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A Second Main Theorem of Nevanlinna Theory for Closed Subschemes in Subgeneral Position*

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Abstract In this paper, by using Seshadri constants for subschemes, the author establishes a second main theorem of Nevanlinna theory for holomorphic curves intersecting closed subschemes in (weak) subgeneral position. As an application of his second main theorem, he obtain a Brody hyperbolicity result for the complement of nef effective divisors. He also give the corresponding Schmidt's subspace theorem and arithmetic hyperbolicity result in Diophantine approximation.

Keywords Second main theorem, In general position, Closed subscheme, Seshadri constant, Schmidt's subspace theorem, Hyperbolicity
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1 Introduction

In higher dimensional Nevanlinna theory, it mainly studies the second main theorem of holomorphic maps between complex manifolds intersecting subvarieties in the target manifold. Cartan [2] established the second main theorem for linearly non-degenerate holomorphic curves into complex projective space intersecting hyperplanes in general position, and Nochka [12] considered the case of subgeneral position. Ru [14–15] established second main theorems for algebraically non-degenerate holomorphic curves into complex projective varieties intersecting hypersurfaces in general position and there are many new developments (see [13, 16]).

Recently, there are many developments in extending the second main theorem to arbitrary subschemes case. Ru and Wang [17] obtained the following second main theorem for holomorphic curves intersecting closed subschemes.

Theorem 1.1 (see [17]) *Let X be a projective variety. Let $Y_1, \cdots, Y_q$ be closed subschemes of X such that, for any $x \in X$, there are at most m subschemes among $Y_1, \cdots, Y_q$ which contains x. Let A be a big Cartier divisor on X. Let $f : \mathbb{C} \to X$ be a holomorphic curve with Zariski-dense image. Let*

$$\beta_{A,Y_j} = \lim_{N\to\infty} \frac{\sum\limits_{m=1}^{\infty} h^0(\widetilde{X}_j, N\pi_j^* A - mE_j)}{N h^0(X, NA)}, \quad j = 1, \cdots, q,$$

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where $\pi_j : \widetilde{X}_j \to X$ *is the blowing-up of* X *along* Y_j*, with associated exceptional divisor* E_j*. Then for every* $\varepsilon > 0$*,*

$$\Big\| \sum_{j=1}^{q} m_f(r, Y_j) \leq m\Big(\max_{1\leq j\leq q}\{\beta_{A,Y_j}^{-1}\} + \varepsilon\Big) T_{f,A}(r), \tag{1.1}$$

where "$\|$" *means the estimate holds for all large* r *outside a set of finite Lebesgue measure.* (*Here we use some notations which will be explained later*).

When the closed subschemes $Y_j = y_j$ are distinct points, the Seshadri constants $\epsilon_{y_j}(A)$ and β_{A,y_j} have the following relation (see [11])

$$\beta_{A,y_j} \geq \frac{n}{n+1}\epsilon_{y_j}(A),$$

where $\dim X = n$. Then one may take $m = 1$ in Theorem 1.1 and obtain the following inequality

$$\Big\| \sum_{j=1}^{q} m_f(r, y_j) \leq \Big(\frac{n+1}{n}\max_{1\leq j\leq q}\Big\{\frac{1}{\epsilon_{y_j}(A)}\Big\} + \varepsilon\Big) T_{f,A}(r). \tag{1.2}$$

More generally, by using the definition of Seshadri constants for general closed subscheme (see Section 2), Heier and Levin [9] obtained the following result.

Theorem 1.2 (see [9, Theorem 1.3]) *Let* X *be a projective variety of dimension* n*. Let* $Y_1, \cdots, Y_q$ *be closed subschemes of* X *such that for every subset* $I \subset \{1, \cdots, q\}$ *with* $\#I \leq n+1$*, we have* $\operatorname{codim} \bigcap_{j\in I} (\operatorname{Supp} Y_j) \geq \#I$*. Let* A *be an ample Cartier divisor on* X*. Let* $f : \mathbb{C} \to X$ *be a holomorphic curve with Zariski-dense image. Then for every* $\varepsilon > 0$*,*

$$\Big\| \sum_{j=1}^{q} \epsilon_{Y_j}(A) m_f(r, Y_j) \leq (n + 1 + \varepsilon) T_{f,A}(r). \tag{1.3}$$

Let Y be a closed subscheme of X of codimension $\operatorname{codim} Y$ in X. Note that the elements of the list obtained by repeating Y up to $\operatorname{codim} Y$ times still satisfy the condition in Theorem 1.2, then from Theorem 1.2 we have the following corollary.

Corollary 1.1 (see [9, Corollary 1.6]) *Let* X *be a projective variety of dimension* n*. Let* Y *be a closed subscheme of* X *of codimension* $\operatorname{codim} Y$ *in* X*. Let* A *be an ample Cartier divisor on* X*. Let* $f : \mathbb{C} \to X$ *be a holomorphic curve with Zariski-dense image. Then for every* $\varepsilon > 0$*,*

$$\| \ \epsilon_Y(A) m_f(r, Y) \leq \Big(\frac{n+1}{\operatorname{codim} Y} + \epsilon\Big) T_{f,A}(r). \tag{1.4}$$

They also compared the constant $\beta_{A,Y}$ and the Seshadri constant $\epsilon_Y(A)$ (see [9, Theorem 4.2]) and obtained that

$$\beta_{A,Y} \geq \frac{\operatorname{codim} Y}{n+1}\epsilon_Y(A) \tag{1.5}$$

under the assumption that X is smooth.

Recently, He and Ru [7] give more general results.

Theorem 1.3 (see [7]) *Let X be a projective variety of dimension n and let $m \geq n$ be a positive integer. Let $Y_1, \cdots, Y_q$ be closed subschemes of X such that for every subset $I \subset \{1, \cdots, q\}$ with $\#I \leq m+1$ we have $\dim \bigcap_{j \in I} (\operatorname{Supp} Y_j) \leq m - \#I$, where we use the convention that $\dim \emptyset = -1$. Let A be an ample Cartier divisor on X. Let $f : \mathbb{C} \to X$ be a holomorphic curve with Zariski-dense image. Then for every $\varepsilon > 0$,*

$$\Big\| \sum_{j=1}^{q} \epsilon_{Y_j}(A) m_f(r, Y_j) \leq [(m-n+1)(n+1) + \varepsilon] T_{f,A}(r). \tag{1.6}$$

We note that, in [7, 9, 17], the conditions on the subschemes $Y_1, \cdots, Y_q$ are called in "(sub)general position", but actually they are different. In this paper, in order to distinguish them, we call the subschemes in "weak (sub)general position" and "(sub)general position".

Let X be a projective variety of dimension n and let $Y_1, \cdots, Y_q$ be closed subschemes of X.

Definition 1.1 (Subgeneral position) *Let m be a positive integer.*

(a) *The closed subschemes $Y_1, \cdots, Y_q$ are called in weak m-subgeneral position if, for any $x \in X$, there are at most m subschemes among $Y_1, \cdots, Y_q$ which contains x. When $m = n$, the subschemes are called in weak general position.*

(b) *The closed subschemes $Y_1, \cdots, Y_q$ are called in m-subgeneral position if for any subset $I \subset \{1, \cdots, q\}$ with $\#I \leq m+1$, we have $\dim \bigcap_{j \in I} (\operatorname{Supp} Y_j) \leq m - \#I$. When $m = n$, the subschemes are called in general position.*

Definition 1.1(a) is used in [17], Definition 1.1(b) is used in [7, 9].

We also note that the condition in (a) is weaker than that in (b), since when $\#I = m + 1$, $\dim \bigcap_{j \in I} (\operatorname{Supp} Y_j) \leq -1$ implies that $\bigcap_{j \in I} (\operatorname{Supp} Y_j)$ must be empty. When $Y_1, \cdots, Y_q$ are hypersurfaces, (a) is equivalent to (b).

Motivated by the main theorem in [3], we give a second main theorem under "weak subgeneral position" condition.

Theorem 1.4 *Let X be a complex projective variety of dimension n and $Y_1, \cdots, Y_q$ be closed subschemes of X. Let A be an ample Cartier divisor on X. Let $f : \mathbb{C} \to X$ be a non-constant holomorphic curve such that $f(\mathbb{C}) \not\subset \operatorname{Supp} Y_j$, for $j = 1, \cdots, q$. Assume that $Y_1, \cdots, Y_q$ are in weak m-subgeneral position. Then, for every $\varepsilon > 0$,*

$$\Big\| \sum_{j=1}^{q} \epsilon_{Y_j}(A) m_f(r, Y_j) \leq [m(n+1) + \varepsilon] T_{f,A}(r). \tag{1.7}$$

Remark 1.1 If $Y_1, \cdots, Y_q$ are general divisors, then Theorem 1.4 is a generalization of [3, Corollary 1.2].

Now we consider the case that $Y_1, \cdots, Y_q$ are divisors. As an application of Theorem 1.4, we can obtain a Brody hyperbolicity result for the complement of nef effective divisors, which is motivated by the work of Heier and Levin [8]. We first recall the definition of Brody hyperbolicity.

Definition 1.2 (Brody hyperbolic) *A complex variety is said to be quasi-Brody hyperbolic if the union of all images of nonconstant holomorphic maps from $\mathbb{C}$ is not Zariski dense in it.*

A complex variety is said to be Brody hyperbolic if it admits no nonconstant holomorphic maps from $\mathbb{C}$.

Let X be a projective variety of dimension n. Let $D_1, \cdots, D_q$ be non-zero effective Cartier divisors in general position on X. For an ample divisor A on X, if there exist positive rational constants $c_1, \cdots, c_q$ such that for all $j = 1, \cdots, q$:

$$A - c_j D_j \text{ is } \mathbb{Q}\text{-nef.}$$

Then (1.3) implies that

$$\Big\| \sum_{j=1}^{q} c_j m_f(r, D_j) \leq (n+1+\varepsilon) T_{f,A}(r). \tag{1.8}$$

In [8], by using (1.8) and choosing appropriate A and c_j, Heier and Levin obtained the following result.

Theorem 1.5 (see [8, The analytic version of Theorem 1.8(a)]) *Let X be a projective variety of dimension n. Let $E_1, \cdots, E_r$ be nef Cartier divisors on X with $\sum\limits_{i=1}^{r} E_i$ ample. Let $D_1, \cdots, D_q$ be non-zero effective (possibly reducible) Cartier divisors in general position on X. Suppose that $D_j \equiv \sum\limits_{i=1}^{r} a_{j,i} E_i$, $j = 1, \cdots, q$, where the coefficients $a_{j,i}$ are non-negative real numbers. Let $P_j = (a_{j,1}, \cdots, a_{j,r}) \in \mathbb{R}^r$, $j = 1, \cdots, q$. Assume that for any proper subset T of the set of standard basis vectors $\{e_1, \cdots, e_r\} \subset \mathbb{R}^r$, at most $(\#T)\lfloor \frac{q}{r} \rfloor$ of the vectors $P_1, \cdots, P_q$ are supported on T. If*

$$q \geq r(n+1) + 1, \quad r = 1, 2,$$
$$q \geq r(n+1) + \frac{(r-1)(r-2)}{2}, \quad r \geq 3,$$

then $X \setminus \sum\limits_{j=1}^{q} D_j$ is quasi-Brody hyperbolic.

If we use our Theorem 1.4 instead of Theorem 1.2, then we can obtain a hyperbolicity result under weak subgeneral poisition condition on the divisors.

Theorem 1.6 *Let X be a projective variety of dimension n. Let $E_1, \cdots, E_r$ be nef Cartier divisors on X with $\sum\limits_{i=1}^{r} E_i$ ample. Let $D_1, \cdots, D_q$ be non-zero effective (possibly reducible) Cartier divisors in weak m-subgeneral position on X. Suppose that $D_j \equiv \sum\limits_{i=1}^{r} a_{j,i} E_i$, $j = 1, \cdots, q$, where the coefficients $a_{j,i}$ are non-negative real numbers. Let $P_j = (a_{j,1}, \cdots, a_{j,r}) \in \mathbb{R}^r$, $j = 1, \cdots, q$. Assume that for any proper subset T of the set of standard basis vectors $\{e_1, \cdots, e_r\} \subset \mathbb{R}^r$, at most $(\#T)\lfloor \frac{q}{r} \rfloor$ of the vectors $P_1, \cdots, P_q$ are supported on T. If*

$$q \geq rm(n+1) + 1, \quad r = 1, 2,$$
$$q \geq rm(n+1) + \frac{(r-1)(r-2)}{2}, \quad r \geq 3,$$

then $X \setminus \sum\limits_{j=1}^{q} D_j$ is Brody hyperbolic.

2 Preliminaries

2.1 Seshadri constants

Let X be a projective variety and Y be a closed subscheme of X, corresponding to a coherent sheaf of ideals $\mathscr{I}$. Let $\mathcal{S} = \bigoplus_{d\geq 0} \mathscr{I}^d$ be the sheaf of graded algebras, where $\mathscr{I}^d$ is the d-th power of $\mathscr{I}$, with the convention that $\mathscr{I}^0 = \mathcal{O}_X$. Then $\widetilde{X} := \operatorname{Proj} \mathcal{S}$ is called the blowing up of X along Y. Let $\pi : \widetilde{X} \to X$ be the corresponding morphism. From [6, Proposition II.7.13(a)], we know that the inverse image ideal sheaf $\widetilde{\mathscr{I}} = \pi^{-1}\mathscr{I} \cdot \mathcal{O}_{\widetilde{X}}$ is an invertible sheaf on $\widetilde{X}$. Let E be the exceptional divisor which is the effective Cartier divisor in $\widetilde{X}$ whose associated invertible sheaf is the dual of $\pi^{-1}\mathscr{I} \cdot \mathcal{O}_{\widetilde{X}}$.

We now introduce the notion of Seshadri constants defined by Heier and Levin in this context as follows. As they said that it is essentially the same definition made in [4, Definition 1.1 and Remark 1.3] by Cutkosky-Ein-Lazarsfeld when X is non-singular.

Definition 2.1 (see [9, Definition 2.3]) *Let Y be a closed subscheme of a projective variety X. Let $\pi : \widetilde{X} \to X$ be the blow-up of X along Y. Let A be a nef Cartier divisor on X. We define the Seshadri constant $\epsilon_Y(A)$ of Y with respect to A to be the real number*

$$\epsilon_Y(A) = \sup\{\gamma \in \mathbb{Q}_{\geq 0} \mid \pi^*A - \gamma E \text{ is } \mathbb{Q}\text{-nef}\}.$$

The Seshadari constants have the following non-decreasing property.

Proposition 2.1 *Let X be a projective variety and X' be a subvariety of X, denote by $i : X' \to X$ the inclusion map. Let A be a nef Cartier divisor on X and Y be a closed subscheme of X. Then we have*

$$\epsilon_{i^*Y}(i^*A) \geq \epsilon_Y(A). \tag{2.1}$$

Proof Let $\pi : \widetilde{X} \to X$ be the blowing up of X along Y and E be the exceptional divisor. Denote by $Y' = i^*Y$. Let $\pi' : \widetilde{X'} \to X'$ be the blowing up of X' along Y' and E' be the exceptional divisor. Since the strict transform of X' in the blowing up $\widetilde{X}$ of X along Y is the blowing up $\widetilde{X'}$ of X' along Y' (see [5, Corollary 5.2(a)], [1, Theorem 1.3.1]), we have $\pi'^*(i^*A) = (\pi^*A)|_{\widetilde{X'}}$ and $E' = E|_{\widetilde{X'}}$. Let $\gamma \in \mathbb{Q}$ such that $\pi^*A - \gamma E$ is $\mathbb{Q}$-nef on $\widetilde{X}$. Note that the restriction of a nef divisor on $\widetilde{X}$ is a nef divisor on $\widetilde{X'}$, it follows that

$$\pi'^*(i^*A) - \gamma E' = (\pi^*A)|_{\widetilde{X'}} - \gamma E|_{\widetilde{X'}} = (\pi^*A - \gamma E)|_{\widetilde{X'}}$$

is $\mathbb{Q}$-nef on $\widetilde{X'}$, which completes the proof.

2.2 Weil functions

We briefly recall the basic definition of Weil functions, one can refer to [19] for more details. Let Y be a closed subscheme of a projective variety X. One can associate a Weil function $\lambda_Y : X \setminus \operatorname{Supp} Y \to \mathbb{R}$, well-defined up to $O(1)$, which satisfies the following properties: If Y and Z are two closed subschemes of X, and $\phi : X' \to X$ is a morphism of projective varieties,

(i) $\lambda_{Y\cap Z} = \min\{\lambda_Y, \lambda_Z\}$;

(ii) $\lambda_{Y+Z} = \lambda_Y + \lambda_Z$;

(iii) $\lambda_Y \le \lambda_Z$, if $Y \subset Z$;

(iv) $\lambda_Y(\phi(\mathbf{x})) = \lambda_{\phi^* Y}(\mathbf{x})$.

In particular, let D be a Cartier divisor on a complex projective variety X. A Weil function with respect to D is a function $\lambda_D : (X \setminus \mathrm{Supp} D) \to \mathbb{R}$ such that for all $\mathbf{x} \in X$ there is an open neighborhood U of $\mathbf{x}$ in X, a nonzero rational function f on X with $D|_U = (f)$, and a continuous function $\alpha : U \to \mathbb{R}$ such that

$$\lambda_D(\mathbf{x}) = -\log |f(\mathbf{x})| + \alpha(\mathbf{x})$$

for all $\mathbf{x} \in (U \setminus \mathrm{Supp} D)$. Note that a continuous fiber metric $\|\cdot\|$ on the line sheaf $\mathcal{O}_X(D)$ determines a Weil function for D given by $\lambda_D(\mathbf{x}) = -\log \|s(\mathbf{x})\|$, where s is the rational section of $\mathcal{O}_X(D)$ such that $D = (s)$. An example of Weil function for the hyperplane $H = \{a_0 x_0 + \cdots + a_n x_n = 0\}$ in $\mathbb{P}^n(\mathbb{C})$ is given by

$$\lambda_H(\mathbf{x}) = \log \frac{\max_{0\le i\le n} |x_i| \max_{0\le i\le n} |a_i|}{|a_0 x_0 + \cdots + a_n x_n|}, \tag{2.2}$$

where $[x_0, \cdots, x_n]$ are homogeneous coordinates for $\mathbf{x}$. The Weil functions with respect to divisors satisfiy the following properties:

(a) Functoriality: If λ is a Weil function for a Cartier divisor D on X, and if $\phi : X' \to X$ is a morphism such that $\phi(X') \not\subset \mathrm{Supp} D$, then $\mathbf{x} \mapsto \lambda(\phi(\mathbf{x}))$ is a Weil function for the Cartier divisor $\phi^* D$ on X'.

(b) Additivity: If λ_1 and λ_2 are Weil functions for Cartier divisors D_1 and D_2 on X, respectively, then $\lambda_1 + \lambda_2$ is a Weil function for $D_1 + D_2$.

(c) Uniqueness: If both λ_1 and λ_2 are Weil functions for a Cartier divisor on X, then $\lambda_1 = \lambda_2 + O(1)$.

(d) Boundedness from below: If D is an effective divisor and λ is a Weil function for D, then λ is bounded from below.

Let X be a projective variety, and let $Y \subset X$ be a closed subscheme.

Lemma 2.1 (see [19, Lemma 2.2]) *There exist effective Cartier divisors $D_1, \cdots, D_\ell$ such that*

$$Y = \bigcap_{i=1}^{\ell} D_i.$$

By Lemma 2.1, we can assume that $Y = D_1 \cap \cdots \cap D_\ell$, where $D_1, \cdots, D_\ell$ are effective Cartier divisors. This means that $\mathscr{I}_Y = \mathscr{I}_{D_1} + \cdots + \mathscr{I}_{D_\ell}$, where $\mathscr{I}_Y, \mathscr{I}_{D_1}, \cdots, \mathscr{I}_{D_\ell}$ are the defining ideal sheaves in $\mathcal{O}_X$. We set

$$\lambda_Y = \min\{\lambda_{D_1}, \cdots, \lambda_{D_\ell}\} + O(1). \tag{2.3}$$

Then we have $\lambda_Y : X \setminus \mathrm{Supp}\, Y \to \mathbb{R}$, which does not depend on the choice of Cartier divisors.

2.3 Nevanlinna functions

In this section, we briefly recall the definitions of characteristic function, proximity function and counting function in Nevanlinna theory.

2.3.1 Characteristic function

Let X be a complex projective variety and $f : \mathbb{C} \to X$ be a holomorphic map. Let $\mathcal{L} \to X$ be an ample line sheaf and ω be its Chern form. We define the characteristic function of f with respect to $\mathcal{L}$ by

$$T_{f,\mathcal{L}}(r) = \int_1^r \frac{\mathrm{d}t}{t} \int_{|z|<t} f^*\omega.$$

Since any line sheaf $\mathcal{L}$ can be written as $\mathcal{L} = \mathcal{L}_1 \otimes \mathcal{L}_2^{-1}$ with $\mathcal{L}_1$, $\mathcal{L}_2$ being both ample, we define $T_{f,\mathcal{L}}(r) = T_{f,\mathcal{L}_1}(r) - T_{f,\mathcal{L}_2}(r)$. A divisor D on X defines a line bundle $\mathcal{O}(D)$, we denote by $T_{f,D}(r) = T_{f,\mathcal{O}(D)}(r)$. If $X = \mathbb{P}^n(\mathbb{C})$ and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(1)$, then we simply write $T_{f,\mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(1)}(r)$ as $T_f(r)$.

The characteristic function satisfies the following properties:

(a) Functoriality: If $\phi : X \to X'$ is a morphism and if $\mathcal{L}$ is a line sheaf on X', then

$$T_{f,\phi^*\mathcal{L}}(r) = T_{\phi\circ f,\mathcal{L}}(r) + O(1).$$

(b) Additivity: If $\mathcal{L}_1$ and $\mathcal{L}_2$ are line sheaves on X, then

$$T_{f,\mathcal{L}_1\otimes\mathcal{L}_2}(r) = T_{f,\mathcal{L}_1}(r) + T_{f,\mathcal{L}_2}(r) + O(1).$$

(c) Positivity: If $\mathcal{L}$ is ample and $f : \mathbb{C} \to X$ is non-constant, then

$$T_{f,\mathcal{L}}(r) \to +\infty \quad \text{as } r \to +\infty.$$

(d) Base locus: If the image of f is not contained in the base locus of $|D|$, then $T_{f,D}(r)$ is bounded from below.

(e) Globally generated line sheaves: If $\mathcal{L}$ is a line sheaf on X, and is generated by its global sections, then $T_{f,\mathcal{L}}(r)$ is bounded from blow.

2.3.2 Counting and proximity functions

Let X be a projective variety and let $Y \subset X$ be a closed subscheme. For a holomorphic curve $f : \mathbb{C} \to X$ with $f(\mathbb{C}) \not\subset \operatorname{Supp} Y$, the proximity function of f with respect to Y is defined by

$$m_f(r, Y) = \int_0^{2\pi} \lambda_Y(f(r\mathrm{e}^{\mathrm{i}\theta}))\frac{\mathrm{d}\theta}{2\pi}.$$

The proximity function satisfies the following properties:

(a) Functoriality: If $\phi : X \to X'$ is a morphism and Y' is a closed subscheme on X' with $\phi \circ f(\mathbb{C}) \not\subset \operatorname{Supp} Y'$, then

$$m_f(r, \phi^*Y') = m_{\phi\circ f}(r, Y') + O(1).$$

(b) Additivity: If Y_1 and Y_2 are two closed subschemes on X, then

$$m_f(r, Y_1 + Y_2) = m_f(r, Y_1) + m_f(r, Y_2) + O(1).$$

(c) Boundedness from below: If D is an effective divisor, then $m_f(r, D)$ is bounded from below.

Next, we introduce the definition of counting function $N_f(r, Y)$ given by Yamanoi in [23]. Assume that Y can be written as $Y = D_1 \cap \cdots \cap D_\ell$ with $D_1, \cdots, D_\ell$ being effective Cartier divisors, then we set

$$\mathrm{ord}_z f^*Y = \min\{\mathrm{ord}_z f^*D_1, \cdots, \mathrm{ord}_z f^*D_\ell\}.$$

When $f(\mathbb{C}) \subset \mathrm{Supp}\, D_i$, we set $\mathrm{ord}_z f^*D_i = +\infty$. The definition of $\mathrm{ord}_z f^*Y$ does not depend on the choice of the Cartier divisors $D_1, \cdots, D_\ell$. We define the counting function by

$$N_f(r, Y) = \int_1^r \Big(\sum_{\{z \in \mathbb{C} \,|\, |z| < t\}} \mathrm{ord}_z f^*Y \Big) \frac{\mathrm{d}t}{t}.$$

For a closed subscheme Y as above, consider the blowing up $\pi : \widetilde{X} \to X$ of X along Y, let $f : \mathbb{C} \to X$ be a holomorphic curve and $\widetilde{f} : \mathbb{C} \to \widetilde{X}$ be its holomorphic lifting. Let A be an ample divisor on X. By using the functoriality property of characteristic function, proximity function and counting function, we have

$$T_{\widetilde{f}, \pi^*A}(r) = T_{f,A}(r) + O(1),$$
$$m_{\widetilde{f}}(r, \pi^*Y) = m_f(r, Y) + O(1),$$
$$N_{\widetilde{f}}(r, \pi^*Y) = N_f(r, Y) + O(1).$$

2.3.3 First main theorem

Let X be a complex projective variety and $f : \mathbb{C} \to X$ be a holomorphic map. Let D be a divisor on X. By using Poincaré-Lelong formula, we have the first main theorem,

$$m_f(r, D) + N_f(r, D) = T_{f,D}(r) + O(1). \tag{2.4}$$

3 Proof of Theorem 1.4

For the nonconstant holomorphic curve $f : \mathbb{C} \to X$ in Theorem 1.4, we consider the Zariski closure $\overline{f(\mathbb{C})}$ of its image. In order to prove Theorem 1.4, we need the following second main theorem for holomorphic curves with Zariski-dense image. (We thank professor Min Ru for pointing out a simple proof of Theorem 3.1.)

Theorem 3.1 *Let X be a complex projective variety of dimension n. Let Y_j be closed subschemes of codimension* $\mathrm{codim}\, Y_j (\geq 1)$ *in X, $j = 1, \cdots, q$. Let A be an ample Cartier divisor on X. Let $f : \mathbb{C} \to X$ be a holomorphic curve with Zariski dense image, assume that $Y_1, \cdots, Y_q$ are in weak m-subgeneral position. Then, for every $\varepsilon > 0$,*

$$\Big\| \sum_{j=1}^{q} \mathrm{codim}\, Y_j \cdot \epsilon_{Y_j}(A) \cdot m_f(r, Y_j) \leq [m(n+1) + \varepsilon] T_{f,A}(r). \tag{3.1}$$

Proof of Theorem 3.1 Denote by $\varrho_j := \mathrm{codim}\, Y_j$, $j = 1, \cdots, q$.

Given $z \in \mathbb{C}$, we arrange $\{1, \cdots, q\}$ as $\{1(z), \cdots, q(z)\}$ so that

$$\begin{aligned} &\varrho_{1(z)} \epsilon_{Y_{1(z)}}(A) \lambda_{Y_{1(z)}}(f(z)) \geq \varrho_{1(z)} \epsilon_{Y_{2(z)}}(A) \lambda_{Y_{2(z)}}(f(z)) \\ &\geq \cdots \geq \varrho_{1(z)} \epsilon_{Y_{m(z)}}(A) \lambda_{Y_{m(z)}}(f(z)) \geq \cdots \geq \varrho_{1(z)} \epsilon_{Y_{q(z)}}(A) \lambda_{Y_{q(z)}}(f(z)). \end{aligned} \tag{3.2}$$

Since $Y_1, \cdots, Y_q$ are in weak m-subgeneral position, we have

$$\begin{aligned}&\sum_{j=1}^{q} \varrho_j \epsilon_{Y_j}(A)\lambda_{Y_j}(f(z))\\ &\le \sum_{j=1}^{m} \varrho_{j(z)}\epsilon_{Y_{j(z)}}(A)\lambda_{Y_{j(z)}}(f(z)) + O(1)\\ &\le m\varrho_{1(z)}\epsilon_{Y_{1(z)}}(A)\lambda_{Y_{1(z)}}(f(z)) + O(1). \end{aligned} \tag{3.3}$$

The proof of the first inequality is similar to that of [21, Lemma 21.7] and is omitted here.

For each $1 \le j \le q$, we observe that according to Definition 1.1(b), the elements of the list obtained by repeating Y_j up to ϱ_j times, i.e., the closed subschemes $Y_{j,1}, Y_{j,2}, \cdots, Y_{j,\varrho}$ with $Y_{j,\ell} = Y_j$ for $\ell = 1, \cdots, \varrho_j$, are in general position (of Definition 1.1(b)). Note that the union of all closed subschemes $Y_{j,\ell}$ ($1 \le j \le q, 1 \le \ell \le \varrho_j$) is a finite set, which may be denoted by $\{\widetilde{Y}_1, \cdots, \widetilde{Y}_T\}$. We rewrite (3.3) as

$$\begin{aligned}&\sum_{j=1}^{q} \varrho_j \epsilon_{Y_j}(A)\lambda_{Y_j}(f(z))\\ &\le m\sum_{\ell=1}^{\varrho_{1(z)}} \epsilon_{Y_{1(z)}}(A)\lambda_{Y_{1(z),\ell}}(f(z)) + O(1)\\ &\le m\max_{\mathcal{K}}\sum_{u\in\mathcal{K}} \epsilon_{\widetilde{Y}_u}(A)\lambda_{\widetilde{Y}_u}(f(z)) + O(1), \end{aligned} \tag{3.4}$$

where the maximum is taken over all subsets $\mathcal{K}$ of $\{1, \cdots, T\}$ such that the closed subschemes $\widetilde{Y}_u, u \in \mathcal{K}$, are in general position.

Now, we need the following general form of second main theorem given in [9].

Theorem 3.2 (see [9, Theorem 1.8]) *Let X be a projective variety of dimension n. Let $Y_1, \cdots, Y_q$ be closed subschemes of X. Let A be an ample Cartier divisor on X. Let $f : \mathbb{C} \to X$ be a holomorphic curve with Zariski-dense image. Then for every $\varepsilon > 0$,*

$$\Big\| \int_0^{2\pi} \max_{\mathcal{J}} \sum_{j\in\mathcal{J}} \epsilon_{Y_j}(A)\lambda_{Y_j}(f(re^{i\theta}))\frac{d\theta}{2\pi} \le (n+1+\varepsilon)T_{f,A}(r), \tag{3.5}$$

where the maximum is taken over all subsets $\mathcal{J}$ of $\{1, \cdots, q\}$ such that the closed subschemes $Y_j, j \in \mathcal{J}$, are in general position.

Then it follows from (3.4) and Theorem 3.2 that, for any given $\varepsilon > 0$,

$$\begin{aligned}&\sum_{j=1}^{q} \varrho_j \epsilon_{Y_j}(A) m_f(r, Y_j)\\ &\le m\int_0^{2\pi} \max_{\mathcal{K}}\sum_{u\in\mathcal{K}} \epsilon_{\widetilde{Y}_u}(A)\lambda_{\widetilde{Y}_u}(f(re^{i\theta}))\frac{d\theta}{2\pi} + O(1)\\ &\le [m(n+1)+\varepsilon]T_{f,A}, \end{aligned} \tag{3.6}$$

This completes the proof of Theorem 3.1.

Remark 3.1 (1) If $q=1$, i.e., there is only one closed subscheme Y, we may take $m=1$, then (3.1) is exactly (1.4). If $Y_1,\cdots,Y_q$ are distinct points, then (1.2) can also be obtained from (3.1).

(2) Recently, Ru and Wang [18] proved that, if $Y_1,\cdots,Y_q$ are intersecting properly on X (which is, by [10, Theorem 17.4], equivalent to in general position when X is smooth), then

$$\Big\|\sum_{j=1}^{q}\beta_{A,Y_j}m_f(r,Y_j)\le(1+\varepsilon)T_{f,A}(r). \tag{3.7}$$

(Or see [22] for more general case.) This gives an improvement of Theorem 1.2 when X is smooth.

By using similar method, one can obtain that, if $Y_1,\cdots,Y_q$ are in weak m-subgeneral position on X, then

$$\Big\|\sum_{j=1}^{q}\beta_{A,Y_j}m_f(r,Y_j)\le m(1+\varepsilon)T_{f,A}(r). \tag{3.8}$$

If X is smooth, then one can also obtain Theorem 3.1 from (3.8) and (1.5). But in the case we are dealing with, $\overline{f(\mathbb{C})}$ may not be smooth.

Proof of Theorem 1.4 Denote by $\varrho:=\min\limits_{1\le j\le q}\{\operatorname{codim}Y_j\}$, then $\varrho\ge1$.

Case 1 If $f(\mathbb{C})$ is Zariski dense in X.

It follows from Theorem 3.1 that

$$\begin{aligned}\Big\|\sum_{j=1}^{q}\epsilon_{Y_j}(A)m_f(r,Y_j)&\le\Big[\frac{m(n+1)}{\varrho}+\varepsilon\Big]T_{f,A}(r)\\&\le[m(n+1)+\varepsilon]T_{f,A}(r).\end{aligned}$$

Case 2 If $f(\mathbb{C})$ is not Zariski dense in X.

Since $f:\mathbb{C}\to X$ is non-constant, we can assume that $f(\mathbb{C})$ is contained in an irreducible subvariety $V(=\overline{f(\mathbb{C})})$ of X of dimension k with $1\le k\le n-1$ such that $f:\mathbb{C}\to V$ is a holomorphic curve with Zariski dense image. Consider the inclusion map $i:V\to X$. The assumption that the subschemes $Y_1,\cdots,Y_q$ are located in weak m-subgeneral position (with respect to X) implies that $i^*Y_1,\cdots,i^*Y_q$ are located in weak m-subgeneral position (with respect to V). Note that if the codimension of i^*Y_j in V is 0 for any $j\in\{1,2,\cdots,q\}$, then $i^*Y_j=V$, which contradicts the assumption that $f(\mathbb{C})\not\subset\operatorname{Supp}Y_j$. Hence, by applying Case 1 to the holomorphic curve $f:\mathbb{C}\to V$, we have, for every $\varepsilon>0$,

$$\begin{aligned}\Big\|\sum_{j=1}^{q}\epsilon_{i^*Y_j}(i^*A)m_f(r,i^*Y_j)&\le[m(k+1)+\varepsilon]T_{f,i^*A}(r)\\&\le[m(n+1)+\varepsilon]T_{f,i^*A}(r).\end{aligned}\tag{3.9}$$

Here,

$$m_f(r,i^*Y_j)=m_{i\circ f}(r,Y_j)+O(1)=m_f(r,Y_j)+O(1), \tag{3.10}$$

$$T_{f,i^*A}(r)=T_{i\circ f,A}(r)+O(1)=T_{f,A}(r)+O(1). \tag{3.11}$$

From Proposition 2.1, we have

$$\epsilon_{i^*Y_j}(i^*A) \geq \epsilon_{Y_j}(A). \tag{3.12}$$

Combining (3.9)–(3.12), we have

$$\Big\| \sum_{j=1}^{q} \epsilon_{Y_j}(A) m_f(r, Y_j) \leq [m(n+1)+\varepsilon] T_{f,A}(r).$$

This completes our proof of Theorem 1.4.

Remark 3.2 Combining the proof of Theorem 1.4 and He-Ru's result (see [7, Main Theorem (Analytic Part)]), one can also obtain a second main theorem for "non-constant" holomorphic curves as follows.

Let X be a complex projective variety of dimension n and $Y_1, \cdots, Y_q$ be closed subschemes of X. Let A be an ample Cartier divisor on X. Let $f : \mathbb{C} \to X$ be a non-constant holomorphic curve. Assume that $Y_1, \cdots, Y_q$ are in m-subgeneral position. Then, for every $\varepsilon > 0$,

$$\Big\| \sum_{j=1}^{q} \epsilon_{Y_j}(A) m_f(r, Y_j) \leq \Big[\Big(m - \min\Big\{n, \frac{m}{2}\Big\} + 1\Big)\Big(\min\Big\{n, \frac{m}{2}\Big\} + 1\Big) + \varepsilon\Big] T_{f,A}(r). \tag{3.13}$$

Though the coefficient on the right-hand side of (3.13) is smaller than that in Theorem 3.1, but in our theorem, the subschemes only need to be in weak m-subgeneral position and m can be smaller than n.

4 Proof of Theorem 1.6

In this section, we give the proof of Theorem 1.6 by using our second main theorem and the method of Heier and Levin [8].

Proof of Theorem 1.6 By assumption, for any proper subset T of the set of standard basis vectors $\{e_1, \cdots, e_r\} \subset \mathbb{R}^r$, at most $(\#T)\lfloor \frac{q}{r} \rfloor$ of the vectors $P_1, \cdots, P_q$ are supported on T, then we may take

$$\alpha_{1,1}(\kappa), \cdots, \alpha_{1,r}(\kappa), \alpha_{2,1}(\kappa), \cdots, \alpha_{2,r}(\kappa), \cdots, \alpha_{q,1}(\kappa), \cdots, \alpha_{q,r}(\kappa)$$

to be (discontinuous) functions of $\kappa \in (0, 1]$ with the following properties.

(i) The function $\alpha_{j,i}(\kappa)$ is identically equal to 0 if $a_{j,i} = 0$. If, on the other hand, $a_{j,i} \neq 0$, then $\alpha_{j,i}(\kappa)$ takes on positive real values such that we have the limits

$$\lim_{\kappa \to 0+} \alpha_{j,i}(\kappa) = 0.$$

(ii) The $\mathbb{R}$-divisors $B_j(\kappa) = \alpha_{j,1}(\kappa)E_1 + \cdots + \alpha_{j,r}(\kappa)E_r$ are such that

$$D'_j(\kappa) := D_j + B_j(\kappa) \equiv \sum_{i=1}^{r} a'_{j,i}(\kappa) E_i, \quad j = 1, \cdots, q,$$

have rational coefficients $a'_{j,i}(\kappa) = a_{j,i} + \alpha_{j,i}(\kappa)$.

(iii) For any proper subset T of the set of standard basis vectors $\{e_1, \cdots, e_r\} \subset \mathbb{R}^r$, at most $(\#T)\lfloor \frac{q}{r} \rfloor$ of the vectors $P'_1(\kappa), \cdots, P'_q(\kappa)$ are supported on T, where $P'_1(\kappa) = (a'_{1,1}(\kappa), \cdots, a'_{1,r}(\kappa)), \cdots, P'_q(\kappa) = (a'_{q,1}(\kappa), \cdots, a'_{q,r}(\kappa))$.

(iv) For all $i \neq i', j \neq j'$,

$$a'_{j,i}(\kappa)a'_{j',i'}(\kappa) - a'_{j,i'}(\kappa)a'_{j'i}(\kappa) \neq 0, \tag{4.1}$$

unless both terms on the left are 0.

For $Q = (b_1, \cdots, b_r) \in \mathbb{R}^r$ with positive coordinates, define

$$n_i(Q, P'_1(\kappa), \cdots, P'_q(\kappa)) = \#\Big\{ j \in \{1, \cdots, q\} \mid \min_{\ell=1,\cdots,r} \frac{b_\ell}{a'_{j,\ell}(\kappa)} = \frac{b_i}{a'_{j,i}(\kappa)} \Big\}, \quad i = 1, \cdots, r.$$

We need the following result in [8].

Lemma 4.1 (see [8, Lemma 2.2]) *Let $P'_j = (a'_{j,1}(\kappa), \cdots, a'_{j,r}(\kappa)) \in \mathbb{R}^r \setminus \{0\}$, $j = 1, \cdots, q$, be vectors with non-negative coordinates. Suppose that for any proper subset T of the set of standard basis vectors $\{e_1, \cdots, e_r\} \subset \mathbb{R}^r$ of cardinality t, at most $t\lfloor \frac{q}{r} \rfloor$ of the vectors P'_j are supported on T. Assume additionally that for all $i \neq i', j \neq j'$, we have*

$$a'_{j,i}(\kappa)a'_{j',i'}(\kappa) - a'_{j,i'}(\kappa)a'_{j',i}(\kappa) \neq 0,$$

unless both terms on the left are 0. Then there exists $Q'(\kappa) = (b'_1(\kappa), \cdots, b'_r(\kappa)) \in \mathbb{Q}^r$ with positive coordinates such that

$$n_i(Q'(\kappa), P'_1(\kappa), \cdots, P'_q(\kappa)) \geq \frac{q}{r} - \frac{r-1}{2}, \quad i = 1, \cdots, r \tag{4.2}$$

and

$$\Big(\frac{1}{2} \min \frac{a'_{j,i}(\kappa)}{a'_{j',i'}(\kappa)}\Big)^r \leq \frac{b'_\ell(\kappa)}{b'_{\ell'}(\kappa)} \leq \Big(2 \max \frac{a'_{j,i}(\kappa)}{a'_{j',i'}(\kappa)}\Big)^r \quad \text{for all } \ell, \ell', \tag{4.3}$$

where the minimum and maximum are taken over all the indexes i, j, i', j' such that $a'_{j,i}(\kappa)a'_{j',i'}(\kappa) \neq 0$.

Then it follows from Lemma 4.1 that for all κ, there exists a vector $Q'(\kappa) = (b'_1(\kappa), \cdots, b'_r(\kappa))$ as in Lemma 4.1 with respect to $P'_1(\kappa), \cdots, P'_q(\kappa)$.

From the definition of $a'_{j,i}(\kappa)$, for a sufficiently small choice of $\widehat{\kappa} > 0$ (we now fix one such choice), there exist positive rational constants γ_1, γ_2 such that for all $0 < \kappa < \widehat{\kappa}$,

$$\gamma_1 < a'_{j,i}(\kappa) < \gamma_2 \tag{4.4}$$

for all i and j such that $a'_{j,i}(\kappa) \neq 0$ (or equivalently, $a_{j,i} \neq 0$).

Since $n_i(Q'(\kappa), P'_1(\kappa), \cdots, P'_q(\kappa)) = n_i(\lambda Q'(\kappa), P'_1(\kappa), \cdots, P'_q(\kappa))$ for any rational number $\lambda > 0$, where $\lambda Q'(\kappa) = (\lambda b'_1(\kappa), \cdots, \lambda b'_r(\kappa))$. We may replace $Q'(\kappa)$ by some $\lambda Q'(\kappa)$, which still satisfies (4.2) and (4.3), and normalize the coordinates so that $b'_1(\kappa) = 1$. Then it follows from (4.3) and (4.4) that, there exist positive rational constants γ_3 and γ_4 such that for all $0 < \kappa < \widehat{\kappa}$,

$$\gamma_3 < b'_i(\kappa) < \gamma_4, \quad i = 1, \cdots, r. \tag{4.5}$$

We now choose a fixed positive rational number $\delta < \min\{\frac{\gamma_1\gamma_3}{2\gamma_2\gamma_4}, \frac{1}{2}\}$ and a fixed $0 < \kappa_0 = \kappa(\delta) < \widehat{\kappa}$ such that

$$\delta\gamma_3 \sum_{i=1}^{r} E_i - \frac{\gamma_4}{\gamma_1} \sum_{j=1}^{q} B_j(\kappa_0) \tag{4.6}$$

is $\mathbb{Q}$-nef.

Set $Q' = Q'(\kappa_0) = (b'_1, \cdots, b'_r)$ with $b'_1 = 1$ and let

$$A' = b'_1 E_1 + \cdots + b'_r E_r.$$

Then A' is $\mathbb{Q}$-ample.

We define positive rational numbers

$$c'_j := \min_{\ell=1,\cdots,r} \frac{b'_\ell}{a'_{j,\ell}(\kappa_0)} < \frac{\gamma_4}{\gamma_1}, \quad j = 1, \cdots, q.$$

Note that

$$\begin{aligned} A' - c'_j D'_j(\kappa_0) &\equiv \sum_{i=1}^{r} (b'_j - c'_j a'_{j,i}(\kappa_0)) E_i \\ &= \sum_{i=1}^{r} a'_{j,i}(\kappa_0) \Big(\frac{b'_j}{a'_{j,i}(\kappa_0)} - c'_j \Big) E_i, \end{aligned}$$

where $\frac{b'_j}{a'_{j,i}(\kappa_0)} - c'_j \geq 0$ for $j = 1, \cdots, q$. Since $c'_j B_j(\kappa_0)$ is a nef $\mathbb{R}$-divisor, it implies that $A' - c'_j D_j$ is a nef $\mathbb{Q}$-divisor for $j = 1, \cdots, q$.

We first deal with the general case $r \geq 3$.

Since

$$q \geq rm(n+1) + \frac{(r-1)(r-2)}{2} = r[m(n+1) - 1] + \frac{r(r-1)}{2} + 1,$$

we have

$$n_i(Q', P'_1(\kappa_0), \cdots, P'_q(\kappa_0)) \geq \frac{q}{r} - \frac{r-1}{2} > m(n+1) - 1, \quad i = 1, \cdots, r.$$

Therefore,

$$n_i(Q', P'_1(\kappa_0), \cdots, P'_q(\kappa_0)) \geq m(n+1), \quad i = 1, \cdots, r \tag{4.7}$$

as $n_i(Q', P'_1(\kappa_0), \cdots, P'_q(\kappa_0))$ is an integer.

For $i \in \{1, \cdots, r\}$, by hypothesis, at most $(r-1)\lfloor \frac{q}{r} \rfloor \leq q - \frac{q}{r}$ of the vectors $P_1, \cdots, P_q$ lie in $\mathrm{Span}(\{e_1, \cdots, e_r\} \backslash \{e_i\})$. Since $q > rm(n+1)$, it follows that there are at least $\lceil \frac{q}{r} \rceil \geq m(n+1)+1$ points $P'_j(\kappa_0)$ with $a'_{j,i}(\kappa_0) > 0$. Combined with (4.7), this implies that

$$\begin{aligned} \sum_{j=1}^{q} c'_j D'_j(\kappa_0) &\geq \sum_{i=1}^{r} m(n+1) b'_i E_i + \sum_{i=1}^{r} \Big(\min_j c'_j \Big) \Big(\min_{j, a'_{j,i}(\kappa_0) \neq 0} a'_{j,i}(\kappa_0) \Big) E_i \\ &\geq m(n+1) A' + \frac{\gamma_1\gamma_3}{\gamma_2} \sum_{i=1}^{r} E_i \end{aligned}$$

$$\geq \Big[m(n+1)+\frac{\gamma_1\gamma_3}{\gamma_2\gamma_4}\Big]A'.$$

Here, for $\mathbb{R}$-divisors F_1 and F_2, we write $F_1 \geq F_2$ if the difference $F_1 - F_2$ is a nef $\mathbb{R}$-divisor.

Finally, we find the inequalities

$$\begin{aligned}&\sum_{j=1}^{q} c_j' D_j - [m(n+1)+\delta]A'\\&=\sum_{j=1}^{q} c_j' D_j'(\kappa_0) - [m(n+1)+2\delta t]A' + \delta A' - \sum_{j=1}^{q} c_j' B_j(\kappa_0)\\&\geq \Big[m(n+1)+\frac{\gamma_1\gamma_3}{\gamma_2\gamma_4}\Big]A' - [m(n+1)+2\delta]A' + \delta\gamma_3\sum_{i=1}^{r} E_i - \frac{\gamma_4}{\gamma_1}\sum_{j=1}^{q} B_j(\kappa_0)\\&\geq \Big(\frac{\gamma_1\gamma_3}{\gamma_2\gamma_4} - 2\delta\Big)A',\end{aligned}$$

where the last inequality is due to (4.6). Therefore, by $\frac{\gamma_1\gamma_3}{\gamma_2\gamma_4} - 2\delta > 0$,

$$\sum_{j=1}^{q} c_j' D_j - [m(n+1)+\delta]A'$$

is $\mathbb{Q}$-ample.

When $r = 1$, since $q \geq rm(n+1)+1$, we have

$$n_i(Q', P_1'(\kappa_0), \cdots, P_q'(\kappa_0)) \geq \frac{q}{r} - \frac{r-1}{2} \geq m(n+1)+1.$$

Thus

$$\begin{aligned}\sum_{j=1}^{q} c_j' D_j'(\kappa_0) &\geq \sum_{i=1}^{r} [m(n+1)+1] b_i' E_i\\&= [m(n+1)+1]A'.\end{aligned}$$

Then we have

$$\begin{aligned}&\sum_{j=1}^{q} c_j' D_j - [m(n+1)+\delta]A'\\&=\sum_{j=1}^{q} c_j' D_j'(\kappa_0) - [m(n+1)+2\delta]A' + \delta A' - \sum_{j=1}^{q} c_j' B_j(\kappa_0)\\&\geq [m(n+1)+1]A' - [m(n+1)+2\delta]A' + \delta\gamma_3\sum_{i=1}^{r} E_i - \frac{\gamma_4}{\gamma_1}\sum_{j=1}^{q} B_j(\kappa_0)\\&\geq (1-2\delta)A'.\end{aligned}$$

Therefore, by $\delta < \frac{1}{2}$,

$$\sum_{j=1}^{q} c_j' D_j - [m(n+1)+\delta]A'$$

is $\mathbb{Q}$-ample.

When $r = 2$, since $q \geq rm(n+1)+1$, we have

$$n_i(Q', P_1'(\kappa_0), \cdots, P_q'(\kappa_0)) \geq \frac{q}{r} - \frac{r-1}{2} \geq m(n+1).$$

Then the proof is the same as the case $r \geq 3$ and we have

$$\sum_{j=1}^{q} c_j' D_j - [m(n+1)+\delta]A'$$

is $\mathbb{Q}$-ample.

Now A' is an ample $\mathbb{Q}$-divisor, there exists a positive integer N big enough such that NA' is an ample integral divisor. Set $A := NA'$ and $c_j := Nc_j'$ for $j = 1, \cdots, q$.

To summarize, there exist an ample divisor A and positive rational constants $c_1, \cdots, c_q, \delta$ such that for all $j = 1, \cdots, q$:

$$A - c_j D_j \text{ is } \mathbb{Q}\text{-nef} \tag{4.8}$$

and

$$\sum_{j=1}^{q} c_j D_j - [m(n+1)+\delta]A \text{ is } \mathbb{Q}\text{-ample.} \tag{4.9}$$

Let $f : \mathbb{C} \to X \setminus \sum_{j=1}^{q} D_j$ be a holomophic curve.

If f is not constant, we may apply Theorem 1.4 to conclude that,

$$\Big\| \sum_{j=1}^{q} c_j m_f(r, D_j) \leq [m(n+1)+\delta] T_{f,A}(r).$$

Since $f(\mathbb{C}) \cap \big(\bigcup_{j=1}^{q} \operatorname{Supp} D_j\big) = \emptyset$ and (2.4), we get

$$\Big\| \sum_{j=1}^{q} c_j T_f(r, D_j) = \sum_{j=1}^{q} c_j m_f(r, D_j) + O(1) \leq [m(n+1)+\delta] T_{f,A}(r) + O(1),$$

which contradicts to our construction that $\sum_{j=1}^{q} c_j D_j - [m(n+1)+\delta]A$ is $\mathbb{Q}$-ample. Thus f must be constant.

5 Schmidt Subspace Theorem

In this section, we introduce the counterpart in number theory of our main results according to Vojta's dictionary which gives an analogue between Nevanlinna theory and Diophantine approximation. The line of reasoning is by now well known and we omit the details here.

Let k be a number field. Denote by M_k the set of places (i.e., equivalence classes of absolute values) of k and write M_k^∞ for the set of archimedean places of k.

Let X be a projective variety defined over k, let $\mathcal{L}$ be a line sheaf on X and let Y be a closed subscheme on X. For every place $v \in M_k$, we can associate the local Weil functions $\lambda_{\mathcal{L},v}$ and

$\lambda_{Y,v}$ with respect to v, which have similar properties as the Weil function introduced in Section 2. For more details, please refer to [20, Section 1.3].

Define

$$h_{\mathcal{L}}(\mathbf{x}) = \sum_{v \in M_k} \lambda_{\mathcal{L},v}(\mathbf{x}) \quad \text{for } \mathbf{x} \in X$$

and

$$m_S(\mathbf{x}, Y) = \sum_{v \in S} \lambda_{Y,v}(\mathbf{x}) \quad \text{for } \mathbf{x} \in X \setminus \operatorname{Supp} \mathrm{Y},$$

where S is a finite subset of M_k containing M_k^∞.

Now, we state the counterparts of Theorems 3.1 and 1.4.

Theorem 5.1 *Let X be a projective variety, defined over a number field k, of dimension n. Let Y_j be closed subschemes of codimension $\operatorname{codim} Y_j (\geq 1)$ in X, $j = 1, \cdots, q$. Let A be an ample Cartier divisor on X. Let S be a finite subset of M_k containing M_k^∞. Assume that $Y_1, \cdots, Y_q$ are in weak m-subgeneral position. Then, for every $\varepsilon > 0$, there exists a proper Zariski-closed subset $Z \subset X$ such that for all points $\mathbf{x} \in X(k) \setminus Z$,*

$$\sum_{j=1}^{q} \operatorname{codim} Y_j \cdot \epsilon_{Y_j}(A) \cdot m_S(\mathbf{x}, Y_j) \leq [m(n+1) + \varepsilon] h_{\mathcal{O}(A)}(\mathbf{x}). \tag{5.1}$$

Theorem 5.2 *Let X be a projective variety, defined over a number field k, and $Y_1, \cdots, Y_q$ be proper closed subschemes of X in weak m-subgeneral position. Let A be an ample Cartier divisor on X. Let S be a finite subset of M_k containing M_k^∞. Then, for every $\varepsilon > 0$, the set of points $\mathbf{x} \in X(k) \setminus \bigcup_{j=1}^{q} \operatorname{Supp} Y_j$ with*

$$\sum_{j=1}^{q} \epsilon_{Y_j}(A) m_S(\mathbf{x}, Y_j) \geq [m(n+1) + \varepsilon] h_{\mathcal{O}(A)}(\mathbf{x}) \tag{5.2}$$

is a finite set.

Now, using the method shown in [8], we give an application of this theorem.

Definition 5.1 (Arithmetically quasi-hyperbolic) *Given a variety $V = X \setminus D$ defined over a number field k.*

(i) *We say that V is arithmetically quasi-hyperbolic if there exists a proper closed subset $Z \subset X$ such that for every number field $k' \supset k$, every finite set of places S of k' containing the archimedean places, and every set R of (k'-rational) (D, S)-integral points on X, the set $R \setminus Z$ is finite.*

(ii) *We say that $X \setminus D$ is arithmetically hyperbolic if all sets of (D, S)-integral points on X are finite (i.e., one may take $Z = \emptyset$ in the definition of quasi-hyperbolicity).*

For the notion of (D, S)-integral sets of points, please refer to [20, *Section* 1.4].

In [8], Heier and Levin showed the following arithmetically quasi-hyperbolicity result as an application of [9, Corollary 1.4].

Theorem 5.3 (see [8, Theorem 1.8(a)] *Let X be a projective variety, defined over a number field k, of dimension n. Let $E_1, \cdots, E_r$ be nef Cartier divisors on X with $\sum_{i=1}^{r} E_i$ ample. Let $D_1, \cdots, D_q$ be non-zero effective (possibly reducible) Cartier divisors in general position on X and let $D = \sum_{j=1}^{q} D_j$. Suppose that $D_j \equiv \sum_{i=1}^{r} a_{j,i} E_i$, $j = 1, \cdots, q$, where the coefficients $a_{j,i}$ are non-negative real numbers. Let $P_j = (a_{j,1}, \cdots, a_{j,r}) \in \mathbb{R}^r$, $j = 1, \cdots, q$. Assume that for any proper subset T of the set of standard basis vectors $\{e_1, \cdots, e_r\} \subset \mathbb{R}^r$, at most $(\#T)\lfloor \frac{q}{r} \rfloor$ of the vectors $P_1, \cdots, P_q$ are supported on T. If*

$$q \geq r(n+1) + 1, \quad r = 1, 2,$$
$$q \geq r(n+1) + \frac{(r-1)(r-2)}{2}, \quad r \geq 3,$$

then $X \setminus D$ is arithmetically quasi-hyperbolic.

As an application of Theorem 5.2, we have an arithmetically hyperbolicity result for divisors in weakly m-subgeneral position as follows.

Theorem 5.4 *Let X be a projective variety, defined over a number field k, of dimension n. Let $E_1, \cdots, E_r$ be nef Cartier divisors on X with $\sum_{i=1}^{r} E_i$ ample. Let $D_1, \cdots, D_q$ be non-zero effective (possibly reducible) Cartier divisors in weak m-subgeneral position on X and let $D = \sum_{j=1}^{q} D_j$. Suppose that $D_j \equiv \sum_{i=1}^{r} a_{j,i} E_i$, $j = 1, \cdots, q$, where the coefficients $a_{j,i}$ are non-negative real numbers. Let $P_j = (a_{j,1}, \cdots, a_{j,r}) \in \mathbb{R}^r$, $j = 1, \cdots, q$. Assume that for any proper subset T of the set of standard basis vectors $\{e_1, \cdots, e_r\} \subset \mathbb{R}^r$, at most $(\#T)\lfloor \frac{q}{r} \rfloor$ of the vectors $P_1, \cdots, P_q$ are supported on T. If*

$$q \geq rm(n+1) + 1, \quad r = 1, 2,$$
$$q \geq rm(n+1) + \frac{(r-1)(r-2)}{2}, \quad r \geq 3,$$

then $X \setminus D$ is arithmetically hyperbolic.

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