Chin. Ann. Math. Ser. B
43(4), 2022, 485–498
DOI: 10.1007/s11401-022-0341-6

Chinese Annals of Mathematics, Series B

Equivariant Mapping Class Group and Orbit Braid Group*

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Abstract Motivated by the work of Birman about the relationship between mapping class groups and braid groups, the authors discuss the relationship between the orbit braid group and the equivariant mapping class group on the closed surface M with a free and proper group action in this paper. Their construction is based on the exact sequence given by the fibration $\mathcal{F}_0^G M \to F(M/G, n)$. The conclusion is closely connected with the braid group of the quotient space. Comparing with the situation without the group action, there is a big difference when the quotient space is $\mathbb{T}^2$.

Keywords Equivariant mapping class group, Orbit braid group, Evaluation map, Center of braid group
2000 MR Subject Classification 20F36, 20F38, 37A20

1 Introduction

Braid groups of the plane were defined by Artin [1] in 1925, and further studied in [2–3]. Braid groups of surfaces were studied by Zariski [21], and were later generalized using the definition from Fox [9]. Bellingeri simplified presentations of braid groups and pure braid groups on surfaces and showed some propertities of surface pure braid groups in [4].

Let M be a closed surface, with a finite set $\mathcal{P}$ of n distinguished points in M. $\mathcal{B}_n M$ (respectively, $\mathcal{F}_n M$) denotes all homeomorphisms between M which preserve the set $\mathcal{P}$ (respectively, each point in $\mathcal{P}$) and preserve the orientation if M is oriented. $\pi_0(\mathcal{F}_n M)$ is the set of all path connected components of $\mathcal{F}_n M$, and the n-th mapping class group denoted by $\mathrm{Mod}(M, n)$ is the set of all path connected components of $\mathcal{B}_n M$. The algebraic structure of the mapping class group is of great importance in the theory of Riemann surfaces. Connections between the mapping class group and the braid group of closed surfaces have been studied, in order to help to find the generators and relations of their mapping class groups.

The (pure) braid group of M on n strands is denoted by $B_n M$ ($P_n M$). Let S_g be an oriented closed surface of genus g, with $g \geq 0$. In 1969, Birman [5–6] gave the basic relationship between mapping class groups and braid groups on S_g.

Manuscript received November 8, 2021. Revised January 11, 2022.
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*This work was supported by the National Natural Science Foundation of China (No. 11971112).

Theorem 1.1 (see [5]) *Let* $i_* : \pi_0(\mathcal{F}_n S_g) \to \pi_0(\mathcal{F}_0 S_g)$ *be the homomorphism induced by inclusion* $\mathcal{F}_n S_g \subset \mathcal{F}_0 S_g$*. Then*

$$\ker i_* \cong P_n S_g \quad \text{if } g \geq 2;$$
$$\ker i_* \cong P_n S_g / Z(P_n S_g) \quad \text{if } g = 1, n \geq 2 \text{ or } g = 0, n \geq 3.$$

Theorem 1.2 (see [5]) *Let* $i_* : \mathrm{Mod}(S_g, n) \to \mathrm{Mod}(S_g, 0)$ *be the homomorphism induced by inclusion* $\mathcal{B}_n S_g \subset \mathcal{B}_0 S_g$*. Then*

$$\ker i_* \cong B_n S_g \quad \text{if } g \geq 2;$$
$$\ker i_* \cong B_n S_g / Z(B_n S_g) \quad \text{if } g = 1, n \geq 2 \text{ or } g = 0, n \geq 3.$$

Using above results, Birman obtained a full set of generators for the mapping class groups of n-punctured oriented 2-manifolds. Then she computed the mapping class group of the n-punctured sphere and gave relations in the mapping class group of torus in a new way.

Denote the nonorientable closed surface of genus k as N_k with $k \geq 1$. S_{k-1} is its orientable double covering and $\pi : S_{k-1} \to N_k$ is a covering map. The induced homomorphism between braid group $\varphi_n : B_n(N_k) \to B_{2n}(S_{k-1})$ is injective (see [13]) on the level of fundamental group. The homomorphism between mapping class groups $\phi_n : \mathrm{Mod}(N_k, n) \to \mathrm{Mod}(S_{k-1}, 2n)$ induced by π is also injective. Furthermore, if $k \geq 3$, then we have a commutative diagram of the following form:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & B_n(N_k) & \longrightarrow & \mathrm{Mod}(N_k, n) & \xrightarrow{\psi_n} & \mathrm{Mod}(N_k, 0) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \phi_n} & & \downarrow{\scriptstyle \phi_0} & & \\ 1 & \longrightarrow & B_{2n}(S_{k-1}) & \longrightarrow & \mathrm{Mod}(S_{k-1}, 2n) & \xrightarrow{\widetilde{\psi}_n} & \mathrm{Mod}(S_{k-1}, 0) & \longrightarrow & 1 \end{array}$$

where ψ_n and $\widetilde{\psi}_n$ are the homomorphisms induced by inclusions $\mathcal{B}_n N_k \subset \mathcal{B}_0 N_k$ and $\mathcal{B}_{2n} S_{k-1} \subset \mathcal{B}_0 S_{k-1}$ (see [14]).

In this paper, we study the relationship between the orbit braid group and the equivariant mapping class group on the closed surface M which admits a group action. Let G be a discrete group and act on M freely and properly. Consider a finite set $\mathcal{P} = \{\mathbf{x}_1, \cdots, \mathbf{x}_n\}$ of n arbitrarily chosen points on M with different orbits. Define $\mathcal{B}_n^G M$ (respectively, $\mathcal{F}_n^G M$) to be the group of all G-homeomorphisms $f : M \to M$ which satisfy $f(G\mathcal{P}) = G\mathcal{P}$ (respectively, $f(G\mathbf{x}_i) = G\mathbf{x}_i, i = 1, \cdots, n$) and preserve the orientation if M is oriented. These two groups are endowed with compact-open topology. $\pi_0(\mathcal{F}_n^G M, \mathrm{id})$ is the set of all path connected components of $\mathcal{F}_n^G M$. The equivariant mapping class group denoted by $M^G(M, n)$ is defined to be the set of all path connected components $\mathcal{B}_n^G M$.

Let M be a connected topological manifold of dimension at least 2 with an effective action of a finite group G. The orbit configuration space of n ordered points in the G-space M is defined as

$$F_G(M, n) = \{(\mathbf{x}_1, \cdots, \mathbf{x}_n) \in M^n : G(\mathbf{x}_i) \cap G(\mathbf{x}_j) = \emptyset \text{ if } i \neq j\}$$

with subspace topology and $G(\mathbf{x})$ denotes the orbit of $\mathbf{x}$. The notion of the orbit configuration space was first defined in [20]. Since then, this subject, with respect to the algebraic topology (especially cohomology) and relative topics of orbit configuration spaces, has been further developed.

The following concept of the orbit braid group has been discussed in [16]. The action of G on M induces a natural action of G^n on $F_G(M,n)$. There is also a canonical free action of the symmetric group Σ_n on $F_G(M,n)$. Generally, these two actions are not commutative. Let $\pi_1^E(F_G(M,n),x,x^{\text{orb}})$ be the set consisting of the homotopy classes relative to ∂I of all paths $\alpha : I \to F_G(M,n)$ with $\alpha(0)=x$ and $\alpha(1)\in x^{\text{orb}}$, where $x^{\text{orb}}=\{gx^{\text{orb}} : g\in G^n, \sigma\in\Sigma_n\}$, which is the orbit set at x under two actions of G^n and Σ_n. The orbit braid group $B_n^{\text{orb}}(M,G)$ bijectively corresponds to $\pi_1^E(F_G(M,n),x,x^{\text{orb}})$. And the pure orbit braid group $P_n^{\text{orb}}(M,G)$ bijectively corresponds to $\pi_1^E(F_G(M,n),x,G^n(x))$. If the action G on M is free, then $P_n^{\text{orb}}(M,G)\cong P_n(M/G)$ and $B_n^{\text{orb}}(M,G)\cong B_n(M/G)$. Thus the orbit braid group $B_n^{\text{orb}}(M,G)$ (respectively, $P_n^{\text{orb}}(M,G)$) we will discuss later corresponds to $B_n(M/G)$ (respectively, $P_n(M/G)$), where M/G is an oriented or nonorientable closed surface (see [15]).

We prove the following results in this article.

Theorem 1.3 (Theorems 3.1–3.3) *Let $i_*^G : \pi_0(\mathcal{F}_n^G M)\to\pi_0(\mathcal{F}_0^G M)$ be the homomorphism induced by inclusion $\mathcal{F}_n^G M\subset\mathcal{F}_0^G M$. Then*

$$\ker i_*^G \cong P_n(M/G) \quad \text{if } M/G \text{ is } S_g,\ g\geq 2 \text{ or } N_k,\ k\geq 2;$$
$$\ker i_*^G \cong P_n(M/G)/Z(P_n(M/G)) \quad \text{if } M/G \text{ is } \mathbb{S}^2 \text{ or } \mathbb{R}P^2.$$

When M/G is $\mathbb{T}^2$,

$$M=\mathbb{T}^2,\quad G=\mathbb{Z}/q\mathbb{Z}\bigoplus\mathbb{Z}/r\mathbb{Z},\quad q,r\geq 1 \text{ and}$$
$$\ker i_*^G\cong P_n(M/G)/\langle\widetilde{a}^q,\widetilde{b}^r : \widetilde{a}^q\widetilde{b}^r=\widetilde{b}^r\widetilde{a}^q\rangle.$$

Theorem 1.4 (Theorem 3.4) *Let M be a closed surface. Let*

$$j_*^G : \mathrm{Mod}_G(M,n)\to\mathrm{Mod}_G(M,0)$$

be the homomorphism induced by inclusion $\mathcal{B}_n^G M\subset\mathcal{B}_0^G M$. Then

$$\ker j_*^G \cong B_n(M/G) \quad \text{if } M/G \text{ is } S_g,\ g\geq 2 \text{ or } N_k,\ k\geq 2;$$
$$\ker j_*^G \cong B_n(M/G)/Z(P_n(M/G)) \quad \text{if } M/G \text{ is } \mathbb{S}^2 \text{ or } \mathbb{R}P^2.$$

When M/G is $\mathbb{T}^2$,

$$M=\mathbb{T}^2,\quad G=\mathbb{Z}/q\mathbb{Z}\bigoplus\mathbb{Z}/r\mathbb{Z},\quad q,r\geq 1 \text{ and}$$
$$\ker j_*^G\cong B_n(M/G)/\langle\widetilde{a}^q,\widetilde{b}^r : \widetilde{a}^q\widetilde{b}^r=\widetilde{b}^r\widetilde{a}^q\rangle.$$

This paper is organized as follows. In Section 2, we introduce the centers of pure braid groups on oriented and nonorientable closed surfaces. Section 3 is the main part of this paper. We establish the relationship between the orbit braid group and the equivariant mapping class group on the closed surface which admits a free and proper group action. The proofs of Theorems 1.3–1.4 are given in this section. The conclusion is closely connected with the quotient space. And comparing with the situation without the group action, there is a big difference when the quotient space is $\mathbb{T}^2$.

2 The Centers of Pure Braid Groups on Closed Surfaces

First, we will briefly recall the definition and properties of braid groups. Let M be a smooth manifold. The configuration space of n ordered points in M, denoted by $F(M,n)$ is defined as:

$$F(M,n) := \{(\mathbf{x}_1, \cdots, \mathbf{x}_n) \in M^n : \mathbf{x}_i \neq \mathbf{x}_j \text{ if } i \neq j\}.$$

There is a natural action of the symmetric group Σ_n on the space $F(M,n)$, given by permuting the coordinates. The configuration space of n unordered points in M is the quotient space:

$$C(M,n) := F(M,n)/\Sigma_n.$$

Following Fox and Neuwirth [9], the n-th pure braid group $P_n(M)$ (respectively, the n-th braid group $B_n(M)$) is defined to be the fundamental group of $F(M,n)$ (respectively, of $C(M,n)$).

If m, n ($m > n$) are positive integers, we can define a homomorphism $\theta_* : P_m(M) \to P_n(M)$ induced by the projective $\theta : F(M,m) \to F(M,n)$ defined by

$$\theta((\mathbf{x}_1, \cdots, \mathbf{x}_m)) = (\mathbf{x}_1, \cdots, \mathbf{x}_n).$$

In [8], Fadell and Neuwirth study the map θ, and show that it is a locally trivial fibration. The fiber over a point $(\mathbf{x}_1, \cdots, \mathbf{x}_n)$ of the base space is $F(M - \{\mathbf{x}_1, \cdots, \mathbf{x}_n\}, m-n)$. Applying the associated long exact homotopy sequence, we obtain the pure braid group exact sequence of Fadell and Neuwirth:

$$\cdots \pi_2(F(M,n)) \to P_{m-n}(M - \{\mathbf{x}_1, \cdots, \mathbf{x}_n\}) \xrightarrow{\delta_*} P_m(M) \xrightarrow{\theta_*} P_n(M) \to 1.$$

The following short exact sequence is proved to be true where $n \geq 3$ if $M = \mathbb{S}^2$, $n \geq 2$ if $M = \mathbb{R}P^2$ and $n \geq 1$ referred to [7]:

$$1 \to P_{m-n}(M - \{\mathbf{x}_1, \cdots, \mathbf{x}_n\}) \xrightarrow{\delta_*} P_m(M) \xrightarrow{\theta_*} P_n(M) \to 1.$$

Let S_g be the oriented closed surface of genus g. Birman computed all the centers of pure braid groups on oriented closed surface in the following theorem.

Theorem 2.1 (see [5–6])

$$\begin{aligned}
&Z(P_n(S_g)) = 1, \quad g \geq 2;\\
&Z(P_n(\mathbb{T}^2)) = \langle \widetilde{a}, \widetilde{b} \mid \widetilde{a}\widetilde{b} = \widetilde{b}\widetilde{a} \rangle;\\
&Z(P_n(\mathbb{S}^2)) = \mathbb{Z}_2^2, \quad n \geq 3;\\
&Z(P_1(\mathbb{S}^2)) = Z(P_2(\mathbb{S}^2)) = 1.
\end{aligned}$$

Let N_k be the nonorientable closed surface of genus k. Paris and Rolfsen obtained the following theorem.

Theorem 2.2 (see [18]) *If $k \geq 2$, then $Z(P_n(N_k)) = 1$.*

Proof We apply an induction on the number n of strands. For $n = 1$,

$$P_1(N_k) = \pi_1(N_k) = \Big\langle \rho_1, \cdots, \rho_k \Big| \prod_{j=1}^{k} \rho_j^2 = 1 \Big\rangle,$$

which is a finitely generated group with a single defining relation. Thus we obtain that $Z(P_1(N_k)) = 1$, if $k \geq 2$ (see [17]). The Fadell-Neuwirth fibration gives us the exact sequence

$$1 \to \pi_1(N_k - \{\mathbf{x}_1, \cdots, \mathbf{x}_n\}) \xrightarrow{\delta_*} P_{n+1}(N_k) \xrightarrow{\theta_*} P_n(N_k) \to 1,$$

where $\pi_1(N_k - \{\mathbf{x}_1, \cdots, \mathbf{x}_n\})$ is a free group for $n \geq 1$. Suppose $Z(P_n(N_k)) = 1$. Since p_* is surjective, $\theta_*(Z(P_{n+1}(N_k))) \subset Z(P_n(N_k)) = 1$. Hence $Z(P_{n+1}(N_k))$ lies in the group of $\ker \theta_* = \operatorname{im} \delta_*$, which is a free group. Thus $Z(P_{n+1}(N_k)) = 1$.

The pure braid group of the projective plane possesses non-trivial center. In [10] and [11], the following theorem has been proved.

Theorem 2.3 *If $n \geq 2$, then $Z(P_n(\mathbb{R}P^2))$ is cyclic of order 2.*

For the proofs of the main theorems, we have to describe the generator of $Z(P_n(\mathbb{R}P^2))$ geometrically and algebraically. Next we will give the generator in a different way. The specific presentation of $P_n(\mathbb{R}P^2)$ is given as follows.

Theorem 2.4 (see [12]) *The group $P_n(\mathbb{R}P^2)$ admits the following presentation:*

- *Generators*: $B_{ij},\ 1 \leq i < j \leq n;\ \rho_k,\ 1 \leq k \leq n.$
- *Relations*:

(a) $B_{rs} B_{ij} B_{rs}^{-1} = \begin{cases} B_{ij}, & i < r < s < j, \\ B_{ij}^{-1} B_{rj}^{-1} B_{ij} B_{rj} B_{ij}, & r < i = s < j, \\ B_{sj}^{-1} B_{ij} B_{sj}, & i = r < s < j, \\ B_{sj}^{-1} B_{rj}^{-1} B_{sj} B_{rj} B_{ij} B_{rj}^{-1} B_{sj}^{-1} B_{rj} B_{sj}, & r < i < s < j; \end{cases}$

(b) $\rho_i \rho_j \rho_i^{-1} = \rho_j^{-1} B_{ij}^{-1} \rho_j^2,\ 1 \leq i < j \leq n;$

(c) $\rho_i^2 = B_{1i} \cdots B_{i-1,i} B_{i,i+1} \cdots B_{in},\ 1 \leq i \leq n;$

(d) *for* $1 \leq i < j \leq n,\ 1 \leq k \leq n,\ k \neq j,$

$$\rho_k B_{ij} \rho_k^{-1} \begin{cases} B_{ij}, & j < k \text{ or } k < i. \\ \rho_j^{-1} B_{ij}^{-1} \rho_j, & k = i. \\ \rho_j^{-1} B_{kj}^{-1} \rho_j B_{kj}^{-1} B_{ij} B_{kj} \rho_j^{-1} B_{kj} \rho_j, & i < k < j. \end{cases}$$

We view the projective plane as the quotient space of the sphere. Then the sketches of B_{ij} and ρ_i are given in Figures 1–2.

From the relations (b) and (d), each generator B_{ij} in $P_n(\mathbb{R}P^2)$ can be presented by ρ_i and ρ_j:

$$B_{ij} = \rho_j \rho_i^{-1} \rho_j^{-1} \rho_i, \quad 1 \leq i < j \leq n.$$

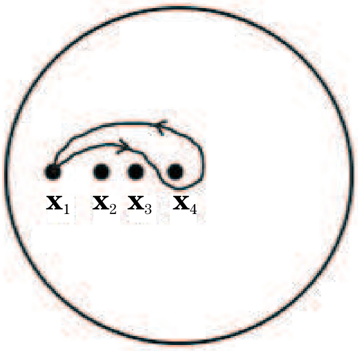

Figure 1

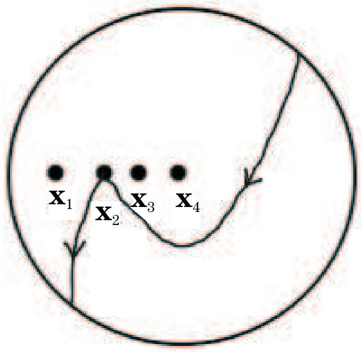

Figure 2

With this presentation and relation (b), we obtain

$$\rho_i\rho_j\rho_i\rho_j = \rho_j\rho_i\rho_j\rho_i, \quad 1 \le i, j \le n.$$

When $n = 1$, $P_1(\mathbb{R}P^2) = \pi_1(\mathbb{R}P^2) = \mathbb{Z}_2$, which is an Abelian group. Next we consider $n \ge 2$.

Lemma 2.1 *For $n \ge 2$, $\tau_n = \tau_{n1} \cdots \tau_{nn}$ lies in the center of $P_n(\mathbb{R}P^2)$, where*

$$\tau_{ni} = B_{i,i+1}B_{i,i+2}\cdots B_{in} = B_{i-1,i}^{-1}\cdots B_{1i}^{-1}\rho_i^2, \quad i = 1, \cdots, n.$$

Proof According to relation (d), $\tau_{ni}\rho_k = \rho_k\tau_{ni}$ for $k < i$. For any $i \ne k$, we have

$$\begin{aligned} B_{ik}^{-1}\rho_k B_{ik} &= (\rho_i^{-1}\rho_k\rho_i\rho_k^{-1})\rho_k(\rho_k\rho_i^{-1}\rho_k^{-1}\rho_i) \\ &= \rho_i^{-1}(\rho_k\rho_i\rho_k\rho_i^{-1})\rho_k^{-1}\rho_i \\ &= \rho_i^{-1}(\rho_i^{-1}\rho_k\rho_i)\rho_k\rho_k^{-1}\rho_i \\ &= \rho_i^{-2}\rho_k\rho_i^2. \end{aligned}$$

Thus we obtain

$$\begin{aligned} \tau_{nk}\rho_k\tau_{nk}^{-1} &= B_{k-1,k}^{-1}\cdots B_{1k}^{-1}\rho_k\cdots B_{1k}\cdots B_{k-1,k} \\ &= \rho_1^{-2}\cdots\rho_{k-1}^{-2}\rho_k\rho_{k-1}^2\cdots\rho_1^2. \end{aligned}$$

From the above results, we obtain

$$\begin{aligned}\tau_n\rho_k &= \tau_{n1}\cdots\tau_{nn}\rho_k\\ &= \tau_{n1}\cdots\tau_{n,k-1}\rho_1^{-2}\cdots\rho_{k-1}^{-2}\rho_k\rho_{k-1}^2\cdots\rho_1^2\tau_{nk}\cdots\tau_{nn}\\ &= (\tau_{n2}\rho_2^{-2})\cdots(\tau_{n,k-1}\rho_{k-1}^{-2})\rho_k\rho_{k-1}^2\cdots\rho_1^2\tau_{nk}\cdots\tau_{nn}\\ &= B_{12}^{-1}(B_{23}^{-1}B_{13}^{-1})\cdots(B_{k-2,k-1}^{-1}\cdots B_{1,k-1}^{-1})\rho_k\rho_{k-1}^2\cdots\rho_1^2\tau_{nk}\cdots\tau_{nn}\\ &= \rho_kB_{12}^{-1}(B_{23}^{-1}B_{13}^{-1})\cdots(B_{k-2,k-1}^{-1}\cdots B_{1,k-1}^{-1})\rho_{k-1}^2\cdots\rho_1^2\tau_{nk}\cdots\tau_{nn}\\ &= \rho_kB_{12}^{-1}(B_{23}^{-1}B_{13}^{-1})\cdots(B_{k-3,k-2}^{-1}\cdots B_{1,k-2}^{-1})\tau_{n,k-1}\rho_{k-2}^2\cdots\rho_1^2\tau_{nk}\cdots\tau_{nn}\\ &= \rho_kB_{12}^{-1}(B_{23}^{-1}B_{13}^{-1})\cdots(B_{k-3,k-2}^{-1}\cdots B_{1,k-2}^{-1})\rho_{k-2}^2\cdots\rho_1^2\tau_{n,k-1}\cdots\tau_{nn}\\ &= \rho_k\tau_n.\end{aligned}$$

Lemma 2.2 *For $n\geq 2$, the center of $P_n(\mathbb{R}P^2)$ is generated by τ_n.*

Proof We apply an induction on the number n of strands. For $n=2, P_2(\mathbb{R}P^2)$ is the quaternion group with the presentation

$$\langle\rho_1,\rho_2\mid\rho_1^2=\rho_2^2,\rho_1^4=1,\rho_1\rho_2\rho_1^{-1}=\rho_2^{-1}\rangle.$$

Then $Z(P_2(\mathbb{R}P^2))=\langle\rho_1^2=\rho_2^2\mid\rho_1^4=1\rangle$, where $\tau_{21}=B_{12}=\rho_1^2$ and $\tau_{22}=B_{12}^{-1}\rho_2^2=1$. The Fadell-Neuwirth fibration gives us the exact sequence

$$1\to\pi_1(\mathbb{R}P^2-\{\mathbf{x}_1,\cdots,\mathbf{x}_n\})\xrightarrow{\delta_*}P_{n+1}(\mathbb{R}P^2)\xrightarrow{\theta_*}P_n(\mathbb{R}P^2)\to 1,$$

where $\pi_1(\mathbb{R}P^2-\{\mathbf{x}_1,\cdots,\mathbf{x}_n\})$ is a free group for $n\geq 2$. Suppose the center of $P_n(\mathbb{R}P^2)$ is generated by τ_n. Since θ_* is surjective and $\theta_*(\tau_{n+1})=\tau_n$, $Z(P_{n+1}(\mathbb{R}P^2))$ is generated by τ_{n+1} and generators of $\ker\theta_*=\operatorname{im}\delta_*$, which is a free group. Thus the center of $P_{n+1}(\mathbb{R}P^2)$ is just generated by τ_{n+1}.

τ_n can be represented by $h(t)$ for $t\in[0,1]$:

$$h(t)=(h_1(t),\cdots,h_n(t)).$$

The components of $h(t)$ are plotted in Figure 3.

3 The Relationship Between Orbit Braid Groups and Equivariant Mapping Class Groups on Compact Closed Surfaces

Let M be a compact closed surface and G be a discrete group acting on M freely and properly. Since M is compact, G is finite. Let $\mathbf{x}_1,\cdots,\mathbf{x}_n$ denote n fixed but arbitrarily chosen points on M, satisfying $G\mathbf{x}_i\bigcap G\mathbf{x}_j=\emptyset$ for $1\leq i\neq j\leq n$.

- Denote $\mathcal{F}_n^GM$ as the group of all G-homeomorphisms $h:M\to M$ which satisfy $h(G\mathbf{x}_i)=G\mathbf{x}_i$, for each i and preserve orientation if M is oriented.
- Denote $\mathcal{B}_n^GM$ as the group of all G-homeomorphisms $h:M\to M$ which satisfy

$$h(G\{\mathbf{x}_1,\cdots,\mathbf{x}_n\})=G\{\mathbf{x}_1,\cdots,\mathbf{x}_n\}$$

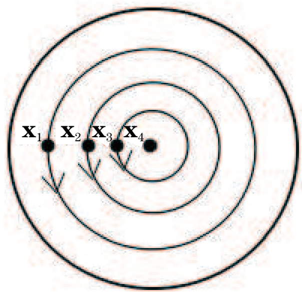

Figure 3

and preserve orientation if M is oriented.

These two groups are to be endowed with compact-open topology. We define equivariant mapping class groups as follows:

• Denote $\pi_0(\mathcal{F}_n^G M, \mathrm{id})$ as the group of all path connected components of $\mathcal{F}_n^G M$.

• Denote $\mathrm{Mod}_G(M,n)$ as the group of all path connected components of $\mathcal{B}_n^G M$ and $\mathrm{Mod}_G(M,n)$ is called the n-th G-equivariant mapping class group of M.

Remark 3.1 The group structures on $\pi_0(\mathcal{F}_n^G M, \mathrm{id})$ and $\mathrm{Mod}_G(M,n)$ inherit the group structures on $\mathcal{F}_n^G M$ and $\mathcal{B}_n^G M$.

Since the action of G on M is free, the (pure) orbit braid group of M is isomorphic to the (pure) braid group of the quotient space (see [16]), which is an oriented or nonorientable closed surface. Hence we only need to consider the (pure) braid group of the quotient space. Similar to the situation without group actions, we define the pure evaluation map as follows.

Definition 3.1

$$\varepsilon^G : \mathcal{F}_0^G M \to F(M/G, n)$$
$$f \mapsto ([f(\mathbf{x}_1)], \cdots, [f(\mathbf{x}_n)]).$$

Observe that we endow $\mathcal{F}_0^G M$ with compact-open topology and $F(M/G,n)$ with subspace topology of $M \times \cdots \times M$. Then ε^G is continuous.

Lemma 3.1 *The pure evaluation map ε^G has a local cross section.*

Proof Let $\boldsymbol{a} = ([\boldsymbol{a}_1], \cdots, [\boldsymbol{a}_n])$ be an arbitrary point in $F(M/G,n)$ and $\boldsymbol{a}_i$ be the representative of each coordinate for $i = 1, \cdots, n$. Since the action of G on M is free and proper, there exist disjoint open sets $\{U_i : \boldsymbol{a}_i \in U_i, i = 1, \cdots, n\}$ such that $U_i \bigcap gU_j = \emptyset$ for every $g \neq e \in G$. We consider an open neighborhood $U(\boldsymbol{a})$ of the point $\boldsymbol{a} \in F(M/G,n)$, where $U(\boldsymbol{a})$ is defined by

$$U(\boldsymbol{a}) = \{([\mathbf{u}_1], \cdots, [\mathbf{u}_n]) : \mathbf{u}_i \in U_i\}.$$

Let $\mathbf{u} = ([\mathbf{u}_1], \cdots, [\mathbf{u}_n])$ be an arbitrary point in $U(\boldsymbol{a})$. Choose $\lambda_{\mathbf{u}}$ to be a G-equivariant homeomorphism from M to M such that

(i) $\lambda_{\mathbf{u}}(\mathbf{x}) = \mathbf{x}$, when $\mathbf{x} \notin \bigcup_{g\in G} \bigcup_{i=1}^{n} gU_i$.

(ii) $\lambda_{\mathbf{u}}$ maps each gU_i into itself homeomorphically.

(iii) $\lambda_{\mathbf{u}}$ fixes the points on the boundary of gU_i.

(iv) $\lambda_{\mathbf{u}}(\boldsymbol{a}_i)$ and $\mathbf{u}_i$ are in the same orbit.

Then the map

$$\chi : U(\boldsymbol{a}) \to \mathcal{F}_0^G M$$

defined by $\chi(\mathbf{u}) = \lambda_{\mathbf{u}}$ is the required cross section.

Lemma 3.2 *The pure evaluation map ε^G is a locally trivial fiber bundle with fiber $\mathcal{F}_n^G M$.*

Proof $\mathcal{F}_n^G M$ is a closed subgroup of $\mathcal{F}_0^G M$ and $F(M/G, n)$ is a paracompact space. Thus we only need to show that the spaces $\mathcal{F}_0^G M/\mathcal{F}_n^G M$ and $F(M/G, n)$ are homeomorphic (see [19]). If $h(\mathbf{x})$ and $h'(\mathbf{x})$ belong to the same right coset of $\mathcal{F}_0^G M$ in $\mathcal{F}_n^G M$, they must have the property $Gh(\mathbf{x}_i) = Gh'(\mathbf{x}_i)$ for $i = 1, \cdots, n$. On one hand, each point $[h] \in \mathcal{F}_0^G M/\mathcal{F}_n^G M$ can be associated in a unique manner with a single point $([h(\mathbf{x}_1)], \cdots, [h(\mathbf{x}_n)]) \in F(M/G, n)$. On the other hand, for each point

$$([\boldsymbol{a}_1], \cdots, [\boldsymbol{a}_n]) \in F(M/G, n),$$

there is a G-equivariant homeomorphism $h \in \mathcal{F}_0^G M$ such that $h(G\boldsymbol{a}_i) = G\mathbf{x}_i$ for $i = 1, \cdots, n$. Finally, it can be established that the topology of $\mathcal{F}_0^G M/\mathcal{F}_n^G M$ coincides with the topology of $F(M/G, n)$. This completes the proof.

Using Lemma 3.2, we obtain an exact sequence as follows:

$$\cdots \to \pi_1(\mathcal{F}_0^G M) \xrightarrow{\varepsilon_*^G} P_n(M/G) \xrightarrow{d_*^G} \pi_0(\mathcal{F}_n^G M) \xrightarrow{i_*^G} \pi_0(\mathcal{F}_0^G M) \to \pi_0(F(M/G, n) = 1 \qquad (3.1)$$

Lemma 3.3 $\ker d_*^G \subset Z(P_n(M/G))$.

Proof For any element $\alpha \in \ker d_*^G = \operatorname{im} \varepsilon_*^G$, there exists an $H \in \pi_1(\mathcal{F}_0^G M)$ such that $\varepsilon_*^G(H) = \alpha$. Now, H can be represented by some loop $H_t : M \to M$, where each H_t is in $\mathcal{F}_0^G M$ and H_0 and H_1 are both identity maps. Then α is represented by $\varepsilon_*^G(H)$ such that

$$\varepsilon_*^G(H)(t) = ([H_t(\mathbf{x}_1)], \cdots, [H_t(\mathbf{x}_n)]).$$

Moreover, each H_t can induce the unique map from M/G to M/G, since it preserves G-action. There is no harm in denoting the induced map by H_t. Choose any element $\beta \in P_n(M/G)$ which can be represented by a loop $(\beta_1, \cdots, \beta_n)$ with $\beta_i(0) = \beta_i(1) = \mathbf{x}_i$ for each $i = 1, \cdots, n$. Use H_t and β to construct $\psi : I^2 \to F_n(M/G)$ by

$$\psi(s, t) = ([H_t(\beta_1(s)], \cdots, [H_t(\beta_n(s)]).$$

Thus, $H(0, t) = H(1, t)$ represents α, while $H(s, 0) = H(s, 1)$ represents β. So $H|_{\partial I^2}$ represents $\alpha\beta\alpha^{-1}\beta^{-1}$. Therefore,

$$\alpha\beta\alpha^{-1}\beta^{-1} = 1.$$

Since $\beta \in P_n(M/G)$ is arbitrary, it follows

$$\alpha \in Z(P_n(M/G)).$$

The proof of Lemma 3.3 is completed.

Suppose G is a finite group of order l, acting freely and properly on M. The natural projection map $p : M \to M/G$ is a regular covering map with the exact sequence

$$1 \to \pi_1(M) \xrightarrow{p_*} \pi_1(M/G) \to G \to 1$$

and G is naturally isomorphic to the group of covering transformation. Furthermore, M/G is a closed compact surface whose Euler characteristic $\chi(M/G)$ satisfies the formula

$$l\chi(M/G) = \chi(M). \tag{3.2}$$

If M is the oriented surface S_g,

$$M/G \cong S_{\frac{g-1}{l}+1} \quad \text{or} \quad N_{\frac{2(g-1)}{l}+2} \quad \text{for } g \geq 1.$$

If M is the nonorientable surface N_k,

$$M/G \cong S_{\frac{k-2}{2l}+1} \quad \text{or} \quad N_{\frac{k-2}{l}+2} \quad \text{for } k \geq 2.$$

When $M/G \cong S_g,\ g \geq 2$ or $M/G \cong N_k,\ k \geq 2$, M can be $S_g,\ g \geq 1$ or $N_k,\ k \geq 2$ and we have

$$\ker d_*^G \subset Z(P_n(M/G)) = 1.$$

We conclude the following theorem.

Theorem 3.1 *Let $i_*^G : \pi_0(\mathcal{F}_n^G M) \to \pi_0(\mathcal{F}_0^G M)$ be the homomorphism induced by inclusion $\mathcal{F}_n^G M \subset \mathcal{F}_0^G M$. Then*

$$\ker i_*^G \cong P_n(M/G) \quad \textit{if } M/G \textit{ is } S_g,\ g \geq 2 \textit{ or } N_k,\ k \geq 2.$$

Consider M is $\mathbb{S}^2$. Because of (3.2), the only nontrivial group $G \neq 1$ that acts freely on the sphere is $\mathbb{Z}_2$. When G is trivial and $n \geq 3$, the discussion is the same as that of [5] and [6]. And since $Z(P_1(\mathbb{S}^2)) = Z(P_2(\mathbb{S}^2)) = 1$, $\ker i_*^G \cong P_n(\mathbb{S}^2)$. We only discuss the situation where the orbit space is $\mathbb{R}P^2$.

Lemma 3.4 *If $G = \mathbb{Z}_2$ acts on $\mathbb{S}^2$ antipodally, then for $n \geq 1$,*

$$Z(P_n(\mathbb{R}P^2)) \subset \ker d_*^G.$$

Proof Since $Z(P_n(\mathbb{R}P^2))$ is generated by τ_n from Theorem 2.4, we only need to prove $d_*^G(\tau_n) = 1$. Define $H_t : \mathbb{S}^2 \to \mathbb{S}^2$ by

$$H_t : (\alpha, \theta) \mapsto (\alpha + 2\pi t, \theta),$$

where the ith component of τ_n can be represented by $[H_t(\mathbf{x}_i)]$, $i = 1, \cdots, n$. Clearly $\tau_n = \varepsilon_*^G([H_t])$. Thus we obtain $\tau_n \in \ker d_*^G$.

As for the nonorientable surface $\mathbb{R}P^2$, G must be trivial because of (3.2).

Lemma 3.5 *If G is trivial and M is $\mathbb{R}P^2$, then $Z(P_n(\mathbb{R}P^2)) \subset \ker d_*^G$.*

Proof Since $Z(P_n(\mathbb{R}P^2))$ is generated by τ_n from Theorem 2.4, we only need to prove $d_*^G(\tau_n) = 1$. View $\mathbb{R}P^2$ as the quotient space of $\mathbb{S}^2$. Define $H_t : \mathbb{R}P^2 \to \mathbb{R}P^2$ by

$$H_t : [(\alpha, \theta)] \mapsto [(\alpha + 2\pi t, \theta)],$$

where the ith component of τ_n can be represented by $H_t(\mathbf{x}_i)$, $i = 1, \cdots, n$. Clearly $\tau_n = \varepsilon_*^G([H_t])$. Thus we obtain $\tau_n \in \ker d_*^G$.

We conclude the situation where the quotient space is $\mathbb{S}^2$ or $\mathbb{R}P^2$.

Theorem 3.2 *Let $i_*^G : \pi_0(\mathcal{F}_n^G M) \to \pi_0(\mathcal{F}_0^G M)$ be the homomorphism induced by inclusion $\mathcal{F}_n^G M \subset \mathcal{F}_0^G M$. Then*

$$\ker i_*^G \cong P_n(M/G)/Z(P_n(M/G)) \quad \textit{if } M/G \textit{ is } \mathbb{S}^2 \textit{ or } \mathbb{R}P^2.$$

Next we consider the situation where the quotient space M/G is $\mathbb{T}^2$. From (3.2), M can be $\mathbb{T}^2$ or Klein bottle N_1. The following lemma rules out one situation.

Lemma 3.6 *The Klein bottle N_1 can not be a cover of the torus $\mathbb{T}^2$.*

Proof $\pi_1(N_1)$ is isomorphic to the non-abelian group $\langle x, y \mid x^2 = y^2 \rangle$ while $\pi_1(\mathbb{T}^2)$ is isomorphic to the Abelian group $\mathbb{Z} \bigoplus \mathbb{Z}$. Suppose N_1 is a cover of $\mathbb{T}^2$, there exists an injective:

$$\pi_1(N_1) \to \pi_1(\mathbb{T}^2).$$

Since the subgroup of an Abelian group is also an Abelian group, the injective map is impossible.

Remark 3.2 This lemma is well known and we just give a proof here.

Thus when M/G is $\mathbb{T}^2$, M can only be $\mathbb{T}^2$. In order to distinguish the original space and the orbit space, we denote the orbit space by $\overline{\mathbb{T}}^2$. Unlike the non-equivariant case, it is possible that $Z(P_n(\overline{\mathbb{T}}^2)) \nsubseteq \ker d_*^G$. There exist examples where $Z(P_n(\overline{\mathbb{T}}^2)) \nsubseteq \ker d_*^G$.

Example 3.1 We view the torus as $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ and denote a point in $\mathbb{T}^2$ by its rectangular coordinates $([u], [v])$, where $[u], [v]$ are real numbers module 1. Denote each fixed point $\mathbf{x}_i$ by $([u_i], [v_i])$ for $i = 1, \cdots, n$. Then $\widetilde{a}$ and $\widetilde{b}$ can be represented by $f(t)$ and $g(t)$ for $t \in [0, 1]$:

$$f(t) = ([u_1 - t], [v_1]), \cdots, ([u_n - t], [v_n]);$$
$$g(t) = ([u_1], [v_1 - t]), \cdots, ([u_n], [v_n - t]).$$

We can plot the components of $f(t)$ and $g(t)$ in Figures 4–5 when n is 4.

Define $\mathbb{Z}_2$-action on $\mathbb{T}^2$ by

$$-1 : \mathbb{T}^2 \to \mathbb{T}^2$$
$$([u], [v]) \mapsto \Big(\Big[u - \frac{1}{2}\Big], [v]\Big).$$

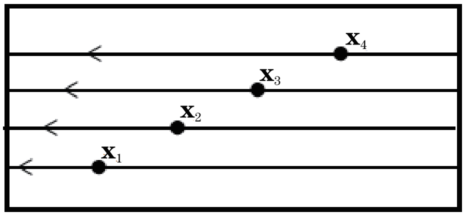

Figure 4

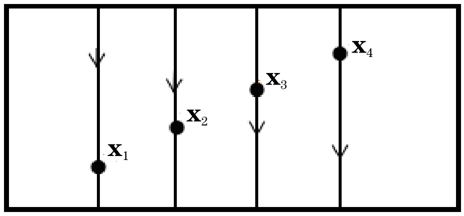

Figure 5

We will compute $d_*^{\mathbb{Z}_2}(\widetilde{a})$. Denote $\widetilde{a}_i$ to be the ith component of $\widetilde{a}$. Construct the homeomorphism H_t for $t \in [0,1]$ by

$$\begin{aligned} H_t &: \mathbb{T}^2 \to \mathbb{T}^2 \\ ([u],[v]) &\mapsto \Big(\Big[u - \frac{1}{2}t\Big],[v]\Big), \end{aligned}$$

which satisfies $H_0 = \mathrm{id}$ and $H_t(x_i) = \widetilde{a}_i$. Since $d_*^{\mathbb{Z}_2}(\widetilde{a}) = H_1 \neq \mathrm{id}$, $Z(P_n(\overline{\mathbb{T}}^2)) \nsubseteq \ker d_*^{\mathbb{Z}_2}$.

In general, $Z(P_n(\overline{\mathbb{T}}^2)) \nsubseteq \ker d_*^{\mathbb{Z}_2}$. However, we can construct the isomorphism between $\ker d_*^G$ and $\mathrm{im}\, p_*$.

Lemma 3.7

$$\ker d_*^G \cong \mathrm{im}\, p_*.$$

Proof Since $\ker d_*^G \subset Z(P_n(\overline{\mathbb{T}}^2)) = \langle \widetilde{a}, \widetilde{b} : \widetilde{a}\widetilde{b} = \widetilde{b}\widetilde{a}\rangle$ and $\mathrm{im}\, p_* \subset \pi_1(\overline{\mathbb{T}}^2) = \langle a, b : ab = ba\rangle$, we can define $\varphi : \ker d_*^G \to \pi_1(\overline{\mathbb{T}}^2)$ by

$$\varphi(\widetilde{a}^m\widetilde{b}^r) = a^m b^r.$$

$\widetilde{a}$ and $\widetilde{b}$ can be represented by loops

$$\begin{aligned} f &: I \to F(\overline{\mathbb{T}}^2, n) \\ &\quad t \mapsto ([u_1 - t],[v_1]), \cdots, ([u_n - t],[v_n]) \end{aligned}$$

and

$$\begin{aligned} g &: I \to F(\overline{\mathbb{T}}^2, n) \\ &\quad t \mapsto ([u_1],[v_1 - t]), \cdots, ([u_n],[v_n - t]), \end{aligned}$$

where $f(0) = f(1) = g(0) = g(1) = (\mathbf{x}_1, \cdots, \mathbf{x}_n)$. According to the construction of d_*^G, there exists the G-homeomorphism $H_t : \mathbb{T}^2 \to \mathbb{T}^2$ for $t \in [0,1]$, such that $H_0 = H_1 = \mathrm{id}$ and

$$(p(H_t(\mathbf{x}_1)), \cdots, p(H_t(\mathbf{x}_n))) = f^m g^r(t)$$

Define the map $H(\mathbf{x}_i) : I \to \mathbb{T}^2$ by $H(\mathbf{x}_i)(t) = H_t(\mathbf{x}_i)$. Then we obtain that $p_*([H(\mathbf{x}_i)]) = a^m b^r$. Thus $a^m b^r \in \mathrm{im}\, p_*$.

Conversely, construct $\psi : \operatorname{im} p_* \to Z(P_n(\overline{\mathbb{T}}^2))$ by

$$\psi(a^m b^r) = \widetilde{a}^m \widetilde{b}^r.$$

So there exists a homotopy class $C \in \pi_1(\mathbb{T}^2)$ such that $p_*(C) = a^m b^r$. Now, C can be represented by some loop $c = ([c_1], [c_2]) : I \to \mathbb{T}^2$, where $c(0) = c(1) = ([0], [0])$. Then construct the G-homeomorphism $H_t : \mathbb{T}^2 \to \mathbb{T}^2$ by

$$H_t([u], [v]) = ([u + c_1], [v + c_2]),$$

which satisfies $H_0 = H_1 = \mathrm{id}$ and $(p(H_t(\mathbf{x}_1)), \cdots, p(H_t(\mathbf{x}_n))) = f^m g^r(t)$. Thus $\widetilde{a}^m \widetilde{b}^r \in \ker d_*^G$.

Since $\mathbb{T}^2$ is the covering space of $\overline{\mathbb{T}}^2$ and $\pi_1(\overline{\mathbb{T}}^2)$ is an Abelian group, each $\operatorname{im} p_*$ is uniquely corresponding to the subgroup of $\pi_1(\overline{\mathbb{T}}^2) \cong \mathbb{Z} \bigoplus \mathbb{Z}$. And because G is finite, there exist positive integers q, r such that $\widetilde{a}^q, \widetilde{b}^r$ generate $\operatorname{im} p_*$. In this case, $G = \mathbb{Z}/q\mathbb{Z} \bigoplus \mathbb{Z}/r\mathbb{Z}$ and $\operatorname{im} p_* = \langle a^q, b^r : a^q b^r = b^r a^q \rangle$. We conclude this situation as follows.

Theorem 3.3 *Let G be a discrete group acting on M freely and properly with the quotient space $\overline{\mathbb{T}}^2$. Then M must be $\mathbb{T}^2$ and each G satisfying the condition is in the form of $\mathbb{Z}/q\mathbb{Z} \bigoplus \mathbb{Z}/r\mathbb{Z}$ for positive integers q, r. Let $i_*^G : \pi_0(\mathcal{F}_n^G M) \to \pi_0(\mathcal{F}_0^G M)$ be the homomorphism induced by inclusion. Then*

$$\ker i_*^G \cong P_n(M/G)/\langle \widetilde{a}^q, \widetilde{b}^r : \widetilde{a}^q \widetilde{b}^r = \widetilde{b}^r \widetilde{a}^q \rangle.$$

Finally we study the relationship between surface orbit braid groups and equivariant mapping class groups. Similarly, we define the evaluation map between $\mathcal{B}_0^G(M) = \mathcal{F}_0^G(M)$ and $F(M/G, n)/\Sigma_n$.

Definition 3.2

$$\eta^G : \mathcal{B}_0^G M \to F(M/G, n)/\Sigma_n$$
$$f \mapsto \{[f(\mathbf{x}_1)], \cdots, [f(\mathbf{x}_n)]\}$$

is called the evaluation map.

Since the evaluation map η^G is the pure evaluation ε^G associated with the projective map $F(M/G, n) \to F(M/G, n)/\Sigma_n$, we obtain the following lemma.

Lemma 3.8 *η^G is a locally trivial fiber bundle with fiber $\mathcal{B}_n^G M$.*

Let $j_*^G : \pi_0(\mathcal{B}_n^G M) \to \pi_0(\mathcal{B}_0^G M)$ be the homomorphism induced by inclusion $\mathcal{B}_n^G M \subset \mathcal{B}_0^G M$. Then there is a long exact sequence

$$\cdots \to \pi_1(\mathcal{B}_0^G M) \xrightarrow{\eta_*^G} B_n(M/G) \xrightarrow{\zeta_*^G} \pi_0(\mathcal{B}_n^G M) \xrightarrow{j_*^G} \pi_0(\mathcal{B}_0^G M) \to \pi_0(F(M/G)/\Sigma_n) = 1. \quad (3.3)$$

By observing the exact sequence (3.1) and (3.3), we obtain:

$$\ker \zeta_*^G = \operatorname{im} \eta_*^G = \operatorname{im} \varepsilon_*^G = \ker d_*^G.$$

From the previous discussion about $\ker d_*^G$ before, we can conclude the following theorem.

Theorem 3.4 *Let $j_*^G : \pi_0(\mathcal{B}_n^G M) \to \pi_0(\mathcal{B}_0^G M)$ be the homomorphism induced by inclusion $\mathcal{B}_n^G M \subset \mathcal{B}_0^G M$. Then*

$$\ker j_*^G \cong B_n(M/G) \quad \text{if } M/G \text{ is } S_g,\ g \geq 2 \text{ or } N_k,\ k \geq 2;$$
$$\ker j_*^G \cong B_n(M/G)/Z(P_n(M/G)) \quad \text{if } M/G \text{ is } \mathbb{S}^2 \text{ or } \mathbb{R}P^2.$$

When M/G is $\mathbb{T}^2$,

$$M = \mathbb{T}^2, \quad G = \mathbb{Z}/q\mathbb{Z} \bigoplus \mathbb{Z}/r\mathbb{Z}, \quad q, r \geq 1 \quad \text{and}$$
$$\ker j_*^G \cong B_n(M/G)/\langle \widetilde{a}^q, \widetilde{b}^r : \widetilde{a}^q \widetilde{b}^r = \widetilde{b}^r \widetilde{a}^q \rangle.$$

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