Chin. Ann. Math. Ser. B
43(3), 2022, 459–472
DOI: 10.1007/s11401-022-0335-4

Chinese Annals of Mathematics, Series B

Four Families of Nontrivial Product Elements in the Stable Homotopy Groups of Spheres*

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Abstract In this paper, the authors introduce a new effective method to compute the generators of the E_1-term of the May spectral sequence. This helps them to obtain four families of non-trivial product elements in the stable homotopy groups of spheres.

Keywords Stable homotopy groups of spheres, Adams spectral sequence, May spectral sequence
2000 MR Subject Classification 55Q45, 55T15, 55S10

1 Introduction

Let S be the sphere spectrum localized at an odd prime p and $\mathcal{A}$ be the mod p Steenrod algebra. To determine the stable homotopy groups of spheres π_*S is one of the central problems in homotopy theory. Up to now, several methods have been found to approach it. For example, we have the classical Adams spectral sequence (ASS for short) (see [1]) $\mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathbb{Z}_p,\mathbb{Z}_p)$ based on the Eilenberg-Maclane spectrum $K\mathbb{Z}_p$, whose E_2-term is the cohomology of $\mathcal{A}$. Thus in order to compute the stable homotopy groups of spheres with the classical ASS, we need to compute the E_2-term of the Adams spectral sequence. We also have the Adams-Novikov spectral sequence (ANSS for short) (see [1, 11]) based on the Brown-Peterson spectrum BP.

From [8], we know that $\mathrm{Ext}_{\mathcal{A}}^{1,*}(\mathbb{Z}_p,\mathbb{Z}_p)$ has $\mathbb{Z}_p$-basis consisting of $a_0\in\mathrm{Ext}_{\mathcal{A}}^{1,1}(\mathbb{Z}_p,\mathbb{Z}_p)$, $h_i\in\mathrm{Ext}_{\mathcal{A}}^{1,p^iq}(\mathbb{Z}_p,\mathbb{Z}_p)$ for all $i\geq 0$ and $\mathrm{Ext}_{\mathcal{A}}^{2,*}(\mathbb{Z}_p,\mathbb{Z}_p)$ has $\mathbb{Z}_p$-basis consisting of α_2, a_0^2, a_0h_i $(i>0)$, g_i $(i\geq 0)$, k_i $(i\geq 0)$, b_i $(i\geq 0)$ and h_ih_j $(j\geq i+2, i\geq 0)$ whose internal degrees are $2q+1$, 2, p^iq+1, $q(p^{i+1}+2p^i)$, $q(2p^{i+1}+p^i)$, $p^{i+1}q$ and $q(p^i+p^j)$, respectively. Aikawa [2] gave the generators of $\mathrm{Ext}_{\mathcal{A}}^{3,*}(\mathbb{Z}_p,\mathbb{Z}_p)$ for $p>2$.

Let $q=2(p-1)$ and M be the mod p Moore spectrum given by the cofibration

$$S\xrightarrow{p}S\xrightarrow{i_0}M\xrightarrow{j_0}\Sigma S. \tag{1.1}$$

Manuscript received January 7, 2020. Revised December 27, 2021.
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*This work was supported by the National Natural Science Foundation of China (Nos. 12001474, 12171165), Guangdong Natural Science Foundation (Nos. 2020A1515011008, 2021A1515010374) and the "13th Five-Year" Science and Technology Project of Jilin Department of Education (No. JJKH20200508KJ).

Let $\alpha : \sum^q M \to M$ be the Adams map and $V(1)$ be its cofibre given by the cofibration

$$\Sigma^q M \xrightarrow{\alpha} M \xrightarrow{i_1} V(1) \xrightarrow{j_1} \Sigma^{q+1} M. \tag{1.2}$$

Let $V(2)$ be the cofibre of $\beta : \sum^{(p+1)q} V(1) \to V(1)$ given by the cofibration

$$\Sigma^{(p+1)q} V(1) \xrightarrow{\beta} V(1) \xrightarrow{i_2} V(2) \xrightarrow{j_2} \Sigma^{(p+1)q+1} V(1). \tag{1.3}$$

Let $V(3)$ be the cofibre of $\gamma : \sum^{(p^2+p+1)q} V(2) \to V(2)$ given by the cofibration

$$\Sigma^{2(p^3-1)} V(2) \xrightarrow{\gamma} V(2) \xrightarrow{i_3} V(3) \xrightarrow{j_3} \Sigma^{2p^3-1} V(2). \tag{1.4}$$

Especially, the composite

$$S^{(1+p)q} \to \Sigma^{(p+1)q} V(1) \xrightarrow{\beta} V(1) \to \Sigma^{q+1} M \to \Sigma^{q+2} S \tag{1.5}$$

is $\beta_1 \in \pi_{pq-2} S$. In [3], Cohen showed that $h_0 b_{n-1}$ survives to E_∞ in the Adams spectral sequence and converges to a non-trivial element $\zeta_n \in \pi_{p^n q+q-3} S$. By the method of ANSS, Lee [4] showed that $\beta_1^{p-1} \zeta_n$ is a non-trivial element of $\pi * S$ for $n \geq 1$, i.e., $b_0^{p-1} h_0 b_n$ is a permanent cycle in the Adams spectral sequence and converges non-trivially to $\beta_1^{p-1} h_0 b_n$.

For $s \geq 1$, we define the α-element $\alpha_s = j_0 \alpha^s i_0 \in \pi_{sq-1} S$, the β-element $\beta_s = j_0 j_1 \beta^s i_1 i_0 \in \pi_{q(sp+(s-1))-2} S$ and the γ-element $\gamma_s = j_0 j_1 j_2 \gamma^s i_2 i_1 i_0 \in \pi_{q(sp^2+(s-1)p+(s-2))-3} S$. In [13], it was shown that these three families of elements are represented by the n-th Greek letter elements $\alpha_s^{(n)}$ in the Adams spectral sequence, where $\alpha_s^{(n)}$ is

$$\alpha_s^{(n)} = \widetilde{i} \wedge \underbrace{\widetilde{v}_n \wedge \cdots \wedge \widetilde{v}_n}_{s} \wedge \widetilde{j} \in \mathrm{Ext}_{\mathcal{A}}^{*,*}(\mathbb{Z}_p, \mathbb{Z}_p), \tag{1.6}$$

where $\wedge$ denotes the Yoneda product and $\widetilde{i} = 1[\ ]1 \in \mathrm{Ext}_{\mathcal{A}}^{0,*}(E(n-1), \mathbb{Z}_p)$, $\widetilde{j} = Q_0 \cdots Q_{n-1}[\ ]1 \in \mathrm{Ext}_{\mathcal{A}}^{0,*}(\mathbb{Z}_p, E(n-1))$. By sending $\alpha_s^{(n)}$ back to the E_1-term of the May spectral sequence, we see that

$$\alpha_s^{(n)} = \frac{s!}{(s-n)!} a_n^{s-n} h_{n,0} h_{n-1,1} \cdots h_{1,n-1}, \tag{1.7}$$

where $\frac{s!}{(s-n)!} \not\equiv 0$ for $s \not\equiv 0, 1, \cdots, n-1 \pmod p$.

Based on the above results, we should naturally ask that whether the product element $\alpha_s^{(n)} x$ for some $x \in \mathrm{Ext}_{\mathcal{A}}^{*,*}(\mathbb{Z}_p, \mathbb{Z}_p)$ converges to a nontrivial element of $\pi_* S$ or not. In [12], it was shown that the product $\widetilde{\beta}_s h_0 b_{n-1}$ ($n \geq 2$, $2 \leq s \leq p-1$) is a permanent cycle in the Adams spectral sequence and converges to a nontrivial element of order p in $\pi_* S$. In [13], Wang-Zheng verified the convergences of $\widetilde{\beta}_s h_0 h_n$ ($p \geq 5, 2 \leq s \leq p-1, n \geq 2$) and $\widetilde{\gamma}_s h_0 h_n$ ($p \geq 7, 3 \leq s \leq p-1, n \geq 3$). Recently, in [5–6], Liu verified the convergences of $\widetilde{\beta}_s h_0 h_n b_0$ ($p > 3, n \geq 3, 1 < s < p-2$) and $\widetilde{\gamma}_s h_0 h_n b_0$ ($p \geq 7, n \geq 4, 3 \leq s < p-2$). In order to detect more nontrivial product elements in $\pi_* S$, we will apply a more effective computation method to generalize the corresponding results in [5–7, 13–14] to obtain four families of nontrivial homotopy elements in $\pi_* S$. Our main results are stated as follows.

Theorem 1.1 (1) *Let $p > 5, k > 0, n \geq 3$ and $s > 1$. Then $\widetilde{\beta}_s h_0 h_n b_0^k$ $(3 < s + 2k + 2 < p)$ and $\widetilde{\beta}_s h_0 b_{n-1} b_0^k$ $(4 < s + 2k + 3 < p)$ are permanent cycles in the Adams spectral sequence and converge to nontrivial elements in $\pi_* S$.*

(2) *Let $p > 7, k > 0, n \geq 4$ and $s > 2$. Then $\widetilde{\gamma}_s h_0 h_n b_0^k$ $(4 < s+2k+2 < p)$ and $\widetilde{\gamma}_s h_0 b_{n-1} b_0^k$ $(5 < s+2k+3 < p)$ are permanent cycles in the Adams spectral sequence and converge to nontrivial elements in $\pi_* S$.*

This paper is organized as follows. In Section 2, after recalling some knowledge on the May spectral sequence, we introduce a better method which is used to compute the generators of E_1-term of the May spectral sequence. Then in Section 3, we use this method to give some important results on Ext-groups which are then applied to give the proof of our main Theorem 1.1.

2 Detecting Generators in E_1-Term of MSS

It is well known that the most successful method for computing $\mathrm{Ext}_{\mathcal{A}}^{*,*}(\mathbb{Z}_p, \mathbb{Z}_p)$ is the May spectral sequences. Let $\mathcal{A}_*$ denote the dual algebra of mod p Steenrod algebra $\mathcal{A}$. Milnor [10] showed that, as a Hopf algebra

$$\mathcal{A}_* = P[\xi_1, \xi_2, \cdots] \otimes E[\tau_0, \tau_1, \cdots],$$

where $P[\ ]$ is the polynomial algebra and $E[\ ]$ is the exterior algebra. The secondary degrees of ξ_i and τ_i are $2(p^i - 1)$ and $2(p^i-1)+1$, respectively. The coproduct $\Delta\colon \mathcal{A}_* \to \mathcal{A}_* \otimes \mathcal{A}_*$ is given by

$$\Delta(\xi_n) = \xi_n \otimes 1 + 1 \otimes \xi_n + \sum_{i=1}^{n-1} \xi_{n-i}^{p^i} \otimes \xi_i$$

and

$$\Delta(\tau_n) = \tau_n \otimes 1 + 1 \otimes \tau_n + \sum_{i=0}^{n-1} \xi_{n-i}^{p^i} \otimes \tau_i.$$

Let $\varepsilon\colon \mathcal{A}_* \to \mathbb{Z}/p$ be the argumentation homomorphism and $\overline{\mathcal{A}}_*$ =Ker ε which is called the argumentation ideal of $\mathcal{A}_*$. It then follows a bigraded cochain complex $(C^{*,*}(H^*S), \mathrm{d}) = (C^{*,*}(\mathbb{Z}/p), \mathrm{d})$, where $C^{*,*}(\mathbb{Z}/p)$ is the cobar construction with s-filtration

$$C^{s,*}(\mathbb{Z}/p) = \underbrace{\overline{\mathcal{A}}_* \otimes \cdots \otimes \overline{\mathcal{A}}_*}_{s}$$

and the differential $\mathrm{d}\colon C^{s,t}(\mathbb{Z}/p) \to C^{s+1,t}(\mathbb{Z}/p)$ is given by

$$\mathrm{d}(\alpha_1 \otimes \cdots \otimes \alpha_s) = \sum_{i=1}^{s} (-1)^{\lambda(i)+1} \alpha_1 \otimes \cdots \otimes (\Delta(\alpha_i) - \alpha_i \otimes 1 - 1 \otimes \alpha_i) \otimes \cdots \otimes \alpha_s, \tag{2.1}$$

where $\lambda(i)$ is the total degree of $\alpha_1 \otimes \cdots \otimes \alpha_i^{'}$ if $\Delta(\alpha_i) - \alpha_i \otimes 1 - 1 \otimes \alpha_i = \Sigma \alpha_i^{'} \otimes \alpha_i^{''}$ (refer to [9]). For example, we have differentials $\mathrm{d}(\xi_2^{p^i}) = \xi_1^{p^{i+1}} \otimes \xi_1^{p^i}$ and $\mathrm{d}(\xi_1^{2p^i}) = 2\xi_1^{p^i} \otimes \xi_1^{p^i}$.

According to the above statements, the cohomology of $C^{*,*}(\mathbb{Z}/p)$ is

$$H^{s,t}(C^{*,*}(\mathbb{Z}/p), \mathrm{d}) = \mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathbb{Z}/p, \mathbb{Z}/p),$$

which is the E_2-term of the ASS. From [8] we know that

$$a_0 = \{\tau_0\}, \quad h_i = \{\xi_1^{p^i}\}, \quad \widetilde{\alpha}_2 = \{2\xi_1 \otimes \tau_1 + \xi_1^2 \otimes \tau_0\},$$

$$g_i = \Big\{\xi_2^{p^i} \otimes \xi_1^{p^i} + \frac{1}{2}\xi_1^{p^{i+1}} \otimes \xi_1^{2p^i}\Big\}, \quad k_i = \Big\{\xi_1^{p^{i+1}} \otimes \xi_2^{p^i} + \frac{1}{2}\xi_1^{2p^{i+1}} \otimes \xi_1^{p^i}\Big\}$$
$$b_i = \Big\{\sum_{j=1}^{p-1} \binom{p}{j}\Big/ p(\xi_1^{p^i(p-j)} \otimes \xi_1^{p^i j})\Big\}$$

are generators of $\mathrm{Ext}_{\mathcal{A}}^{s,t}(\mathbb{Z}/p, \mathbb{Z}/p)$.

Based on [11, Theorem 3.2.5], we can set a May filtration on $\mathcal{A}_*$ by letting $M(\tau_{i-1}) = M(\xi_i^{p^j}) = 2i-1$. It induces a corresponding filtration

$$F^0 \subseteq F^1 \subseteq \cdots \subseteq F^{M-1} \subseteq F^M \subseteq \cdots \subseteq \mathcal{A}_*. \tag{2.2}$$

It was shown that for the associated bigraded Hopf algebra

$$E^0\mathcal{A}_* = \bigoplus (F^M/F^{M-1}),$$

there is an isomorphism

$$E^0\mathcal{A}_* \cong E[\tau_i | i \geq 0] \otimes T[\xi_{i,j} | i > 0, j \geq 0],$$

where $T[\ ]$ denotes the truncated polynomial algebra of height p on the indicated generators, τ_i and $\xi_{i,j}$ are the projections of τ_i and $\xi_i^{p^i}$, respectively. Applying the filtration (2.2) to the cobar construction $C^{*,*}(\mathbb{Z}/p)$, we obtain a filtration

$$F^{*,*,0} \subseteq F^{*,*,1} \subseteq \cdots \subseteq F^{*,*,M-1} \subseteq F^{*,*,M} \subseteq \cdots \subseteq C^{*,*}(\mathbb{Z}/p). \tag{2.3}$$

It follows an tri-graded exact couple which induces the so-called May spectral sequence (MSS for short)

$$\{E_r^{s,t,M}, \mathrm{d}_r\} \Longrightarrow \mathrm{Ext}_{\mathcal{A}}^{*,*}(\mathbb{Z}/p, \mathbb{Z}/p),$$

where $\mathrm{d}_r \colon E_r^{s,t,M} \to E_r^{s+1,t,M-r}$ is the r^{th} differential of the MSS. Since the MSS converges to the E_2-term of the ASS, the nontriviality of the elements in $\mathrm{Ext}_{\mathcal{A}}^{*,*}(\mathbb{Z}/p, \mathbb{Z}/p)$ is equivalent to showing that its representation in the MSS is an infinite cycle.

The E_0-term of the MSS is $C^{*,*}(E^0\mathcal{A}_*) = \bigoplus(F^{*,*,M}/F^{*,*,M})$ and the E_1-term $E_1 = H^*(E_0\mathcal{A}_*, \mathrm{d}_0)$ is isomorphic to

$$E[h_{i,j} | i > 0, j \geq 0] \otimes P[b_{i,j} | i > 0, j \geq 0] \otimes P[a_i | i \geq 0],$$

where

$$h_{i,j} \in E_1^{1,2(p^i-1)p^j,2i-1}, \quad b_{i,j} \in E_1^{2,2(p^i-1)p^{j+1},p(2i-1)}$$

and

$$a_i \in E_1^{1,2(p^i-1)+1,2i+1}.$$

In the filtrated cobar complexes, $h_{i,j}$, $b_{i,j}$ and a_i are represented by

$$\xi_i^{p^j}, \quad \sum_{k=1}^{p-1} \binom{p}{k}\Big/ p\xi_i^{kp^j} \otimes \xi_i^{(p-k)p^j} \quad \text{and} \quad \tau_i,$$

respectively. It is known that the generators $h_{1,i}$, $b_{1,i}$ and a_0 converge to h_i, b_i, $a_0 \in \mathrm{Ext}_{\mathcal{A}}^{*,*}(\mathbb{Z}/p, \mathbb{Z}/p)$, respectively. In the May spectral sequence, one has the r-th May differentail

$$\mathrm{d}_r(xy) = \mathrm{d}_r(x)y + (-1)^s x\mathrm{d}_r(y)$$

for $x \in E_r^{s,t,M}$ and $y \in E_r^{s',t',M'}$. The first May differential d_1 is given by

$$\mathrm{d}_1(h_{i,j}) = \sum_{0<k<i} h_{i-k,k+j}h_{k,j}, \quad \mathrm{d}_1(a_i) = \sum_{0\le k<i} h_{i-k,k}a_k, \quad \mathrm{d}_1(b_{i,j}) = 0.$$

There exists the graded commutativity in the E_1-term of MSS:

$$x \cdot y = (-1)^{(s+t)(s'+t')} y \cdot x \quad \text{for } x \in E_1^{s,t,*} \text{ and } y \in E_1^{s',t',*}.$$

For each element $x \in E_1^{s,t,M}$, we define $\dim x = s$, $\deg x = t$. Then we have

$$\begin{cases} \dim h_{i,j} = \dim a_i = 1, \dim b_{i,j} = 2, \\ \deg h_{i,j} = 2(p^i - 1)p^j = q(p^{i+j-1} + \cdots + p^j), \\ \deg b_{i,j} = 2(p^i - 1)p^{j+1} = q(p^{i+j} + \cdots + p^{j+1}), \\ \deg a_i = 2p^i - 1 = q(p^{i-1} + \cdots + 1) + 1, \\ \deg a_0 = 1, \end{cases} \tag{2.4}$$

where $i \ge 1$, $j \ge 0$.

In the follows, we introduce a new method to compute the generators of the E_1-term of MSS. According to [15], we denote $a_i, h_{i,j}$ and $b_{i,j}$ by x, y and z, respectively. By the graded commutativity of $E_1^{*,*,*}$, we consider a generator

$$g = (x_1, \cdots, x_b)(y_1, \cdots, y_m)(z_1, \cdots, z_l) \in E_1^{b+m+2l,t+b,*},$$

where $t = (\overline{c}_0 + \overline{c}_1 p + \cdots + \overline{c}_n p^n)q$ with $0 \le \overline{c}_i < p$ $(\overline{c}_n > 0), 0 < b < q$.

We define the polynomial algebra

$$\widetilde{E}_1^{*,*,*} = P[h_{i,j} | i > 0, j \ge 0] \otimes P[b_{i,j} | i > 0, j \ge 0] \otimes P[a_i | i \ge 0].$$

Then there is the obvious identification $E_1^{*,*,*} = \widetilde{E}_1^{*,*,*}/(h_{i,j}^2)$. Furthermore, if $b_{i,j}$ is replaced by $h_{i,j+1}$, then we get

$$F_1^{*,*,*} = P[a_i | i \ge 0] \otimes P[h_{i,j} | i > 0, j \ge 0].$$

By the graded commutativity of $F_1^{*,*,*}$, we can consider a generator

$$g = (x_1, \cdots, x_b)(y_1, \cdots, y_m) \in F_1^{b+m,t+b,*},$$

where $t = (\overline{c}_0 + \overline{c}_1 p + \cdots + \overline{c}_n p^n)q$ with $0 \le \overline{c}_i < p$ $(\overline{c}_n > 0), 0 < b < q$. Note that the degrees of x_i and y_i can be uniquely expressed as:

$$\begin{aligned} \deg x_i &= q(x_{i,0} + x_{i,1}p + \cdots + x_{i,n}p^n) + 1, \\ \deg y_i &= q(y_{i,0} + y_{i,1}p + \cdots + y_{i,n}p^n). \end{aligned}$$

Then the generator g determines a matrix

$$\begin{array}{c} \qquad A \qquad\qquad\qquad B \\ \left(\begin{array}{ccc|ccc} x_{1,0} & \cdots & x_{b,0} & y_{1,0} & \cdots & y_{m,0} \\ x_{1,1} & \cdots & x_{b,1} & y_{1,1} & \cdots & y_{m,1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_{1,n} & \cdots & x_{b,n} & y_{1,n} & \cdots & y_{m,n} \end{array}\right) \begin{array}{c} c_0 \\ c_1 \\ \vdots \\ c_n, \end{array} \end{array} \tag{2.5}$$

where $(x_{i,0}, x_{i,1}, \cdots, x_{i,n})$ is of the form $(1, \cdots, 1, 0, \cdots, 0)$ and the transposition $(\overbrace{1, \cdots, 1}^{i}, 0, \cdots, 0)^{\mathrm{T}}$ represents a_i in part A. Meanwhile, $(y_{i,0}, y_{i,1}, \cdots, y_{i,n})$ is of the form $(0, \cdots, 0, 1, \cdots, 1, 0, \cdots, 0)$ and $(\overbrace{0, \cdots, 0}^{j}, \overbrace{1, \cdots, 1}^{i}, 0, \cdots, 0)^T$ represents $h_{i,j}$ in part B.

By the graded commutativity of $F_1^{*,*,*}$, matrix (2.5) can always be transformed into a new one by the interchange of columns in part A and B respectively, and the entries $x_{i,j}, y_{i,j}$ in the new matrix satisfy the following conditions:

$$
\begin{aligned}
&(1)\ x_{1,j} \geq x_{2,j} \geq \cdots \geq x_{b,j}, x_{i,0} \geq x_{i,1} \geq x_{i,n} \text{ for } i \leq b \text{ and } j \leq n.\\
&(2)\ \text{If } y_{i,j-1} = 0 \text{ and } y_{i,j} = 1, \text{ then for all } k < j, y_{i,k} = 0.\\
&(3)\ \text{If } y_{i,j} = 1 \text{ and } y_{i,j+1} = 0, \text{ then for all } k > j, y_{i,k} = 0.\\
&(4)\ y_{1,0} \geq y_{2,0} \geq \cdots \geq y_{m,0}.\\
&(5)\ \text{If } y_{i,0} = y_{i+1,0}, y_{i,1} = y_{i+1,1}, \cdots, y_{i,j} = y_{i+1,j}, \text{ then } y_{i,j+1} \geq y_{i+1,j+1}.
\end{aligned}
\tag{2.6}
$$

By the properties of the p-adic number and the reason of the second degree, we have the following equations

$$
\begin{cases}
x_{1,0} + \cdots + x_{b,0} + y_{1,0} + \cdots + y_{m,0} = \overline{c}_0 + \lambda_1 p = c_0\\
x_{1,1} + \cdots + x_{b,1} + y_{1,1} + \cdots + y_{m,1} = \overline{c}_1 - \lambda_1 + \lambda_2 p = c_1\\
\qquad\qquad\qquad\qquad\vdots\\
x_{1,n-1} + \cdots + x_{b,n-1} + y_{1,n-1} + \cdots + y_{m,n-1} = \overline{c}_{n-1} - \lambda_{n-1} + \lambda_n p = c_{n-1},\\
x_{1,n} + \cdots + x_{b,n} + y_{1,n} + \cdots + y_{m,n} = \overline{c}_n - \lambda_n = c_n,
\end{cases}
\tag{2.7}
$$

where $\lambda_i \geq 0$, the integer sequence $c = (c_0, c_1, \cdots, c_n)$ is determined by $(\overline{c}_0, \overline{c}_1, \cdots, \overline{c}_n)$ and the carry sequence $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$.

We want to get the solutions of (2.7) which satisfy the conditions (2.6). Unfortunately, the matrix solutions do not always detect generators of $E_1^{b+m,t+b,*}$. In order to achieve our object, we introduce the following diagram which shows our manipulation during computing the generators of E_1-term:

$$
\begin{tikzcd}
 & & & F_1^{b+\widetilde{m}} \arrow[dlll, "*_1"'] \arrow[dll, "*_1"'] \arrow[d, "*_1"] \arrow[drr, "*_1"] & & \\
F_1^{b+\widetilde{m}} \arrow[r] \arrow[drrr, "*_2"'] & F_1^{b+\widetilde{m}+1} \arrow[r] \arrow[drr, "*_2"'] & \cdots \arrow[r] & F_1^{b+\widetilde{m}+k} \arrow[r] \arrow[d, "*_2"] & \cdots \arrow[r] & F_1^s, \arrow[dll, "*_2"] \\
 & & & E_1^s & &
\end{tikzcd}
\tag{2.8}
$$

where $*_1$ denotes the resolution $h_{i,j} \to h_{i-k,j+k} h_{k,j}$ and $a_i \to a_{i-j} h_{j,i-j}$, $*_2$ denotes the replacement $h_{i,j+1} \to b_{i,j}$, $k \geq 0$, $\widetilde{m} \leq s - b$.

From the discussion above, the determination of $E_1^{s,t+b,*}$ is reduced to the following steps:

S1: Express $\frac{t}{q}$ by the p-adic number, and then $t = (\overline{c}_0 + \overline{c}_1 p + \cdots + \overline{c}_n p^n) q$.

S2: List up all the possibles of sequence λ in the corresponding sequence c.

S3: For each sequence c, we can solve (2.7) which satisfy the conditions (2.6). Thus, we get all generators of $F_1^{b+\widetilde{m},t+b,*}$.

S4: Through replacement $h_{i,j+1} \to b_{i,j}$ and resolution

$$h_{i,j} \to h_{i-k,j+k}h_{k,j} \quad \text{or} \quad a_i \to a_{i-j}h_{j,i-j},$$

we can get all generators of $E_1^{s,t+b,*}$.

In what follows, we will apply the above method to compute the generators of certain E_1-terms of MSS. For convenience, we write $t_1 = p^nq + (s+k)pq + sq$ and $t_2 = p^nq + sp^2q + (s+k-1)pq + (s-1)q$ with $k > 0$.

Proposition 2.1 *Let $p > 5$ and $n \geq 3$. Then we have*
(1) *for $s > 1$, $3 < s + 2k + 2 < p$ and $1 \leq r \leq s + 2k + 2$, there is*

$$E_1^{s+2k+2-r,t_1+(s-2)-(r-1),*} = 0;$$

(2) *for $s > 1$, $4 < s + 2k + 3 < p$ and $1 \leq r \leq s + 2k + 3$, there is*

$$E_1^{s+2k+3-r,t_1+(s-2)-(r-1),*} = \begin{cases} \mathbb{Z}_p\{a_2^{s-2}h_{2,0}h_{1,1}h_{1,0}h_{1,n}b_{1,0}^k\}, & r = 1; \\ 0, & \text{others.} \end{cases}$$

Proof (1) Consider the generator $g \in E_1^{s+2k+2-r,t_1+(s-2)-(r-1),*}$ for $1 \leq r \leq s + 2k + 2$, where $t_1 = p^nq + (s+k)pq + sq$ with

$$(\overline{c}_0, \cdots, \overline{c}_n) = (s, s+k, 0, \cdots, 0, 1).$$

Then we have dim $g = s + 2k + 2 - r$ and deg $g = t_1 + (s - r - 1)$.

If $s - 1 < r \leq s + 2k + 2$, then $s - r - 1 < 0$. So there are at least $(s - r - 1 + q)$ $a_i^{'}$s in g, by the reason of dimension, this is impossible. Now, we can assume that $1 \leq r \leq s - 1$, which follows $s - r - 1 \geq 0$.

Since $s+2k+2-r < s-r-1+q$, the number of x_i in g is $(s-r-1)$. Consider the generator $g = x_1 \cdots x_{s-r-1}y_1 \cdots y_m$. From (2.7), one easily gets that the sequence $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ equals $(0, 0, \cdots, 0)$ and the corresponding $c = (s, s+k, 0, \cdots, 1)$.

Case 1.1 $r = 1$. Solve (2.7) by virtue of (2.6), we get the following matrix:

$$\begin{array}{c} \begin{array}{ccccccccc} & s-2 & & & & & s+k & & \end{array} \\ \left(\begin{array}{ccc|ccccccc} 1 & \cdots & 1 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{array}\right) \begin{array}{l} s \\ s+k \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1. \end{array} \end{array}$$

It detects the generator $a_2^{s-2}h_{2,0}h_{2,0}h_{1,1}^kh_{1,n} \in F_1^{s+k+1,t_1+(s-2),*}$. By the replacement $h_{11}^k \to b_{10}^k$, we get the generator $a_2^{s-2}h_{2,0}h_{2,0}b_{1,0}^kh_{1,n}$, which is zero for $h_{2,0}^2 = 0$ in $E_1^{s+2k+1,t_1+(s-2),*}$.

Case 1.2 $r \geq 2$. Similar to Case 1.1, we detect the generator

$$a_2^{s-r-1}h_{2,0}^{r+1}h_{1,1}^kh_{1,n} \in F_1^{s+k+1,t_1+(s-2),*}.$$

By the replacement $h_{11}^k \to b_{10}^k$, we get the generator $a_2^{s-r-1}h_{2,0}^{r+1}b_{1,0}^k h_{1,n}$, which is zero due to $h_{2,0}^2 = 0$ in $E_1^{s+2k+1,t_1+(s-2),*}$.

Combining Cases 1.1 and 1.2 gives the desired result.

(2) Similar to (1), we consider the generator

$$g = x_1 \cdots x_{s-r-1} y_1 \cdots y_m \in E_1^{s+2k+3-r,t_1+(s-r-1),*}$$

for $1 \le r \le s+2k+2$, where $t_1 = p^n q + (s+k)pq + sq$ with

$$(\overline{c}_0, \cdots, \overline{c}_n) = (s, s+k, 0, \cdots, 0, 1).$$

Then we have $\dim g = s+2k+3-r$ and $\deg g = t_1 + (s-r-1)$. From (2.7), one can easily get the carry sequence

$$\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) = (0, 0, \cdots, 0)$$

and correspondingly $c = (s, s+k, 0, \cdots, 0)$.

Case 2.1 $r = 1$. Solve (2.7) by virtue of (2.6), we get the generator

$$a_2^{s-2}h_{2,0}h_{2,0}h_{1,1}^k h_{1,n} \in F_1^{s+k+1,t_1+s-2,*}.$$

By the resolution $h_{2,0} \to h_{1,1}h_{1,0}$, we get the generator

$$a_2^{s-2}h_{2,0}h_{1,0}h_{1,1}^{k+1}h_{1,n} \in F_1^{s+k+2,t_1+s-2,*}.$$

By the replacement $h_{1,1}^k \to b_{1,0}^k$, we get the generator

$$a_2^{s-2}h_{2,0}h_{1,0}h_{1,1}b_{1,0}^k h_{1,n} \in E_1^{s+2k+2,t_1+s-2,*}.$$

Case 2.2 $r \ge 2$. Similar to Case 2.1, we detect the generator $a_2^{s-r-1}h_{2,0}^{r+1}h_{1,1}^k h_{1,n} \in F_1^{s+k+1,t_1+(s-2),*}$. By the replacement $h_{1,1}^k \to b_{1,0}^k$ and $h_{2,0} \to h_{1,1}h_{1,0}$, we get the generator $a_2^{s-r-1}h_{2,0}^r h_{1,1}h_{1,0}b_{1,0}^k h_{1,n}$, which is zero for $h_{2,0}^2 = 0$ in $E_1^{s+2k+2,t_1+(s-2),*}$.

Combining Case 2.1 with Case 2.2 gives the desired result.

Proposition 2.2 *Let $p > 7$, $n \ge 4$ and $s > 2$. Then we have*

(1) *for $4 < s+2k+2 < p$ and $1 \le r \le s+2k+2$, there is*

$$E_1^{s+2k+2-r,t_2+(s-3)-(r-1),*} = \begin{cases} \mathbb{Z}_p\{a_3^{s-3}h_{3,0}h_{2,1}h_{1,2}h_{1,1}h_{1,0}h_{1,n}b_{1,0}^{k-1}\}, & r = 1; \\ 0, & \text{others.} \end{cases}$$

(2) *for $5 < s+2k+3 < p$ and $1 \le r \le s+2k+3$, there is*

$$E_1^{s+2k+3-r,t_2+(s-3)-(r-1),*} = \begin{cases} \mathbb{Z}_p\{a_3^{s-3}h_{3,0}h_{2,1}h_{1,2}h_{1,0}h_{1,n}b_{1,0}^{k}\}, & r = 1; \\ 0, & \text{others,} \end{cases}$$

Proof (1) Consider the generator $g \in E_1^{s+2k+2-r,t_2+(s-r-2),*}$ for $1 \le r \le s+2k+2$. Then we have

$$(\overline{c}_0, \cdots, \overline{c}_n) = (s-1, s+k-1, s, 0, \cdots, 0, 1).$$

Meanwhile, we have $\dim g = s+2k+2-r$ and $\deg g = t_2 + (s-r-2)$. One can easily get $s-r-2 \ge 0$.

Since $s+2k+2-r<s-r-2+q$, the number of x_i in g is $(s-r-2)$. Thus we can consider the generator $g=x_1\cdots x_{s-r-2}y_1\cdots y_m$. From (2.7), we get the carry sequence $\lambda=(\lambda_1,\lambda_2,\cdots,\lambda_n)=(0,0,\cdots,0)$ and the corresponding $c=(s-1,s+k-1,s,0,\cdots,1)$.

Case 1.1 $r=1$. Solving (2.7) by virtue of (2.6) deduces three possible $k-1$ rows as follows:

$$\begin{array}{cc}
\qquad\qquad\qquad s-3 \qquad\qquad\qquad\qquad\qquad\qquad s+k-1 & \\
\left(\begin{array}{cccccccc|cccccccccccc}
1 & \cdots & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
1 & \cdots & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\
\hline
1 & \cdots & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
1 & \cdots & 1 & 1 & \cdots & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
1 & \cdots & 1 & 1 & \cdots & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & \cdots & 1 & 0 & \cdots & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 0 & \cdots & 0
\end{array}\right)
&
\begin{array}{lr}
s-1 & \\
s+k-1 & \\
\hline
s & \cdots\ (1) \\
 & \cdots\ (2) \\
 & \cdots\ (3) \\
 & \vdots \\
 & \cdots\ (k-1).
\end{array}
\end{array}$$

If we choose (1) as the third row, then we obtain the following matrix:

$$\begin{array}{cc}
\qquad\qquad s-3 \qquad\qquad\qquad\qquad s+k & \\
\left(\begin{array}{ccc|cccccccc}
1 & \cdots & 1 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
1 & \cdots & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 0 \\
1 & \cdots & 1 & 1 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\
0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1
\end{array}\right)
&
\begin{array}{l}
s-1 \\ s+k-1 \\ s \\ 0 \\ \vdots \\ 0 \\ 1.
\end{array}
\end{array}$$

It detects the generator

$$a_3^{s-3}h_{3,0}h_{3,0}h_{2,1}h_{1,1}^{k-1}h_{1,n}\in F_1^{s+k,t_2+(s-3),*}.$$

By the resolution $h_{3,0}\to h_{1,2}h_{2,0}$, we get the generator

$$a_3^{s-3}h_{3,0}h_{2,1}h_{2,0}h_{1,2}h_{1,1}^{k-1}h_{1,n}\in F_1^{s+k+1,t_2+(s-3),*}.$$

By the replacement $h_{1,1}^{k-1}\to b_{1,0}^{k-1}$, we get the generator

$$a_3^{s-3}h_{3,0}h_{2,1}h_{2,0}h_{1,2}h_{1,n}b_{1,0}^{k-1}\in F_1^{s+2k,t_2+(s-3),*}.$$

By the resolution $h_{2,0}\to h_{1,1}h_{1,0}$, we get the generator

$$a_3^{s-3}h_{3,0}h_{2,1}h_{1,1}h_{1,0}h_{1,2}h_{1,n}b_{1,0}^{k-1}\in E_1^{s+2k+1,t_2+(s-3),*}.$$

If we choose (2) as the third row, then we get the following matrix:

$$\begin{array}{cc}
\qquad\qquad s-3 \qquad\qquad\qquad\qquad s+k & \\
\left(\begin{array}{cccc|cccccccc}
1 & \cdots & 1 & 1 & 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
1 & \cdots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 0 \\
1 & \cdots & 1 & 0 & 1 & 1 & 1 & 1 & 0 & \cdots & 0 & 0 \\
0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1
\end{array}\right)
&
\begin{array}{l}
s-1 \\ s+k-1 \\ s \\ 0 \\ \vdots \\ 0 \\ 1.
\end{array}
\end{array}$$

It detects the generator

$$a_3^{s-3}h_{3,0}h_{3,0}h_{2,0}h_{2,0}h_{1,1}^{k-2}h_{1,n} \in F_1^{s+k,t_2+(s-3),*}.$$

By the resolution $h_{2,0} \to h_{1,1}h_{1,0}$ and $h_{3,0} \to h_{2,1}h_{1,0}$ or $h_{1,2}h_{2,0}$, we get the generator

$$a_3^{s-3}h_{3,0}h_{2,1}h_{2,0}h_{1,1}^{k-1}h_{1,0}h_{1,0}h_{1,n} \in F_1^{s+k+2,t_2+(s-3),*}$$

and

$$a_3^{s-3}h_{3,0}h_{2,0}^2h_{1,2}h_{1,1}^{k-1}h_{1,0}h_{1,n} \in F_1^{s+k+2,t_2+(s-3),*},$$

which are both zeroes due to $h_{1,0}^2 = 0$ and $h_{2,0}^2 = 0$ in $E_1^{s+k+2,t_2+(s-3),*}$. Thus such g is impossible to exist.

Similarly, if we choose (3)–(k-1) as the third row, respectively, it is easy to know that g does not exist either.

Case 1.2 $r \geq 2$. Since $\sum\limits_{i=1}^{s-r-2} x_i \leq s-r-2 \leq s-4 < \overline{c}_0 = s-1$, the first equation of (2.7) has no solution. Thus such g is impossible to exist.

Combining Cases 1.1 and 1.2 gives the desired result.

(2) Consider the generator $g \in E_1^{s+2k+3-r,t_2+(s-r-2),*}$ for $1 \leq r \leq s+2k+2$. Then we have $(\overline{c}_0, \cdots, \overline{c}_n) = (s-1, s+k-1, s, 0, \cdots, 0, 1)$. Meanwhile, there is dim $g = s+2k+3-r$ and deg $g = t_2 + (s-r-2)$ with $s-r-2 \geq 0$. Since $s+2k+3-r < s-r-2+q$, we know that the number of x_i in g is $(s-r-2)$, thus we can consider the generator $g = x_1 \cdots x_{s-r-2}y_1 \cdots y_m$. From (2.7), one easily gets the carry sequence $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) = (0, 0, \cdots, 0)$ and the corresponding $c = (s-1, s+k-1, s, 0, \cdots, 1)$.

Case 2.1 $r = 1$. Similar to Case 1.1 in Proposition 2.2(1), solving (2.7) by virtue of (2.6) follows the generator

$$a_3^{s-3}h_{3,0}h_{3,0}h_{2,1}h_{1,1}^{k-1}h_{1,n} \in F_1^{s+k,t_2+(s-3),*}.$$

By the resolution $h_{3,0} \to h_{1,2}h_{2,0} \to h_{1,1}h_{1,0}h_{1,2}$, we get the generator

$$a_3^{s-3}h_{3,0}h_{2,1}h_{1,2}h_{1,0}h_{1,1}^{k}h_{1,n} \in F_1^{s+k+2,t_2+(s-3),*}.$$

By the replacement $h_{1,1}^k \to b_{1,0}^k$, we get the generator

$$a_3^{s-3}h_{3,0}h_{2,1}h_{1,2}h_{1,0}h_{1,n}b_{1,0}^k \in E_1^{s+2k+2,t_2+(s-3),*}.$$

Case 2.2 $r \geq 2$. Since $\sum\limits_{i=1}^{s-r-2} x_i \leq s-r-2 \leq s-4 < \overline{c}_0 = s-1$, the first equation of (2.7) has no solution. Thus such g does not exist.

Combining Case 2.1 with Case 2.2 gives the desired result.

3 Proof of Theorem 1.1

In this section, we give some results on Ext-groups which will be used in the proof of the main theorem. For simplicity, we still let $t_1 = p^nq + (s+k)pq + sq$ and $t_2 = p^nq + sp^2q + (s+k-1)pq + (s-1)q$ with $k > 0$.

Lemma 3.1 *Let $p > 5$, $n \geq 3$ and $s > 1$. Then we have*

(1) *when $3 < s+2k+2 < p$, then $\mathrm{Ext}_{\mathcal{A}}^{s+2k+2,t_1+(s-2)}(\mathbb{Z}_p,\mathbb{Z}_p)$ contains one nonzero element $h_0h_nb_0^k\widetilde{\beta}_s$ and $\mathrm{Ext}_{\mathcal{A}}^{s+2k+2-r,t_1+(s-2)-(r-1)}(\mathbb{Z}_p,\mathbb{Z}_p)$ becomes trivial for $r \geq 2$;*

(2) *when $4 < s+2k+3 < p$, then $\mathrm{Ext}_{\mathcal{A}}^{s+2k+3,t_1+(s-2)}(\mathbb{Z}_p,\mathbb{Z}_p)$ contains one nonzero element $h_0b_{n-1}b_0^k\widetilde{\beta}_s$ and $\mathrm{Ext}_{\mathcal{A}}^{s+2k+3-r,t_1+(s-2)-(r-1)}(\mathbb{Z}_p,\mathbb{Z}_p)$ becomes trivial for $r \geq 2$.*

Proof (1) It is known that $h_{1,0}, h_{1,n}(n \geq 0), b_{1,0}^k, \frac{s!}{(s-2)!}a_2^{s-2}h_{2,0}h_{1,1} \in E_1^{*,*,*}$ are permanent cocycles and converge nontrivially to $h_0, h_n(n \geq 0), b_0^k$ and $\widetilde{\beta}_s \in \mathrm{Ext}_{\mathcal{A}}^{*,*}(\mathbb{Z}_p,\mathbb{Z}_p)$, respectively. It follows that

$$h_{1,0}h_{1,n}b_{1,0}^k\frac{s!}{(s-2)!}a_2^{s-2}h_{2,0}h_{1,1} \in E_1^{s+2k+2,t_1+(s-2),*}$$

is a permanent cocycle in the May spectral sequence and converges to

$$h_0h_nb_0^k\widetilde{\beta}_s \in \mathrm{Ext}_{\mathcal{A}}^{s+2k+2,t_1+(s-2)}(\mathbb{Z}_p,\mathbb{Z}_p).$$

According to Proposition 2.1(1), we have

$$E_1^{s+2k+1,t_1+(s-2),*} = 0,$$

which follows $E_r^{s+2k+1,t_1+(s-2),*}$ is trivial for $r \geq 1$. Thus the permanent cycle

$$h_{1,0}h_{1,n}b_{1,0}^k\frac{s!}{(s-2)!}a_2^{s-2}h_{2,0}h_{1,1} \in E_r^{s+2k+2,t_1+(s-2),*}$$

cannot be hit by any differential in the May spectral sequence. Thus

$$h_0h_nb_0^k\widetilde{\beta}_s \neq 0 \in \mathrm{Ext}_{\mathcal{A}}^{s+2k+2,t_1+(s-2)}(\mathbb{Z}_p,\mathbb{Z}_p).$$

When $r \geq 2$, by Proposition 2.1(1), we see that $E_1^{s+2k+2-r,t_1+(s-2)-(r-1),*}$ is trivial. It follows that

$$\mathrm{Ext}_{\mathcal{A}}^{s+2k+2-r,t_1+(s-2)-(r-1)}(\mathbb{Z}_p,\mathbb{Z}_p) = 0.$$

(2) In the May spectral sequence $h_{1,0}, b_{1,n-1}, b_{1,0}^k, \frac{s!}{(s-2)!}a_2^{s-2}h_{2,0}h_{1,1} \in E_1^{*,*,*}$ are permanent cocycles and converge nontrivially to $h_0, b_{n-1}, b_0^k, \widetilde{\beta}_s \in \mathrm{Ext}_{\mathcal{A}}^{*,*}(\mathbb{Z}_p,\mathbb{Z}_p)$ for $n \geq 0$, respectively. It follows that

$$h_{1,0}b_{1,n-1}b_{1,0}^k\frac{s!}{(s-2)!}a_2^{s-2}h_{2,0}h_{1,1} \in E_1^{s+2k+3,t_1+(s-2),*}$$

is a permanent cocycle in the May spectral sequence and converges to

$$h_0b_{n-1}b_0^k\widetilde{\beta}_s \in \mathrm{Ext}_{\mathcal{A}}^{s+2k+3,t_1+(s-2)}(\mathbb{Z}_p,\mathbb{Z}_p).$$

According to Proposition 2.1(2), $E_1^{s+2k+2,t_1+(s-2),*}$ contains one generator

$$a_2^{s-2}h_{2,0}h_{1,0}h_{1,1}b_{1,0}^kh_{1,n}.$$

Since

$$\mathrm{d}_r(a_2^{s-2}h_{2,0}h_{1,0}h_{1,1}b_{1,0}^kh_{1,n}) = 0 \quad \text{for } r \geq 1,$$

the permanent cocycle $h_{1,0}b_{1,n-1}b_{1,0}^k\frac{s!}{(s-2)!}a_2^{s-2}h_{2,0}h_{1,1} \in E_r^{s+2k+2,*,*}$ does not bound in the May spectral sequence. It follows that $h_0b_{n-1}b_0^k$ is nonzero in the $\mathrm{Ext}_{\mathcal{A}}^{s+2k+3,t_1+(s-2)}(\mathbb{Z}_p,\mathbb{Z}_p)$.

When $r \geq 2$, by Proposition 2.1(2), we see that $E_1^{s+2k+3-r,t_1+(s-2)-(r-1),*}$ is trivial. It follows that

$$\mathrm{Ext}_{\mathcal{A}}^{s+2k+3-r,t_1+(s-2)-(r-1)}(\mathbb{Z}_p,\mathbb{Z}_p) = 0.$$

Lemma 3.2 *Let $p > 7$, $n \geq 4$ and $s > 2$. Then we have*

(1) *when $4 < s+2k+2 < p$, then $\mathrm{Ext}_{\mathcal{A}}^{s+2k+2,t_1+(s-3)}(\mathbb{Z}_p,\mathbb{Z}_p)$ contains one nonzero element $h_0h_nb_0^k\widetilde{\gamma}_s$ and $\mathrm{Ext}_{\mathcal{A}}^{s+2k+2-r,t_1+(s-3)-(r-1)}(\mathbb{Z}_p,\mathbb{Z}_p)$ becomes trivial for $r \geq 2$;*

(2) *when $5 < s+2k+3 < p$, then $\mathrm{Ext}_{\mathcal{A}}^{s+2k+3,t_1+(s-3)}(\mathbb{Z}_p,\mathbb{Z}_p)$ contains one nonzero element $h_0b_{n-1}b_0^k\widetilde{\gamma}_s$ and $\mathrm{Ext}_{\mathcal{A}}^{s+2k+3-r,t_1+(s-3)-(r-1)}(\mathbb{Z}_p,\mathbb{Z}_p)$ becomes trivial for $r \geq 2$.*

Proof (1) It is known that $h_{1,0}, h_{1,n}(n \geq 0), b_{1,0}^k, \frac{s!}{(s-3)!}a_3^{s-3}h_{3,0}h_{2,1}h_{1,2} \in E_1^{*,*,*}$ are permanent cocycles and converge nontrivially to $h_0, h_n(n \geq 0), b_0^k$ and $\widetilde{\gamma}_s \in \mathrm{Ext}_{\mathcal{A}}^{*,*}(\mathbb{Z}_p,\mathbb{Z}_p)$, respectively. It follows that

$$h_{1,0}h_{1,n}b_{1,0}^k\frac{s!}{(s-3)!}a_3^{s-3}h_{3,0}h_{2,1}h_{1,2} \in E_1^{*,*,*}$$

is a permanent cocycle and converges to $h_0h_nb_0^k\widetilde{\gamma}_s \in \mathrm{Ext}_{\mathcal{A}}^{*,*}(\mathbb{Z}_p,\mathbb{Z}_p)$ in the May spectral sequence.

According to Proposition 2.2(1), there is one generator

$$a_3^{s-3}h_{3,0}h_{2,1}h_{1,2}h_{1,1}h_{1,0}h_{1,n}b_{1,0}^{k-1} \in E_1^{s+2k+1,t_2+(s-3),*}.$$

Since

$$\mathrm{d}_r(a_3^{s-3}h_{3,0}h_{2,1}h_{1,2}h_{1,1}h_{1,0}h_{1,n}b_{1,0}^{k-1}) = 0 \quad \text{for } r \geq 1,$$

the permanent cocycle $h_{1,0}h_{1,n}b_{1,0}^k\frac{s!}{(s-3)!}a_3^{s-3}h_{3,0}h_{2,1}h_{1,2} \in E_r^{s+2k+2,*,*}$ does not bound in the May spectral sequence. It follows that

$$h_0h_nb_0^k\widetilde{\gamma}_s \neq 0 \in \mathrm{Ext}_{\mathcal{A}}^{s+2k+2,t_2+(s-3)}(\mathbb{Z}_p,\mathbb{Z}_p).$$

For $r \geq 2$, according to Proposition 2.2(1), we have

$$E_1^{s+2k+2-r,t_2+(s-3)-(r-1),*} = 0.$$

Thus

$$\mathrm{Ext}_{\mathcal{A}}^{s+2k+2-r,t_2+(s-3)-(r-1)}(\mathbb{Z}_p,\mathbb{Z}_p) = 0.$$

(2) In the May spectral sequence $h_{1,0}, b_{1,n-1}(n \geq 0), b_{1,0}^k, \frac{s!}{(s-3)!}a_3^{s-3}h_{3,0}h_{2,1}h_{1,2} \in E_1^{*,*,*}$ are permanent cocycles and converge nontrivially to $h_0, b_{n-1}(n \geq 0), b_0^k$ and $\widetilde{\gamma}_s \in \mathrm{Ext}_{\mathcal{A}}^{*,*}(\mathbb{Z}_p,\mathbb{Z}_p)$, respectively. It follows that

$$h_{1,0}b_{1,n-1}b_{1,0}^k\frac{s!}{(s-3)!}a_3^{s-3}h_{3,0}h_{2,1}h_{1,2} \in E_1^{*,*,*}$$

is a permanent cocycle and converges to $h_0b_{n-1}b_0^k\widetilde{\gamma}_s \in \mathrm{Ext}_{\mathcal{A}}^{*,*}(\mathbb{Z}_p,\mathbb{Z}_p)$ in the May spectral sequence.

According to Proposition 2.2(2), there is one generator

$$a_3^{s-3}h_{3,0}h_{2,1}h_{1,2}h_{1,0}h_{1,n}b_{1,0}^k \in E_1^{s+2k+3,t_2+(s-3),*}.$$

Since

$$\mathrm{d}_r(a_3^{s-3}h_{3,0}h_{2,1}h_{1,2}h_{1,0}h_{1,n}b_{1,0}^k) = 0 \quad \text{for } r \geq 1,$$

the permanent cocycle $h_{1,0}b_{1,n-1}b_{1,0}^k\frac{s!}{(s-3)!}a_3^{s-3}h_{3,0}h_{2,1}h_{1,2} \in E_r^{s+2k+3,*,*}$ does not bound in the May spectral sequence. It follows that

$$h_0b_{n-1}b_0^k\widetilde{\gamma}_s \neq 0 \in \mathrm{Ext}_{\mathcal{A}}^{s+2k+3,t_2+(s-3)}(\mathbb{Z}_p,\mathbb{Z}_p).$$

For $r \geq 2$, according to Proposition 2.2(2), we have

$$E_1^{s+2k+3-r,t_2+(s-3)-(r-1),*} = 0.$$

Thus

$$\mathrm{Ext}_{\mathcal{A}}^{s+2k+3-r,t_2+(s-3)-(r-1)}(\mathbb{Z}_p,\mathbb{Z}_p) = 0.$$

In what follows, we give the proof of our main theorem.

Proof of Theorem 1.1 (1) It is well known that $b_0 \in \mathrm{Ext}_{\mathcal{A}}^{2,pq}(\mathbb{Z}_p,\mathbb{Z}_p)$ is a permanent cycle in the Adams spectral sequence and converges nontrivially to the β-element $\beta_1 = j_0j_1\beta i_1i_0 \in \pi_{pq-2}S$. From [3], $(i_0)_*(h_0h_n) \in \mathrm{Ext}_{\mathcal{A}}^{2,p^nq+q}(H^*M,\mathbb{Z}_p)$ is a permanent cycle in the Adams spectral sequence and converges to a non-trivial element $\xi_n \in \pi_{p^nq+q-2}(M)$. Therefore the following composite

$$\Sigma^{p^nq+kpq+q-2-2k}S \xrightarrow{\xi_n\beta_1^k} M \xrightarrow{i_1} V(1) \xrightarrow{\beta^s} \Sigma^{-s(p+1)q}V(1) \xrightarrow{j_0j_1} \Sigma^{-s(p+1)q+q+2}S$$

is represented up to nonzero scalar by

$$(j_0j_1\beta^si_1i_0)_*(h_0h_n)b_0^k \in \mathrm{Ext}_{\mathcal{A}}^{s+2k+2,t_1+(s-2)}(\mathbb{Z}_p,\mathbb{Z}_p)$$

in the Adams spectral sequence.

By using the Yoneda products, we know that the composite

$$\mathrm{Ext}_{\mathcal{A}}^{0,*}(\mathbb{Z}_p,\mathbb{Z}_p) \xrightarrow{(i_1i_0)_*} \mathrm{Ext}_{\mathcal{A}}^{0,*}(H^*V(1),\mathbb{Z}_p) \xrightarrow{(j_0j_1)_*(\beta^s)_*} \mathrm{Ext}_{\mathcal{A}}^{s,*+spq+(s-1)q+(s-2)}(\mathbb{Z}_p,\mathbb{Z}_p)$$

is a multiplication by

$$\widetilde{\beta}_s \in \mathrm{Ext}_{\mathcal{A}}^{s,spq+(s-1)q+(s-2)}(\mathbb{Z}_p,\mathbb{Z}_p).$$

Hence $(j_0j_1\beta^si_1i_0)_*(h_0h_n)\beta_1^k$ is represented by

$$h_0h_nb_0^k\widetilde{\beta}_s \in \mathrm{Ext}_{\mathcal{A}}^{s+2k+2,t_1+(s-2)}(\mathbb{Z}_p,\mathbb{Z}_p)$$

in the Adams spectral sequence.

Moreover, from Lemma 3.1(1) $\mathrm{Ext}_{\mathcal{A}}^{s+2k+2-r,t_1+(s-2)-(r-1)}(\mathbb{Z}_p,\mathbb{Z}_p) = 0$ $(r \geq 2)$, it follows that $h_0h_nb_0^k\widetilde{\beta}_s$ can not be hit by any differential in the Adams spectral sequence. Thus $h_0h_nb_0^k\widetilde{\beta}_s$ survives non-trivially to a homotopy element of π_*S.

Similarly, by the virtue of the convergence of h_0b_{n-1} (see [3]) and Lemma 3.1(1), we can get that $h_0b_{n-1}b_0^k\widetilde{\beta}_s$ survives non-trivially to a homotopy element of π_*S.

(2) Consider the following composite

$$\Sigma^{p^nq+kpq+q-2-2k}S \xrightarrow{\xi_n\beta_1^k} M \xrightarrow{i_2i_1} V(2) \xrightarrow{\gamma^s} \Sigma^{-s(p^2+p+1)q}V(2) \xrightarrow{j_0j_1j_2} \Sigma^{-s(p^2+p+1)q+(p+2)q+3}S$$

is represented up to nonzero scalar by

$$(j_2j_0j_1\gamma^si_2i_1i_0)_*(h_0h_n)b_0^k \in \mathrm{Ext}_{\mathcal{A}}^{s+2k+2,t_2+(s-3)}(\mathbb{Z}_p,\mathbb{Z}_p)$$

in the Adams spectral sequence. By using the Yoneda products, we know that the composite

$$\mathrm{Ext}_{\mathcal{A}}^{0,*}(\mathbb{Z}_p,\mathbb{Z}_p) \xrightarrow{(i_2i_1i_0)_*} \mathrm{Ext}_{\mathcal{A}}^{0,*}(H^*V(1),\mathbb{Z}_p) \xrightarrow{(j_0j_1j_2)_*(\gamma^s)_*} \mathrm{Ext}_{\mathcal{A}}^{s,*+sp^2q+(s-1)pq+(s-2)q+s-3}(\mathbb{Z}_p,\mathbb{Z}_p)$$

is a multiplication by

$$\widetilde{\gamma}_s \in \mathrm{Ext}_{\mathcal{A}}^{s,sp^2q+(s-1)pq+(s-2)q+s-3}(\mathbb{Z}_p,\mathbb{Z}_p).$$

Hence the composite $(j_0j_1j_2\gamma^s i_2i_1i_0)_*(h_0h_n)\beta_1^k$ is represented by

$$h_0h_nb_0^k\widetilde{\gamma}_s \in \mathrm{Ext}_{\mathcal{A}}^{s+2k+2,t_2+(s-3)}(\mathbb{Z}_p,\mathbb{Z}_p)$$

in the Adams spectral sequence.

Moreover, from Lemma 3.2(1) $\mathrm{Ext}_{\mathcal{A}}^{s+2k+2-r,t_2+(s-2)-(r-1)}(\mathbb{Z}_p,\mathbb{Z}_p) = 0$ $(r \geq 2)$, it follows that $h_0h_nb_0^k\widetilde{\gamma}_s$ can not be hit by any differential in the Adams spectral sequence. Thus $h_0h_nb_0^k\widetilde{\gamma}_s$ survives non-trivially to a homotopy element of π_*S.

By virtue of the convergence of h_0b_{n-1} (see [3]) and Lemma 3.2(2), we see that $h_0b_{n-1}b_0^k\widetilde{\gamma}_s$ survives non-trivially to a homotopy element of π_*S.

Acknowledgement The authors would like to give their thanks to the referee for his/her many helpful suggestions on the original version of this paper.

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