Chin. Ann. Math. Ser. B
43(3), 2022, 473–484
DOI: 10.1007/s11401-022-0336-3

Chinese Annals of Mathematics, Series B

A Characterization of the Standard Tori in $\mathbb{C}^2$ as Compact Lagrangian ξ-Submanifolds*

Xingxiao LI[1] Ruiwei XU[2]

Abstract In this paper, the authors give a characterization theorem for the standard tori $\mathbb{S}^1(a)\times\mathbb{S}^1(b)$, $a,b>0$, as the compact Lagrangian ξ-submanifolds in the two-dimensional complex Euclidean space $\mathbb{C}^2$, and obtain the best version of a former rigidity theorem for compact Lagrangian ξ-submanifold in $\mathbb{C}^2$. Furthermore, their argument in this paper also proves a new rigidity theorem which is a direct generalization of a rigidity theorem by Li and Wang for Lagrangian self-shrinkers in $\mathbb{C}^2$.

Keywords ξ-Submanifold, the Second fundamental form, Mean curvature vector, Standard tori
2000 MR Subject Classification 53A30, 53B25

1 Introduction

An m-dimensional submanifold $x: M^m\to\mathbb{R}^{m+p}$ in the $(m+p)$-dimensional Euclidean space $\mathbb{R}^{m+p}$ is called a self-shrinker (to the mean curvature flow) if its mean curvature vector field H satisfies

$$H+x^{\perp}=0, \tag{1.1}$$

where $x^{\perp}$ is the orthogonal projection of the position vector x to the normal space $T^{\perp}M^m$ of x.

As we know, self-shrinkers play an important role in the study of the mean curvature flow. In fact, they correspond to the self-shrinking solutions to the mean curvature flow, and describe all possible Type I singularities of the flow (see [10, 12]). Up to now, various results of classification or rigidity theorems on the self-shrinkers have been obtained (see [2, 14, 17]). In particular, there are also interesting results about the Lagrangian self-shrinkers in the complex Euclidean m-space $\mathbb{C}^m$. For example, in [1], Anciaux gives new examples of self-shrinking and self-expanding Lagrangian solutions to the mean curvature flow. In [3], the authors classify all Hamiltonian stationary Lagrangian surfaces in the complex plane $\mathbb{C}^2$, which are self-similar solutions of the mean curvature flow and, in [4], several rigidity results for Lagrangian mean curvature flow are

Manuscript received November 13, 2019. Revised December 30, 2021.
[1]School of Mathematics and Information Sciences, Henan Normal University, Xinxiang 453007, Henan, China. E-mail: xxl@htu.edu.cn
[2]Corresponding author. School of Mathematics and Information Sciences, Henan Normal University, Xinxiang 453007, Henan, China. E-mail: rwxu@htu.edu.cn
*This work was supported by the National Natural Science Foundation of China (Nos. 11671121, 11871197).

obtained. In [5], Cheng-Hori-Wei obtain a complete classification for 2-dimensional complete Lagrangian self-shrinkers in Euclidean space $\mathbb{R}^4$ with constant squared norm of the second fundamental form. In 2017, Li and Wang prove a rigidity theorem (see [13]) which gives a new characterization of the Clifford torus $\mathbb{S}^1(1) \times \mathbb{S}^1(1)$ as a Lagrangian self-shrinker, improving a previous theorem by Castro and Lerma in [4], without the additional condition that the Gauss curvature K of M^2 is either non-negative or non-positive.

Theorem 1.1 (see [4, 13]) *Let $x : M^2 \to \mathbb{C}^2$ be a compact oriented Lagrangian self-shrinker with h its second fundamental form. If $|h|^2 \leq 2$, then $|h|^2 = 2$ and $x(M^2)$ is the Clifford torus $\mathbb{S}^1(1) \times \mathbb{S}^1(1)$, up to a holomorphic isometry on $\mathbb{C}^2$.*

On the other hand, the first author and one of his coauthors made in 2016 a natural generalization of self-shrinkers to the concept of ξ-submanifolds (see [15]): An immersed submanifold $x : M^n \to \mathbb{R}^{n+p}$ is called a ξ-submanifold if there is a parallel normal vector field ξ such that the mean curvature vector field H satisfies (see [15])

$$H + x^{\perp} = \xi. \tag{1.2}$$

Obviously, the affine plane $\mathbb{R}^2$, the standard tori $\mathbb{S}^1(a) \times \mathbb{S}^1(b)$ for $a, b > 0$, and the circular cylinders $\mathbb{S}^1(a) \times \mathbb{R}^1$ are canonical examples of Lagrangian ξ-submanifold in $\mathbb{C}^2$. Some other examples of ξ-submanifolds can be seen in [16]. The variation characterization and the stability properties of ξ-submanifolds were also systematically studied in [16]. In particular, a submanifold $x : M^m \to \mathbb{R}^{m+p}$ is a ξ-submanifold if and only if its modified mean curvature is parallel when viewed as a submanifold in the Gaussian space $(\mathbb{R}^{m+p}, \mathrm{e}^{-\frac{1}{m}|x|^2}\langle\cdot,\cdot\rangle)$. Extending the result of Theorem 1.1, we also proved in [15] the following rigidity theorem for Lagrangian ξ-surfaces in $\mathbb{C}^2$.

Theorem 1.2 (see [15]) *Let $x : M^2 \to \mathbb{C}^2$ be a compact oriented Lagrangian ξ-submanifold with the second fundamental form h and mean curvature vector H. Assume that*

$$|h|^2 + |H - \xi|^2 \leq |\xi|^2 + 4.$$

Then $|h|^2 + |H - \xi|^2 \equiv |\xi|^2 + 4$ and $x(M^2) = T^2$ is a topological torus.

Furthermore, if $\langle H, \xi\rangle$ is constant and one of the following four conditions holds:

$$(1)\ |h|^2 \geq 2, \quad (2)\ |H|^2 \geq 2, \quad (3)\ |h|^2 \geq \langle H, H - \xi\rangle, \quad (4)\ \langle H, \xi\rangle \geq 0, \tag{1.3}$$

then, up to a holomorphic isometry on $\mathbb{C}^2$, $x(M^2)$ must be a standard torus $\mathbb{S}^1(a) \times \mathbb{S}^1(b)$, where a and b are positive numbers satisfying $a^2 + b^2 \geq 2a^2b^2$.

Then we obtain the following corollary.

Corollary 1.1 (see [15]) *Let $x : M^2 \to \mathbb{C}^2$ be a compact oriented Lagrangian self-shrinker. If*

$$|h|^2 + |H|^2 \leq 4,$$

then $|h|^2 + |H|^2 \equiv 4$ and $x(M^2) = \mathbb{S}^1(1) \times \mathbb{S}^1(1)$ up to a holomorphic isometry on $\mathbb{C}^2$.

This corollary gives a new rigidity theorem for self-shrinkers that is parallel to Theorem 1.1.

Clearly, ξ-submanifolds are also an extension to higher codimension of the concept of λ-hypersurface which were introduced in [8] by Cheng and Wei. According to [8], a hypersurface $x : M^m \to \mathbb{R}^{m+1}$ is called a λ-hypersurface if its scalar mean curvature H satisfies

$$H + \langle x, n\rangle = \lambda \tag{1.4}$$

for some constant λ, where n is the unit normal vector of x. Some rigidity or classification results for λ-hypersurfaces are obtained, for example, in [6–7, 11], and more recently, [9].

Note that the conditions in Theorem 1.2 seem rather tedious anyway. Thus it is interesting if one can make some simplification of the theorem. In the present paper, we are able to greatly improve Theorem 1.2 by proving the following best version of it, giving a new characterization of the standard tori as the compact Lagrangian ξ-submanifolds in $\mathbb{C}^2$.

Theorem 1.3 *Let $x : M^2 \to \mathbb{C}^2$ be a compact oriented Lagrangian ξ-submanifold with the second fundamental form h and mean curvature vector H. If*

$$|h|^2 + |H - \xi|^2 \le |\xi|^2 + 4,$$

then $|h|^2 + |H - \xi|^2 \equiv |\xi|^2 + 4$ and, up to a holomorphic isometry on $\mathbb{C}^2$, $x(M^2)$ must be a standard torus $\mathbb{S}^1(a) \times \mathbb{S}^1(b)$ for some positive numbers a and b.

Furthermore, by a careful check of our argument in the proof of Theorem 1.3, one easily sees that the following interesting theorem also holds, which is apparently a direct generalization of the rigidity theorem by Li and Wang mentioned above (Theorem 1.1).

Theorem 1.4 *Let $x : M^2 \to \mathbb{C}^2$ be a compact oriented Lagrangian ξ-submanifold with the second fundamental form h and mean curvature vector H. If*

$$|h|^2 - \langle H, \xi\rangle \le 2,$$

then $|h|^2 - \langle H, \xi\rangle \equiv 2$ and, up to a holomorphic isometry on $\mathbb{C}^2$, $x(M^2)$ must be a standard torus $\mathbb{S}^1(a) \times \mathbb{S}^1(b)$ for some positive numbers a and b with

$$\xi = \Big(\frac{1}{a} - a\Big)n_1 + \Big(\frac{1}{b} - b\Big)n_2,$$

where n_1, n_2 are the inwards unit normals of $\mathbb{S}^1(a) \subset \mathbb{C}^1$ and $\mathbb{S}^1(b) \subset \mathbb{C}^1$, respectively.

2 Lagrangian Submanifolds in $\mathbb{C}^m$

Let $\mathbb{C}^m$ be the complex Euclidean m-space with the canonical complex structure J. Through out this paper, $x : M^m \to \mathbb{C}^m$ always denotes an m-dimensional Lagrangian submanifold, and ∇, D, $\nabla^\perp$ denote, respectively, the Levi-Civita connections on M^m, $\mathbb{C}^m$, and the normal connection on the normal bundle $T^\perp M^m$. Then the formulas of Gauss and Weingarten are respectively given by

$$D_X Y = \nabla_X Y + h(X, Y), \quad D_X \eta = -A_\eta X + \nabla^\perp_X \eta,$$

where X, Y are tangent vector fields on M^m and η is a normal vector field of x. The Lagrangian condition implies that

$$\nabla^{\perp}_X JY = J\nabla_X Y, \quad A_{JX}Y = -Jh(X,Y) = A_{JY}X,$$

where h and A are the second fundamental form and the shape operator of x, respectively. In particular, $\langle h(X,Y), JZ\rangle$ is totally symmetric as a 3-form, namely,

$$\langle h(X,Y), JZ\rangle = \langle h(X,Z), JY\rangle = \langle h(Y,Z), JX\rangle. \tag{2.1}$$

From now on, we agree with the following convention on the ranges of indices:

$$1 \le i, j, \cdots \le m, \quad m+1 \le \alpha, \beta, \cdots \le 2m, \quad 1 \le A, B, \cdots \le 2m, \quad i^* = m+i.$$

For a Lagrangian submanifold $x : M^m \to \mathbb{C}^m$, there are orthonormal frame fields of the form $\{e_i, e_{i^*}\}$ for $\mathbb{C}^m$ along x, where $e_i \in TM^m$ and $e_{i^*} = Je_i$. Such a frame is called an adapted Lagrangian frame field in the literature. The dual frame field is always denoted by $\{\theta^i, \theta^{i^*}\}$, where $\theta^{i^*} = -J\theta^i$. Write

$$h = \sum h^{k^*}_{ij}\theta^i\theta^j e_{k^*}, \quad \text{where } h^{k^*}_{ij} = \langle h(e_i, e_j), e_{k^*}\rangle,$$

or equivalently,

$$h(e_i, e_j) = \sum h^{k^*}_{ij} e_{k^*}.$$

Then (2.1) is equivalent to

$$h^{k^*}_{ij} = h^{i^*}_{kj} = h^{j^*}_{ik}. \tag{2.2}$$

If θ^j_i and $\theta^{j^*}_{i^*}$ denote the connection forms of ∇ and $\nabla^{\perp}$, respectively, then the components $h^{k^*}_{ij,l}$, $h^{k^*}_{ij,lp}$ of the covariant derivatives of h are given respectively by

$$\sum h^{k^*}_{ij,l}\theta^l = \mathrm{d}h^{k^*}_{ij} - \sum h^{k^*}_{lj}\theta^l_i - \sum h^{k^*}_{il}\theta^l_j + \sum h^{m^*}_{ij}\theta^{k^*}_{m^*}; \tag{2.3}$$

$$\sum h^{k^*}_{ij,lp}\theta^p = \mathrm{d}h^{k^*}_{ij,l} - \sum h^{k^*}_{pj,l}\theta^p_i - \sum h^{k^*}_{ip,l}\theta^p_j - \sum h^{k^*}_{ij,p}\theta^p_l + \sum h^{p^*}_{ij,l}\theta^{k^*}_{p^*}. \tag{2.4}$$

Moreover, the equations of motion are as follows:

$$\mathrm{d}x = \sum \theta^i e_i, \quad \mathrm{d}e_i = \sum \theta^j_i e_j + \sum h^{k^*}_{ij}\theta^j e_{k^*}, \tag{2.5}$$

$$\mathrm{d}e_{k^*} = -\sum h^{k^*}_{ij}\theta^j e_i + \sum \theta^{l^*}_{k^*} e_{l^*}. \tag{2.6}$$

Let R_{ijkl} and $R_{i^*j^*kl}$ denote the components of curvature operators of ∇ and $\nabla^{\perp}$, respectively. Then the equations of Gauss, Codazzi and Ricci are as follows:

$$R_{mijk} = \sum (h^{l^*}_{mk}h^{l^*}_{ij} - h^{l^*}_{mj}h^{l^*}_{ik}), \tag{2.7}$$

$$h^{k^*}_{ij,l} = h^{k^*}_{il,j}, \tag{2.8}$$

$$R_{i^*j^*kl} = \sum (h^{i^*}_{ml}h^{j^*}_{mk} - h^{i^*}_{mk}h^{j^*}_{ml}). \tag{2.9}$$

The scalar curvature of ∇ is

$$R = |H|^2 - |h|^2 \quad \text{with } |H|^2 = \sum_k \Big(\sum h^{k^*}_{ii}\Big)^2,\ |h|^2 = \sum_{i,j,k} (h^{k^*}_{ij})^2, \tag{2.10}$$

where the mean curvature vector field H is defined by

$$H = \sum H^{k^*} e_{k^*} = \sum h^{k^*}_{ii} e_{k^*}.$$

Combining (2.2) and (2.8), we know that $h^{k^*}_{ij,l}$ is totally symmetric, namely,

$$h^{k^*}_{ij,l} = h^{i^*}_{jl,k} = h^{j^*}_{lk,i} = h^{l^*}_{ki,j}, \tag{2.11}$$

and the Ricci identities are as follows:

$$h^{k^*}_{ij,lp} - h^{k^*}_{ij,pl} = \sum h^{k^*}_{mj} R_{imlp} + \sum h^{k^*}_{im} R_{jmlp} + \sum h^{m^*}_{ij} R_{k^*m^*lp}. \tag{2.12}$$

Note that, with respect to the adapted Lagrangian frame $\{e_i, e_{i^*}\}$, the connection forms $\theta_{i^*j^*} = \theta_{ij}$. It follows that

$$R_{m^*i^*jk} = R_{mijk}. \tag{2.13}$$

Furthermore, the first and second derivatives $H^{k^*}_{,i}$, $H^{k^*}_{,ij}$ of the mean curvature vector field H are given as

$$H^{k^*}_{,i} = \sum h^{k^*}_{jj,i}, \quad H^{k^*}_{,ij} = \sum h^{k^*}_{ll,ij}. \tag{2.14}$$

For any smooth function f on M^m, the covariant derivatives $f_{,i}$, $f_{,ij}$ of f, the Laplacian of f are respectively defined as follows:

$$\mathrm{d}f = \sum f_{,i}\theta^i, \quad \sum f_{,ij}\theta^j = \mathrm{d}f_{,i} - \sum f_{,j}\theta^j_i, \quad \triangle f = \sum_i f_{,ii}. \tag{2.15}$$

The well-known operator $\mathcal{L}$ acting on smooth functions is defined by (see [10])

$$\mathcal{L} = \triangle - \langle x, \nabla\cdot\rangle = \mathrm{e}^{\frac{|x|^2}{2}} \operatorname{div}\big(\mathrm{e}^{-\frac{|x|^2}{2}} \nabla\cdot\big), \tag{2.16}$$

which has been proven to be one of the most effect tools in the study of self-shrinkers and, more generally, of λ-hypersurfaces.

3 Proof of Main Theorem

Let $x : M^m \to \mathbb{C}^m$ be a Lagrangian ξ-submanifold. Then, with respect to an orthonormal frame field $\{e_i\}$, the defining equation (1.2) is equivalent to

$$H^{k^*} = -\langle x, e_{k^*}\rangle + \xi^{k^*}, \quad 1 \le k \le m, \tag{3.1}$$

where $\xi = \sum \xi^{k^*} e_{k^*}$ is a parallel normal vector field. By a direct computation using (3.1) we can get the following lemma.

Lemma 3.1 (see [13]) *Let $x : M^2 \to \mathbb{C}^2$ be a Lagrangian ξ-submanifold. Then*

$$H^{k^*}_{,i} = \sum h^{k^*}_{ij} \langle x, e_j\rangle, \tag{3.2}$$

$$H^{k^*}_{,ij} = \sum h^{k^*}_{im,j}\langle x, e_m\rangle + h^{k^*}_{ij} - \sum (H-\xi)^{p^*} h^{k^*}_{im} h^{p^*}_{mj}, \tag{3.3}$$

$$\begin{aligned} H^{l^*}_{,ijk} &= \sum h^{l^*}_{im,jk}\langle x, e_m\rangle + \sum h^{l^*}_{im,j}\langle e_k, e_m\rangle - \sum h^{l^*}_{im,j} h^{p^*}_{mk}(H-\xi)^{p^*} + h^{l^*}_{ij,k} \\ &\quad - \sum h^{l^*}_{im} h^{p^*}_{mj} H^{p^*}_{,k} - \sum (H-\xi)^{p^*} h^{l^*}_{im,k} h^{p^*}_{mj} - \sum (H-\xi)^{p^*} h^{l^*}_{im} h^{p^*}_{mj,k}. \end{aligned} \tag{3.4}$$

The following two lemmas are also obtained by direct computations, see also [15].

Lemma 3.2 (see [15]) *Let K be the Gauss curvature of the induce metric on M^2 via x. Then it holds that*

$$\begin{aligned} \frac{1}{2}\mathcal{L}|h|^2 &= |\nabla h|^2 + |h|^2 + K(3|h|^2 - 2|H|^2 + \langle H, H-\xi\rangle) \\ &\quad - \sum H^{k^*} h^{k^*}_{ij} h^{l^*}_{ij} (H-\xi)^{l^*}; \end{aligned} \tag{3.5}$$

$$\frac{1}{2}\mathcal{L}(|H|^2) = |\nabla^\perp H|^2 + |H|^2 - \sum h^{k^*}_{ij} h^{l^*}_{ij} H^{k^*} (H-\xi)^{l^*}. \tag{3.6}$$

Lemma 3.3 (see [15]) *It holds that*

$$\begin{aligned} &\frac{1}{2}\mathcal{L}(|h|^2 + |H-\xi|^2) \\ &= |\nabla h|^2 + |\nabla^\perp H|^2 + |h|^2 + K(3|h|^2 - 2|H|^2 + \langle H, H-\xi\rangle) \\ &\quad + \langle H, H-\xi\rangle - \sum h^{k^*}_{ij} h^{l^*}_{ij} (H-\xi)^{k^*} (H-\xi)^{l^*} - \sum h^{k^*}_{ij} h^{l^*}_{ij} H^{k^*} (H-\xi)^{l^*}. \end{aligned} \tag{3.7}$$

The following two propositions are important in the proof of Theorem 1.3, which are also key to the argument in [15].

Proposition 3.1 (see [15]) *Let $x : M^2 \to \mathbb{C}^2$ be an oriented and compact Lagrangian ξ-submanifold. If*

$$|h|^2 + |H-\xi|^2 \le |\xi|^2 + 4,$$

then

$$|h|^2 + |H-\xi|^2 \equiv |\xi|^2 + 4 \tag{3.8}$$

and $x(M^2)$ is a topological torus.

Proposition 3.2 (see [15]) *Let $x : M^m \to N^m$ be a Lagrangian submanifold in a Kähler manifold N^m with the second fundamental form h. If both M^m and N^m are flat, then around each point $p \in M^m$, there exists an orthonormal frame field $\{e_i, e_{i^*}\}$ with $e_{i^*} = Je_i$ $(1 \le i \le m)$ such that, at the point p,*

$$h^{k^*}_{ij} := \langle h(e_i, e_j), e_{k^*}\rangle = \lambda^{k^*}_i \delta_{ij}, \quad 1 \le i, j, k \le m. \tag{3.9}$$

Now we are in the position to give the proof of Theorem 1.3.

Firstly, by Proposition 3.1,

$$|h|^2 + |H - \xi|^2 \equiv |\xi|^2 + 4 = \text{constant}. \tag{3.10}$$

So, with respect to any local orthonormal frame field $\{e_1, e_2\}$, it holds that

$$\sum h^{k^*}_{ij} h^{k^*}_{ij,l} + \sum (H^{k^*} - \xi^{k^*}) H^{k^*}_{,l} = 0, \quad l = 1, 2. \tag{3.11}$$

Secondly, it is much more convenient for us to simply consider the following two cases.

Case 1: $\xi = 0$.

In this case, Theorem 1.3 reduces to Corollary 1.1.

Case 2: $\xi \neq 0$.

In this case, we easily find that the normal bundle $T^\perp M^2$ has an orthonormal parallel frame field $\{e_3, e_4\}$, say, we can take e_3 to be parallel to ξ. It follows that the immersion $x : M^2 \to \mathbb{C}^2$ is of flat normal bundle. Since x is Lagrangian, its tangent bundle TM^2 must also be flat (i.e. $K \equiv 0$) because the complex structure $J : TM^2 \to T^\perp M$ is a bundle isomorphism which preserves both the bundle metric and the bundle connection. Then it follows from the Gauss equation that $|h|^2 \equiv |H|^2$. Consequently, for any local orthonormal frame field $\{e_i, e_{i^*}\}$, we have $|h|^2_i \equiv |H|^2_i$, $i = 1, 2$, and $\mathcal{L}(|h|^2) = \mathcal{L}(|H|^2)$. Equivalently,

$$\sum h^{k^*}_{ij} h^{k^*}_{ij,l} = \sum H^{k^*} H^{k^*}_{,l}, \quad l = 1, 2 \tag{3.12}$$

and

$$|\nabla h|^2 \equiv |\nabla^\perp H|^2, \tag{3.13}$$

by (3.5)–(3.6) and the fact that

$$K = |H|^2 - |h|^2 \equiv 0. \tag{3.14}$$

To proceed, we need to prove the following proposition.

Proposition 3.3 *Let $x : M^2 \to \mathbb{C}^2$ be a flat Lagrangian ξ-submanifold in $\mathbb{C}^2$. Then for any orthonormal frame field $\{e_1, e_2\}$, we have*

$$\begin{aligned}\frac{1}{2}\mathcal{L}(|\nabla h|^2) = {} & |\nabla^2 h|^2 + 2|\nabla h|^2 - 2\sum h^{p^*}_{ij,k} h^{p^*}_{ij,l} h^{q^*}_{kl} (H - \xi)^{q^*} \\ & - \sum h^{p^*}_{ij,k} h^{p^*}_{il} h^{q^*}_{jk,l} (H - \xi)^{q^*} - \sum h^{p^*}_{ij,k} h^{p^*}_{il} h^{q^*}_{jl} H^{q^*}_{,k}.\end{aligned} \tag{3.15}$$

Proof Since x is flat and Lagrangian, the normal bundle $T^\perp M^2$ is also flat. So by (2.2), (2.11) and Ricci identities, $\nabla^r h$ is totally symmetric for any $r \geq 0$. It then follows from (3.4) that

$$\begin{aligned}\frac{1}{2}\mathcal{L}(|\nabla h|^2) &= \frac{1}{2}\Delta(|\nabla h|^2) - \frac{1}{2}\langle x, \nabla(|\nabla h|^2)\rangle \\ &= |\nabla^2 h|^2 + \sum h^{p^*}_{ij,k} h^{p^*}_{ij,kll} - \langle x, e_l\rangle \sum h^{p^*}_{ij,k} h^{p^*}_{ij,kl} \\ &= |\nabla^2 h|^2 + \sum h^{p^*}_{ij,k} H^{p^*}_{,ijk} - \langle x, e_l\rangle \sum h^{p^*}_{ij,k} h^{p^*}_{ij,kl}\end{aligned}$$

$$\begin{aligned}
&= |\nabla^2 h|^2 + \sum h^{l^*}_{ij,k} h^{l^*}_{im,jk} \langle x, e_m\rangle + \sum h^{l^*}_{ij,k} h^{l^*}_{im,j}\langle e_k, e_m\rangle \\
&\quad - \sum h^{l^*}_{ij,k} h^{l^*}_{im,j} h^{p^*}_{mk} (H-\xi)^{p^*} + \sum h^{l^*}_{ij,k} h^{l^*}_{ij,k} - \sum h^{l^*}_{ij,k} h^{l^*}_{im} h^{p^*}_{mj} H^{p^*}_{,k} \\
&\quad - \sum (H-\xi)^{p^*} h^{l^*}_{ij,k} h^{l^*}_{im,k} h^{p^*}_{mj} - \sum (H-\xi)^{p^*} h^{l^*}_{ij,k} h^{l^*}_{im} h^{p^*}_{mj,k} \\
&\quad - \langle x, e_l\rangle \sum h^{p^*}_{ij,k} h^{p^*}_{ij,kl} \\
&= |\nabla^2 h|^2 + 2|\nabla h|^2 - 2\sum h^{l^*}_{ij,k} h^{l^*}_{im,j} h^{p^*}_{mk} (H-\xi)^{p^*} \\
&\quad - \sum h^{l^*}_{ij,k} h^{l^*}_{im} h^{p^*}_{mj} H^{p^*}_{,k} - \sum (H-\xi)^{p^*} h^{l^*}_{ij,k} h^{l^*}_{im} h^{p^*}_{mj,k}. \qquad (3.16)
\end{aligned}$$

Thus Proposition 3.3 is proved.

Now, from (3.11) and (3.12) we find

$$\sum h^{k^*}_{ij} h^{k^*}_{ij,l} = \sum H^{k^*} H^{k^*}_{,l} = \frac{1}{2}\sum H^{k^*}_{,l} \xi^{k^*}, \quad l = 1, 2. \qquad (3.17)$$

Next we are to prove that h is parallel, i.e., $\nabla h \equiv 0$. For this end we first assume the contrary. Then $|\nabla h|^2$ must be positive at some point on M^2. Let $p \in M^2$ be the point such that

$$|\nabla h|^2(p) = \max_{M^2} |\nabla h|^2.$$

Then we have that $|\nabla h|^2(p) > 0$ and

$$\nabla(|\nabla h|^2)(p) = 0, \quad \mathcal{L}(|\nabla h|^2)(p) \le 0. \qquad (3.18)$$

Furthermore, by Proposition 3.2, we can choose an orthonormal frame field $\{e_i, e_{i^*}\}$ such that

$$h^{k^*}_{ij} = \lambda^{k^*}_i \delta_{ij} \quad \text{at } p. \qquad (3.19)$$

Since $h^{k^*}_{ij}$ is totally symmetric in i, j, k, we know that

$$h^{k*}_{ij}(p) = 0 \quad \text{if } (i,j,k) \ne (1,1,1) \text{ or } (i,j,k) \ne (2,2,2). \qquad (3.20)$$

This with (3.2) shows that

$$H^{j^*}_{,i}(p) = \sum h^{j^*}_{ik}(p)\langle x, e_k\rangle(p) = 0, \quad i \ne j.$$

Therefore

$$h^{1^*}_{11,2} + h^{1^*}_{22,2} = 0, \quad h^{2^*}_{11,1} + h^{2^*}_{22,1} = 0 \qquad (3.21)$$

at p.

On the other hand, from (3.12), (3.17) and (3.20), we obtain at p

$$\begin{aligned}
h^{1^*}_{11} h^{1^*}_{111} + h^{2^*}_{22} h^{2^*}_{221} &= h^{1^*}_{11}(h^{1^*}_{111} + h^{1^*}_{221}) + h^{2^*}_{22}(h^{2^*}_{111} + h^{2^*}_{221}), \\
h^{1^*}_{11} h^{1^*}_{112} + h^{2^*}_{22} h^{2^*}_{222} &= h^{1^*}_{11}(h^{1^*}_{112} + h^{1^*}_{222}) + h^{2^*}_{22}(h^{2^*}_{112} + h^{2^*}_{222}), \\
2h^{1^*}_{11} h^{1^*}_{111} + 2h^{2^*}_{22} h^{2^*}_{221} &= \xi^{1^*}(h^{1^*}_{111} + h^{1^*}_{221}) + \xi^{2^*}(h^{2^*}_{111} + h^{2^*}_{221}), \\
2h^{1^*}_{11} h^{1^*}_{112} + 2h^{2^*}_{22} h^{2^*}_{222} &= \xi^{1^*}(h^{1^*}_{112} + h^{1^*}_{222}) + \xi^{2^*}(h^{2^*}_{112} + h^{2^*}_{222}).
\end{aligned}$$

It then follows from (2.11) and (3.21) that, at p,

$$h^{1^*}_{11}h^{2^*}_{112}+h^{2^*}_{22}h^{2^*}_{111}=0,\quad -h^{1^*}_{11}h^{2^*}_{111}+h^{2^*}_{22}h^{2^*}_{112}=0, \tag{3.22}$$

$$2h^{1^*}_{11}h^{1^*}_{111}-2h^{2^*}_{22}h^{2^*}_{111}=\xi^{1^*}(h^{1^*}_{111}+h^{2^*}_{112}), \tag{3.23}$$

$$2h^{1^*}_{11}h^{2^*}_{111}+2h^{2^*}_{22}h^{2^*}_{222}=\xi^{2^*}(h^{2^*}_{112}+h^{2^*}_{222}). \tag{3.24}$$

Noting that

$$|h|^2(p)=(h^{1^*}_{11}(p))^2+(h^{2^*}_{22}(p))^2\neq 0 \tag{3.25}$$

due to (3.10), we find from (2.11) and (3.21)–(3.22) that at p,

$$h^{k^*}_{ij,l}=0\quad \text{if } (i,j,k,l)\neq(1,1,1,1) \text{ or } (i,j,k,l)\neq(2,2,2,2). \tag{3.26}$$

It follows that

$$H^{1^*}_{,1}(p)=h^{1^*}_{11,1}(p),\quad H^{2^*}_{,2}(p)=h^{2^*}_{22,2}(p). \tag{3.27}$$

Inputting (3.26) into (3.23) and (3.24), we obtain at p,

$$h^{1^*}_{111}(2h^{1^*}_{11}-\xi^{1^*})=0,\quad h^{2^*}_{222}(2h^{2^*}_{22}-\xi^{2^*})=0. \tag{3.28}$$

Without loss of generality, by the contrary assumption we can assume that $h^{1^*}_{11,1}(p)\neq 0$. Then from (3.28) we have

$$2h^{1^*}_{11}(p)=\xi^{1^*}(p). \tag{3.29}$$

We need to consider the following two cases separately:

(1) $h^{2^*}_{222}(p)\neq 0$.

In this case, we also have

$$2h^{2^*}_{22}(p)=\xi^{2^*}(p). \tag{3.30}$$

Thus

$$H(p)=h^{1^*}_{11}(p)e_{1^*}(p)+h^{2^*}_{22}(p)e_{2^*}(p)=\frac{1}{2}(\xi^{1^*}(p)e_{1^*}(p)+\xi^{2^*}(p)e_{2^*}(p))=\frac{1}{2}\xi,$$

and thus $H(p)-\xi(p)=-\frac{1}{2}\xi$. It follows that

$$|h|^2=|H|^2=|H(p)-\xi(p)|^2=\frac{1}{4}|\xi|^2,$$

which contradicts to (3.10).

(2) $h^{2^*}_{222}(p)=0$.

In this case, we have

$$H^{j^*}_{,i}(p)=h^{j^*}_{11,i}+h^{j^*}_{22,i}=0\quad \text{if } (i,j)\neq(1,1). \tag{3.31}$$

On the other hand, from (3.7), (3.10) and (3.13)–(3.14), we find

$$2|\nabla h|^2=-(|h|^2+\langle H,H-\xi\rangle)+\sum h^{k^*}_{ij}h^{l^*}_{ij}(H-\xi)^{k^*}(H-\xi)^{l^*}+\sum h^{k^*}_{ij}h^{l^*}_{ij}H^{k^*}(H-\xi)^{l^*}$$

$$\begin{aligned}&=-(|h|^2+|H-\xi|^2)-\langle\xi,H-\xi\rangle+\sum h_{ij}^{k^*}h_{ij}^{l^*}(H-\xi)^{k^*}(H-\xi)^{l^*}\\&\quad+\sum h_{ij}^{k^*}h_{ij}^{l^*}H^{k^*}(H-\xi)^{l^*}\\&=-(|\xi|^2+4)-\langle\xi,H-\xi\rangle+\sum h_{ij}^{k^*}h_{ij}^{l^*}(H-\xi)^{k^*}(H-\xi)^{l^*}\\&\quad+\sum h_{ij}^{k^*}h_{ij}^{l^*}H^{k^*}(H-\xi)^{l^*}.\end{aligned}\tag{3.32}$$

Thus, using (3.32), we directly find

$$\begin{aligned}2|\nabla h|^2_{,q}&=-\langle\xi,H_{,q}\rangle+2\Big(\sum h_{ij}^{k^*}h_{ij,q}^{l^*}(H-\xi)^{k^*}(H-\xi)^{l^*}+\sum h_{ij}^{k^*}h_{ij}^{l^*}(H-\xi)^{k^*}H^{l^*}_{,q}\Big)\\&\quad+\Big(\sum h_{ij,q}^{k^*}h_{ij}^{l^*}H^{k^*}(H-\xi)^{l^*}+\sum h_{ij}^{k^*}h_{ij,q}^{l^*}H^{k^*}(H-\xi)^{l^*}\\&\quad+\sum h_{ij}^{k^*}h_{ij}^{l^*}H^{k^*}_{,q}(H-\xi)^{l^*}+\sum h_{ij}^{k^*}h_{ij}^{l^*}H^{k^*}H^{l^*}_{,q}\Big).\end{aligned}\tag{3.33}$$

Inserting (3.18), (3.26)–(3.27), (3.29) and (3.31) into (3.33), we obtain that

$$0=2|\nabla h|^2_{,1}(p)=-2h_{11}^{1^*}(p)h_{11,1}^{1^*}(p)(1+(h_{11}^{1^*})^2(p)).$$

So that $\xi^{1^*}(p)=2h_{11}^{1*}(p)=0$. This with (3.17), (3.26), (3.31), Proposition 3.3 and the assumption that $h_{222}^{2^*}(p)=0$ shows that

$$0\geq\mathcal{L}(|\nabla h|^2)(p)\geq 2|\nabla h|^2(p)+2(h_{11,1}^{1^*})^2(h_{11}^{1^*})^2=2(h_{11,1}^{1^*})^2>0.$$

This contradiction shows that $\nabla h\equiv 0$.

Since M^2 is flat, it is locally isometric to $\mathbb{R}^2$. Therefore, the fact that $\nabla h\equiv 0$ and (3.19) guarantee that, each point $p\in M^2$ has an open neighbourhood U_p on which a parallel orthonormal frame field $\{e_1,e_2\}$ is defined such that the corresponding components $h_{ij}^{k^*}$ satisfy

$$h_{ij}^{k^*}=\lambda_i^{k^*}\delta_{ij}\tag{3.34}$$

on U_p. Moreover, $h_{ij}^{k^*}$ are all constant.

Now let

$$\tilde e_1=e_1\cos\theta-e_2\sin\theta,\quad \tilde e_2=e_1\sin\theta+e_2\cos\theta$$

be another oriented frame field such that the corresponding components $\tilde h_{ij}^{k^*}$ also satisfy (3.34). Then a direct computation shows that we obtain

$$0=\tilde h_{12}^{1^*}=\sin\theta\cos\theta(h_{11}^{1^*}\cos\theta+h_{22}^{2^*}\sin\theta),\quad 0=\tilde h_{12}^{2^*}=\sin\theta\cos\theta(h_{11}^{1^*}\sin\theta-h_{22}^{2^*}\cos\theta)\tag{3.35}$$

and

$$\tilde h_{11}^{1^*}=h_{11}^{1^*}\cos^3\theta-h_{22}^{2^*}\sin^3\theta,\quad \tilde h_{22}^{2^*}=h_{11}^{1^*}\sin^3\theta+h_{22}^{2^*}\cos^3\theta,\tag{3.36}$$

$$\tilde h_{11}^{1^*}\tilde h_{22}^{2^*}=(\sin\theta\cos\theta)^3((h_{11}^{1^*})^2-(h_{22}^{2^*})^2)+h_{11}^{1^*}h_{22}^{2^*}(\cos^6\theta-\sin^6\theta).\tag{3.37}$$

Since $|h|^2=(h_{11}^{1^*})^2+(h_{22}^{2^*})^2\neq 0$ by (3.10), we see from (3.35) that $\sin 2\theta=0$, namely, $\theta=0$, or $\frac{\pi}{2}$ or π. Then it follows from (3.36)–(3.37) that, by choosing $\theta=\pi$, we can change the sign of both $h_{11}^{1^*}$ and $h_{22}^{2^*}$; while by choosing $\theta=\frac{\pi}{2}$, we can change the sign of $h_{11}^{1^*}h_{22}^{2^*}$. Thus, without loss of generality, we can always assume that $h^{1^*}_{11} > 0$ and $h^{2^*}_{22} \geq 0$. This shows that the frame field $\{e_1, e_2\}$ satisfying (3.34) and $h^{1^*}_{11} > 0$, $h^{2^*}_{22} \geq 0$ can be uniquely determined and, therefore, the neighbourhood U_p can be extended to the total space M^2.

Let $\{\omega^1, \omega^2\}$ be the dual frame of $\{e_1, e_2\}$. Then we have

$$\begin{aligned} \mathrm{d}x_*(e_1) &= h^{1^*}_{11}\omega^1 e_{1^*}, \quad \mathrm{d}e_{1^*} = -h^{1^*}_{11}\omega^1 x_*(e_1), \\ \mathrm{d}x_*(e_2) &= h^{2^*}_{22}\omega^2 e_{2^*}, \quad \mathrm{d}e_{2^*} = -h^{2^*}_{22}\omega^2 x_*(e_2). \end{aligned} \tag{3.38}$$

We claim that $h^{2^*}_{22} > 0$. In fact, if $h^{2^*}_{22} = 0$, then $x_*(e_2)$ is a constant vector in $\mathbb{C}^2$ along M^2 which means that $x(M)$ contains a family of parallel straight lines, contradicting to the compactness of M.

Define

$$V_1 = \mathrm{Span}_{\mathbb{R}}\{x_*(e_1), e_{1^*}\}, \quad V_2 = \mathrm{Span}_{\mathbb{R}}\{x_*(e_2), e_{2^*}\}.$$

Then (3.38) shows that V_1, V_2 are two orthogonal and constant complex line bundles on M^2, corresponding to two orthogonal, one dimensional complex subspaces of $\mathbb{C}^2$. So, up to some holomorphic isometry of $\mathbb{C}^2$, we can identify both V_1 and V_2 with $\mathbb{C}^1$. Due to the flatness of M^2, the isometric immersion x can be locally represented by $x \equiv (\gamma_1, \gamma_2) : (a_1, b_1) \times (a_2, b_2) \to \mathbb{C}^1 \times \mathbb{C}^1 \equiv \mathbb{C}^2$, where

$$\gamma_1 : (a_1, b_1) \to \mathbb{C}^1, \quad \gamma_2 : (a_2, b_2) \to \mathbb{C}^1$$

are two unit-speed curves (i.e., with arc-length parameters). Moreover, as plane curves, the curvatures of γ_1 and γ_2 are respectively the constants $h^{1^*}_{11} > 0$ and $h^{2^*}_{22} > 0$. So both $\gamma_1(a_1, b_1)$ and $\gamma_2(a_2, b_2)$ are circle arcs of radius $a := (h^{1^*}_{11})^{-1}$ and $b := (h^{2^*}_{22})^{-1}$, respectively.

Finally, by the compactness of M^2, we obtain that $x(M^2) = S^1(a) \times S^1(b)$. Therefore, Theorem 1.3 is proved.

Proof of Theorem 1.4:

Case (1): $\xi = 0$. In this case, the theorem reduces to the theorem of Li and Wang (Theorem 1.1).

Case (2): $\xi \neq 0$. In this case, (3.14) still holds and thus the inequality $|h|^2 - \langle H, \xi\rangle \leq 2$ is equivalent to that $|h|^2 + |H - \xi|^2 \leq |\xi|^2 + 4$. Therefore, Theorem 1.4 is equivalent to Theorem 1.3 in the present case.

Acknowledgement The authors would like to thank the reviewer's careful revisions and suggestions, which are helpful for improving our manuscript.

References

[1] Anciaux, H., Construction of Lagrangian self-similar solutions to the mean curvature flow in $\mathbb{C}^n$, *Geom. Dedicata*, **120**, 2006, 37–48.

[2] Cao, H.-D. and Li, H. Z., A gap theorem for self-shrinkers of the mean curvature flow in arbitrary codimension, *Calc. Var. Partial Differential Equations*, **46**, 2013, 879–889.

[3] Castro, I. and Lerma, A. M., Hamiltonian stationary self-similar solutions for Lagrangian mean curvature flow in the complex Euclidean plane, *Proc. Amer. Math. Soc.*, **138**, 2010, 1821–1832.

[4] Castro, I. and Lerma, A. M., The Clifford torus as a self-shrinker for the Lagrangian mean curvature flow, *Int. Math. Res. Not.*, **6**, 2014, 1515–1527.

[5] Cheng, Q.-M., Hori, H. and Wei, G. X., Complete Lagrangian self-shrinkers in $\mathbb{R}^4$, arXiv [math.DG]: 1802.02396v2, 2018.

[6] Cheng, Q.-M., Ogata, S. and Wei, G. X., Rigidity theorems of λ-hypersurfaces, *Comm. Anal. Geom.*, **24**, 2016, 45–58.

[7] Cheng, Q.-M. and Wei, G. X., The Gauss image of λ-hypersurfaces and a Bernstein type problem, arXiv [math.DG]: 1410.5302v1, 2014.

[8] Cheng, Q.-M. and Wei, G. X., Complete λ-hypersurfaces of the weighted volume-preserving mean curvature flow, *Calc. Var. Partial Differential Equations*, **57**(2), 2018, 21 pp.

[9] Cheng, Q.-M. and Wei, G. X., Complete λ-surfaces in $\mathbb{R}^3$, *Calc. Var. Partial Differential Equations*, **60**(1), 2021, 19 pp.

[10] Colding, T. H. and Minicozzi II, W. P., Generic mean curvature flow I; Generic singularities, *Ann. of Math.*, **175**, 2012, 755–833.

[11] Guang, Q., Gap and rigidity theorems of λ-hypersurfaces, *Proc. Amer. Math. Soc.*, **146**, 2018, 4459–4471.

[12] Huisken, G., Asymptotic behavior for singularities of the mean curvature flow, *J. Differential Geom.*, **31**, 1990, 285–299.

[13] Li, H. Z. and Wang, X. F., New characterizations of the clifford torus as a Lagrangian self-shrinker, *J. Geom. Anal.*, **27**, 2017, 1393–1412.

[14] Li, H. Z. and Wei, Y., Classification and rigidity of self-shrinkers in the mean curvature flow, *J. Math. Soc. Japan*, **66**, 2014, 709–734.

[15] Li, X. X. and Chang, X. F., A rigidity theorem of ξ-submanifolds in $\mathbb{C}^2$, *Geom. Dedicata*, **185**, 2016, 155–169.

[16] Li, X. X. and Li, Z. P., Variational characterizations of ξ-submanifolds in the Eulicdean space $\mathbb{R}^{m+p}$, *Ann. Mat. Pura Appl.*, **199**, 2020, 1491–1518.

[17] Smoczyk, K., Self-shrinkers of the mean curvature flow in arbitrary codimension, *Int. Math. Res. Not.*, **48**, 2005, 2983–3004.