Chin. Ann. Math. Ser. B
43(4), 2022, 585–600
DOI: 10.1007/s11401-022-0347-0

Chinese Annals of Mathematics, Series B

Continuity of Almost Harmonic Maps with the Perturbation Term in a Critical Space

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Abstract The authors study the continuity estimate of the solutions of almost harmonic maps with the perturbation term f in a critical integrability class (Zygmund class) $L^{\frac{n}{2}}\log^q L$, n is the dimension with $n \geq 3$. They prove that when $q > \frac{n}{2}$ the solution must be continuous and they can get continuity modulus estimates. As a byproduct of their method, they also study boundary continuity for the almost harmonic maps in high dimension.

Keywords Harmonic maps, Nonlinear elliptic PDE, Boundary regularity
2000 MR Subject Classification 58E20, 35B65, 35J60, 35S05, 35S15

1 Introduction

Let $B \subset \mathbf{R}^n$ be an open ball and (N, h) be a smooth Riemannian manifold which is compact and without boundary. We may assume that N is isometrically embedded into the Euclidean space $\mathbf{R}^m$ by the Nash's embedding theorem. Consider the Dirichlet functional

$$E(u) = \frac{1}{2}\int_B |\nabla u|^2 \mathrm{d}x.$$

Its critical points are called harmonic maps and satisfy the Euler-Lagrange equation

$$\triangle u + A(u)(\nabla u, \nabla u) = 0, \tag{1.1}$$

where A is the trace of second fundamental form of (N, h).

The study of regularity for harmonic maps has a long history which can be traced to Morrey [12] for two dimension case and to Schoen and Unlenbeck [22] for higher dimensions. In the two dimensional case, because of the conformal invariance property, the analysis for regularity of weak solutions of harmonic maps was pioneered by Hélein [5–6] who proved that every weakly harmonic map from a surface into a compact manifold is always smooth. Later, these results were extended to higher dimensions by Evans [3] for the target manifold which is a sphere, and Bethuel [1] for the general case, they proved partial regularity results for stationary harmonic maps by using similar ideas of Hélein. Recently, Rivière [19] found a new approach to study the regularity of the solution of conformally invariant two dimensional geometric

Manuscript received January 5, 2021. Revised January 20, 2022.
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variational problems, which include harmonic maps from two dimensional domain and the famous Hildebrandt's conjectures. In this new approach (see [19]), a key observation is that system (1.1) can also be written as the following more general form

$$-\Delta u = \Omega \cdot \nabla u, \tag{1.2}$$

where Ω is an antisymmetric matrix and (1.2) is called as Rivière's equation, please see [16] and [24] for more details. In a similar way, Rivière and Struwe [20] extended this method to high dimensional regularity of harmonic maps under the assumption of smallness of the solution in some homogeneous Morrey space. This method also has some other applications (see [8, 16, 26]). Another kind of elliptic systems sharing the structure like (1.2) are so called Dirac-harmonic map, which is inspired by the supersymmetric nonlinear sigma model from the quantum field theory, and is a natural and interesting extension of harmonic maps in an analytic literature. Related studies for regularity of Dirac-harmonic map are referred to [2, 29].

Almost harmonic maps (Approximation of harmonic map), mean harmonic maps with a perturbation (or a potential) term f:

$$-\Delta u = A(\nabla u, \nabla u) + f \tag{1.3}$$

in B, a bounded domain of $\mathbf{R}^n$. Here $f : B \to \mathbf{R}^m$ is a vector function in some suitable Euclidean space $\mathbf{R}^m$. Actually, to compare with (1.2), we can study more general elliptic systems as Rivière's equation by adding a potential term f:

$$-\Delta u = \Omega \cdot \nabla u + f, \tag{1.4}$$

where Ω is an antisymmetric one form valued matrix and belongs to L^2.

The study of almost harmonic maps, to our knowledge, comes from two aspects.

On one hand, from the definition of the harmonic maps, it is natural to find critical points of the Dirichlet energy. However, the classical variational methods cannot be used to the Dirichlet energy because $E(u)$ does not satisfy the Palais-Smale condition. Sacks and Uhlenbeck [21], Lamm [7] introduced a regularization of the Dirichlet energy to overcome this difficulty. Later, Lin and Wang [10–11] used a Ginzburg-Landau approximation to regularize the Dirichlet energy and proved the energy monotonicity formula in this case. In this paper, we consider the equation in a bounded domain B,

$$\Delta u + A(u)(\nabla u, \nabla u) = f.$$

The energy functional $F(u)$ of this Euler-Lagrange equation is

$$F(u) = E(u) + \int_B V(u)\mathrm{d}x$$

for some $V \in C^1(N)$, in this case, $f = \nabla V(u)$.

On the other hand, it is well-known that for the functional $E(u)$, the harmonic map heat flow is the L^2 gradient flow. The corresponding equation is

$$u_t - \Delta u = A(u)(\nabla u, \nabla u). \tag{1.5}$$

Due to its restriction of weak solution, we consider some class of weak solutions which satisfies an energy identity. Then it holds

$$E(u(t_2,\cdot)) + \int_{t_1}^{t_2}\int_B \Big|\frac{\partial u}{\partial t}\Big|^2 \mathrm{d}x\mathrm{d}t \le E(u(t_1,\cdot)).$$

If some solution of (1.5) satisfies this inequality and the initial data has finite energy, we have that almost any time slice satisfies

$$\Delta u + A(u)(\nabla u, \nabla u) \in L^2(B; R^n).$$

From this result, it is of our interest to study the almost harmonic maps with perturbation in the different spaces. Moser [14] considered the perturbation term $f \in L^p$, $p > \frac{n}{2}$, and proved Hölder continuity for weak solutions under a suitable smallness condition. Similarly, for the same case, Sharp and Topping [24] used a type of "geometric bootstrapping" and iteration method which can show that the solutions have regularity property in two dimension. Also in high dimensions, Sharp [23] used the coulomb gauge to show the improved regularity. For $p = \frac{n}{2}$, Moser [15] obtained an inequality in an Orlicz space belonging to a function with exponential growth. Later, the regularity results were extended to higher dimensions with $p \in (1,\infty)$, under an appropriate smallness condition, a certain degree of regularity follows in [16]. Li and Zhu [9] considered $f \in L\ln^+ L$ and proved the compactness of mapping from Riemannian surface with tension fields which are bounded in $L\ln^+ L$. Later, Sharp and Topping [24] extended the results of Li and Zhu and showed the stronger compactness results under the condition of f merely bounded in $L\ln L$.

For the almost harmonic maps in high dimensions, the proof holds always with the help of a suitable smallness conditions. We know that the well-known monotonicity formula (see [18]) can be applied to prove the stationary condition changing to smallness of the energy of solutions, this way would not have an influence on the expected results. In general, there is no monotonicity formula for the almost harmonic maps, however, Struwe [25] found that monotonicity formula can be viewed as a parabolic version for the harmonic map heat flow.

In this paper, we consider the regularity properties for the weak solutions of almost harmonic maps with perturbation in a critical Zygmund class, or specific Orlicz space $L^p\log^q L$. We show that we have this type of regularity which is similar to the regularity results of harmonic maps under suitable smallness conditions.

Our main results are as follows.

Theorem 1.1 *Let $u \in W^{1,2}(B,N)$ be a solution of almost harmonic map systems* (1.3) (*in the sense of distribution*). *B is a bounded domain of $\mathbf{R}^n$ and (N,h) is a compact Riemannian manifold with $f \in L^{\frac{n}{2}}\log^q L, q > \frac{n}{2}$. There exists $\varepsilon_0 > 0$ such that if $\|\nabla u\|_{M_{2,n-2}(B)} \le \varepsilon_0$, then u is continuous in the interior of B.*

We can also prove the following continuity regularity result up to the boundary.

Theorem 1.2 *Let $u \in W^{1,2}(B,N)$ be a solution of almost harmonic map systems* (1.3) (*in the sense of distribution*). *B is a bounded domain of $\mathbf{R}^n$ and (N,h) is a compact Riemannian manifold with $f \in L^{\frac{n}{2}}\log^q L, q > \frac{n}{2}$. Assume that the trace $u|_{\partial B} = \phi$ is continuous. There exists $\varepsilon_0 > 0$ such that whenever $\|\nabla u\|_{M_{2,n-2}(B)} \le \varepsilon_0$, u is continuous up to the boundary of B.*

Remark 1.1 Here the exponent $\frac{n}{2}$ is critical in the sense that even for linear equation, $\Delta u = f$, we cannot expect the continuity of the solution. In this critical exponent level, we consider continuity problem by assuming that f belongs to a Zygmund space $L^{\frac{n}{2}} \log^q L$. Indeed, we prove the related regularity result for system (1.4), see Theorems 2.1 and 3.1.

Throughout this paper, we use the convention of the summation. The standard Lebesgue spaces are denoted by $L^p(B)$ ($p \geq 1$ and B is a domain of $\mathbf{R}^n$). $B_r(x)$ denotes the ball of radius $r > 0$ around the center $x \in \mathbf{R}^n$ and $|B_r(x)|$ denotes Lebesgue measure (volume). The mean value of some function $f(x)$ over $B_r(x)$ is defined as

$$[f]_{B_r(x)} = \frac{1}{|B_r(x)|} \int_{B_r(x)} f(x).$$

Various constants arise in our paper unless indicated otherwise, they are always absolute constants. The symbol C denotes a generic constant and its value may change from line to line.

2 Interior Regularity and Proof of Theorem 1.1

At the analytical level, our motives, to derive the regularity property from the log part integrability factor q, come from the following improved Morrey lemma.

Lemma 2.1 *Suppose that $p \geq 1$ and $\alpha > 1$. There exists a constant C_0, depending only on n, α and A, such that the following holds. Suppose $u \in W^{1,p}(B_{2R}(x_0))$ satisfies*

$$\int_{B_r(x_1)} |\nabla u|^p \mathrm{d}V \leq A r^{n-p} \frac{1}{\log^{p\alpha} \frac{1}{r}} \tag{2.1}$$

for every $x_1 \in B_R(x_0)$ and $0 < r \leq R$. Then for almost all $y_1, y_2 \in B_R(x_0)$,

$$|u(y_1) - u(y_2)| \leq C_0 \Big(1 + \log \frac{1}{|y_1 - y_2|}\Big)^{1-\alpha}, \tag{2.2}$$

so u is continuous in $B_{\frac{R}{2}}(x_0)$.

Proof First noting that the Hölder inequality and (2.1) imply that

$$\int_{B_r(x_1)} |\nabla u| \mathrm{d}V \leq C r^{n-1} \frac{1}{\log^{\alpha} \frac{1}{r}} \tag{2.3}$$

is true. Now for any given pair of points $y_1, y_2 \in B_R(x_0)$, set $r_0 = \frac{1}{2}|y_1 - y_2|$ and $\overline{y} = \frac{1}{2}(y_1 + y_2)$. Then from (2.3) and using the Poincaré inequality, it holds

$$\begin{aligned}
&\Big|\frac{1}{|B_{r_0}(y_1)|} \int_{B_{r_0}(y_1)} u - \frac{1}{|B_{r_0}(y_2)|} \int_{B_{r_0}(y_2)} u\Big| \\
&\leq C r_0^{1-n} \int_{B_{2r_0}(\overline{y})} |\nabla u| \mathrm{d}V \leq C \frac{1}{\log^{\alpha} \frac{1}{2r_0}}.
\end{aligned} \tag{2.4}$$

Now letting $r_k = 2^{-k} r_0$, similarly we have

$$\Big|\frac{1}{|B_{r_k}(y_1)|} \int_{B_{r_k}(y_1)} u - \frac{1}{|B_{r_{k-1}}(y_1)|} \int_{B_{r_{k-1}}(y_1)} u\Big|$$

$$\leq Cr_{k-1}^{1-n}\int_{B_{r_{k-1}}(y_1)}|\nabla u| \leq C\frac{1}{\big((k-1)\log 2+\log\frac{1}{r_0}\big)^{\alpha}}, \tag{2.5}$$

and similar estimates hold for y_2 instead of y_1. By Lebesgue's differential theorem, we know that for almost every y_1 (and similarly for almost every y_2)

$$\frac{1}{|B_{r_k}(y_1)|}\int_{B_{r_k}(y_1)} u \to u(y_1) \quad \text{as } k\to\infty.$$

Then by summing up the above inequalities, we obtain

$$\begin{aligned}|u(y_1)-u(y_2)| &= \lim_{k\to\infty}\Big|\frac{1}{|B_{r_k}(y_1)|}\int_{B_{r_k}(y_1)} u - \frac{1}{|B_{r_k}(y_2)|}\int_{B_{r_k}(y_2)} u\Big| \\ &\leq C\sum_{k=1}^{\infty}\frac{1}{\big(k\log 2+\log\frac{1}{r_0}\big)^{\alpha}} \\ &\leq C\Big[1+\log\frac{1}{2r_0}\Big]^{1-\alpha}. \end{aligned}\tag{2.6}$$

This implies (2.2) and completes the proof.

Let $p\geq 1$ and $q\in\mathbf{R}$. We define the Orlicz norm as

$$\|f\|_{L^p\log^q L(\Omega)} = \inf\{\lambda>0 : [\lambda^{-1}f]_{L^p\log^q L(\Omega)}\leq 1\}$$

and

$$[f]_{L^p\log^q L(\Omega)} = \int_{\Omega}|f|^p\log^q(\mathrm{e}+|f|).$$

Lemma 2.2 *Let $f\in L^{\frac{n}{2}}\log^q L(B_R)$, $n>2$, $q\geq 0$ and $\varphi\in L^{\infty}(B_R)$. Then there exists $R_0>0$ such that when $0<R\leq R_0$, we have*

$$\int_{B_R} f\varphi \leq C\frac{R^{n-2}}{\big(1+\log\frac{1}{R}\big)^{\frac{2q}{n}}}\|f\|_{L^{\frac{n}{2}}\log^q L(B_R)}\|\varphi\|_{L^{\infty}}. \tag{2.7}$$

Proof By the well-known Hölder inequality for $L^p\log^q L$ space (or the duality of Orlicz space), we have

$$\begin{aligned}\int_{B_R}|f\varphi| &\leq C\|f\|_{L^{\frac{n}{2}}\log^q L(B_R)}\|\varphi\|_{L^{\frac{n}{n-2}}\log^{-\frac{2q}{n-2}}L(B_R)} \\ &\leq C\|\chi(B_R)\|_{L^{k\frac{n}{n-2}}\log^{-\frac{2q}{n-2}}L(B_R)}\|f\|_{L^{\frac{n}{2}}\log^q L(B_R)}\|\varphi\|_{L^{\infty}}. \end{aligned}\tag{2.8}$$

here $\chi(B_R)$ denotes the characteristics function of B_R. Now we look for the solution of the equation

$$t^{\frac{n}{n-2}}\log^{-\frac{2q}{n-2}}(\mathrm{e}+t) = \frac{1}{|B_R|}. \tag{2.9}$$

Let $y(t) = t^{\frac{n}{n-2}}\log^{-\frac{2q}{n-2}}(\mathrm{e}+t)$, then

$$y'(t) = t^{\frac{2}{n-2}}\log^{-\frac{2q}{n-2}}(\mathrm{e}+t)\Big[\frac{n}{n-2}-\frac{2q}{n-2}\log(\mathrm{e}+t)\frac{t}{\mathrm{e}+t}\Big],$$

it is easy to see that there exist numbers a, b, $0 < a < b < \infty$ such that when $t \notin [a,b]$, it holds that $y'(t) > 0$. Denote $y_0 = \sup\limits_{t\in[a,b]} y(t)$, and R_0 is chosen so that $\frac{1}{|B_{R_0}|} = y_0$. Then when $0 < R \le R_0$, (2.9) has only one solution t_0. We claim that there exist C_1 and C_2 such that

$$C_1\Big(\frac{1}{|B_R|}\Big)^{\frac{n-2}{n}} \log^{\frac{2q}{n}}\Big(\mathrm{e}+\frac{1}{|B_R|}\Big) \le t_0 \le C_2\Big(\frac{1}{|B_R|}\Big)^{\frac{n-2}{n}} \log^{\frac{2q}{n}}\Big(\mathrm{e}+\frac{1}{|B_R|}\Big). \tag{2.10}$$

Denoting by $t^* = \gamma\big(\frac{1}{|B_R|}\big)^{\frac{n-2}{n}} \log^{\frac{2q}{n}}\big(\mathrm{e}+\frac{1}{|B_R|}\big)$, γ is a positive number, then

$$y(t^*) = \gamma^{\frac{n}{n-2}}\Big(\frac{1}{|B_R|}\Big)\frac{\log^{\frac{2q}{n-2}}\big(\mathrm{e}+\frac{1}{|B_R|}\big)}{\log^{\frac{2q}{n-2}}\big(\mathrm{e}+\gamma\big(\frac{1}{|B_R|}\big)^{\frac{n-2}{n}}\log^{\frac{2q}{n}}\big(\mathrm{e}+\frac{1}{|B_R|}\big)\big)},$$

however it is easy to see that

$$\lim_{R\to 0}\frac{\gamma^{\frac{n}{n-2}}\log^{\frac{2q}{n-2}}\big(\mathrm{e}+\frac{1}{|B_R|}\big)}{\log^{\frac{2q}{n-2}}\big(\mathrm{e}+\gamma\big(\frac{1}{|B_R|}\big)^{\frac{n-2}{n}}\log^{\frac{2q}{n}}\big(\mathrm{e}+\frac{1}{|B_R|}\big)\big)} = \gamma^{\frac{n}{n-2}}\Big(\frac{n}{n-2}\Big)^{\frac{2q}{n-2}},$$

so by using the intermediate value theorem, we conclude (2.10) is true. Then by (2.8) and the definition of Orlicz norm for $\chi(B_R)$, we have

$$\int_{B_R}|f\varphi| \le C\frac{R^{n-2}}{\big(1+\log\frac{1}{R}\big)^{\frac{2q}{n}}}\|f\|_{L^{\frac{n}{2}}\log^q L(B_R)}\|\varphi\|_{L^\infty},$$

this completes the proof.

Let B be a bounded domain in $\mathbf{R}^n$, recall that a function $f \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ belongs to the space $BMO(B)$ if

$$\|f\|_{BMO} = \sup_{x_0\in B, r>0}\Big(\int_{B_r(x_0)\cap B}|f-f_{B_r(x_0)\cap B}|\mathrm{d}x\Big) < \infty.$$

We need the following lemma by Unlenbeck [28] or Rivière and Struwe [20] (optimal gauge transformation).

Lemma 2.3 *There exist $\varepsilon(n) > 0$ and $C(n)$ such that, for every $\Omega = (\Omega_{\alpha\beta})_{1\le\alpha,\beta\le m}$ in $L^2(B_1, so(n)\otimes\mathbf{R}^m)$ satisfying*

$$\|\Omega\|_{M^{2,n-2}(B)} < \varepsilon(n),$$

there exist $\xi \in W^{1,2}(B_1, so(n)\otimes\Lambda^{n-2}\mathbf{R}^n)$ and $P \in W^{1,2}(B_1, SO(n))$ such that

1)

$$P^{-1}\Omega P + P^{-1}\mathrm{d}P = *\mathrm{d}\xi, \tag{2.11}$$

2)

$$\xi = 0 \quad \textit{on } \partial B_1,$$

3)

$$\|\nabla\xi\|_{M^{2,n-2}(B_1)} + \|\nabla P\|_{M^{2,n-2}(B_1)} \le C(n)\|\Omega\|_{M^{2,n-2}(B_1)}. \tag{2.12}$$

First we notice that the harmonic map equation (1.3) can be rewritten as the elliptic system (1.4), the details of these deductions we refer to [20] for codimension one case and [16] for general case. So we just focus on proving the regularity of elliptic system (1.4).

We have the following result.

Theorem 2.1 *Let $u \in W^{1,2}(B, N)$ be a weak solution of* (1.4). *B is a bounded domain of $\mathbf{R}^n$ and (N, h) is a compact Riemannian manifold with $\nabla u \in M^{2,n-2}(B)$ and $f \in L^{\frac{n}{2}} \log^q L(B)$, $q > \frac{n}{2}$. There exists $\varepsilon_0 > 0$ such that if*

$$\sup_{B_r(x)\subset B} \frac{1}{r^{n-2}} \int_{B_r(x)} |\Omega|^2 \le \varepsilon_0,$$

then u is continuous in the interior of B.

Proof The proof will be divided into several steps. Since the regularity is a local property, we assume for simplicity that $(B, g) = (B_1, g_0)$, where $B_1 \subset \mathbf{R}^n$ is the unit ball with the standard Euclidean metric g_0 in $\mathbf{R}^n$. Let u be a weak solution of (1.4),

$$-\mathrm{div}(\nabla u^\alpha) = \sum_\beta (\Omega^\alpha_\beta) \cdot \nabla u^\beta + f^\alpha, \tag{2.13}$$

here (Ω^α_β) is an $(m \times m)$ 1-form valued antisymmetric matrix. Let $x_0 \in B_1$ and $R_0 > 0$ such that $B_{R_0}(x_0) \subset B_1$. For any $z \in B_1$ and $R > 0$ with $B_{2R}(z) \subset B_{R_0}(x_0)$, and $0 < r < R$, applying the optimal gauge transformation on $B_R(z)$, then the Hodge decomposition implies

$$P^{-1}\mathrm{d}u = \mathrm{d}F + *\mathrm{d}G + h,$$

where $F \in W^{1,2}_0(B_R(z))$ and $G \in W^{1,2}_0(B_R(z), \mathbf{R}^m \otimes \Lambda^{n-2}\mathbf{R}^n)$ and with a harmonic 1-form $h \in L^2(B_R(z), \mathbf{R}^m \otimes \Lambda^1\mathbf{R}^n)$. From (2.13), it is easy to see that

$$\begin{cases} -\Delta F = -\mathrm{div}(P^{-1}\nabla u) = *\mathrm{d}\xi \cdot P^{-1}\mathrm{d}u + P^{-1}f, \\ F|_{\partial B_R(z)} = 0 \end{cases}$$

and

$$\begin{cases} -\Delta G = *\mathrm{d}(P^{-1}\mathrm{d}u) = *(\mathrm{d}P^{-1} \wedge \mathrm{d}u), \\ G|_{\partial B_R(z)} = 0. \end{cases} \tag{2.14}$$

Fix a number $1 < p < \frac{n}{n-1}$ and let $p' > n$ be the conjugate exponent. Now we estimate the L^p norm of ∇F, ∇G and ∇h, respectively. The estimate for G is similar to [20]. By duality one has

$$\|\mathrm{d}G\|_{L^p} \le C \sup_{\substack{\varphi \in W^{1,p'}_0(B_R(z); \wedge^{n-2}\mathbf{R}^{n-2}) \\ \|\mathrm{d}\varphi\|_{L^{p'}} \le 1}} \int_{B_R(z)} \langle \mathrm{d}G, \mathrm{d}\varphi \rangle.$$

Note that $W^{1,p'}_0(B_R(z)) \hookrightarrow C^{1-\frac{2}{p'}}(B_R(z))$ and for $\varphi \in W^{1,p'}_0(B_R)$ there holds

$$\|\varphi\|_{L^\infty} \le CR^{1-\frac{n}{p'}} \|\varphi\|_{W^{1,p'}(B_R(z))}, \tag{2.15}$$

and also the Hölder inequality implies

$$\|\nabla\varphi\|_{L^2(B_R(z))} \le CR^{\frac{n}{p}-\frac{n}{2}} \|\nabla\varphi\|_{L^{p'}(B_R(z))}. \tag{2.16}$$

Then using integration by parts and the duality of Hardy space $\mathcal{H}$ and BMO space, we have

$$\begin{aligned}\int_{B_R(z)}\langle \mathrm{d}G, \mathrm{d}\varphi\rangle &= -\int_{B_R(z)}\varphi\Delta G\\ &= \int_{B_R(z)}\varphi \mathrm{d}P^{-1}\wedge \mathrm{d}u\\ &= -\int_{B_R(z)}\mathrm{d}P^{-1}\wedge \mathrm{d}\varphi(u-u_{B_R(z)})\\ &\le C\|\mathrm{d}P\|_{L^2}\|\mathrm{d}\varphi\|_{L^{p'}}\|u-u_{B_R(z)}\|_{L^s}\\ &\le CR^{\frac{n}{p}-1}\varepsilon_0\|u\|_{BMO},\end{aligned} \tag{2.17}$$

here $u_{B_R(z)}$ represents the average of u over $B_R(z)$, i.e., $u_{B_R(z)} = \frac{1}{|B_R(z)|}\int_{B_R(z)} u$, $\varepsilon(n)$ is the same number as in (2.3) and s satisfies $\frac{1}{2}+\frac{1}{p'}+\frac{1}{s}=1$. Hence

$$\|G\|_{W^{1,p}(B_R(z))} \le C\varepsilon_0 R^{\frac{n}{p}-1}\|u\|_{BMO(B_R(z))}. \tag{2.18}$$

Now we proceed to estimate $\|F\|_{W^{1,p}(B_R(z))}$. Using duality again, one has

$$\|F\|_{W^{1,p}(B_R(z))} \le C\sup_{\|\varphi\|_{L^{p'}}\le 1}\int_{B_R(z)}\langle \mathrm{d}F, \mathrm{d}\varphi\rangle,$$

then from (2.14),

$$\begin{aligned}&\int_{B_R(z)}\langle \mathrm{d}F, \mathrm{d}\varphi\rangle\\ &= -\int_{B_{R(z)}}\varphi\Delta F\\ &= \int_{B_{R(z)}}\varphi \mathrm{d}\xi\wedge P^{-1}\mathrm{d}u + \int_{B_{R(z)}}P^{-1}f\varphi\\ &= \mathrm{I}+\mathrm{II}.\end{aligned} \tag{2.19}$$

By using the integration by parts and combining with (2.12), (2.15) and (2.16),

$$\begin{aligned}\mathrm{I} &= \int_{B_{R(z)}}\varphi \mathrm{d}\xi\wedge P^{-1}\mathrm{d}u\\ &= \int_{B_{R(z)}}(u-u_{B_R(z)})\mathrm{d}\xi\wedge \mathrm{d}(\varphi P^{-1})\\ &\le C\|\mathrm{d}\xi\wedge \mathrm{d}(\varphi P^{-1})\|_{\mathcal{H}^1}\|u\|_{BMO(B_R(z))}\\ &\le C\|\mathrm{d}\xi\|_{L^2}(\|\mathrm{d}P\|_{L^2}\|\varphi\|_{L^\infty}+\|\mathrm{d}\varphi\|_{L^2})\|u\|_{BMO}\\ &\le C\varepsilon_0 R^{\frac{n}{p}-1}\|u\|_{BMO(B_R(z))},\end{aligned} \tag{2.20}$$

and Lemma 2.2 implies that

$$\begin{aligned}\mathrm{II} &= \int_{B_{R(z)}}P^{-1}f\varphi\\ &\le C\frac{R^{n-1-\frac{n}{p'}}}{\left(1+\log\frac{1}{R}\right)^{\frac{2q}{n}}}\|f\|_{L^{\frac{n}{2}}\log^q L(B_R(z))}\end{aligned}$$

$$= C\frac{R^{\frac{n}{p}-1}}{\left(1+\log\frac{1}{R}\right)^{\frac{2q}{n}}}\|f\|_{L^{\frac{n}{2}}\log^{q}L(B_{R_0}(x_0))}. \tag{2.21}$$

Finally we establish the estimate for the harmonic 1-form h in a standard way by using the Campanato estimates result for harmonic functions of Giaguinta, see [4, Theorem 2.1 on p. 78], which yields for any $0 < r \le R$,

$$\int_{B_r(z)}|h|^p \le C\left(\frac{r}{R}\right)^n\int_{B_R(z)}|h|^p. \tag{2.22}$$

Then combining (2.18), (2.20) and (2.21) together, we obtain

$$\begin{aligned}\int_{B_r(z)}|\nabla u|^p &\le C_p\int_{B_r(z)}|h|^p + C_p\int_{B_r(z)}(|\mathrm{d}f|^p+|\mathrm{d}g|^p)\\ &\le C\left(\frac{r}{R}\right)^n\int_{B_R(z)}|h|^p + C_p\int_{B_R(z)}(|\mathrm{d}f|^p+|\mathrm{d}g|^p)\\ &\le C\left(\frac{r}{R}\right)^n\int_{B_R(z)}|\nabla u|^p\mathrm{d}V + C\varepsilon_0R^{n-p}\|u\|^p_{BMO(B_R(z))}\\ &\quad + C\frac{R^{n-p}}{\left(1+\log\frac{1}{R}\right)^{\frac{2pq}{n}}}\|f\|^p_{L^{\frac{n}{2}}\log^{q}L(B_{R_0}(x_0))}.\end{aligned} \tag{2.23}$$

Multiplying by r^{p-n} and, for brevity, denoting by

$$\Phi(z,r) = \frac{1}{r^{n-p}}\int_{B_r(z)}|\nabla u|^p$$

and

$$\|\nabla u\|^p_{M_{p,n-p}(B_r(x_0))} = \sup_{\substack{z\in B_r(x_0)\\ \rho<r-|z-x_0|}}\frac{1}{\rho^{n-p}}\int_{B_\rho(z)}|\nabla u|^p,$$

then (2.23) implies that

$$\begin{aligned}\Phi(z,r) &\le C\left(\frac{r}{R}\right)^p\Phi(z,R) + C\left(\frac{r}{R}\right)^{p-n}\varepsilon_0\|u\|^p_{BMO}\\ &\quad + C\left(\frac{r}{R}\right)^{p-n}\frac{1}{\left(1+\log\frac{1}{R}\right)^{\frac{2pq}{n}}}\|f\|^p_{L^{\frac{n}{2}}\log^{q}L(B_{R_0}(x_0))}.\end{aligned} \tag{2.24}$$

On the other hand, we can estimate the term $\|u\|_{BMO(B_R(z))}$ by a well-known fact, see [20],

$$\sup_z\|u\|_{BMO(B_R(z))} \le C\|\nabla u\|_{M_{p,n-p}(B_{R_0}(x_0))}. \tag{2.25}$$

Hence we obtain

$$\begin{aligned}\Phi(z,r) &\le C\left(\frac{r}{R}\right)^p\Phi(z,R) + C\left(\frac{r}{R}\right)^{p-n}\varepsilon_0\|\nabla u\|^p_{M_{p,n-p}(B_{R_0}(x_0))}\\ &\quad + C\left(\frac{r}{R}\right)^{p-n}\frac{1}{\left(1+\log\frac{1}{R}\right)^{\frac{2pq}{n}}}\|f\|^p_{L^{\frac{n}{2}}\log^{q}L(B_{R_0}(x_0))}.\end{aligned} \tag{2.26}$$

Now, pick some $0 < \theta_0 < \frac{1}{2}$ to be fixed later and set $r = \theta_0 R$, from (2.26) we have

$$\begin{aligned}\Phi(z, \theta_0 R) \le C_1 \theta_0^p (1 + \theta_0^{-n}\varepsilon_0) \|\nabla u\|^p_{M_{p,n-p}(B_{R_0}(x_0))} \\ + C\theta_0^{p-n} \frac{1}{\big(1 + \log \frac{1}{R}\big)^{\frac{2pq}{n}}} \|f\|^p_{L^{\frac{n}{2}} \log^q L(B_{R_0}(x_0))}.\end{aligned} \tag{2.27}$$

Choosing θ_0 small enough to ensure $C_1\theta_0^p \le \frac{1}{4}$, and then choosing ε_0 small enough such that $\varepsilon_0 \le \theta_0^n$, we get the estimate

$$\begin{aligned}\Phi(z, \theta_0 R) \le \frac{1}{2} \|\nabla u\|^p_{M_{p,n-p}(B_{R_0}(x_0))} \\ + C(p, \theta_0) \frac{1}{\big(1 + \log \frac{1}{R}\big)^{\frac{2pq}{n}}} \|f\|^p_{L^{\frac{n}{2}} \log^q L(B_{R_0}(x_0))}\end{aligned} \tag{2.28}$$

for all $z \in B_1$ and $R > 0$ with $B_{2R}(z) \subset B_{R_0}(x_0) \subset B_1$. Taking the supremum over all those z, r such that $B_r(z) \subset B_{\frac{\theta_0}{2} R_0}(x_0)$, then from (2.28) we obtain

$$\begin{aligned}\|\nabla u\|^p_{M_{p,n-p}(B_{\frac{\theta_0}{2} R_0}(x_0))} \le \frac{1}{2} \|\nabla u\|^p_{M_{p,n-p}(B_{R_0}(x_0))} \\ + C(p, \theta_0) \frac{1}{\big(1 + \log \frac{2}{R_0}\big)^{\frac{2pq}{n}}} \|f\|^p_{L^{\frac{n}{2}} \log^q L(B_{R_0}(x_0))}\end{aligned} \tag{2.29}$$

for all x_0, R_0 with $B_{R_0}(x_0) \subset B_1$.

Now we proceed to get a decay type estimate by using a standard iteration tricks. For brevity denoting by $\frac{\theta_0}{2} = \sigma_0$, for any $0 < r < R_0$, let $i \in \mathbf{N}$ (the set of all natural numbers) be chosen such that

$$\sigma_0^{i+1} R_0 < r \le \sigma_0^i R_0$$

is satisfied. Hence from (2.29) and the monotonicity property of $\|\nabla u\|^p_{M_{p,n-p}(B_r(x_0))}$ with the variable r, it holds

$$\begin{aligned}\|\nabla u\|^p_{M_{p,n-p}(B_r(x_0))} &\le \Big(\frac{1}{2}\Big)^i \|\nabla u\|^p_{M_{p,n-p}(B_{\sigma_0^i R_0}(x_0))} \\ &\quad + C\|f\|^p_{L^{\frac{n}{2}} \log^q L(B_{R_0}(x_0))} \sigma^i_{l=1} \frac{1}{\big(1 + \log \frac{1}{\sigma_0^{i-l} R_0}\big)^{\frac{2pq}{n}}} \Big(\frac{1}{2}\Big)^{l-1} \\ &\le 2(\sigma_0^\alpha)^{i+1} \|\nabla u\|^p_{M_{p,n-p}(B_{R_0}(x_0))} \\ &\quad + C \frac{1}{\big(1 + \log \frac{1}{R_0}\big)^{\frac{2pq}{n}}} \|f\|^p_{L^{\frac{n}{2}} \log^q L(B_{R_0}(x_0))} \\ &\le 2\Big(\frac{r}{R_0}\Big)^\alpha \|\nabla u\|^p_{M_{p,n-p}(B_{R_0}(x_0))} \\ &\quad + C \frac{1}{\big(1 + \log \frac{1}{R_0}\big)^{\frac{2pq}{n}}} \|f\|^p_{L^{\frac{n}{2}} \log^q L(B_{R_0}(x_0))},\end{aligned} \tag{2.30}$$

here the positive number α is chosen as $\alpha = \frac{\log \frac{1}{\sigma_0}}{\log 2} > 0$.

With this decay estimate (2.30) of the Morrey norm $\|\nabla u\|^p_{M_{p,n-p}(B_{R_0}(x_0))}$ in hand, finally we can conclude the continuity of the solution of (1.3) or (1.4). We cannot use the Morrey lemma or Lemma 2.1 directly, because the right hand side of (2.30) consists of two ingredients with different scales. However, this can be handled by the same argument as in the proof of Lemma 2.1. For any given pair of points $x_1, x_2 \in B_{\frac{1}{2}}$, set $r_0 = \frac{1}{2}|x_1 - x_2|$ and $\overline{x} = \frac{1}{2}(x_1 + x_2)$. Then from (2.30) and using the Poincaré inequality we have

$$\begin{aligned}
&\Big|\frac{1}{|B_{r_0}(x_1)|}\int_{B_{r_0}(x_1)} u - \frac{1}{|B_{r_0}(x_2)|}\int_{B_{r_0}(x_2)} u\Big| \\
&\le C r_0^{1-n}\int_{B_{2r_0}(\overline{x})} |\nabla u| \\
&\le C\Big(r_0^{p-n}\int_{B_{2r_0}(\overline{x})} |\nabla u|^p\Big)^{\frac{1}{p}} \\
&\le 2C\Big(\frac{1}{2}\Big)^{\frac{\alpha}{p}} \|\nabla u\|_{M_{p,n-p}(B_{4r_0}(\overline{x}))} \\
&\quad + C\frac{1}{\big(1+\log\frac{1}{2r_0}\big)^{\frac{2q}{n}}}\|f\|_{L^{\frac{n}{2}}\log^q L(B_{4r_0}(\overline{x}))}.
\end{aligned} \tag{2.31}$$

Similarly letting $r_k = 2^{-k} r_0$, we have

$$\begin{aligned}
&\Big|\frac{1}{|B_{r_k}(x_1)|}\int_{B_{r_k}(x_1)} u - \frac{1}{|B_{r_{k-1}}(x_1)|}\int_{B_{r_{k-1}}(x_1)} u\Big| \\
&\le C\Big(r_{k-1}^{p-n}\int_{B_{r_{k-1}}(x_1)} |\nabla u|^p\Big)^{\frac{1}{p}} \\
&\le 2C\Big(\frac{1}{2}\Big)^{\frac{k\alpha}{p}} \|\nabla u\|_{M_{p,n-p}(B_{4r_0}(\overline{x}))} \\
&\quad + C\frac{1}{\big((k-1)\log 2+\log\frac{1}{r_0}\big)^{\frac{2q}{n}}}\|f\|_{L^{\frac{n}{2}}\log^q L(B_{4r_0}(\overline{x}))},
\end{aligned} \tag{2.32}$$

and similar estimates hold for x_2 instead of x_1. By Lebesgue's differential theorem we know that for almost every x_1 (and similarly for almost every x_2)

$$\frac{1}{|B_{r_k}(x_1)|}\int_{B_{r_k}(x_1)} u \to u(x_1) \quad \text{as } k\to\infty.$$

Then summing up above inequalities together and using (2.30) again, we obtain

$$\begin{aligned}
|u(x_1)-u(x_2)| &= \lim_{k\to\infty}\Big|\frac{1}{|B_{r_k}(x_1)|}\int_{B_{r_k}(x_1)} u - \frac{1}{|B_{r_k}(x_2)|}\int_{B_{r_k}(x_2)} u\Big| \\
&\le C\|\nabla u\|_{M_{p,n-p}(B_{4r_0}(\overline{x}))}\sum_{k=1}^{\infty}\Big(\frac{1}{2}\Big)^{\frac{k\alpha}{p}} \\
&\quad + C\|f\|_{L^{\frac{n}{2}}\log^q L(B_{4r_0}(\overline{x}))}\sum_{k=1}^{\infty}\frac{1}{\big(k\log 2+\log\frac{1}{r_0}\big)^{\frac{2q}{n}}} \\
&\le C(\alpha,p)\|\nabla u\|_{M_{p,n-p}(B_{4r_0}(\overline{x}))} \\
&\quad + C\Big[1+\log\frac{1}{2r_0}\Big]^{1-\frac{2q}{n}}\|f\|_{L^{\frac{n}{2}}\log^q L(B_1)}
\end{aligned}$$

$$\begin{aligned}&\le C\|\nabla u\|_{M_{2,n-2}(B_{R_1}(\overline{x}))}\Big(\frac{r_0}{R_1}\Big)^{\alpha}\\&\quad+C\Big[1+\log\frac{1}{2r_0}\Big]^{1-\frac{2q}{n}}\|f\|_{L^{\frac{n}{2}}\log^q L(B_1)}.\end{aligned}\tag{2.33}$$

(2.33) is satisfied for all x_1 $x_2\in B_1$, $4r_0<R_1$, and $B_{R_1}(\bar{x})\subset B_1$. Because $\frac{2q}{n}>1$, (2.33) implies that u is continuous on $B_{\frac{1}{2}}$ and this completes the proof.

3 Continuity Estimate up to the Boundary and Proof of Theorem 1.2

In this section, we establish the regularity of the solution of (1.3) or (1.4) up to the boundary. To derive the continuity of u up to boundary we first need to get the following variant Dirichlet type growth theorem, which gives an appropriate estimate for the modulus of continuity for u.

Proposition 3.1 *Let $u\in W^{1,2}(B_1,N)$ be a solution of* (1.3) *or* (1.4) (*in the sense of distribution*). *B_1 is the unit ball of $\mathbf{R}^n$ and (N,h) is a compact Riemannian manifold with $f\in L^{\frac{n}{2}}\log^q L(B_1), q>\frac{n}{2}$. Let $x_0\in B_1$, $0<r_0<1$ such that $B_{r_0}(x_0)\subset B_1$ and $1<p<\frac{n}{n-1}$, then there exists $\varepsilon_0>0$ such that if $\nabla u\in M_{2,n-2}(B_1)$ and $\|\Omega\|_{M_{2,n-2}}\le\varepsilon_0$, the inequality*

$$|u(x)-u(y)|\le C(n,q)(\|\nabla u\|_{M_{p,n-p}(B_{r_0}(x_0))}+\|f\|_{L^{\frac{n}{2}}\log^q L(B_{r_0}(x_0))})\tag{3.1}$$

holds true for any x $y\in B_{\frac{r_0}{2}}(x_0)$.

Remark 3.1 Indeed, we can prove this result by using similar argument as in the proof of (2.30) by combining with the argument for the proof of Lemma 2.1, however, here we use a technique from Morrey in [13], which is also used by Müller and Schikorra [17] for proving the boundary regularity result for similar problem in the two-dimensional case.

Proof For any $z\in B_{r_0}(x_0)$, we have

$$|u(x)-u(y)|\le|u(x)-u(z)|+|u(y)-u(z)|,$$

hence

$$\begin{aligned}&|u(x)-u(y)|\\&\le\frac{1}{|B_{r_0}(x_0)|}\int_{B_{r_0}(x_0)}|u(x)-u(z)|\mathrm{d}z+\frac{1}{|B_{r_0}(x_0)|}\int_{B_{r_0}(x_0)}|u(y)-u(z)|\mathrm{d}z.\end{aligned}$$

Denoting by $x_t=x+t(x_0-x)$ for $t\in[0,1]$, a direct calculation and using (2.30) we obtain

$$\begin{aligned}&\frac{1}{|B_{r_0}(x_0)|}\int_{B_{r_0}(x_0)}|u(x)-u(z)|\mathrm{d}z\\&\le C\frac{1}{r_0^{n-1}}\int_{B_{r_0}(x_0)}\int_0^1|\nabla u(x+t(z-x))|\mathrm{d}t\mathrm{d}z\\&\le C\frac{1}{r_0^{n-1}}\int_0^1\Big(\int_{B_{r_0}(x_0)}|\nabla u(x+t(z-x))|^p\mathrm{d}z\Big)^{\frac{1}{p}}r_0^{\frac{n(p-1)}{p}}\mathrm{d}t\\&\le C\int_0^1 r_0^{1-\frac{n}{p}}t^{-\frac{n}{p}}\Big(\int_{B_{r_0t}(x_t)}|\nabla u|^p\Big)^{\frac{1}{p}}\mathrm{d}t\end{aligned}$$

$$
\begin{aligned}
&\le C\int_0^1 r_0^{1-\frac{n}{p}}t^{-\frac{n}{p}}(r_0t)^{\frac{n-p}{p}}\|\nabla u\|_{M_{p,n-p}(B_{r_0t}(x_t))}\mathrm{d}t\\
&= C\int_0^1 t^{-1}\|\nabla u\|_{M_{p,n-p}(B_{r_0t}(x_t))}\mathrm{d}t\\
&\le C\int_0^1 t^{-1}\Big(\frac{r_0t}{2r_0t^\gamma}\Big)^{\frac{\alpha}{p}}\|\nabla u\|_{M_{p,n-p}(B_{r_0}(x_t))}\mathrm{d}t\\
&\quad + C\int_0^1 t^{-1}\frac{1}{\big(1+\log\frac{1}{2r_0t^\gamma}\big)^{\frac{2q}{n}}}\|f\|_{L^{\frac{n}{2}}\log^q L(B_{2r_0t^\gamma}(x_t))}\mathrm{d}t\\
&\le C\int_0^1 t^{-1+\frac{\alpha}{p}(1-\gamma)}\mathrm{d}t\|\nabla u\|_{M_{2,n-2}(B_{2r_0t^\gamma}(x_0))}\\
&\quad + C\|f\|_{L^{\frac{n}{2}}\log^q L(B_{2r_0}(x_0))}\int_0^1 t^{-1}\frac{1}{\big(1+\log\frac{1}{2r_0t^\gamma}\big)^{\frac{2q}{n}}}\mathrm{d}t\\
&\le C\|\nabla u\|_{M_{p,n-p}(B_{2r_0}(x_0))} + C\frac{1}{\big(1+\log\frac{1}{2r_0}\big)^{\frac{2q}{n}-1}}\|f\|_{L^{\frac{n}{2}}\log^q L(B_{2r_0}(x_0))},
\end{aligned}
$$

here we choose $0<\gamma<1$, so $r_0t\le 2r_0t^\gamma$ for all $t\in[0,1]$, which implies (3.1).

Now we are in a position to give the proof of Theorem 1.2. Similarly as for interior regularity, we prove the following theorem for system (1.4), then Theorem 1.2 can be deduced as an application.

Theorem 3.1 *Let $u\in W^{1,2}(B_1,N)$ be a solution of almost harmonic map system* (1.4) (*in the sense of distribution*). *B_1 is the unit ball of $\mathbf{R}^n$ and (N,h) is a compact Riemannian manifold, $f\in L^{\frac{n}{2}}\log^q L(B_1), q>\frac{n}{2}$. Assume that $\nabla u\in M_{2,n-2}(B_1)$ and the trace $u|_{\partial B_1}=\phi$ is continuous. Then there exists $\varepsilon_0>0$ such that if for all $x\in B_1$,*

$$
\sup_{B_r(x)\cap B}\frac{1}{r^{n-2}}\int_{B_r(x)\cap B}|\Omega|^2\le\varepsilon_0,
$$

then u is continuous up to the boundary of B_1.

Proof We use the spherical coordinate, for any $x\in B_1$ denoting $x=(r,\Theta)$, $r=|x|$ and $\Theta=\frac{x}{|x|}\in S^{n-1}$, $r\in[0,1]$. Let

$$
u(x)=u(r,\Theta).
$$

By assumption of theorem, representation of the trace $u|_{\partial B_1}=\phi(\Theta)$ is continuous. Let us fix $x_0=\Theta_0\in\partial B_1$ and let $x_1=(r_1,\Theta_1)$ be an interior point in B_1. Let $x^*=(r_1,\Theta^*)\in B_{\delta^2}(x_1)$, where Θ^* will be chosen later and $\delta=1-r_1$. Denoting by

$$
x_P^*=\frac{x^*}{|x^*|}=\Theta^*\in\partial B_1,
$$

then we have

$$
\begin{aligned}
|u(x_1)-\phi(x_0)| &\le |u(x_1)-u(x^*)|+|u(x^*)-\phi(x_P^*)|+|\phi(\Theta^*)-\phi(x_0)|\\
&= \mathrm{I}+\mathrm{II}+\mathrm{III}.
\end{aligned}
$$

It is easy to see that for small enough δ and small $|\Theta_0 - \Theta_1|$, the term III becomes small. From Proposition 3.1 and (2.30), we have

$$\begin{aligned}
\mathrm{I} &= |u(x_1) - u(x^*)| \\
&\leq C(\|\nabla u\|_{M_{p,n-p}(B_{2\delta^2}(x_1))} + \|f\|_{L^{\frac{n}{2}}\log^q L(B_{2\delta^2}(x_1))}) \\
&\leq C\Big(\delta^\alpha \|\nabla u\|_{M_{2,n-2}(B_\delta(x_1))} + \frac{1}{\left(1+\log\frac{1}{\delta}\right)^{\frac{2q}{n}-1}}\|f\|_{L^{\frac{n}{2}}\log^q L(B_1)}\Big),
\end{aligned}$$

this implies that, for small $\delta = 1 - r_1$, the term I also becomes small. Now we prove that II is small when δ is small enough. First we need to choose specific point $x^* = (r_1, \Theta^*)$, which can be chosen in a similar way as for the two-dimensional case elliptic systems boundary regularity problem, see for example [27]. Denoting by $E(\delta) = \int_{1-\delta\leq|x|\leq 1} |\nabla u|^2$, it is easy to see that

$$\int_{S^{n-1}}\int_{1-\delta}^1 |u_r|^2 r^{n-1}\mathrm{d}r\mathrm{d}\Theta \leq E(\delta),$$

so we have

$$\int_{S^{n-1}}\int_{1-\delta}^1 |u_r|^2 \mathrm{d}r\mathrm{d}\Theta \leq \frac{E(\delta)}{(1-\delta)^{n-1}}. \tag{3.2}$$

Hence for any positive number $0 < \eta < \omega_{n-1} = \int_{S^{n-1}} \mathrm{d}\Theta$, we argue that there exists a set $U(\eta) \subset S^{n-1}$ with positive $(n-1)$-dimensional Lebesgue measure satisfying

$$\int_{1-\delta}^1 |u_r|^2(r, \Theta^\#)\mathrm{d}r \leq \frac{E(\delta)}{(1-\delta)^{n-1}\eta} \tag{3.3}$$

for all $\Theta^\# \in U(\eta)$. This can be done by contradiction. Setting $G(\Theta) = \int_{1-\delta}^1 |u_r|^2(r,\Theta)\mathrm{d}r$ and $U(\eta) = \big\{\Theta \in S^{n-1}, G(\Theta) \leq \frac{E(\delta)}{(1-\delta)^{n-1}\eta}\big\}$, otherwise, we have

$$\begin{aligned}
\int_{\{S^{n-1}\setminus U(\eta)\}}\int_{1-\delta}^1 |u_r|^2 \mathrm{d}r\mathrm{d}\Theta &> \frac{E(\delta)}{(1-\delta)^{n-1}\eta}\omega_{n-1} \\
&> \frac{E(\delta)}{(1-\delta)^{n-1}},
\end{aligned}$$

this contradicts with (3.2). Hence we have

$$\begin{aligned}
\mathrm{II} &= |u(x^*) - \phi(x_P^*)| \\
&= |u(x^*) - u(x_P^*)| \\
&\leq \int_{r_1}^1 |u_r(r,\Theta^*)|\mathrm{d}r \\
&\leq \delta^{\frac{1}{2}}\Big(\int_{r_1}^1 |u_r(r,\Theta^*)|^2\mathrm{d}r\Big)^{\frac{1}{2}} \\
&\leq \delta^{\frac{1}{2}}\Big(\frac{E(\delta)}{(1-\delta)^{n-1}\eta}\Big)^{\frac{1}{2}}.
\end{aligned} \tag{3.4}$$

Setting $\eta = \frac{\delta}{4}$ and choosing $\Theta^* \in U\big(\frac{\delta}{4}\big)$, from (3.4) we obtain that

$$\mathrm{II} \leq 2\Big(\frac{E(\delta)}{(1-\delta)^{n-1}}\Big)^{\frac{1}{2}} \to 0 \quad \text{as } \delta \to 0,$$

and this completes the proof.

Remark 3.2 When dimension is two, similar problems for the study of (1.3) and (1.4) were investigated widely, for example see [9, 17, 23], however we cannot use our results directly to the case dim $= 2$, this is due to that in Lemma 2.2 we need the condition $\frac{n}{2} > 1$. We study the regularity up to boundary and its global compactness properties of (1.4) in two dimensional case in a forthcoming paper.

Acknowledgement The authors would like to thank Prof. Congming Li for his encouragements and valuable suggestion.

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