^2&0\end{pmatrix}\begin{pmatrix}e_1\\e_2\\e_3\end{pmatrix}.\tag{2.22}$$

If (G_1, g, J) is the first kind algebraic Ricci soliton associated to the connection ∇^0, then $\mathrm{Ric}^0 = c\mathrm{Id} + D$, so

$$\begin{cases}De_1=-\Big(\alpha^2+\dfrac{\beta^2}{2}+c\Big)e_1,\\De_2=-\Big(\alpha^2+\dfrac{\beta^2}{2}+c\Big)e_2,\\De_3=\dfrac{\alpha\beta}{2}e_1+\alpha^2e_2-ce_3.\end{cases}\tag{2.23}$$

By (2.15) and (2.23), we get $\alpha^2 + c = 0$, $\beta = 0$. Then we have the following theorem.

Theorem 2.1 *(G_1, g, J) is the first kind algebraic Ricci soliton associated to the connection ∇^0 if and only if $\alpha^2 + c = 0$, $\beta = 0$, $\alpha \neq 0$. In particular,*

$$\mathrm{Ric}^0\begin{pmatrix}e_1\\e_2\\e_3\end{pmatrix}=\begin{pmatrix}-\alpha^2&0&0\\0&-\alpha^2&0\\0&\alpha^2&0\end{pmatrix}\begin{pmatrix}e_1\\e_2\\e_3\end{pmatrix},\quad D\begin{pmatrix}e_1\\e_2\\e_3\end{pmatrix}=\begin{pmatrix}0&0&0\\0&0&0\\0&\alpha^2&\alpha^2\end{pmatrix}\begin{pmatrix}e_1\\e_2\\e_3\end{pmatrix}.\tag{2.24}$$

By (2.12)–(2.13) and (2.22), we have

$$\widetilde{\mathrm{Ric}}^0 \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} -\big(\alpha^2+\frac{\beta^2}{2}\big) & 0 & -\frac{\alpha\beta}{4} \\ 0 & -\big(\alpha^2+\frac{\beta^2}{2}\big) & -\frac{\alpha^2}{2} \\ \frac{\alpha\beta}{4} & \frac{\alpha^2}{2} & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{2.25}$$

If (G_1, g, J) is the second kind algebraic Ricci soliton associated to the connection ∇^0, then $\widetilde{\mathrm{Ric}}^0 = c\mathrm{Id} + D$, so

$$\begin{cases} De_1 = -\Big(\alpha^2+\dfrac{\beta^2}{2}+c\Big)e_1 - \dfrac{\alpha\beta}{4}e_3, \\ De_2 = -\Big(\alpha^2+\dfrac{\beta^2}{2}+c\Big)e_2 - \dfrac{\alpha^2}{2}e_3, \\ De_3 = \dfrac{\alpha\beta}{4}e_1 + \dfrac{\alpha^2}{2}e_2 - ce_3. \end{cases} \tag{2.26}$$

By (2.15) and (2.26), we get $\frac{\alpha^2}{2}+c=0$, $\beta=0$. Then we have the following theorem.

Theorem 2.2 *(G_1, g, J) is the second kind algebraic Ricci soliton associated to the connection ∇^0 if and only if $\frac{\alpha^2}{2}+c=0$, $\beta=0$, $\alpha\neq 0$. In particular,*

$$\widetilde{\mathrm{Ric}}^0 \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} -\alpha^2 & 0 & 0 \\ 0 & -\alpha^2 & -\frac{\alpha^2}{2} \\ 0 & \frac{\alpha^2}{2} & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix},$$

$$D \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} -\frac{\alpha^2}{2} & 0 & 0 \\ 0 & -\frac{\alpha^2}{2} & -\frac{\alpha^2}{2} \\ 0 & \frac{\alpha^2}{2} & \frac{\alpha^2}{2} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{2.27}$$

By (2.6) and (2.19)–(2.20), we have that the Kobayashi-Nomizu connection ∇^1 of (G_1, J) is given by

$$\begin{aligned} &\nabla^1_{e_1}e_1 = -\alpha e_2, \quad \nabla^1_{e_1}e_2 = \alpha e_1, \quad \nabla^1_{e_1}e_3 = 0, \\ &\nabla^1_{e_2}e_1 = 0, \quad \nabla^1_{e_2}e_2 = 0, \quad \nabla^1_{e_2}e_3 = \alpha e_3, \\ &\nabla^0_{e_3}e_1 = \alpha e_1 + \beta e_2, \quad \nabla^1_{e_3}e_2 = -\alpha e_2 - \beta e_1, \quad \nabla^1_{e_3}e_3 = 0. \end{aligned} \tag{2.28}$$

By (2.8) and (2.28), we have that the curvature R^1 of the Kobayashi-Nomizu connection ∇^1 of (G_1, J) is given by

$$\begin{aligned} &R^1(e_1,e_2)e_1 = \alpha\beta e_1 + (\alpha^2+\beta^2)e_2, \quad R^1(e_1,e_2)e_2 = -(\alpha^2+\beta^2)e_1 - \alpha\beta e_2, \\ &R^1(e_1,e_2)e_3 = 0, \quad R^1(e_1,e_3)e_1 = -3\alpha^2 e_2, \quad R^1(e_1,e_3)e_2 = -\alpha^2 e_1, \\ &R^1(e_1,e_3)e_3 = \alpha\beta e_3, \quad R^1(e_2,e_3)e_1 = -\alpha^2 e_1, \\ &R^1(e_2,e_3)e_2 = \alpha^2 e_2, \quad R^1(e_2,e_3)e_3 = -\alpha^2 e_3. \end{aligned} \tag{2.29}$$

By (2.10)–(2.11) and (2.29), we get

$$\mathrm{Ric}^1 \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} -(\alpha^2+\beta^2) & \alpha\beta & \alpha\beta \\ \alpha\beta & -(\alpha^2+\beta^2) & -\alpha^2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{2.30}$$

If (G_1, g, J) is the first kind algebraic Ricci soliton associated to the connection ∇^1, then $\mathrm{Ric}^1 = c\mathrm{Id} + D$, so

$$\begin{cases} De_1 = -(\alpha^2+\beta^2+c)e_1 + \alpha\beta e_2 + \alpha\beta e_3, \\ De_2 = \alpha\beta e_1 - (\alpha^2+\beta^2+c)e_2 - \alpha^2 e_3, \\ De_3 = -ce_3. \end{cases} \tag{2.31}$$

By (2.15) and (2.31), we get $\beta = 0$, $c = 0$. Then we have the following theorem.

Theorem 2.3 *(G_1, g, J) is the first kind algebraic Ricci soliton associated to the connection ∇^1 if and only if $\beta = 0$, $c = 0$, $\alpha \neq 0$. In particular,*

$$\begin{aligned} \mathrm{Ric}^1 \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} &= \begin{pmatrix} -\alpha^2 & 0 & 0 \\ 0 & -\alpha^2 & -\alpha^2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}, \\ D \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} &= \begin{pmatrix} -\alpha^2 & 0 & 0 \\ 0 & -\alpha^2 & -\alpha^2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \end{aligned} \tag{2.32}$$

By (2.12)–(2.13) and (2.30), we have

$$\widetilde{\mathrm{Ric}}^1 \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} -(\alpha^2+\beta^2) & \alpha\beta & \frac{\alpha\beta}{2} \\ \alpha\beta & -(\alpha^2+\beta^2) & -\frac{\alpha^2}{2} \\ -\frac{\alpha\beta}{2} & \frac{\alpha^2}{2} & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{2.33}$$

If (G_1, g, J) is the second kind algebraic Ricci soliton associated to the connection ∇^1, then $\widetilde{\mathrm{Ric}}^1 = c\mathrm{Id} + D$, so

$$\begin{cases} De_1 = -(\alpha^2+\beta^2+c)e_1 + \alpha\beta e_2 + \dfrac{\alpha\beta}{2} e_3, \\ De_2 = \alpha\beta e_1 - (\alpha^2+\beta^2+c)e_2 - \dfrac{\alpha^2}{2} e_3, \\ De_3 = -\dfrac{\alpha\beta}{2} e_1 + \dfrac{\alpha^2}{2} e_2 - ce_3. \end{cases} \tag{2.34}$$

By (2.15) and (2.34), we get $\frac{\alpha^2}{2} + c = 0$, $\beta = 0$. Then we have the following theorem.

Theorem 2.4 *(G_1, g, J) is the second kind algebraic Ricci soliton associated to the connection ∇^1 if and only if $\frac{\alpha^2}{2} + c = 0$, $\beta = 0$, $\alpha \neq 0$. In particular,*

$$\begin{aligned} \widetilde{\mathrm{Ric}}^1 \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} &= \begin{pmatrix} -\alpha^2 & 0 & 0 \\ 0 & -\alpha^2 & -\frac{\alpha^2}{2} \\ 0 & \frac{\alpha^2}{2} & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}, \\ D \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} &= \begin{pmatrix} -\frac{\alpha^2}{2} & 0 & 0 \\ 0 & -\frac{\alpha^2}{2} & -\frac{\alpha^2}{2} \\ 0 & \frac{\alpha^2}{2} & \frac{\alpha^2}{2} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \end{aligned} \tag{2.35}$$

By [1, Lemma 3.5], we have that for G_2, there exists a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$ with e_3 timelike such that the Lie algebra of G_2 satisfies

$$[e_1, e_2] = \gamma e_2 - \beta e_3, \quad [e_1, e_3] = -\beta e_2 - \gamma e_3, \quad [e_2, e_3] = \alpha e_1, \quad \gamma \neq 0. \tag{2.36}$$

Similar to the case of G_1, we have the following theorem.

Theorem 2.5 (1) *(G_2, g, J) is the first kind algebraic Ricci soliton associated to the connection ∇^0 if and only if $\alpha = \beta = 0$, $\gamma^2 + c = 0$, $\gamma \neq 0$. In particular,*

$$\mathrm{Ric}^0 \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} -\gamma^2 & 0 & 0 \\ 0 & -\gamma^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}, \quad D \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \gamma^2 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{2.37}$$

(2) *(G_2, g, J) is the second kind algebraic Ricci soliton associated to the connection ∇^0 if and only if $\alpha = \beta = 0$, $\gamma^2 + c = 0$, $\gamma \neq 0$. In particular,*

$$\widetilde{\mathrm{Ric}}^0 \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} -\gamma^2 & 0 & 0 \\ 0 & -\gamma^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}, \quad D \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \gamma^2 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{2.38}$$

(3) *(G_2, g, J) is the first kind algebraic Ricci soliton associated to the connection ∇^1 if and only if $\alpha = \beta = 0$, $\gamma^2 + c = 0$, $\gamma \neq 0$. In particular,*

$$\mathrm{Ric}^1 \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} -\gamma^2 & 0 & 0 \\ 0 & -\gamma^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}, \quad D \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \gamma^2 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{2.39}$$

(4) *(G_2, g, J) is the second kind algebraic Ricci soliton associated to the connection ∇^1 if and only if $\alpha = \beta = 0$, $\gamma^2 + c = 0$, $\gamma \neq 0$. In particular,*

$$\widetilde{\mathrm{Ric}}^1 \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} -\gamma^2 & 0 & 0 \\ 0 & -\gamma^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}, \quad D \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \gamma^2 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{2.40}$$

Proof The canonical connection ∇^0 of (G_2, J) is given by

$$\begin{aligned} &\nabla^0_{e_1} e_1 = 0, \quad \nabla^0_{e_1} e_2 = 0, \quad \nabla^0_{e_1} e_3 = 0, \\ &\nabla^0_{e_2} e_1 = -\gamma e_2, \quad \nabla^0_{e_2} e_2 = \gamma e_1, \quad \nabla^0_{e_2} e_3 = 0, \\ &\nabla^0_{e_3} e_1 = \frac{\alpha}{2} e_2, \quad \nabla^0_{e_3} e_2 = -\frac{\alpha}{2} e_1, \quad \nabla^0_{e_3} e_3 = 0. \end{aligned} \tag{2.41}$$

By (2.7) and (2.41), we have that the curvature R^0 of the canonical connection ∇^0 of (G_2, J) is given by

$$\begin{aligned} &R^0(e_1, e_2)e_1 = \Big(\gamma^2 + \frac{\alpha\beta}{2}\Big) e_2, \quad R^0(e_1, e_2)e_2 = -\Big(\gamma^2 + \frac{\alpha\beta}{2}\Big) e_1, \quad R^0(e_1, e_2)e_3 = 0, \\ &R^0(e_1, e_3)e_1 = \Big(\frac{\alpha\gamma}{2} - \beta\gamma\Big) e_2, \quad R^0(e_1, e_3)e_2 = -\Big(\frac{\alpha\gamma}{2} - \beta\gamma\Big) e_1, \quad R^0(e_1, e_3)e_3 = 0, \\ &R^0(e_2, e_3)e_1 = 0, \quad R^0(e_2, e_3)e_2 = 0, \quad R^0(e_2, e_3)e_3 = 0. \end{aligned} \tag{2.42}$$

By (2.9), (2.11) and (2.42), we get for (G_2, ∇^0),

$$\mathrm{Ric}^0 \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} -\big(\gamma^2 + \frac{\alpha\beta}{2}\big) & 0 & 0 \\ 0 & -\big(\gamma^2 + \frac{\alpha\beta}{2}\big) & 0 \\ 0 & \beta\gamma - \frac{\alpha\gamma}{2} & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{2.43}$$

If (G_2, g, J) is the first kind algebraic Ricci soliton associated to the connection ∇^0, then $\mathrm{Ric}^0 = c\mathrm{Id} + D$, so

$$\begin{cases} De_1 = -\Big(\gamma^2 + \dfrac{\alpha\beta}{2} + c\Big)e_1, \\ De_2 = -\Big(\gamma^2 + \dfrac{\alpha\beta}{2} + c\Big)e_2, \\ De_3 = \Big(\beta\gamma - \dfrac{\alpha\gamma}{2}\Big)e_2 - ce_3. \end{cases} \tag{2.44}$$

By (2.15) and (2.44), we get $\alpha = \beta = 0$, $\gamma^2 + c = 0$. Then case (1) holds. The other three cases hold similarly.

By [1, Lemma 3.8], we have that for G_3, there exists a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$ with e_3 timelike such that the Lie algebra of G_3 satisfies

$$[e_1, e_2] = -\gamma e_3, \quad [e_1, e_3] = -\beta e_2, \quad [e_2, e_3] = \alpha e_1. \tag{2.45}$$

Theorem 2.6 (1) *(G_3, g, J) is the first kind algebraic Ricci soliton associated to the connection ∇^0 if and only if*

(i) *$\alpha = \beta = \gamma = 0$. In particular,*

$$D\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} -c & 0 & 0 \\ 0 & -c & 0 \\ 0 & 0 & -c \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{2.46}$$

(ii) *$\alpha = \beta = 0$, $\gamma^2 = c$. In particular,*

$$D\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} -\frac{\gamma^2}{2} & 0 & 0 \\ 0 & -\frac{\gamma^2}{2} & 0 \\ 0 & 0 & -\gamma^2 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.$$

(iii) *$\alpha \neq 0$ or $\beta \neq 0$, $\gamma = 0$, $c = 0$. In particular,*

$$D\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.$$

(iv) *$\alpha \neq 0$ or $\beta \neq 0$, $\gamma = \alpha + \beta$, $c = 0$. In particular,*

$$D\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.$$

(2) *(G_3, g, J) is the first kind algebraic Ricci soliton associated to the connection ∇^1 if and only if*

(i) *$\alpha\beta \neq 0$, $\gamma = 0$, $c = 0$. In particular,*

$$D\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.$$

(ii) $\alpha=\beta=\gamma=0$, $c\neq 0$. *In particular,*

$$D\begin{pmatrix}e_1\\e_2\\e_3\end{pmatrix}=\begin{pmatrix}-c&0&0\\0&-c&0\\0&0&-c\end{pmatrix}\begin{pmatrix}e_1\\e_2\\e_3\end{pmatrix}.$$

(iii) $\alpha=0$, $\gamma\beta+c=0$. *In particular,*

$$D\begin{pmatrix}e_1\\e_2\\e_3\end{pmatrix}=\begin{pmatrix}0&0&0\\0&-c&0\\0&0&-c\end{pmatrix}\begin{pmatrix}e_1\\e_2\\e_3\end{pmatrix}.$$

(iv) $\beta=0$, $\gamma\alpha+c=0$. *In particular,*

$$D\begin{pmatrix}e_1\\e_2\\e_3\end{pmatrix}=\begin{pmatrix}-c&0&0\\0&0&0\\0&0&-c\end{pmatrix}\begin{pmatrix}e_1\\e_2\\e_3\end{pmatrix}.$$

Proof The canonical connection ∇^0 of (G_3,J) is given by

$$\begin{aligned}&\nabla^0_{e_1}e_1=0,\quad \nabla^0_{e_1}e_2=0,\quad \nabla^0_{e_1}e_3=0,\\&\nabla^0_{e_2}e_1=0,\quad \nabla^0_{e_2}e_2=0,\quad \nabla^0_{e_2}e_3=0,\\&\nabla^0_{e_3}e_1=a_3e_2,\quad \nabla^0_{e_3}e_2=-a_3e_1,\quad \nabla^0_{e_3}e_3=0.\end{aligned}\tag{2.47}$$

By (2.7) and (2.47), we have the curvature R^0 of the canonical connection ∇^0 of (G_3,J) is given by

$$\begin{aligned}&R^0(e_1,e_2)e_1=\gamma a_3e_2,\quad R^0(e_1,e_2)e_2=-\gamma a_3e_1,\quad R^0(e_1,e_2)e_3=0,\\&R^0(e_1,e_3)e_1=0,\quad R^0(e_1,e_3)e_2=0,\quad R^0(e_1,e_3)e_3=0,\\&R^0(e_2,e_3)e_1=0,\quad R^0(e_2,e_3)e_2=0,\quad R^0(e_2,e_3)e_3=0.\end{aligned}\tag{2.48}$$

By (2.9), (2.11) and (2.48), we get for (G_3,∇^0),

$$\mathrm{Ric}^0\begin{pmatrix}e_1\\e_2\\e_3\end{pmatrix}=\begin{pmatrix}-\gamma a_3&0&0\\0&-\gamma a_3&0\\0&0&0\end{pmatrix}\begin{pmatrix}e_1\\e_2\\e_3\end{pmatrix}.\tag{2.49}$$

If (G_3,g,J) is the first kind algebraic Ricci soliton associated to the connection ∇^0, then $\mathrm{Ric}^0=c\mathrm{Id}+D$, so

$$\begin{cases}De_1=-(\gamma a_3+c)e_1,\\De_2=-(\gamma a_3+c)e_2,\\De_3=-ce_3.\end{cases}\tag{2.50}$$

By (2.15) and (2.50), we get the case (1). Similarly the case (2) holds.

By [1, Lemma 3.11], we have that for G_4, there exists a pseudo-orthonormal basis $\{e_1,e_2,e_3\}$ with e_3 timelike such that the Lie algebra of G_4 satisfies

$$[e_1,e_2]=-e_2+(2\eta-\beta)e_3,\quad \eta=1\text{ or }-1,\quad [e_1,e_3]=-\beta e_2+e_3,\quad [e_2,e_3]=\alpha e_1.\tag{2.51}$$

Theorem 2.7 (1) (G_4, g, J) *is the first kind algebraic Ricci soliton associated to the connection* ∇^0 *if and only if*

(i) $\alpha = 0$, $\beta = 1$, $c = 0$, $\eta = 1$. *In particular,*

$$D\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{2.52}$$

(ii) $\alpha = 0$, $c = -1$, $\beta = 2\eta$. *In particular,*

$$D\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\eta & 1 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{2.53}$$

(2) (G_4, g, J) *is the second kind algebraic Ricci soliton associated to the connection* ∇^0 *if and only if* $\alpha = 0$, $\beta = \eta$, $c = 0$. *In particular,*

$$D\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{2.54}$$

(3) (G_4, g, J) *is not the first kind algebraic Ricci soliton associated to the connection* ∇^1.

(4) (G_4, g, J) *is not the second kind algebraic Ricci soliton associated to the connection* ∇^1.

Proof The canonical connection ∇^0 of (G_4, J) is given by

$$\begin{aligned} &\nabla^0_{e_1}e_1 = 0, \quad \nabla^0_{e_1}e_2 = 0, \quad \nabla^0_{e_1}e_3 = 0, \\ &\nabla^0_{e_2}e_1 = e_2, \quad \nabla^0_{e_2}e_2 = -e_1, \quad \nabla^0_{e_2}e_3 = 0, \\ &\nabla^0_{e_3}e_1 = b_3e_2, \quad \nabla^0_{e_3}e_2 = -b_3e_1, \quad \nabla^0_{e_3}e_3 = 0. \end{aligned} \tag{2.55}$$

By (2.7) and (2.55), we have the curvature R^0 of the canonical connection ∇^0 of (G_4, J) is given by

$$\begin{aligned} &R^0(e_1, e_2)e_1 = [(\beta - 2\eta)b_3 + 1]e_2, \quad R^0(e_1, e_2)e_2 = [b_3(2\eta - \beta) - 1]e_1, \\ &R^0(e_1, e_2)e_3 = 0, \quad R^0(e_1, e_3)e_1 = (\beta - b_3)e_2, \quad R^0(e_1, e_3)e_2 = (b_3 - \beta)e_1, \\ &R^0(e_1, e_3)e_3 = 0, \quad R^0(e_2, e_3)e_1 = 0, \quad R^0(e_2, e_3)e_2 = 0, \quad R^0(e_2, e_3)e_3 = 0. \end{aligned} \tag{2.56}$$

By (2.9), (2.11) and (2.56), we get for (G_4, ∇^0),

$$\mathrm{Ric}^0\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} [b_3(2\eta - \beta) - 1] & 0 & 0 \\ 0 & [b_3(2\eta - \beta) - 1] & 0 \\ 0 & b_3 - \beta & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{2.57}$$

If (G_4, g, J) is the first kind algebraic Ricci soliton associated to the connection ∇^0, then $\mathrm{Ric}^0 = c\mathrm{Id} + D$, so

$$\begin{cases} De_1 = [b_3(2\eta - \beta) - 1 - c]e_1, \\ De_2 = [b_3(2\eta - \beta) - 1 - c]e_2, \\ De_3 = (b_3 - \beta)e_2 - ce_3. \end{cases} \tag{2.58}$$

By (2.15) and (2.58), we get

$$\begin{cases}(2\eta-\beta)(2b_3-\beta)-c-1=0,\\ (2b_3(2\eta-\beta)-2-c)(2\eta-\beta)=0,\\ 2(b_3-\beta)-c\beta=0,\\ c\alpha=0.\end{cases} \tag{2.59}$$

Solving (2.59), we get the case (1). The other cases hold similarly.

By [1, Lemma 4.1], we have that for G_5, there exists a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$ with e_3 timelike such that the Lie algebra of G_5 satisfies

$$[e_1,e_2]=0,\quad [e_1,e_3]=\alpha e_1+\beta e_2,\quad [e_2,e_3]=\gamma e_1+\delta e_2,\quad \alpha+\delta\neq 0,\quad \alpha\gamma+\beta\delta=0. \tag{2.60}$$

Theorem 2.8 (1) (G_5, g, J) *is the first kind algebraic Ricci soliton associated to the connection* ∇^0 *if and only if* $c = 0$. *In particular,*

$$D\begin{pmatrix}e_1\\ e_2\\ e_3\end{pmatrix}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}e_1\\ e_2\\ e_3\end{pmatrix}. \tag{2.61}$$

(2) (G_5, g, J) *is the first kind algebraic Ricci soliton associated to the connection* ∇^1 *if and only if* $c = 0$. *In particular,*

$$D\begin{pmatrix}e_1\\ e_2\\ e_3\end{pmatrix}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}e_1\\ e_2\\ e_3\end{pmatrix}. \tag{2.62}$$

Proof The canonical connection ∇^0 of (G_5, J) is given by

$$\begin{aligned}&\nabla^0_{e_1}e_1=0,\quad \nabla^0_{e_1}e_2=0,\quad \nabla^0_{e_1}e_3=0,\\ &\nabla^0_{e_2}e_1=0,\quad \nabla^0_{e_2}e_2=0,\quad \nabla^0_{e_2}e_3=0,\\ &\nabla^0_{e_3}e_1=-\frac{\beta-\gamma}{2}e_2,\quad \nabla^0_{e_3}e_2=\frac{\beta-\gamma}{2}e_1,\quad \nabla^0_{e_3}e_3=0.\end{aligned} \tag{2.63}$$

By (2.7) and (2.63), we have that the curvature R^0 of the canonical connection ∇^0 of (G_5, J) is flat, that is $R^0(e_i, e_j)e_k = 0$. So we get for (G_5, ∇^0),

$$\mathrm{Ric}^0\begin{pmatrix}e_1\\ e_2\\ e_3\end{pmatrix}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}e_1\\ e_2\\ e_3\end{pmatrix}. \tag{2.64}$$

If (G_5, g, J) is the first kind algebraic Ricci soliton associated to the connection ∇^0, then $\mathrm{Ric}^0 = c\mathrm{Id} + D$, so

$$\begin{cases}De_1=-ce_1,\\ De_2=-ce_2,\\ De_3=-ce_3.\end{cases} \tag{2.65}$$

By (2.15) and (2.65), we get the case (1). Similarly the case (2) holds.

By [1, Lemma 4.3], we have that for G_6, there exists a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$ with e_3 timelike such that the Lie algebra of G_6 satisfies

$$[e_1, e_2] = \alpha e_2 + \beta e_3, \quad [e_1, e_3] = \gamma e_2 + \delta e_3, \quad [e_2, e_3] = 0, \quad \alpha + \delta \neq 0, \quad \alpha\gamma - \beta\delta = 0. \tag{2.66}$$

Theorem 2.9 (1) (G_6, g, J) *is the first kind algebraic Ricci soliton associated to the connection* ∇^0 *if and only if*

(i) $\alpha = \beta = \gamma = c = 0$, $\delta \neq 0$. *In particular,*

$$D\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{2.67}$$

(ii) $\alpha \neq 0$, $\beta = \gamma = 0$, $\alpha^2 + c = 0$, $\alpha + \delta \neq 0$. *In particular,*

$$D\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha^2 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{2.68}$$

(iii) $\alpha \neq 0$, $\beta \neq 0$, $\gamma = \delta = 0$, $\beta^2 = 2\alpha^2$, $c = 0$. *In particular,*

$$D\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{2.69}$$

(2) (G_6, g, J) *is the second kind algebraic Ricci soliton associated to the connection* ∇^0 *if and only if*

(i) $\beta = \gamma = 0$, $\alpha^2 + c = 0$, $\alpha + \delta \neq 0$. *In particular,*

$$D\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -c \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{2.70}$$

(ii) $\gamma = \delta = c = 0$, $\alpha \neq 0$, $\beta \neq 0$, $\beta^2 = 2\alpha^2$. *In particular,*

$$D\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{2.71}$$

(3) (G_6, g, J) *is the first kind algebraic Ricci soliton associated to the connection* ∇^1 *if and only if*

(i) $\alpha = \beta = c = 0$, $\delta \neq 0$. *In particular,*

$$D\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{2.72}$$

(ii) $\alpha \neq 0$, $\beta = \gamma = 0$, $\alpha^2 + c = 0$, $\alpha + \delta \neq 0$. *In particular,*

$$D\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha^2 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{2.73}$$

Proof The canonical connection ∇^0 of (G_6, J) is given by

$$\begin{aligned}
&\nabla^0_{e_1}e_1 = 0, \quad \nabla^0_{e_1}e_2 = 0, \quad \nabla^0_{e_1}e_3 = 0,\\
&\nabla^0_{e_2}e_1 = -\alpha e_2, \quad \nabla^0_{e_2}e_2 = \alpha e_1, \quad \nabla^0_{e_2}e_3 = 0,\\
&\nabla^0_{e_3}e_1 = \frac{\beta-\gamma}{2}e_2, \quad \nabla^0_{e_3}e_2 = -\frac{\beta-\gamma}{2}e_1, \quad \nabla^0_{e_3}e_3 = 0.
\end{aligned} \tag{2.74}$$

By (2.7) and (2.74), we have that the curvature R^0 of the canonical connection ∇^0 of (G_6, J) is given by

$$\begin{aligned}
&R^0(e_1,e_2)e_1 = \Big[\alpha^2 - \frac{1}{2}\beta(\beta-\gamma)\Big]e_2, \quad R^0(e_1,e_2)e_2 = \Big[-\alpha^2 + \frac{1}{2}\beta(\beta-\gamma)\Big]e_1,\\
&R^0(e_1,e_2)e_3 = 0,\\
&R^0(e_1,e_3)e_1 = \Big[\gamma\alpha - \frac{1}{2}\delta(\beta-\gamma)\Big]e_2, \quad R^0(e_1,e_3)e_2 = \Big[-\gamma\alpha + \frac{1}{2}\delta(\beta-\gamma)\Big]e_1,\\
&R^0(e_1,e_3)e_3 = 0,\\
&R^0(e_2,e_3)e_1 = 0, \quad R^0(e_2,e_3)e_2 = 0, \quad R^0(e_2,e_3)e_3 = 0.
\end{aligned} \tag{2.75}$$

By (2.9), (2.11) and (2.75), we get for (G_6, ∇^0),

$$\mathrm{Ric}^0\begin{pmatrix} e_1\\ e_2\\ e_3\end{pmatrix} = \begin{pmatrix} \frac{1}{2}\beta(\beta-\gamma)-\alpha^2 & 0 & 0\\ 0 & \frac{1}{2}\beta(\beta-\gamma)-\alpha^2 & 0\\ 0 & -\gamma\alpha+\frac{1}{2}\delta(\beta-\gamma) & 0\end{pmatrix}\begin{pmatrix} e_1\\ e_2\\ e_3\end{pmatrix}. \tag{2.76}$$

If (G_6, g, J) is the first kind algebraic Ricci soliton associated to the connection ∇^0, then $\mathrm{Ric}^0 = c\mathrm{Id} + D$, so

$$\begin{cases}
De_1 = \Big[\frac{1}{2}\beta(\beta-\gamma) - \alpha^2 - c\Big]e_1,\\
De_2 = \Big[\frac{1}{2}\beta(\beta-\gamma) - \alpha^2 - c\Big]e_2,\\
De_3 = \Big[-\gamma\alpha + \frac{1}{2}\delta(\beta-\gamma)\Big]e_2 - ce_3.
\end{cases} \tag{2.77}$$

By (2.15) and (2.77), we get

$$\begin{cases}
\alpha\Big[\frac{1}{2}\beta(\beta-\gamma) - \alpha^2 - c\Big] - \beta\Big[-\gamma\alpha + \frac{1}{2}\delta(\beta-\gamma)\Big] = 0,\\
\beta[\beta(\beta-\gamma) - 2\alpha^2 - c] = 0,\\
-c\gamma + \Big[-\gamma\alpha + \frac{1}{2}\delta(\beta-\gamma)\Big](\alpha-\delta) = 0,\\
\delta\Big[\frac{1}{2}\beta(\beta-\gamma) - \alpha^2 - c\Big] + \beta\Big[-\gamma\alpha + \frac{1}{2}\delta(\beta-\gamma)\Big] = 0.
\end{cases} \tag{2.78}$$

Solving (2.78), then the case (1) holds. The other cases hold similarly.

By [1, Lemma 4.5], we have that for G_7, there exists a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$ with e_3 timelike such that the Lie algebra of G_7 satisfies

$$\begin{aligned}
&[e_1,e_2] = -\alpha e_1 - \beta e_2 - \beta e_3, \quad [e_1,e_3] = \alpha e_1 + \beta e_2 + \beta e_3,\\
&[e_2,e_3] = \gamma e_1 + \delta e_2 + \delta e_3, \quad \alpha+\delta \neq 0,\ \alpha\gamma = 0.
\end{aligned} \tag{2.79}$$

Theorem 2.10 (1) (G_7, g, J) *is the first kind algebraic Ricci soliton associated to the connection* ∇^0 *if and only if*

(i) $\alpha = \gamma = c = 0$, $\beta \neq 0$, $\delta \neq 0$. *In particular,*

$$D\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{2.80}$$

(ii) $\beta = \gamma = 0$, $\alpha^2 + c = 0$, $\alpha + \delta \neq 0$. *In particular,*

$$D\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \alpha^2 & \alpha^2 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{2.81}$$

(2) (G_7, g, J) *is the second kind algebraic Ricci soliton associated to the connection* ∇^0 *if and only if*

(i) $\alpha = \gamma = c = 0$, $\delta \neq 0$. *In particular,*

$$D\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{2.82}$$

(ii) $\alpha \neq 0$, $\beta = \gamma = 0$, $\frac{\alpha^2}{2} + c = 0$, $\alpha + \delta \neq 0$. *In particular,*

$$D\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} -\frac{\alpha^2}{2} & 0 & 0 \\ 0 & -\frac{\alpha^2}{2} & -\frac{\alpha^2}{2} \\ 0 & \frac{\alpha^2}{2} & \frac{\alpha^2}{2} \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{2.83}$$

(3) (G_7, g, J) *is the first kind algebraic Ricci soliton associated to the connection* ∇^1 *if and only if* $\alpha \neq 0$, $\beta = \gamma = 0$, $\alpha = 2\delta$, $c = -3\delta^2$. *In particular*

$$D\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} -\delta^2 & 0 & 0 \\ 0 & -\delta^2 & -\delta^2 \\ 0 & 3\delta^2 & 3\delta^2 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{2.84}$$

(4) (G_7, g, J) *is the second kind algebraic Ricci soliton associated to the connection* ∇^1 *if and only if* $\alpha \neq 0$, $\beta = \gamma = 0$, $\alpha = 2\delta$, $\alpha^2 + 2c = 0$. *In particular*

$$D\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} -\frac{\alpha^2}{2} & 0 & 0 \\ 0 & -\frac{\alpha^2}{2} & -\frac{\alpha^2}{2} \\ 0 & \frac{\alpha^2}{2} & \frac{\alpha^2}{2} \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{2.85}$$

Proof The canonical connection ∇^0 of (G_7, J) is given by

$$\begin{aligned} &\nabla^0_{e_1} e_1 = \alpha e_2, \quad \nabla^0_{e_1} e_2 = -\alpha e_1, \quad \nabla^0_{e_1} e_3 = 0, \\ &\nabla^0_{e_2} e_1 = \beta e_2, \quad \nabla^0_{e_2} e_2 = -\beta e_1, \quad \nabla^0_{e_2} e_3 = 0, \\ &\nabla^0_{e_3} e_1 = -\Big(\beta - \frac{\gamma}{2}\Big) e_2, \quad \nabla^0_{e_3} e_2 = \Big(\beta - \frac{\gamma}{2}\Big) e_1, \quad \nabla^0_{e_3} e_3 = 0. \end{aligned} \tag{2.86}$$

By (2.7) and (2.86), we have that the curvature R^0 of the canonical connection ∇^0 of (G_7, J) is given by

$$\begin{aligned} &R^0(e_1,e_2)e_1=\Big(\alpha^2+\frac{\beta\gamma}{2}\Big)e_2, \quad R^0(e_1,e_2)e_2=-\Big(\alpha^2+\frac{\beta\gamma}{2}\Big)e_1, \quad R^0(e_1,e_2)e_3=0,\\ &R^0(e_1,e_3)e_1=-\Big(\alpha^2+\frac{\beta\gamma}{2}\Big)e_2, \quad R^0(e_1,e_3)e_2=\Big(\alpha^2+\frac{\beta\gamma}{2}\Big)e_1, \quad R^0(e_1,e_3)e_3=0,\\ &R^0(e_2,e_3)e_1=-\Big(\gamma\alpha+\frac{\delta\gamma}{2}\Big)e_2, \quad R^0(e_2,e_3)e_2=\Big(\gamma\alpha+\frac{\delta\gamma}{2}\Big)e_1, \quad R^0(e_2,e_3)e_3=0. \end{aligned} \tag{2.87}$$

By (2.9), (2.11) and (2.87), we get for (G_7, ∇^0),

$$\mathrm{Ric}^0\begin{pmatrix} e_1\\ e_2\\ e_3 \end{pmatrix}=\begin{pmatrix} -(\alpha^2+\frac{\beta\gamma}{2}) & 0 & 0\\ 0 & -(\alpha^2+\frac{\beta\gamma}{2}) & 0\\ -(\gamma\alpha+\frac{\delta\gamma}{2}) & (\alpha^2+\frac{\beta\gamma}{2}) & 0 \end{pmatrix}\begin{pmatrix} e_1\\ e_2\\ e_3 \end{pmatrix}. \tag{2.88}$$

If (G_7, g, J) is the first kind algebraic Ricci soliton associated to the connection ∇^0, then $\mathrm{Ric}^0 = c\mathrm{Id} + D$, so

$$\begin{cases} De_1=-\Big(\alpha^2+\dfrac{\beta\gamma}{2}+c\Big)e_1,\\ De_2=-\Big(\alpha^2+\dfrac{\beta\gamma}{2}+c\Big)e_2,\\ De_3=-\Big(\gamma\alpha+\dfrac{\delta\gamma}{2}\Big)e_1+\Big(\alpha^2+\dfrac{\beta\gamma}{2}\Big)e_2-ce_3. \end{cases} \tag{2.89}$$

By (2.15) and (2.89), we get

$$\begin{cases} \alpha\Big(\alpha^2+\dfrac{\beta\gamma}{2}+c\Big)-\beta\Big(\gamma\alpha+\dfrac{\delta\gamma}{2}\Big)=0,\\ \beta(2\alpha^2+\beta\gamma+c)=0,\\ c\gamma+(\alpha-\delta)\Big(\gamma\alpha+\dfrac{\delta\gamma}{2}\Big)=0,\\ \Big(\alpha^2+\dfrac{\beta\gamma}{2}+c\Big)\delta+\beta\Big(\gamma\alpha+\dfrac{\delta\gamma}{2}\Big)=0. \end{cases} \tag{2.90}$$

Solving (2.90), we get the case (1). The other cases hold similarly.

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