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The Ground State Solutions for Kirchhoff-Schrödinger Type Equations with Singular Exponential Nonlinearities in $\mathbb{R}^N$ *

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Abstract In this paper, the authors consider the following singular Kirchhoff-Schrödinger problem

$$M\Big(\int_{\mathbb{R}^N}|\nabla u|^N+V(x)|u|^N\mathrm{d}x\Big)(-\Delta_N u+V(x)|u|^{N-2}u)=\frac{f(x,u)}{|x|^\eta}\quad \text{in } \mathbb{R}^N, \qquad (P_\eta)$$

where $0<\eta<N$, M is a Kirchhoff-type function and $V(x)$ is a continuous function with positive lower bound, $f(x,t)$ has a critical exponential growth behavior at infinity. Combining variational techniques with some estimates, they get the existence of ground state solution for (P_η). Moreover, they also get the same result without the A-R condition.

Keywords Ground state solutions, Singular elliptic equations, Critical exponential growth, Kirchhoff-Schrödinger equations, Singular Trudinger-Moser inequality

2000 MR Subject Classification 35J60, 35B33, 46E30

1 Introduction

The research of nonlinear Kirchhoff equations has attracted a lot of attention and a classical Kirchhoff equation is given by

$$\begin{cases}-\Big(a+b\displaystyle\int_\Omega|\nabla u|^2\mathrm{d}x\Big)\Delta u=f(x,u) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^N$ is a bounded domain, $a,b>0$ and $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a continuous function. It is related to the stationary analogue of the following equation

$$u_{tt}-\Big(a+b\int_\Omega|\nabla u|^2\mathrm{d}x\Big)\Delta u=f(x,u), \tag{1.2}$$

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which was proposed by Kirchhoff in [21] as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings. The problem (1.1) is nonlocal since the appearance of the integration term $\int_\Omega |\nabla u|^2 \mathrm{d}x$. In [28], Lions proposed an abstract framework for this kind problems and after that, the Kirchhoff problem began to receive a lot of attention, see [3–4, 6, 8, 11, 32] and the references therein. For Kirchhoff type equation with a singular nonlinear term, in [33], by proving the mountain pass structure of the related functional and using the concentration compactness principle, the authors obtained the existence of a nontrivial solution of the following polyharmonic Kirchhoff type problem

$$\begin{cases} -M\Big(\displaystyle\int_\Omega |\nabla^m u|^{\frac{N}{m}} \mathrm{d}x\Big)\Delta^m_{\frac{N}{m}} u = \dfrac{f(x,u)}{|x|^\eta} & \text{in } \Omega, \\ u = \nabla u = \nabla^2 u = \cdots = \nabla^{m-1} u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where Δ_p with $p \geq 2$ is the p-Laplacian operator, $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, m is an integer and $2 \leq 2m \leq N$, $0 < \eta < N$, $M(\cdot)$ is a Kirchhoff-type function, $f(x,t)$ has critical exponential growth behavior at infinity. Moreover, they also discuss the above problem with convex–concave type sign changing nonlinearity. All theses results are based on Trudinger-Moser inequality (see [34, 36]) and critical point theory. In the case of $M = 1$ and $2m = N$, it becomes to the following Polyharmonic problem

$$\begin{cases} (-\Delta)^m u = f(x,u) & \text{in } \Omega, \\ u = \nabla u = \nabla^2 u = \cdots = \nabla^{m-1} u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

When $m = 1$, the above problem was investigated in [1, 13, 15]. In [22], Lam and Lu studied the above polyharmonic equation in both cases of f satisfying the well-known Ambrosetti-Rabinowitz condition and without the Ambrosetti-Rabinowitz condition. One can also refer to [24–25] for some relevant results.

A singular Schrödinger equation is written as

$$-\mathrm{div}(|\nabla u|^{N-2}\nabla u) + V(x)|u|^{N-2}u = \frac{f(x,u)}{|x|^\eta}, \quad x \in \mathbb{R}^N, \tag{1.5}$$

where $N \geq 2$, $0 \leq \eta < N$, $V : \mathbb{R}^N \to \mathbb{R}$ is a continuous function, $f(x,s)$ is continuous in $\mathbb{R}^N \times \mathbb{R}$ and behaves like $\mathrm{e}^{\alpha|s|^{\frac{N}{N-1}}}$ as $|s| \to \infty$. When $\eta = 0$, (1.5) is a usual elliptic equation with no singular term and it was first studied by Cao [7] for the case $N = 2$. It was studied by Panda [35], do Ó [16] and Alves [5] for the general dimensional cases. When $0 < \eta < N$, (1.5) becomes an elliptic equation with a singular term. From the literatures one can see that to deal with the singular Schrödinger equations is closely related to the singular Trudinger-Moser type inequality (see for example [2, Theorem 1.1] or [14, Theorem 3]).

For the following perturbation problem

$$-\mathrm{div}(|\nabla u|^{N-2}\nabla u) + V(x)|u|^{N-2}u = \frac{f(x,u)}{|x|^\eta} + \varepsilon h(x), \quad x \in \mathbb{R}^N, \tag{1.6}$$

when $\eta = 0$, using Ekeland variational principle and mountain-pass theorem, do Ó et al [17] got the multiplicity of solutions. When $\eta > 0$, the existence of nontrivial weak solution of the problem (1.6) was proved by Adimurthi and Yang [2]. In [37], Yang derived some similar results for the equations with bi-Laplacian operator in dimensional four, and proved the existence and multiplicity of weak solutions for N-Laplacian elliptic equations in his another paper (see [38]). Lam and Lu [23] considered the existence and multiplicity of nontrivial weak solution for non-uniformly N-Laplacian elliptic equations. Motivated by the work of Lions [29], do Ó et al. [18] improved the Trudinger-Moser inequality in $\mathbb{R}^N$ and obtained a ground state solution for the quasilinear elliptic equation

$$-\mathrm{div}(|\nabla u|^{N-2}\nabla u)+|u|^{N-2}u=f(x,u), \quad x\in\mathbb{R}^N \tag{1.7}$$

with the function $f(x,s)$ satisfying the so called exponential critical growth condition at $+\infty$, i.e., there exists $\alpha_0>0$ such that

$$\lim_{|s|\to\infty} f(x,s)\mathrm{e}^{-\alpha|s|^{\frac{N}{N-1}}}=\begin{cases}0, & \forall\alpha>\alpha_0,\\ +\infty, & \forall\alpha<\alpha_0.\end{cases}$$

For Kirchhoff-Schrödinger type equation, Li and Yang [26] studied the following problem

$$\begin{cases}\Big(\displaystyle\int_{\mathbb{R}^N}(|\nabla u|^N+V(x)|u|^N)\mathrm{d}x\Big)^k(-\Delta_N u+V(x)|u|^{N-2}u)=\lambda Q(x)|u|^{p-2}u+f(u),\\ u\in W^{1,N}(\mathbb{R}^N),\end{cases} \tag{1.8}$$

where $\Delta_N u=\mathrm{div}(|\nabla u|^{N-2}\nabla u)$, $k>0$, $1<p<N$, $V:\mathbb{R}^N\to(0,\infty)$ is a continuous function with positive lower bound and coercive, $\lambda>0$ is a real parameter, $Q(x)$ is a positive function in $L^{\frac{N}{N-p}}(\mathbb{R}^N)$ and $f(t)$ satisfies exponential growth condition. Recently, Furtado and Zanata [20] studied the following Schrödinger-Kirchhoff type equation

$$M\Big(\int_{\mathbb{R}^2}|\nabla u|^2+V(x)u^2\mathrm{d}x\Big)(-\Delta u+V(x)u)=A(x)f(u) \quad \text{in } \mathbb{R}^2, \tag{1.9}$$

where $M(t)$ is a Kirchhoff-type function and $V(x)$ may vanish on a set of positive measure and may take negative values in somewhere. The function $A(x)$ is locally bounded and the function $f(t)$ has critical exponential growth. Applying variational method, they got the existence of ground state solution. Moreover, in the local case $M\equiv 1$, they also got some relevant results. In [12], the existence and multiplicity of solutions were investigated for the elliptic systems involving Kirchhoff equations.

In this paper, we consider the following singular Kirchhoff-Schrödinger

$$M\Big(\int_{\mathbb{R}^N}|\nabla u|^N+V(x)|u|^N\mathrm{d}x\Big)(-\Delta_N u+V(x)|u|^{N-2}u)=\frac{f(x,u)}{|x|^\eta} \quad \text{in } \mathbb{R}^N, \tag{1.10}$$

where $N\geq 2$, $0<\eta<N$. The potential V is a continuous function with positive lower bound, $f(x,t)$ has a critical exponential growth behavior at infinity. Let $\mathfrak{M}(t)=\int_0^t M(s)\mathrm{d}s$, we always assume that $M:\mathbb{R}^+\to\mathbb{R}^+$ is a continuous function with $M(0)=0$, and satisfies

(M_1) for any $d>0$ there exists $\kappa:=\kappa(d)>0$ such that $M(t)\geq\kappa$ for all $t\geq d$;

(M_2) for any t_1, $t_2\geq 0$, it holds

$$\mathfrak{M}(t_1+t_2)\geq\mathfrak{M}(t_1)+\mathfrak{M}(t_2);$$

(M_3) there exists $\theta>1$ such that $\frac{M(t)}{t^{\theta-1}}$ is decreasing in $(0,\infty)$.

Remark 1.1 A typical example of $M(t)$ is given by $M(t) = a + bt^{\theta-1}$ for all $t \geq 0$ and some $\theta > 1$, where $a, b \geq 0$ and $a + b > 0$.

Remark 1.2 By (M_3), we can obtain that $\theta\mathfrak{M}(t) - M(t)t$ is nondecreasing for $t > 0$. In particular,

$$\theta\mathfrak{M}(t) - M(t)t \geq 0, \quad \forall t \geq 0. \tag{1.11}$$

Since it is concerned with nonnegative weak solutions, we require that $f(x,t) = 0$ for all $(x,t) \in \mathbb{R}^N \times (-\infty, 0]$. Furthermore, we assume the function f satisfying:

(f_0) f is a continuous function and $f(x,t) > 0$ for all $t > 0$ and $x \in \mathbb{R}^N$.

(f_1) There exist positive constants t_0 and M_0 such that

$$0 < F(x,t) := \int_0^t f(x,s)\mathrm{d}s \leq M_0 f(x,t), \quad \forall (x,t) \in \mathbb{R}^N \times [t_0, +\infty).$$

(f_2) There exist constants $\alpha_0, c_1, c_2 > 0$ such that for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}^+$,

$$f(x,t) \leq c_1|t|^{\theta N-1} + c_2[\mathrm{e}^{\alpha_0|t|^{\frac{N}{N-1}}} - S_{N-2}(\alpha_0, t)],$$

where

$$S_{N-2}(\alpha_0, t) = \sum_{k=0}^{N-2} \frac{\alpha_0^k |t|^{\frac{kN}{N-1}}}{k!}.$$

(f_3) There exists $\mu > \theta N$ such that

$$0 < \mu F(x,t) \leq t f(x,t), \quad F(x,t) = \int_0^t f(x,s)\mathrm{d}s, \tag{1.12}$$

where $x \in \mathbb{R}^N$ and $t \in \mathbb{R}^+$.

(f_3) is the well known Ambrosetti-Rabinowitz condition (A-R condition, for short).

We also give the following conditions on the potential $V(x)$:

(V_1) V is a continuous function satisfying $V(x) \geq V_0 > 0$.

Define a function space

$$E = \Big\{u \in W^{1,N}(\mathbb{R}^N) : \int_{\mathbb{R}^N} (|\nabla u|^N + V(x)|u|^N)\mathrm{d}x < \infty\Big\}$$

equipped with the norm

$$\|u\|_E = \Big(\int_{\mathbb{R}^N} (|\nabla u|^N + V(x)|u|^N)\mathrm{d}x\Big)^{\frac{1}{N}}.$$

The condition (V_1) implies that E is a reflexive Banach space. For any $p \geq N$, we define

$$S_p = \inf_{u \in E\backslash\{0\}} \frac{\|u\|_E}{\big(\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^\eta}\mathrm{d}x\big)^{\frac{1}{p}}} \tag{1.13}$$

and

$$\lambda_\eta = \inf_{u \in E\backslash\{0\}} \frac{\|u\|_E^{\theta N}}{\int_{\mathbb{R}^N} \frac{|u|^{\theta N}}{|x|^\eta}\mathrm{d}x}.$$

The continuous embedding of $E \hookrightarrow W^{1,N}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ $(p \geq N)$ and Hölder inequality imply

$$\begin{aligned}&\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^\eta}\mathrm{d}x\\ \leq &\int_{\{|x|>1\}} |u|^p \mathrm{d}x + \Big(\int_{\{|x|\leq 1\}} |u|^{pt'}\mathrm{d}x\Big)^{\frac{1}{t'}}\Big(\int_{\{|x|\leq 1\}} \frac{1}{|x|^{\eta t}}\mathrm{d}x\Big)^{\frac{1}{t}}\\ \leq &C\|u\|_E^p,\end{aligned}$$

where $\frac{1}{t} + \frac{1}{t'} = 1$ and $t > 1$ such that $\eta t < N$. Thus we have $S_p > 0$. We now introduce the following conditions.

(f_4) $\limsup\limits_{t\to 0^+} \frac{NF(x,t)}{|t|^{\theta N}} < \mathfrak{M}(1)\lambda_\eta$ uniformly in $\mathbb{R}^N$.

(f_5) There exist constants $q > \theta N$ and C_q such that for all $(x, t) \in \mathbb{R}^N \times (0, \infty)$,

$$f(x,t) > C_q t^{q-1},$$

where

$$C_q := \inf\Big\{C > 0 : q\mathfrak{M}(t^N S_q^N) - NCt^q \leq q\mathfrak{M}\Big(\Big(\Big(1-\frac{\eta}{N}\Big)\frac{\alpha_N}{\alpha_0}\Big)^{N-1}\Big)\Big\}.$$

(f_6) $\frac{f(x,t)}{t^{\theta N-1}}$ is strictly increasing in $t > 0$.

Our main results can be stated as follows.

Theorem 1.1 *Suppose V satisfies (V_1) and f satisfies (f_0)–(f_6). Then problem* (1.10) *possesses a positive ground state solution.*

Remark 1.3 The main difficulty is how to obtain a strong convergence subsequence from a (PS) sequence and prove that the limit is a ground state solution of problem (1.10), which can be overcome by the concentration compactness principle and the singular Trudinger-Moser inequality.

Now instead the conditions (f_1) and (f_3), we assume that

(f_1') there exists constant $c > 0$ such that $F(x,t) \leq c|t|^{\theta N} + cf(x,t)$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}^+$.

(f_3') $\lim\limits_{|t|\to+\infty} \frac{F(x,t)}{|t|^{\theta N}} = +\infty$ uniformly on $x \in \mathbb{R}^N$.

Theorem 1.2 *Suppose V satisfies (V_1), f satisfies (f_0), (f_1'), (f_2), (f_3') and (f_4)–(f_6). Then problem* (1.10) *has a positive ground state solution.*

Remark 1.4 In Theorem 1.2, we study the ground state solution of Kirchhoff-Schrödinger equation without the A-R condition. For Schrödinger equation with exponential growth and singular term, the A-R condition was weakened in [31]. Instead of using the mountain-pass theorem of (PS) sequence, we use the mountain-pass theorem of Cerami sequence and obtain the boundedness of Cerami subsequence for the energy functional.

The rest of the paper is organized as follows. In Section 2, some preliminary results are introduced. In Section 3, we study the functionals related to (1.10). In Section 4, we give a proof of Theorem 1.1. In Section 5, we give a proof of Theorem 1.2, which is a key step to prove the boundedness of $(C)_c$ sequence.

2 Preliminaries

In this section, we give some preliminaries for our use later.

Lemma 2.1 (see [39, p. 5]) *Suppose $q \geq N$ and $0 < \eta < N$. Then E can be compactly embedded into $L^q(\mathbb{R}^N, |x|^{-\eta}\mathrm{d}x)$.*

Notice that $0 < \eta < N$, we have the singular Trudinger-Moser inequality.

Lemma 2.2 (see [2, p. 2397]) *For all $\alpha > 0$, $0 < \eta < N$, and $u \in W^{1,N}(\mathbb{R}^N)$, $N \geq 2$, there holds*

$$\int_{\mathbb{R}^N} \frac{\mathrm{e}^{\alpha|u|^{\frac{N}{N-1}}} - S_{N-2}(\alpha, u)}{|x|^\eta}\mathrm{d}x < +\infty. \tag{2.1}$$

Furthermore, for all $\alpha \leq \big(1 - \frac{\eta}{N}\big)\alpha_N$ and $\tau > 0$, there holds

$$\sup_{\|u\|_{1,\tau} \leq 1} \int_{\mathbb{R}^N} \frac{\mathrm{e}^{\alpha|u|^{\frac{N}{N-1}}} - S_{N-2}(\alpha, u)}{|x|^\eta}\mathrm{d}x < +\infty, \tag{2.2}$$

where $\|u\|_{1,\tau} = \big(\int_{\mathbb{R}^N}(|\nabla u|^N + \tau|u|^N)\mathrm{d}x\big)^{\frac{1}{N}}$.

Lemma 2.3 *Let $\beta > 0, 0 < \eta < N$ and $\|u\|_E \leq M$ such that $\beta M^{\frac{N}{N-1}} < \big(1 - \frac{\eta}{N}\big)\alpha_N$ and $q > N$, then*

$$\int_{\mathbb{R}^N} \frac{\mathrm{e}^{\beta|u|^{\frac{N}{N-1}}} - S_{N-2}(\beta, u)}{|x|^\eta}|u|^q\mathrm{d}x \leq C(\beta, N)\|u\|_E^q.$$

Proof Set $R(\beta, u) = \mathrm{e}^{\beta|u|^{\frac{N}{N-1}}} - S_{N-2}(\beta, u)$, by using the Hölder inequality and Lemma 2.1, we have

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{R(\beta, u)}{|x|^\eta}|u|^q\mathrm{d}x &\leq \Big(\int_{\mathbb{R}^N} \frac{R(p\beta, u)}{|x|^\eta}\mathrm{d}x\Big)^{\frac{1}{p}}\Big(\int_{\mathbb{R}^N} \frac{|u|^{qp'}}{|x|^\eta}\mathrm{d}x\Big)^{\frac{1}{p'}} \\ &\leq \Big(\int_{\mathbb{R}^N} \frac{R(p\beta M^{\frac{N}{N-1}}, \widetilde{u})}{|x|^\eta}\mathrm{d}x\Big)^{\frac{1}{p}}\|u\|_E^q \\ &\leq C(\beta, N)\|u\|_E^q,\end{aligned}$$

where $\widetilde{u} = \frac{u}{\|u\|_E}$ and $p > 1$ is sufficiently close to 1 such that $\beta p M^{\frac{N}{N-1}} \leq \big(1 - \frac{\eta}{N}\big)\alpha_N$, $\frac{1}{p} + \frac{1}{p'} = 1$. The last estimate is a direct consequence of Lemma 2.2 since $\|\widetilde{u}\|_{1,\tau} \leq \|\widetilde{u}\|_E = 1$ for any positive $\tau \leq V_0$.

Next, we claim that from the condition (f_6) one can get the following condition.

(f_6') $H(x, t) = tf(x, t) - \theta NF(x, t)$ is strictly increasing in $t > 0$. In fact, we have the following lemma.

Lemma 2.4 *If (f_6) holds, then for all $x \in \mathbb{R}^N$, we have that $tf(x, t) - \theta NF(x, t)$ is strictly increasing in $t > 0$.*

Proof Let $0 < t_1 < t_2$ be fixed. It follows from (f_6) that

$$t_1 f(x, t_1) - \theta NF(x, t_1) < \frac{f(x, t_2)}{t_2^{\theta N-1}}t_1^{\theta N} - \theta NF(x, t_2) + \theta N\int_{t_1}^{t_2} f(x, s)\mathrm{d}s. \tag{2.3}$$

On the other hand,

$$\theta N \int_{t_1}^{t_2} f(x,s)\mathrm{d}s < \theta N \frac{f(x,t_2)}{t_2^{\theta N-1}} \int_{t_1}^{t_2} s^{\theta N-1}\mathrm{d}s = \frac{f(x,t_2)}{t_2^{\theta N-1}}(t_2^{\theta N} - t_1^{\theta N}). \tag{2.4}$$

From (2.3)–(2.4), we derive that

$$t_1 f(x,t_1) - \theta N F(x,t_1) < t_2 f(x,t_2) - \theta N F(x,t_2).$$

This completes the proof.

3 Functionals and Compactness Analysis

We say that $u \in E$ is a positive weak solution of problem (1.10) if $u > 0$ in $\mathbb{R}^N$, and for all $\phi \in E$,

$$M(\|u\|_E^N)\int_{\mathbb{R}^N}(|\nabla u|^{N-2}\nabla u\nabla\phi + V(x)|u|^{N-2}u\phi)\mathrm{d}x - \int_{\mathbb{R}^N}\frac{f(x,u)}{|x|^\eta}\phi\mathrm{d}x = 0.$$

Define the functional $I : E \to \mathbb{R}$ by

$$I(u) = \frac{1}{N}\mathfrak{M}(\|u\|_E^N) - \int_{\mathbb{R}^N}\frac{F(x,u)}{|x|^\eta}\mathrm{d}x, \tag{3.1}$$

where $F(x,t) = \int_0^t f(x,s)\mathrm{d}s$. I is well defined and due to the Trudinger-Moser inequality $I \in C^1(E,\mathbb{R})$. A straightforward calculation shows that

$$\langle I'(u),\phi\rangle = M(\|u\|_E^N)\int_{\mathbb{R}^N}(|\nabla u|^{N-2}\nabla u\nabla\phi + V(x)|u|^{N-2}u\phi)\mathrm{d}x - \int_{\mathbb{R}^N}\frac{f(x,u)}{|x|^\eta}\phi\mathrm{d}x \tag{3.2}$$

for all $u,\phi \in E$. Hence a critical point of I defined in (3.1) is a weak solution of (1.10).

Next, we will check the geometry of the functional I.

Lemma 3.1 *Assume that* (V_1), (f_2) *and* (f_4) *hold. Then there exist positive constants* δ *and* r *such that*

$$I(u) \geq \delta \quad \text{for } \|u\|_E = r.$$

Proof From (f_4), there exist $\sigma, \varepsilon > 0$, such that if $|u| \leq \varepsilon$,

$$F(x,u) \leq \frac{\mathfrak{M}(1)\lambda_\eta - \sigma}{N}|u|^{\theta N}$$

for all $x \in \mathbb{R}^N$. On the other hand, using (f_2) for each $q > \theta N$, we have

$$\begin{aligned} F(x,u) &\leq \frac{c_1}{\theta N}|u|^{\theta N} + c_2|u|[\mathrm{e}^{\alpha_0|u|^{\frac{N}{N-1}}} - S_{N-2}(\alpha_0,u)] \\ &\leq C|u|^q[\mathrm{e}^{\alpha_0|u|^{\frac{N}{N-1}}} - S_{N-2}(\alpha_0,u)] \end{aligned}$$

for $|u| \geq \varepsilon$ and $x \in \mathbb{R}^N$. Combining the above estimates, we obtain

$$F(x,u) \leq \frac{\mathfrak{M}(1)\lambda_\eta - \sigma}{N}|u|^{\theta N} + C|u|^q[\mathrm{e}^{\alpha_0|u|^{\frac{N}{N-1}}} - S_{N-2}(\alpha_0,u)]$$

for all $(x,u)\in\mathbb{R}^N\times\mathbb{R}$. Furthermore, (1.11) gives $\mathfrak{M}(t)\geq\mathfrak{M}(1)t^{\theta},\ t\in[0,1]$. Fix $r\in(0,1)$ such that $\alpha_0 r^{\frac{N}{N-1}}<\alpha_N\big(1-\frac{\eta}{N}\big)$. For any $u\in E$ with $\|u\|_E=r$, then by Lemma 2.3 we have

$$\begin{aligned}
I(u)&=\frac{1}{N}\mathfrak{M}(r^N)-\int_{\mathbb{R}^N}\frac{F(x,u)}{|x|^\eta}\mathrm{d}x\\
&\geq\frac{\mathfrak{M}(1)}{N}r^{\theta N}-\frac{\mathfrak{M}(1)\lambda_\eta-\sigma}{N}\int_{\mathbb{R}^N}\frac{|u|^{\theta N}}{|x|^\eta}\mathrm{d}x-C\int_{\mathbb{R}^N}\frac{|u|^q[\mathrm{e}^{\alpha_0|u|^{\frac{N}{N-1}}}-S_{N-2}(\alpha_0,u)]}{|x|^\eta}\mathrm{d}x\\
&\geq\frac{\mathfrak{M}(1)}{N}r^{\theta N}-\frac{\mathfrak{M}(1)\lambda_\eta-\sigma}{N}\int_{\mathbb{R}^N}\frac{|u|^{\theta N}}{|x|^\eta}\mathrm{d}x-C\|u\|_E^q\\
&\geq\frac{\mathfrak{M}(1)}{N}r^{\theta N}-\frac{\mathfrak{M}(1)\lambda_\eta-\sigma}{N\lambda_\eta}\|u\|_E^{\theta N}-Cr^q\\
&=\frac{\sigma}{N\lambda_\eta}r^{\theta N}-Cr^q.
\end{aligned}$$

We now choose sufficiently small $r>0$ such that

$$\frac{\sigma}{N\lambda_\eta}r^{\theta N}-Cr^q\geq\frac{\sigma}{2N\lambda_\eta}r^{\theta N}.$$

So we derive that

$$I(u)\geq\tfrac{\sigma}{2N\lambda_\eta}r^{\theta N}:=\delta>0\quad\text{for }\|u\|_E=r.$$

This completes the proof.

Lemma 3.2 *If the condition (f_3) is satisfied, then there exists $e\in E$ with $\|e\|_E>r$ such that*

$$I(e)<\inf_{\|u\|_E=r}I(u),$$

where r is given in Lemma 3.1.

Proof From (1.11), we have $\mathfrak{M}(t)\leq\mathfrak{M}(1)t^{\theta},\ t\geq1$. Let $u\in E\setminus\{0\}$, $u\geq0$ with compact support $\Omega=\mathrm{supp}(u)$ and $\|u\|_E=1$. From (f_3) or (1.12), for $\mu>\theta N$, there exist $C_1,C_2>0$ such that for all $(x,t)\in\Omega\times\mathbb{R}^+$,

$$F(x,s)\geq C_1t^\mu-C_2.$$

Then for all $t\geq1$, there holds

$$I(tu)\leq\frac{\mathfrak{M}(1)t^{\theta N}}{N}\|u\|_E^{\theta N}-C_1t^\mu\int_\Omega\frac{|u|^\mu}{|x|^\eta}\mathrm{d}x+C_2|\Omega|.$$

Hence, $I(tu)\to-\infty$ as $t\to\infty$. Setting $e=tu$ with t sufficiently large, the proof of Lemma 3.2 is completed.

From Lemmas 3.1–3.2, we get a $(PS)_c$ sequence $\{u_n\}\subset E$, i.e.,

$$I(u_n)\to c>0\quad\text{and}\quad I'(u_n)\to0\quad\text{as }n\to\infty,\tag{3.3}$$

where

$$c=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}I(\gamma(t))\tag{3.4}$$

and

$$\Gamma =: \{\gamma \in C([0,1]:E) : \gamma(0)=0, \gamma(1)=e\}.$$

Lemma 3.3 *Suppose that the condition* (f_5) *is satisfied, then for the min–max level* c *defined in* (3.4), *there holds* $c \in \big(0, \frac{1}{N}\mathfrak{M}\big(\big((1-\frac{\eta}{N})\frac{\alpha_N}{\alpha_0}\big)^{N-1}\big)\big)$.

Proof Firstly, we claim the best constant S_p defined in (1.13) can be achieved at an element $u_0 \in E$. In fact, since

$$S_q = \inf_{u\in E\setminus\{0\}} \frac{\|u\|_E}{\big(\int_{\mathbb{R}^N} \frac{|u|^q}{|x|^\eta}\mathrm{d}x\big)^{\frac{1}{q}}},$$

we can choose u_n such that

$$\int_{\mathbb{R}^N} \frac{|u_n|^q}{|x|^\eta}\mathrm{d}x = 1 \quad \text{and} \quad \|u_n\|_E \to S_q \quad \text{as } n\to\infty,$$

so u_n is bounded in E. From Lemma 2.1, there exists $u_0 \in E$ such that up to a subsequence $u_n \rightharpoonup u_0$ in E, $u_n \to u_0$ in $L^q(\mathbb{R}^N, |x|^{-\eta}\mathrm{d}x)$ and $u_n(x) \to u_0(x)$ almost everywhere in $\mathbb{R}^N$. This implies

$$\int_{\mathbb{R}^N} \frac{|u_0|^q}{|x|^\eta}\mathrm{d}x = \lim_{n\to\infty}\int_{\mathbb{R}^N} \frac{|u_n|^q}{|x|^\eta}\mathrm{d}x = 1.$$

We also have $\|u_0\|_E \le \lim\limits_{n\to\infty} \|u_n\|_E = S_q$, thus $\|u_0\|_E = S_q$. With this $u_0 \in E$, by condition (f_5) and following the argument in the proof of Lemma 3.2, we see that there exists a positive number t_0 such that $I(t_0u_0) < 0$.

The inequality $c \ge \delta$ can be easily proved as in the proof of Lemma 3.1. From the definition of c, take $\gamma : [0,1] \to E, \gamma(t) = te$, where $e = t_0u_0$ with $I(t_0u_0) < 0$ as explained above. We have $\gamma \in \Gamma$ and by using the condition (f_5), we have

$$c \le \max_{t\in[0,1]} I(\gamma(t)) \le \max_{t\ge 0} I(tu_0) < \max_{t\ge 0}\Big(\frac{\mathfrak{M}(t^N S_q^N)}{N} - \frac{t^q}{q}C_q\Big) \le \frac{1}{N}\mathfrak{M}\Big(\Big(\Big(1-\frac{\eta}{N}\Big)\frac{\alpha_N}{\alpha_0}\Big)^{N-1}\Big).$$

The proof of Lemma 3.3 is completed.

Lemma 3.4 *Suppose that the conditions* (V_1), (f_1)*–*(f_4) *and* (f_6) *are satisfied. Let* $\{u_n\} \subset E$ *be an arbitrary* $(PS)_c$ *sequence of* I*. Then there exist a subsequence of* $\{u_n\}$ (*still denoted by* $\{u_n\}$) *and* $u \in E$ *such that*

$$\begin{cases} \dfrac{f(x,u_n)}{|x|^\eta} \to \dfrac{f(x,u)}{|x|^\eta} & \textit{strongly in } L^1_{\mathrm{loc}}(\mathbb{R}^N), \\ \dfrac{F(x,u_n)}{|x|^\eta} \to \dfrac{F(x,u)}{|x|^\eta} & \textit{strongly in } L^1(\mathbb{R}^N). \end{cases}$$

Proof Let $\{u_n\} \subset E$ be an arbitrary $(PS)_c$ sequence of I, i.e.,

$$I(u_n) \to c > 0 \quad \text{and} \quad I'(u_n) \to 0 \quad \text{as } n \to \infty. \tag{3.5}$$

We shall prove that the sequence $\{u_n\}$ is bounded in E. Arguing by contradiction, suppose that $\{u_n\}$ is unbounded in E. Then up to a subsequence, we have $\|u_n\|_E \to \infty$ and $d := \inf\limits_{n\ge 1} \|u_n\|_E^N > 0$. Since $\mu > \theta N$, combining (1.11) and (M_1), we have

$$c + o(1) + o(1)\|u_n\|_E$$

$$
\begin{aligned}
&\geq I(u_n) - \frac{1}{\mu}\langle I'(u_n), u_n\rangle \\
&= \frac{1}{N}\mathfrak{M}(\|u_n\|_E^N) - \frac{1}{\mu}M(\|u_n\|_E^N)\|u_n\|_E^N - \frac{1}{\mu}\int_{\mathbb{R}^N}\frac{\mu F(x,u_n) - f(x,u_n)u_n}{|x|^\eta}\mathrm{d}x \\
&\geq \Big(\frac{1}{\theta N} - \frac{1}{\mu}\Big)M(\|u_n\|_E^N)\|u_n\|_E^N - \frac{1}{\mu}\int_{\mathbb{R}^N}\frac{\mu F(x,u_n) - f(x,u_n)u_n}{|x|^\eta}\mathrm{d}x \\
&\geq \Big(\frac{1}{\theta N} - \frac{1}{\mu}\Big)\kappa\|u_n\|_E^N.
\end{aligned}
$$

Divide the above inequality by $\|u_n\|_E^N$ and let $n \to \infty$, we have

$$
0 \geq \Big(\frac{1}{\theta N} - \frac{1}{\mu}\Big)\kappa > 0,
$$

which is impossible. Therefore $\{u_n\}$ is bounded in E.

It then follows from (3.3) that

$$
\int_{\mathbb{R}^N}\frac{f(x,u_n)u_n}{|x|^\eta}\mathrm{d}x \leq C, \quad \int_{\mathbb{R}^N}\frac{F(x,u_n)}{|x|^\eta}\mathrm{d}x \leq C.
$$

By [13, Lemma 2.1, p. 143], we get

$$
\frac{f(x,u_n)}{|x|^\eta} \to \frac{f(x,u)}{|x|^\eta} \quad \text{strongly in } L^1_{\mathrm{loc}}(\mathbb{R}^N). \tag{3.6}
$$

By (f_1) and (f_2), there exists $C > 0$ such that

$$
F(x,u_n) \leq C|u_n|^{\theta N} + Cf(x,u_n).
$$

From Lemma 2.1 and the generalized Lebesgue's dominated convergence theorem in [27, p. 20], arguing as [39, Lemma 4.6, p. 13], we derive that

$$
\frac{F(x,u_n)}{|x|^\eta} \to \frac{F(x,u)}{|x|^\eta} \quad \text{strongly in } L^1(\mathbb{R}^N). \tag{3.7}
$$

This completes the proof of Lemma 3.4.

4 Proof of Theorem 1.1

Proof of Theorem 1.1 By the process as in the proof of Lemma 3.4, we have that the $(PS)_c$ sequence $\{u_n\}$ is bounded in E. We claim that $I(u) \geq 0$. Indeed, suppose by contradiction that $I(u) < 0$. Then $u \neq 0$, set $r(t) := I(tu)$, $t \geq 0$, we have $r(0) = 0$ and $r(1) < 0$. As the proof of Lemma 3.1, for $t > 0$ small enough, it holds $r(t) > 0$. So there exists $t_0 \in (0,1)$ such that

$$
r(t_0) = \max_{t\in[0,1]} r(t), \quad r'(t_0) = \langle I'(t_0u), u\rangle = 0.
$$

By Remark 1.1 and Lemma 2.4, we have

$$
\begin{aligned}
c \leq I(t_0u) &= I(t_0u) - \frac{1}{\theta N}\langle I'(t_0u), u\rangle \\
&= \frac{1}{N}\mathfrak{M}(\|t_0u\|^N) - \frac{1}{\theta N}M(\|t_0u\|^N)\|t_0u\|^N
\end{aligned}
$$

$$+\frac{1}{\theta N}\int_{\mathbb{R}^N}\frac{f(x,t_0u)t_0u-\theta NF(x,t_0u)}{|x|^\eta}\mathrm{d}x$$
$$<\frac{1}{N}\mathfrak{M}(\|u\|^N)-\frac{1}{\theta N}M(\|u\|^N)\|u\|^N$$
$$+\frac{1}{\theta N}\int_{\mathbb{R}^N}\frac{f(x,u)u-\theta NF(x,u)}{|x|^\eta}\mathrm{d}x.$$

Furthermore, by the weak lower semi-continuity of the norm and Fatou's lemma, we have

$$\begin{aligned}c&<\liminf_{n\to\infty}\Big(\frac{1}{N}\mathfrak{M}(\|u_n\|^N)-\frac{1}{\theta N}M(\|u_n\|^N)\|u\|^N\Big)\\&\quad+\frac{1}{\theta N}\liminf_{n\to\infty}\int_{\mathbb{R}^N}\frac{f(x,u_n)u_n-\theta NF(x,u_n)}{|x|^\eta}\mathrm{d}x\\&\le\liminf_{n\to\infty}\Big(I'(u_n)-\frac{1}{\theta N}\langle I'(u_n),u_n\rangle\Big)=c,\end{aligned}$$

which is impossible. Thus the claim is true.

Now we show that $I'(u)=0$ and $I(u)=c$. In fact, from the lower semi-continuity of the norm in E, we have $\|u\|_E\le\lim_{n\to\infty}\|u_n\|_E$. Suppose, by contradiction, that $\|u\|_E<\lim_{n\to\infty}\|u_n\|_E:=\xi$. Set $v_n:=\frac{u_n}{\|u_n\|_E}$ and $v:=\frac{u}{\xi}$, then $v_n\rightharpoonup v$ weakly in E and $\|v\|_E<1$. From $I(u)\ge0$ and Lemma 3.4, we have

$$\begin{aligned}\mathfrak{M}(\xi^N)&=\lim_{n\to\infty}\mathfrak{M}(\|u_n\|_E^N)=\lim_{n\to\infty}N\Big(I(u_n)+\int_{\mathbb{R}^N}\frac{F(x,u_n)}{|x|^\eta}\mathrm{d}x\Big)\\&=Nc+N\int_{\mathbb{R}^N}\frac{F(x,u)}{|x|^\eta}\mathrm{d}x=Nc+\mathfrak{M}(\|u\|_E^N)-NI(u)\\&<\mathfrak{M}\Big(\Big(\Big(1-\frac{\eta}{N}\Big)\frac{\alpha_N}{\alpha_0}\Big)^{N-1}\Big)+\mathfrak{M}(\|u\|_E^N)\\&\le\mathfrak{M}\Big(\Big(\Big(1-\frac{\eta}{N}\Big)\frac{\alpha_N}{\alpha_0}\Big)^{N-1}+\|u\|_E^N\Big).\end{aligned}$$

Here, we have used the condition (M_2) in the last inequality. By (M_1), it holds $\xi^N<\big(\big(1-\frac{\eta}{N}\big)\frac{\alpha_N}{\alpha_0}\big)^{N-1}+\|u\|^N$. Notice that

$$\xi^N=\frac{\xi^N-\|u\|_E^N}{1-\|v\|^N}.$$

Thus

$$\xi^N<\frac{\big(\big(1-\frac{\eta}{N}\big)\frac{\alpha_N}{\alpha_0}\big)^{N-1}}{1-\|v\|_E^N}.$$

Choosing $q>1$ sufficiently close to 1 and $\beta_0>0$ such that for large n,

$$q\alpha_0\|u_n\|_E^{\frac{N}{N-1}}\le\beta_0<\frac{\big(1-\frac{\eta}{N}\big)\alpha_N}{(1-\|v\|_E^N)^{\frac{1}{N-1}}}.$$

By using concentration compactness principle (see [19, p. 230], [39, p. 3]), together with singular Trudinger-Moser inequality, it holds

$$\int_{\mathbb{R}^N}\frac{\mathrm{e}^{q\alpha_0|u_n|^{\frac{N}{N-1}}}-S_{N-2}(q\alpha_0,|u_n|)}{|x|^\eta}\mathrm{d}x$$

$$\leq \int_{\mathbb{R}^N} \frac{\mathrm{e}^{\beta_0|v_n|^{\frac{N}{N-1}}} - S_{N-2}(\beta_0, |v_n|)}{|x|^\eta}\mathrm{d}x \leq C. \tag{4.1}$$

From (f_2) and Hölder inequality, there holds

$$\begin{aligned} &\Big|\int_{\mathbb{R}^N} \frac{f(x,u_n)(u_n-u)}{|x|^\eta}\mathrm{d}x\Big| \\ &\leq c_1\Big(\int_{\mathbb{R}^N} \frac{|u_n|^{\theta N}}{|x|^\eta}\mathrm{d}x\Big)^{\frac{\theta N-1}{N}}\Big(\int_{\mathbb{R}^N} \frac{|u_n-u|^{\theta N}}{|x|^\eta}\mathrm{d}x\Big)^{\frac{1}{\theta N}} \\ &\quad + c_2\Big(\int_{\mathbb{R}^N} \frac{|u_n-u|^{q'}}{|x|^\eta}\mathrm{d}x\Big)^{\frac{1}{q'}}\Big(\int_{\mathbb{R}^N} \frac{\mathrm{e}^{q\alpha_0|u_n|^{\frac{N}{N-1}}} - S_{N-2}(q\alpha_0, |u_n|)}{|x|^\eta}\mathrm{d}x\Big)^{\frac{1}{q}}, \end{aligned} \tag{4.2}$$

where $\frac{1}{q'} + \frac{1}{q} = 1$. In view of Lemma 2.1, combining (4.1) with (4.2), we obtain

$$\int_{\mathbb{R}^N} \frac{f(x,u_n)(u_n-u)}{|x|^\eta}\mathrm{d}x \to 0. \tag{4.3}$$

Since $I'(u_n)(u_n-u) \to 0$, there holds

$$M(\|u_n\|_E^N)\int_{\mathbb{R}^N}(|\nabla u_n|^{N-2}\nabla u_n\nabla(u_n-u) + V(x)|u_n|^{N-2}u_n(u_n-u))\mathrm{d}x \to 0. \tag{4.4}$$

On the other hand, by $u_n \rightharpoonup u$ in E, we have

$$M(\|u_n\|_E^N)\int_{\mathbb{R}^N}(|\nabla u|^{N-2}\nabla u\nabla(u_n-u) + V(x)|u|^{N-2}u(u_n-u))\mathrm{d}x \to 0. \tag{4.5}$$

(4.4) minus (4.5) and applying with the next inequality

$$2^{2-N}|\nabla u_n - \nabla u|^N \leq \langle|\nabla u_n|^{N-2}\nabla u_n - |\nabla u|^{N-2}\nabla u, \nabla u_n - \nabla u\rangle$$

and

$$2^{2-N}|u_n - u|^N \leq \langle|u_n|^{N-2}u_n - |u|^{N-2}u, u_n - u\rangle,$$

we can derive

$$\begin{aligned} &\lim_{n\to\infty} M(\|u_n\|_E^N)\|u_n-u\|_E^N \\ &\leq 2^{N-2}M(\|u_n\|_E^N)\int_{\mathbb{R}^N}(|\nabla u_n|^{N-2}\nabla u_n\nabla(u_n-u) + V(x)|u_n|^{N-2}u_n(u_n-u))\mathrm{d}x \\ &\quad + 2^{N-2}M(\|u_n\|_E^N)\int_{\mathbb{R}^N}(|\nabla u|^{N-2}\nabla u\nabla(u_n-u) + V(x)|u|^{N-2}u(u_n-u))\mathrm{d}x = 0. \end{aligned} \tag{4.6}$$

It is in contradiction with the fact $\|u\|_E < \lim_{n\to\infty}\|u_n\|_E := \xi$. Thus, we have $\|u\|_E = \xi = \lim_{n\to\infty}\|u_n\|_E$. Since $\{u_n\}$ is bounded in E, we can apply Brezis-Lieb lemma to obtain $u_n \to u$ strongly in E. Since $I \in C^1(E,\mathbb{R})$, we have $I'(u) = 0$ and $I(u) = c$.

Next, we show that u is nonzero. If $u \equiv 0$, since $F(x,0) = 0$ for all $x \in \mathbb{R}^N$, from Lemma 3.4, we have

$$\lim_{n\to\infty}\frac{1}{N}\mathfrak{M}(\|u_n\|_E^N)c < \mathfrak{M}\Big(\Big(\Big(1-\frac{\eta}{N}\Big)\frac{\alpha_N}{\alpha_0}\Big)^{N-1}\Big), \tag{4.7}$$

Thus, there exist $\varepsilon_0>0$ and $n_*>0$ such that $\|u_n\|_E^N \le \big(\frac{N-\eta}{N}\frac{\alpha_N}{\alpha_0}-\varepsilon_0\big)^{N-1}$ for all $n>n_*$. Choose $q>1$ sufficiently close to 1 such that $q\alpha_0\|u_n\|_E^{\frac{N}{N-1}} \le \big(1-\frac{\eta}{N}\big)\alpha_N-\varepsilon_0\alpha_0$ for all $n>n_*$. By (f_2), there holds

$$|f(x,u_n)u_n| \le c_1|u_n|^{\theta N} + c_2|u_n|[\mathrm{e}^{\alpha_0|u_n|^{\frac{N}{N-1}}} - S_{N-2}(\alpha_0,|u_n|)].$$

Thus by using singular Trudinger-Moser inequality, it holds

$$\begin{aligned}
&\int_{\mathbb{R}^N}\frac{|f(x,u_n)u_n|}{|x|^\eta}\mathrm{d}x\\
&\le c_1\int_{\mathbb{R}^N}\frac{|u_n|^{\theta N}}{|x|^\eta}\mathrm{d}x + c_2\int_{\mathbb{R}^N}\frac{|u_n|[\mathrm{e}^{\alpha_0|u_n|^{\frac{N}{N-1}}} - S_{N-2}(\alpha_0,|u_n|)]}{|x|^\eta}\mathrm{d}x\\
&\le c_1\int_{\mathbb{R}^N}\frac{|u_n|^{\theta N}}{|x|^\eta}\mathrm{d}x + c_2\Big(\int_{\mathbb{R}^N}\frac{\mathrm{e}^{q\alpha_0|u_n|^{\frac{N}{N-1}}} - S_{N-2}(q\alpha_0,|u_n|)}{|x|^\eta}\mathrm{d}x\Big)^{\frac{1}{q}}\Big(\int_{\mathbb{R}^N}\frac{|u_n|^{q'}}{|x|^\eta}\mathrm{d}x\Big)^{\frac{1}{q'}}\\
&\le c_1\int_{\mathbb{R}^N}\frac{|u_n|^{\theta N}}{|x|^\eta}\mathrm{d}x + C\Big(\int_{\mathbb{R}^N}\frac{|u_n|^{q'}}{|x|^\eta}\mathrm{d}x\Big)^{\frac{1}{q'}} \to 0,
\end{aligned}$$

here we have used Lemma 2.1 in the last estimate. From $I'(u_n)u_n\to 0$, we have

$$\lim_{n\to\infty} M(\|u_n\|_E^N)\|u_n\|_E^N = 0. \tag{4.8}$$

From the condition (M_1), we can get $\|u_n\|\to 0$. Then $I(u_n)\to 0$, which contradicts the fact that $I(u_n)\to c>0$, so u is nonzero.

Finally, we are ready to show the existence of positive ground state solution for (1.10). Setting

$$m=\inf_{u\in\Lambda} I(u), \quad \Lambda := \{u\in E\backslash\{0\} : I'(u)u=0\}.$$

Let c be the mountain-pass level, obviously $m\le c$.

On the other hand, for any $u\in\Lambda$, there holds $u>0$. In fact, denote $u^- := \min\{u,0\}$, from $I'(u)u^-=0$, we get $\|u^-\|=0$. Since f is nonnegative, by applying the Harnack inequality, we have $u>0$ in $\mathbb{R}^N$. Define $h:(0,+\infty)\to\mathbb{R}$ by $h(t)=I(tu)$. We have that h is differentiable and

$$h'(t)=I'(tu)u=M(t^N\|u\|^N)t^{N-1}\|u\|^N - \int_{\mathbb{R}^N}\frac{f(x,tu)u}{|x|^\eta}\mathrm{d}x, \quad \forall t\ge 0.$$

From $I'(u)u=0$, we get

$$h'(t)=I'(tu)u - t^{\theta N-1}I'(u)u,$$

so

$$\begin{aligned}
h'(t) &= t^{\theta N-1}\|u\|_E^{\theta N}\Big[\frac{M(t^N\|u\|^N)}{t^{(\theta-1)N}\|u\|_E^{(\theta-1)N}} - \frac{M(\|u\|^N)}{\|u\|_E^{(\theta-1)N}}\Big]\\
&\quad + t^{\theta N-1}\int_{\mathbb{R}^N}\Big(\frac{f(x,u)}{u^{\theta N-1}} - \frac{f(x,tu)}{(tu)^{\theta N-1}}\Big)u^{\theta N}\mathrm{d}x.
\end{aligned}$$

By (M_3), (f_6) and $u>0$, we conclude that $h'(t)>0$ for $0<t<1$ and $h'(t)<0$ for $t>1$. From $h'(1)=0$, we have

$$I(u)=\max_{t\ge 0} I(tu).$$

From the above argument, we see that $h'(t) < 0$ is strongly decreasing in $t \in (1, +\infty)$, so $h(t) \to -\infty$ as $t \to +\infty$. Now, define $\gamma : [0, 1] \to E$, $\gamma(t) = tt_0u$, where t_0 is a real number which satisfies $I(t_0u) < 0$, we have $\gamma \in \Gamma$, and therefore

$$c \leq \max_{t\in[0,1]} I(\gamma(t)) \leq \max_{t\geq 0} I(tu) = I(u).$$

Since $u \in \Lambda$ is arbitrary, we have $c \leq m$, thus $c = m$. This ends the proof of Theorem 1.1.

5 The Ground State Solution Without the A-R Condition

In this section, insteading the conditions (f_1) and (f_3), we assume following condition on the function f.

(f_1') There exists constant $c > 0$ such that $F(x, t) \leq c|t|^N + cf(x, t)$ for all $(x, t) \in \mathbb{R}^N \times \mathbb{R}^+$.

(f_3') $\lim\limits_{|t|\to+\infty} \frac{F(x,t)}{|t|^{\theta N}} = \infty$ uniformly on $x \in \mathbb{R}^N$, where $F(x, t) = \int_0^t f(x, s)\mathrm{d}s$.

We will use a Cerami's mountain pass theorem which was introduced in [9–10]. For readers' convenience, we give a brief introduction here.

Definition A *Let $(E, \|\cdot\|_E)$ be a real Banach space with its dual space $(E^*, \|\cdot\|_{E^*})$. Suppose $I \in C^1(E, \mathbb{R})$. For $c \in \mathbb{R}$, we call $\{u_n\} \subset E$ a $(\mathrm{C})_c$ sequence of the functional I, if*

$$I(u_n) \to c \ \ \text{and} \ \ (1 + \|u_n\|_E)\|I'(u_n)\|_{E^*} \to 0 \quad \text{as } n \to \infty.$$

Proposition A *Let $(E, \|\cdot\|_E)$ be a real Banach space. $I \in C^1(E, \mathbb{R})$, $I(0) = 0$ and satisfies:*

(i) *There exist positive constants δ and r such that*

$$I(u) \geq \delta \quad \text{for } \|u\|_E = r$$

and

(ii) *there exists $e \in E$ with $\|e\|_E > r$ such that*

$$I(e) \leq 0.$$

Define c by

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)),$$

where

$$\Gamma =: \{\gamma \in C([0, 1] : E) : \gamma(0) = 0, \gamma(1) = e\}.$$

Then I possesses a $(C)_c$ sequence.

In order to prove Theorem 1.2, along the line of the proof of Theorem 1.1, we get into two steps. Firstly, we check the mountain pass geometry of the functional I under the weak condition. Secondly, it is the key step to establish that any $(C)_c$ sequence is bounded.

Lemma 5.1 *Assume that (V_1), (f_2), (f_3'), (f_4) hold. Then*

(i) *there exist positive constants δ and r such that*

$$I(u) \geq \delta \quad \text{for } \|u\|_E = r.$$

(ii) *there exists $e \in E$ with $\|e\|_E > r$ such that*

$$I(e) < \inf_{\|u\|_E = r} I(u).$$

Proof The proof of (i) is similar as that in Lemma 3.1. From (M_3), we have $\mathfrak{M}(t) \le \mathfrak{M}(1)t^{\theta},\ t \ge 1$. Let $u \in E \setminus \{0\}$, $u \ge 0$ with compact support $\Omega = \mathrm{supp}(u)$, by (f_3'), for any $L > 0$, there exists $d > 0$ such that for all $(x,s) \in \Omega \times \mathbb{R}^+$,

$$F(x,s) \ge Ls^{\theta N} - d.$$

Then

$$\begin{aligned} I(tu) &\le \frac{\mathfrak{M}(1)t^{\theta N}}{N}\|u\|_E^{\theta N} - Lt^{\theta N}\int_{\Omega}\frac{|u|^{\theta N}}{|x|^{\eta}}\mathrm{d}x + O(1) \\ &\le t^{\theta N}\Big(\frac{\mathfrak{M}(1)\|u\|_E^{\theta N}}{N} - L\int_{\Omega}\frac{|u|^{\theta N}}{|x|^{\eta}}\mathrm{d}x\Big) + O(1). \end{aligned}$$

Now choosing $L > \frac{\mathfrak{M}(1)\|u\|_E^{\theta N}}{N\int_{\Omega}\frac{|u|^{\theta N}}{|x|^{\eta}}\mathrm{d}x}$, it implies that $I(tu) \to -\infty$ as $t \to \infty$. Setting $e = tu$ with t sufficiently large, the proof of (ii) is completed.

From Lemmas 3.1, 5.1 and Proposition A, we get a $(C)_c$ sequence $\{u_n\} \subset E$, i.e.,

$$I(u_n) \to c > 0 \quad \text{and} \quad (1 + \|u_n\|_E)\|I'(u_n)\|_{E^*} \to 0 \quad \text{as } n \to \infty. \tag{5.1}$$

Lemma 5.2 *Suppose that the conditions (f_2), (f_5) and (f_6) are satisfied. Let $\{u_n\} \subset E$ be an arbitrary $(C)_c$ sequence of I. Then $\{u_n\}$ is bounded up to a subsequence.*

Proof Let $\{u_n\} \subset E$ be a $(C)_c$ sequence of I, i.e.,

$$\frac{\mathfrak{M}(\|u_n\|_E^N)}{N} - \int_{\mathbb{R}^N}\frac{F(x,u_n)}{|x|^{\eta}}\mathrm{d}x \to c \quad \text{as } n \to \infty \tag{5.2}$$

and

$$(1 + \|u_n\|_E)|\langle I'(u_n), \varphi\rangle| \le \tau_n\|\varphi\|_E \quad \text{for all } \varphi \in E, \tag{5.3}$$

where $\tau_n \to 0$ as $n \to \infty$. We shall prove that the sequence $\{u_n\}$ is bounded in E. Indeed, suppose by contradiction that

$$\|u_n\|_E \to +\infty$$

and set

$$v_n = \frac{u_n}{\|u_n\|_E}.$$

Combining with Lemma 2.1 and the similar argument as in [30, Lemma 5, p. 12], we can get $v_n^+ \rightharpoonup 0$ in E.

Let $t_n \in [0,1]$ be such that

$$I(t_n u_n) = \max_{t\in[0,1]} I(tu_n).$$

For any given $A \in \big(0, \big(\frac{N-\eta}{N}\frac{\alpha_N}{\alpha_0}\big)^{\frac{N-1}{N}}\big)$, for the sake of simplicity, let

$$\varepsilon = \frac{\big(1 - \frac{\eta}{N}\big)\alpha_N}{A^{\frac{N}{N-1}}} - \alpha_0 > 0.$$

In the following argument we will take $A \to \big(\frac{N-\eta}{N}\frac{\alpha_N}{\alpha_0}\big)^{\frac{N-1}{N}}$ and so we have $\varepsilon \to 0$.

By condition (f_2), there exists $C > 0$ such that

$$F(x,t) \le C|t|^{\theta N} + \varepsilon R(\alpha_0+\varepsilon, |t|), \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}^+, \tag{5.4}$$

where $R(\alpha,s) = \mathrm{e}^{\alpha s^{\frac{N}{N-1}}} - S_{N-2}(\alpha,s)$. In fact, from condition (f_2), there holds

$$F(x,t) \le \frac{C}{N}|t|^{\theta N} + |t|R(\alpha_0,|t|).$$

By using Young inequality, for $\frac{1}{p}+\frac{1}{q}=1$, $p,q>1$, there holds

$$ab \le \varepsilon\frac{a^p}{p} + \varepsilon^{-\frac{q}{p}}\frac{b^q}{q}.$$

So we have

$$F(x,t) \le \frac{C}{N}|t|^{\theta N} + \frac{\varepsilon R(\alpha_0,|t|)^p}{p} + \varepsilon^{-\frac{q}{p}}\frac{|t|^q}{q}.$$

Now we take $p = \frac{\alpha_0+\varepsilon}{\alpha_0}$ and $q = \frac{\alpha_0+\varepsilon}{\varepsilon} > \theta N$. One can see that near infinity $|t|^q$ can be estimated from above by $R(\alpha_0+\varepsilon,|t|)$, and near the origin $|t|^q$ can be estimated from above by $|t|^{\theta N}$, thus we obtain (5.4). We also have $\frac{A}{\|u_n\|} \in (0,1]$ with sufficient large n, so by using (5.4), we have

$$\begin{aligned}
I(t_nu_n) &\ge I\Big(\frac{A}{\|u_n\|}u_n\Big) = I(Av_n) = \frac{\mathfrak{M}(A^N)}{N} - \int_{\mathbb{R}^N}\frac{F(x,Av_n)}{|x|^\eta}\mathrm{d}x\\
&= \frac{\mathfrak{M}(A^N)}{N} - \int_{\mathbb{R}^N}\frac{F(x,Av_n^+)}{|x|^\eta}\mathrm{d}x\\
&\ge \frac{\mathfrak{M}(A^N)}{N} - CA^{\theta N}\int_{\mathbb{R}^N}\frac{|v_n^+|^{\theta N}}{|x|^\eta}\mathrm{d}x - \varepsilon\int_{\mathbb{R}^N}\frac{R(\alpha_0+\varepsilon, Av_n^+)}{|x|^\eta}\mathrm{d}x\\
&\ge \frac{\mathfrak{M}(A^N)}{N} - CA^{\theta N}\int_{\mathbb{R}^N}\frac{|v_n^+|^{\theta N}}{|x|^\eta}\mathrm{d}x - \varepsilon\int_{\mathbb{R}^N}\frac{R((\alpha_0+\varepsilon)A^{\frac{N}{N-1}}, v_n^+)}{|x|^\eta}\mathrm{d}x\\
&\ge \frac{\mathfrak{M}(A^N)}{N} - CA^{\theta N}\int_{\mathbb{R}^N}\frac{|v_n^+|^{\theta N}}{|x|^\eta}\mathrm{d}x - \varepsilon\int_{\mathbb{R}^N}\frac{R\big((1-\frac{\eta}{N})\alpha_N, v_n^+\big)}{|x|^\eta}\mathrm{d}x.
\end{aligned}$$

Since $v_n^+ \rightharpoonup 0$ in E and the embedding $E \hookrightarrow L^q(\mathbb{R}^N, |x|^{-\eta}\mathrm{d}x)$ $(q \ge N)$ is compact, by using the Hölder inequality, we have $\int_{\mathbb{R}^N}\frac{|v_n^+|^N}{|x|^\eta}\mathrm{d}x \to 0$. By singular Trudinger-Moser inequality, $\int_{\mathbb{R}^N}\frac{R((1-\frac{\eta}{N})\alpha_N, v_n^+)}{|x|^\eta}\mathrm{d}x$ is bounded. When $A \to \big(\frac{N-\eta}{N}\frac{\alpha_N}{\alpha_0}\big)^{\frac{N-1}{N}}$, from Lemma 3.3, we can show

$$\liminf_{n\to\infty} I(t_nu_n) \ge \frac{1}{N}\mathfrak{M}\Big(\Big(\Big(1-\frac{\eta}{N}\Big)\frac{\alpha_N}{\alpha_0}\Big)^{N-1}\Big) > c. \tag{5.5}$$

Since $I(0)=0$ and $I(u_n) \to c$, we can assume $t_n \in (0,1)$, and so $I'(t_nu_n)t_nu_n = 0$, it follows from Lemma 2.4 and Remark 1.1,

$$\begin{aligned}
\theta NI(t_nu_n) &= \theta NI(t_nu_n) - I'(t_nu_n)t_nu_n\\
&= \theta\mathfrak{M}(\|t_nu_n\|^N) - \theta N\int_{\mathbb{R}^N}\frac{F(x,t_nu_n)}{|x|^\eta}\mathrm{d}x\\
&\quad - M(\|t_nu_n\|^N)\|t_nu_n\|^N + \int_{\mathbb{R}^N}\frac{f(x,t_nu_n)t_nu_n}{|x|^\eta}\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&= \theta\mathfrak{M}(\|t_nu_n\|^N) - M(\|t_nu_n\|^N)\|t_nu_n\|^N + \int_{\mathbb{R}^N} \frac{f(x,t_nu_n)t_nu_n - \theta NF(x,t_nu_n)}{|x|^\eta}\mathrm{d}x \\
&\le \theta\mathfrak{M}(\|u_n\|^N) - M(\|u_n\|^N)\|t_nu_n\|^N + \int_{\mathbb{R}^N} \frac{f(x,u_n)u_n - \theta NF(x,u_n)}{|x|^\eta}\mathrm{d}x \\
&= \theta NI(u_n) - I'(u_n)u_n \\
&= \theta NI(u_n) + o_n(1) = \theta Nc + o_n(1),
\end{aligned}
$$

which is a contradiction to (5.5). This proves that $\{u_n\}$ is bounded in E.

Proof of Theorem 1.2 From Lemma 5.2, we have that the $(C)_c$ sequence $\{u_n\}$ of I is bounded in E. Applying the same procedure as in proof of Theorem 1.1, we derive that $I'(u) = 0$ and $I(u) = c$. Moreover, we also get that u is positive and u is a ground state solution of (1.10).

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