Chin. Ann. Math. Ser. B
43(4), 2022, 601–620
DOI: 10.1007/s11401-022-0348-z

Chinese Annals of Mathematics, Series B

Translating Solutions of the Nonparametric Mean Curvature Flow with Nonzero Neumann Boundary Data in Product Manifold $M^n \times \mathbb{R}$ *

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Abstract In this paper, the authors can prove the existence of translating solutions to the nonparametric mean curvature flow with nonzero Neumann boundary data in a prescribed product manifold $M^n \times \mathbb{R}$, where M^n is an n-dimensional ($n \geq 2$) complete Riemannian manifold with nonnegative Ricci curvature, and $\mathbb{R}$ is the Euclidean 1-space.

Keywords Translating solutions, Singularity, Nonparametric mean curvature flow, Convexity, Ricci curvature.
2020 MR Subject Classification 53C42, 53B20, 35B50, 35K93

1 Introduction

The mean curvature flow (MCF for short) is one of the most important extrinsic curvature flows and has many nice applications. For instance, by using the curve shortening flow (i.e., the lower-dimensional case of MCF), Topping [21] successfully gave an isoperimetric inequality on simply connected surfaces with Gaussian curvature satisfying some integral precondition. This result extends those isoperimetric inequalities (introduced in detail in, e.g., [6, 19]) obtained separately by Alexandrov, Fiala-Huber, Bol, and Bernstein-Schmidt. Applying the long-time existence and convergence conclusions of graphic MCF of any codimension in prescribed product manifolds (see [23]), Wang [24] showed that for a bounded C^2 convex domain $\mathbb{D}$ (with diameter δ and boundary $\partial\mathbb{D}$) in the Euclidean n-space $\mathbb{R}^n$ and $\phi : \partial\mathbb{D} \to \mathbb{R}^m$ a continuous map, there exists a map $\psi : \mathbb{D} \to \mathbb{R}^m$, with $\psi|_{\partial\mathbb{D}} = \phi$ and with the graph of ψ a minimal submanifold in $\mathbb{R}^{n+m}$, provided $\psi|_{\partial\mathbb{D}}$ is a smooth map and $8n\delta \sup\limits_{\mathbb{D}} |D^2\psi| + \sqrt{2} \sup\limits_{\partial\mathbb{D}} |D\psi| < 1$. This conclusion provides classical solutions to the Dirichlet problem for minimal surface systems in arbitrary codimensions for a class of boundary maps. Specially, when $m = 1$, the existence of ψ was obtained by Jenkins and Serrin [14] already. Inspired by Wang's work mentioned above, by applying the spacelike MCF in the Minkowski space $\mathbb{R}^{n+m,n}$, Mao [16] can successfully get the existence of ψ for maximal spacelike submanifolds (with index n) in $\mathbb{R}^{n+m,n}$.

Manuscript received December 29, 2020. Revised December 24, 2021.
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*This work was supported by the National Natural Science Foundation of China (Nos. 11801496, 11926352), the Fok Ying-Tung Education Foundation (China) and Hubei Key Laboratory of Applied Mathematics (Hubei University).

In the early study of the theory of MCF, a classical result from Huisken [11] says that a given compact strictly convex hypersurface M^n in $\mathbb{R}^{n+1}$ evolving along the MCF would contract to a single point at finite time. More precisely, let $X(\cdot,t)=X_t$ be a one-parameter family of immersions $X_t : M^n \to \mathbb{R}^{n+1}$ whose images $M_t^n = X_t(M^n)$ satisfy

$$\frac{\partial}{\partial t}X(x,t)=\vec{H} \quad \text{on } M^n\times[0,T) \tag{1.1}$$

for some $T>0$, with the initial condition $X(x,0)=X_0(x)$ on M^n, where $\vec{H}$ is the mean curvature vector of the evolving hypersurface M_t^n, by using the method of L^p estimates, Huisken [11] proved that if M^n is a compact strictly convex hypersurface in $\mathbb{R}^{n+1}$, the MCF equation (1.1), with the initial condition, has a unique smooth solution on the finite time interval $[0,T_{\max})$ with $T_{\max}<\infty$, and the evolving hypersurfaces M_t^n contract to a single point as $t\to T_{\max}$. By imposing a pinching condition on the second fundamental form of the initial hypersurface, Huisken [12] has extended the above conclusion to a more general setting that the ambient space $\mathbb{R}^{n+1}$ was replaced by a class of smooth complete Riemannian manifolds N^{n+1} having some uniform bounds for curvatures and injectivity radius (of course, N^{n+1} covers $\mathbb{R}^{n+1}$ as a special case). From these two facts, one might know that generally the MCF would occur singularity at finite time. A natural question is in the following.

Problem 1 When does the MCF exist for all the time?

That is to say, under specified setting, there is no singularity formed during the evolution of MCF.

If there exists a constant vector V such that

$$\vec{H}=V^{\perp},$$

then the evolving submanifold $X_t : M^n \to \mathbb{R}^{n+m}$ is called a translating soliton of the MCF equation (1.1). Here $(\cdot)^{\perp}$ denotes the normal projection of a prescribed vector to the normal bundle of M_t^n in $\mathbb{R}^{m+n}$. It is easy to see that the translating soliton gives an eternal solution $X_t = X_0 + tV$ to (1.1), which is called the translating solution. Translating solitons play an important role in the study of type-II singularities of the MCF. For instance, Angenent and Velázquez [4–5] gave some examples of convergence which implies that type-II singularities of the MCF are modeled by translating surfaces. Clearly, the existence of translation solutions to (1.1) can give a positive answer to Problem 1.

Huisken [13] considered the evolution of graphic hypersurfaces over a bounded domain (with smooth boundary) in $\mathbb{R}^n$ under the MCF with a vanishing Neumann boundary condition (NBC for short), and proved that the flow exists for all the time and evolving graphic hypersurfaces in $\mathbb{R}^{n+1}$ converge to the graph of a constant function as $t\to\infty$. The vanishing NBC here has strong geometric meaning, that is, the evolving graphic hypersurface is perpendicular with the parabolic boundary during the evolution (or the contact angle between the evolving graphic hypersurface and parabolic boundary is $\frac{\pi}{2}$). Is the vanishing NBC necessary? What about the non-vanishing case? There are many literatures working on this direction and we would like to mention some of them. When the dimension n satisfies $n=1$ or $n=2$, Altschuler and Wu [2–3] gave a positive answer to these questions. In fact, they proved:

• When $n=1$, a graphic curve defined over an open bounded interval evolves along the flow given by a class of quasilinear parabolic equations (of course, including the MCF as a special case), with arbitrary contact angle (i.e., with nonzero NBC), would exist for all the time, and the evolving curves converge as $t \to \infty$ to a solution moving by translation with speed uniquely determined by the boundary data.

• When $n = 2$, a graphic surface defined over a compact strictly convex domain (with smooth boundary) in $\mathbb{R}^2$ evolves along the MCF, with arbitrary contact angle (i.e., with nonzero NBC), would exist for all the time, and the evolving surfaces converge as $t \to \infty$ to a surface (unique up to translation) which moves at a constant speed (uniquely determined by the boundary data).

For the higher dimensional case, Guan [10] has given a partial answer. In fact, he can get the long-time existence of the evolution of graphic hypersurfaces, defined over a bounded domain (with smooth boundary) in $\mathbb{R}^n$, under a nonparametric mean curvature type flow (i.e., the MCF with a forcing term given by an admissible function defined therein) with nonzero NBC. However, the asymptotic behavior of the flow cannot be obtained in his setting. Zhou [25] extended Altschuler-Wu's conclusion (see [3]) to the situation that graphic surfaces were defined over a compact strictly convex domain (with smooth boundary) in 2-dimensional Riemannian surfaces M^2 with nonnegative Ricci curvature, and extended Guan's conclusion (see [10]) to the situation that graphic hypersurfaces were defined over a bounded domain (with smooth boundary) in n-dimensional ($n \geq 2$) Riemannian manifolds M^n. However, similar to Guan's work (see [10]), Zhou [25] also cannot give the asymptotic behavior of the MCF with a forcing term (given by an admissible function) and with nonzero NBC in product manifolds $M^n \times \mathbb{R}$. Recently, Ma, Wang and Wei [15] improved Huisken's work (see [13]) to a more general setting that the vanishing NBC therein can be replaced by a nonzero NBC of specialized type.

Our purpose here is trying to extend the main conclusion in [15] to a more general case – the ambient space $\mathbb{R}^{n+1}$ will be replaced by product manifolds of type $M^n \times \mathbb{R}$, where M^n is a complete Riemannian manifold of nonnegative Ricci curvature.

Throughout this paper, let (M^n, σ) be a complete n-manifold ($n \geq 2$) with the Riemannian metric σ, and let $\Omega \subset M^n$ be a compact strictly convex domain with smooth boundary $\partial\Omega$. Denote by $(U_A; w_A^1, w_A^2, \cdots, w_A^n)$ the local coordinate coverings of M, and $\frac{\partial}{\partial w_A^i}$, $i = 1, 2, \cdots, n$, the corresponding coordinate vector fields, where $A \in I \subseteq N$ with N the set of all positive integers. For simplicity, we just write $\{w_A^1, w_A^2, \cdots, w_A^n\}$ as $\{w^1, w^2, \cdots, w^n\}$ to represent the local coordinates on M, and write $\frac{\partial}{\partial w_A^i}$ as $\frac{\partial}{\partial w^i}$ or ∂_i. In this setting, the metric σ should be $\sigma = \sum\limits_{i,j=1}^{n} \sigma_{ij} r \mathrm{d}w^i \otimes \mathrm{d}w^j$ with $\sigma_{ij} = \sigma(\partial_i, \partial_j)$. Denote by D the covariant derivative on Ω. Now, we would like to consider, along the MCF (1.1) with nonzero NBC, the evolution of graphic hypersurfaces, defined over Ω, in product manifold $M^n \times \mathbb{R}$ with the product metric $\overline{g} = \sigma_{ij} \mathrm{d}w^i \otimes \mathrm{d}w^j + \mathrm{d}s \otimes \mathrm{d}s$. More precisely, given a smooth[1] graphic hypersurface $\mathcal{G} \subset M^n \times \mathbb{R}$ defined over Ω, then there exists a smooth function $u_0 \in C^\infty(\overline{\Omega})$ such that $\mathcal{G}$ can be represented by $\mathcal{G} := \{(x, u_0(x)) \mid x \in \Omega\}$. It is not hard to know that the metric of $\mathcal{G}$ is given by $g = i^*\overline{g}$, where i^* is the pullback mapping of the immersion $i : \mathcal{G} \hookrightarrow M^n \times \mathbb{R}$, tangent vectors are given by

$$\vec{e}_i = \partial_i + D_i u \partial_s, \quad i = 1, 2, \cdots, n,$$

[1] In fact, it is not necessary to impose smoothness assumption on the initial hypersurface $\mathcal{G}$. The $C^{2,\alpha}$-regularity for $\mathcal{G}$ is enough to get all the estimates in the sequel. However, in order to avoid the boring regularity arguments, which is not necessary, here we assume $\mathcal{G}$ is smooth.

and the corresponding upward unit normal vector is given by

$$\vec{\gamma} = -\frac{\sum_{i=1}^{n} D^i u \partial_i - \partial_s}{\sqrt{1+|Du|^2}},$$

where $D^j u = \sum_{i=1}^{n} \sigma^{ij} D_i u$. Denote by ∇ the covariant derivative operator on $M^n \times \mathbb{R}$, and then the second fundamental form $h_{ij} \mathrm{d}\omega^i \otimes \mathrm{d}\omega^j$ of $\mathcal{G}$ is given by

$$h_{ij} = \langle \nabla_{\vec{e}_i} \vec{e}_j, \vec{\gamma} \rangle_{\overline{g}} = \frac{D_i D_j u}{\sqrt{1+|Du|^2}}.$$

Moreover, the scalar mean curvature[2] of $\mathcal{G}$ is

$$H = \sum_{i=1}^{n} h_i^i = \frac{\sum_{i,k=1}^{n} g^{ik} D_i D_k u}{\sqrt{1+|Du|^2}} = \frac{\sum_{i,k=1}^{n} \left(\sigma^{ik} - \frac{D^i u D^k u}{1+|Du|^2}\right) D_i D_k u}{\sqrt{1+|Du|^2}}. \tag{1.2}$$

Hence, in our situation here, the evolution of $\mathcal{G}$ under the MCF with nonzero NBC in $M^n \times \mathbb{R}$ with the metric $\overline{g}$ can be reduced to solvability of the following initial-boundary value problem (IBVP for short)

$$(\sharp) \quad \begin{cases} \dfrac{\partial u}{\partial t} = \sum_{i,j=1}^{n} \left(\sigma^{ij} - \dfrac{D^i u D^j u}{1+|Du|^2}\right) D_i D_j u & \text{in } \Omega \times [0,T), \\ D_{\vec{\nu}} u = \phi(x) & \text{on } \partial\Omega \times [0,T), \\ u(\cdot, 0) = u_0(\cdot) & \text{on } \Omega_0, \end{cases}$$

where $\vec{\nu}$ is the inward unit normal vector of $\partial\Omega$, $\Omega_t = \Omega \times \{t\}$ is a slice in $\Omega \times [0,T)$, $u_0(x) \in C^\infty(\overline{\Omega})$ and $\phi(x) \in C^\infty(\overline{\Omega})$ are smooth functions satisfying

$$u_{0,\vec{\nu}} = \phi(x) \quad \text{on } \partial\Omega. \tag{1.3}$$

Here (1.3) is called compatibility condition of system ($\sharp$), and a comma "," in the subscript means doing covariant derivative with respect to a prescribed tensor. This convention will also be used in the sequel. For the IBVP ($\sharp$), we can prove the following theorem.

Theorem 1.1 *If the Ricci curvature of M^n is nonnegative, $\Omega \subset M^n$ is a compact strictly convex domain with smooth boundary $\partial\Omega$, then for the* IBVP ($\sharp$)*, we have*

(1) *the* IBVP ($\sharp$) *has a smooth solution $u(x,t)$ on $\overline{\Omega} \times [0,\infty)$;*

(2) *the smooth solution $u(x,t)$ converges as $t \to \infty$ to $\lambda t + w(x)$, i.e.,*

$$\lim_{t\to\infty} \|u(x,t) - (\lambda t + w(x))\|_{C^0(\overline{\Omega})} = 0,$$

[2] In fact, the mean curvature H is computed as $H = \sum_{i,j=1}^{n} g^{ij} h_{ij}$ with the second fundamental form h_{ij} given by $h_{ij} = \langle \nabla_{\vec{e}_i} \vec{e}_j, \vec{\gamma} \rangle_{\overline{g}} = \frac{D_i D_j u}{\sqrt{1+|Du|^2}}$. However, if one uses another definition for h_{ij}, that is, $h_{ij} = -\langle \nabla_{\vec{e}_i} \vec{e}_j, \vec{\gamma} \rangle_{\overline{g}}$ (equivalently, choosing an opposite orientation for the unit normal vector), then $H = -\mathrm{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)$, and consequently, the evolution equation in ($\sharp$) does not change. Obviously, there is no essential difference between these two settings.

where $\lambda \in \mathbb{R}$ *and* $w \in C^{2,\alpha}(\overline{\Omega})$ (*unique up to a constant*) *solving the following boundary value problem* (BVP *for short*)

$$(\ddagger)\quad \begin{cases} \sum\limits_{i,j=1}^{n} \Big(\sigma^{ij} - \dfrac{D^i u D^j u}{1+|Du|^2}\Big) D_i D_j u = \lambda & \text{in } \Omega, \\ D_{\vec{\nu}} u = \phi(x) & \text{on } \partial\Omega. \end{cases}$$

Here $0 < \alpha < 1$ *and* λ *is called the additive eigenvalue of the* BVP ($\ddagger$).

Remark 1.1 (I) By (1.2), it is easy to know that

$$\sum_{i,j=1}^{n} \Big(\sigma^{ij} - \frac{D^i u D^j u}{1+|Du|^2}\Big) D_i D_j u = H \cdot \sqrt{1+|Du|^2} = \operatorname{div}\Big(\frac{Du}{\sqrt{1+|Du|^2}}\Big) \cdot \sqrt{1+|Du|^2},$$

which, substituting into the first equation of ($\ddagger$), implies

$$\operatorname{div}\Big(\frac{Dw}{\sqrt{1+|Dw|^2}}\Big) = \frac{\lambda}{\sqrt{1+|Dw|^2}},$$

where $u = w(x)$ is the solution to the BVP ($\ddagger$). Integrating the above equality and using the divergence theorem, one can get

$$\lambda = -\frac{\int_{\partial\Omega} \frac{\phi(x)}{\sqrt{1+|Dw|^2}} \mathrm{d}x}{\int_{\Omega} (1+|Dw|^2)^{-\frac{1}{2}} \mathrm{d}x}.$$

Clearly, if $\phi(x) \equiv 0$, then $\lambda = 0$. Moreover, in this setting, for the IBVP ($\sharp$), as $t \to \infty$, its smooth solution $u(x,t)$ would converge to a constant function defined over $\Omega \subset M^n$.

(II) We would like to mention one thing, that is, if $M^n = \mathbb{R}^n$ and $\phi(x) \equiv 0$, then Theorem 1.1 here degenerates into Huisken's main conclusion in [13]; if $M^n = \mathbb{R}^n$, our main conclusion here becomes exactly [15, Theorems 1.1–1.2].

(III) Recent years, the study of submanifolds of constant curvature in product manifolds attracts many geometers' attention. For instance, Hopf in 1955 discovered that the complexification of the traceless part of the second fundamental form of an immersed surface Σ^2, with constant mean curvature (CMC for short) H, in $\mathbb{R}^3$ is a holomorphic quadratic differential Q on Σ^2, and then he used this observation to get his well-known conclusion that any immersed CMC sphere $\mathbb{S}^2 \hookrightarrow \mathbb{R}^3$ is a standard distance sphere with radius $\frac{1}{H}$. By introducing a generalized quadratic differential $\widetilde{Q}$ for immersed surfaces Σ^2 in product spaces $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$, with $\mathbb{S}^2$, $\mathbb{H}^2$ the 2-dimensional sphere and hyperbolic surface, respectively, Abresch and Rosenberg [1] can extend Hopf's result to CMC spheres in these target spaces. Meeks and Rosenberg [18] successfully classified stable properly embedded orientable minimal surfaces in the product space $M \times \mathbb{R}$, where M is a closed orientable Riemannian surface. In fact, they proved that such a surface must be a product of a stable embedded geodesic on M with $\mathbb{R}$, a minimal graph over a region of M bounded by stable geodesics, $M \times \{t\}$ for some $t \in \mathbb{R}$, or is in a moduli space of periodic multigraphs parameterized by $P \times \mathbb{R}^+$, where P is the set of primitive (non-multiple) homology classes in $H_1(M)$. Mazet, Rodríguez and Rosenberg [17] analyzed properties of periodic minimal or constant mean curvature surfaces in the product manifold $\mathbb{H}^2 \times \mathbb{R}$, and they also constructed examples of periodic minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$. In [20], Rosenberg, Schulze and Spruck showed that a properly immersed minimal hypersurface in $M\times\mathbb{R}^+$ equals some slice $M\times\{c\}$ when M is a complete, recurrent n-dimensional Riemannian manifold with bounded curvature. Of course, for more information, readers can check references therein of these papers. Hence, it is interesting and important to consider submanifolds of constant curvature in the product manifold of type $M^n\times\mathbb{R}$. Based on this reason, in our setting here, it should be interesting and important to consider the following CMC equation with nonzero NBC:

$$(\natural)\qquad \begin{cases} H=\operatorname{div}\Big(\dfrac{Du}{\sqrt{1+|Du|^2}}\Big)=\lambda & \text{in } \Omega,\\ D_{\vec{\nu}}u=\phi(x) & \text{on } \partial\Omega. \end{cases}$$

Of course, all the symbols in the above system have the same meaning as those in (‡). The existence and uniqueness of solution to the BVP ($\natural$) have been obtained recently (see [9] for details).

(IV) The evolution of space-like surfaces in the Lorentz 3-manifold $M^2\times\mathbb{R}$ under the MCF with arbitrary contact angle (of course, in this situation, the NBC is nonzero) has been investigated in [7], and the long-time existence and the existence of translating solutions to the flow have been obtained.

(V) As we know, if the warping function was chosen to be a constant function, then warped product manifolds would degenerate into product manifolds. Hence, one might ask "whether one could expect to get a similar conclusion to Theorem 1.1 in warped product manifolds or not?". By constructing an interesting graphic hypersurface example in a prescribed warped product (see [25, Appendix A]), Zhou gave a negative answer to this question. Speaking in other words, he showed that the MCF with nonzero NBC in warped product manifolds would form singularities within finite time.

(VI) In fact, Huisken [13] considered the following IBVP:

$$\begin{cases} \dfrac{\partial u}{\partial t}=\displaystyle\sum_{i,j=1}^{n}\Big(\delta^{ij}-\frac{D^iuD^ju}{1+|Du|^2}\Big)D_iD_ju & \text{in } \Omega\times[0,T),\\ D_{\vec{\nu}}u=0 & \text{on } \partial\Omega\times[0,T),\\ u(\cdot,0)=u_0(\cdot) & \text{on } \Omega_0, \end{cases}$$

which, as mentioned before, describes the evolution of graphic hypersurfaces over $\Omega\subset\mathbb{R}^n$ under the MCF with a zero NBC, and obtained the long-time existence, i.e., $T=\infty$. The vanishing NBC here means that

$$\langle\vec{\gamma},\vec{\nu}\rangle_{\overline{g}}=\frac{D_{\vec{\nu}}u}{\sqrt{1+|Du|^2}}=0,$$

which is to say $\vec{\gamma}\perp\vec{\nu}$, i.e., the contact angle between $\vec{\gamma}$ and $\vec{\nu}$ is $\frac{\pi}{2}$. If the contact angle is arbitrary, then the corresponding NBC should have the form $D_{\vec{\nu}}u|_{\partial\Omega}=\varphi(x)\cdot\sqrt{1+|Du|^2}$ for some $\varphi(x)\in C^\infty(\overline{\Omega})$, $|\varphi(x)|\le 1$ on $\partial\Omega$, and $\varphi(x)=u_{0,\vec{\nu}}$ on $\partial\Omega$. Based on this reason, we can say that although the IBVP ($\sharp$) has nonzero NBC, the geometric meaning of the NBC in ($\sharp$) is not sufficient. Can we deal with the IBVP ($\sharp$) if the RHS of the nonzero NBC therein contains Du also? Inspired by a recent work (see [22]), Gao and Mao [8] considered a generalization of the IBVP ($\sharp$) where the NBC can be replaced by

$$D_{\vec{\nu}}u=\phi(x)\cdot(\sqrt{1+|Du|^2})^{\frac{1-q}{2}}$$

for any $q > 0$, and similar conclusion to Theorem 1.1 could be derived.

This paper is organized as follows. In Section 2, the time-derivative estimate, the gradient estimate, and the estimate for higher-order derivatives of u will be shown in detail, which naturally lead to the long-time existence of the IBVP (♯). In Section 3, by using an approximating approach, the solvability of the BVP (‡) can be given first, which will be used later to get the asymptotic behavior of solutions u to the IBVP (♯).

2 The Long-Time Existence

For convenience, we use several notations as follows:

$$\begin{aligned} v &= \sqrt{1+|Du|^2}, \\ g_{ij} &= \sigma_{ij} + D_i u D_j u, \\ g^{ij} &= \sigma^{ij} - \frac{D^i u D^j u}{1+|Du|^2}, \\ u_t &= \frac{\partial u}{\partial t}. \end{aligned}$$

For vectors V, W or matrices A, B, we shall use the shorthand as follows:

$$\langle V, W\rangle_g = \sum_{i,j=1}^n g^{ij} V_i W_j, \quad \langle V, W\rangle_\sigma = \sum_{i,j=1}^n \sigma^{ij} V_i W_j, \quad \langle A, B\rangle_{g,\sigma} = \sum_{i,j,k,l=1}^n g^{ij}\sigma^{kl} A_{ik} B_{jl}.$$

First, by applying a similar method to that in the proof of [3, Lemma 2.2], we would like to show the time-derivative estimate for u.

Lemma 2.1 *For the* IBVP (♯), *we have*

$$\sup_{\overline{\Omega}\times[0,T]} |u_t|^2 = \sup_{\Omega_0} |u_t|^2.$$

That is to say, there exists some positive constant $c_0 = c_0(u_0) \in \mathbb{R}^+$ such that for any $(x,t) \in \overline{\Omega} \times [0,T]$, we have

$$|u_t|^2(x,t) \le c_0.$$

Proof We first show that the maximum of $|u_t|$ must occur on $(\partial\Omega \times [0,T]) \cup \Omega_0$. By a direct computation, we have

$$\begin{aligned} \frac{\partial}{\partial t}|u_t|^2 &= 2u_t \frac{\partial u_t}{\partial t} \\ &= \sum_{i,j=1}^n 2u_t \Big(\frac{\partial g^{ij}}{\partial t} D_i D_j u + g^{ij} D_i D_j u_t \Big) \\ &= \sum_{i,j,k=1}^n 2u_t \frac{\partial g^{ij}}{\partial D^k u} \frac{\partial D^k u}{\partial t} D_i D_j u + \sum_{i,j=1}^n g^{ij} (D_i D_j |u_t|^2 - 2 D_i u_t D_j u_t) \\ &= \sum_{i,j,k=1}^n 2u_t \frac{\partial g^{ij}}{\partial D^k u} \frac{\partial D^k u}{\partial t} D_i D_j u + \sum_{i,j=1}^n g^{ij} D_i D_j |u_t|^2 - 2\langle Du_t, Du_t\rangle_\sigma \end{aligned}$$

$$= \sum_{i,j,k,l=1}^{n} 2u_t \frac{\partial g^{ij}}{\partial D^k u} \frac{\partial(\sigma^{kl} D_l u)}{\partial t} D_i D_j u + \sum_{i,j=1}^{n} g^{ij} D_i D_j |u_t|^2 - 2\langle Du_t, Du_t\rangle_\sigma$$
$$= \sum_{i,j,k,l=1}^{n} \frac{\partial g^{ij}}{\partial D^k u} \sigma^{kl} D_i D_j u D_l |u_t|^2 + \sum_{i,j=1}^{n} g^{ij} D_i D_j |u_t|^2 - 2\langle Du_t, Du_t\rangle_\sigma,$$

which implies

$$\sup_{\overline{\Omega}\times[0,T]} |u_t|^2 = \sup_{(\partial\Omega\times[0,T])\cup\Omega_0} |u_t|^2$$

by directly applying the weak maximum principle.

Next, we expel the possibility that the maximum occurs at $\partial\Omega \times [0,T]$. Assume that

$$\max_{\Omega\times\{t\}} |u_t|^2 = |u_t|^2(\xi,\tau) > 0$$

for some $(\xi,\tau) \in \partial\Omega \times [0,T]$. By the Hopf lemma, it follows that $\frac{\partial|u_t|^2}{\partial\vec{\nu}}\big|_{(\xi,\tau)} < 0$. But by the boundary condition of the IBVP ($\sharp$), one has $\frac{\partial|u_t|^2}{\partial\vec{\nu}}\big|_{(\xi,\tau)} = \frac{\partial}{\partial t}(D_{\vec{\nu}} u|_{(\xi,\tau)}) = \frac{\partial}{\partial t}(\phi(x)) = 0$. It is a contradiction. Therefore, the maximum cannot be achieved at $\partial\Omega \times [0,T]$. The conclusion of Lemma 2.1 follows.

We know that if Ω is a strictly convex domain with smooth boundary $\partial\Omega$, then there exists a smooth function β on Ω such that $\beta|_\Omega < 0$, $\beta|_{\partial\Omega} = 0$, $\sup_\Omega |D\beta| \le 1$,

$$(\beta_{ij})_{n\times n} \ge k_0 (\delta_{ij})_{n\times n}$$

for some positive constant $k_0 > 0$, $\beta_{\vec{\nu}} = D_{\vec{\nu}}\beta = -1$ and $|D\beta| = 1$ on $\partial\Omega$. Besides, since Ω is strictly convex, we have

$$(h_{ij}^{\partial\Omega})_{(n-1)\times(n-1)} \ge \kappa_1 (\delta_{ij})_{(n-1)\times(n-1)},$$

where $h_{ij}^{\partial\Omega}$, $1 \le i,j \le n-1$, is the second fundamental form of the boundary $\partial\Omega$, and $\kappa_1 > 0$ is the minimal principal curvature of $\partial\Omega$.

Lemma 2.2 *Assume that $u(x,t) \in C^{3,2}(\Omega \times [0,T))$ is a solution to the* IBVP ($\sharp$)*, and the Ricci curvature of M^n is nonnegative. Then there exists a constant $c_1 := c_1(n,\Omega,u_0,\phi(x))$ such that*

$$\sup_{\Omega\times[0,T)} |Du| \le c_1.$$

Proof To reach the conclusion of this lemma, we only need to prove that for $0 < T' < T$, we can bound $|Du|$ on $\overline{\Omega \times [0,T')}$ independent of T' and then take a limit argument.

Let

$$\Phi(x) := \log |D\omega|^2 + f(\beta),$$

where

$$\omega = u + \phi(x)\beta, \quad f = \zeta\beta,$$

and ζ is a positive constant which will be determined later. For convenience, denote by $G = -\phi(x)\beta$.

We firstly show that the maximum of $\Phi(x)$ on $\overline{\Omega}\times[0,T']$ cannot be achieved at the boundary $\partial\Omega\times[0,T']$.

Choose a suitable local coordinate around a point $x_0\in\overline{\Omega}$ such that τ_n is the inward unit normal vector of $\partial\Omega$, and τ_i, $i=1,2,\cdots,n-1$ are the unit smooth tangent vectors of $\partial\Omega$. Denote by $D_{\tau_i}u := u_i$, $D_{\tau_j}u := u_j$, $D_{\tau_i}D_{\tau_j}u := u_{ij}$ for $1\le i,j\le n$.[3] By the boundary condition, one has

$$D_{\tau_n}\omega|_{\partial\Omega} = \omega_n|_{\partial\Omega} = u_n|_{\partial\Omega} + (\phi_n\beta+\beta_n\phi)|_{\partial\Omega} = 0.$$

If $\Phi(x,t)$ attains its maximum at $(x_0,t_0)\in\partial\Omega\times[0,T']$, then at (x_0,t_0), we have

$$\begin{aligned}
0\ge\Phi_n &= \frac{|D\omega|_n^2}{|D\omega|^2}-\zeta \\
&= \sum_{k=1}^{n-1}\frac{2\omega^k D_{\tau_n}D_{\tau_k}\omega}{|D\omega|^2}-\zeta \\
&= \sum_{k=1}^{n-1}\frac{2\omega^k[\tau_k(\tau_n(\omega))-(D_{\tau_k}\tau_n)\omega]}{|D\omega|^2}-\zeta \\
&= -\sum_{k=1}^{n-1}\frac{2\omega^k(D_{\tau_k}\tau_n)(\omega)}{|D\omega|^2}-\zeta \\
&= -\sum_{k=1}^{n-1}\frac{2\omega^k\omega_j\langle D_{\tau_k}\tau_n,\tau_j\rangle_\sigma}{|D\omega|^2}-\zeta \\
&= \sum_{k=1}^{n-1}\frac{2\omega^k\omega_j\langle D_{\tau_k}\tau_j,\tau_n\rangle_\sigma}{|D\omega|^2}-\zeta \\
&= \sum_{k,j=1}^{n-1}\frac{2\omega^k\omega_j h_{kj}^{\partial\Omega}}{|D\omega|^2}-\zeta \\
&\ge 2\kappa_1-\zeta.
\end{aligned}\tag{2.1}$$

Hence, by taking $0<\zeta<2\kappa_1$, the maximum of Φ can only be achieved in $\Omega\times[0,T']$. By the way, there is one thing we would like to mention here, that is, in (2.1), the relation

$$w^k=\sum_{l=1}^{n}\sigma^{kl}w_l=\sum_{l=1}^{n-1}\sigma^{kl}w_l$$

holds. Here we have used the convention in Riemannian geometry to deal with the subscripts and superscripts, and this convention will also be used in the sequel.

Assume that $\Phi(x,t)$ attains its maximum at $(x_0,t_0)\in\Omega\times[0,T']$. By direct calculation, we have

$$\Phi_t(x_0,t_0)=\frac{|D\omega|_t^2}{|D\omega|^2},$$

[3] Covariant derivatives of other tensors can be simplified similarly. For instance, one has $\omega_i = D_{\tau_i}\omega$, $\omega_{ij} = D_{\tau_i}D_{\tau_j}\omega$.

$$\Phi_i(x_0,t_0)=\frac{|D\omega|_i^2}{|D\omega|^2}+\zeta\beta_i=0 \tag{2.2}$$

and

$$\begin{aligned}\Phi_{ij}(x_0,t_0)&=\frac{|D\omega|_{ij}^2}{|D\omega|^2}-\frac{|D\omega|_i^2|D\omega|_j^2}{|D\omega|^4}+\zeta\beta_{ij}\\&=\frac{|D\omega|_{ij}^2}{|D\omega|^2}+\zeta\beta_{ij}-\zeta^2\beta_i\beta_j.\end{aligned} \tag{2.3}$$

Since $g^{ij}=\sigma^{ij}-\frac{D^iuD^ju}{1+|Du|^2}$, we have

$$\begin{aligned}0&\geq\sum_{i,j=1}^n g^{ij}\Phi_{ij}-\Phi_t\\&=\sum_{i,j=1}^n g^{ij}\frac{|D\omega|_{ij}^2}{|D\omega|^2}-\frac{|D\omega|_t^2}{|D\omega|^2}+\zeta\sum_{i,j=1}^n g^{ij}\beta_{ij}-\zeta^2\sum_{i,j=1}^n g^{ij}\beta_i\beta_j\\&\triangleq I_1+I_2,\end{aligned} \tag{2.4}$$

where

$$I_1=\sum_{i,j=1}^n g^{ij}\frac{|D\omega|_{ij}^2}{|D\omega|^2}-\frac{|D\omega|_t^2}{|D\omega|^2}$$

and

$$I_2=\sum_{i,j=1}^n(\zeta g^{ij}\beta_{ij}-\zeta^2g^{ij}\beta_i\beta_j).$$

At (x_0,t_0), one can make a suitable change[4] to the coordinate vector fields $\{\tau_1,\tau_2,\cdots,\tau_n\}$ such that $|Du|=u_1$, $(u_{ij})_{2\leq i,j\leq n}$ is diagonal, and $(\sigma_{ij})_{2\leq i,j\leq n}$ is diagonal. Clearly, in this setting, $\sigma^{11}=1$. Besides, we have

$$\begin{aligned}g^{11}&=\frac{1}{v^2},\quad g^{ij}=0\quad\text{for } 2\leq i,j\leq n, i\neq j\quad\text{and}\quad g^{ii}=\sigma^{ii}\quad\text{for } i\geq 2,\\(g^{ij})_k&=\Big(\sigma^{ij}-\frac{D^iuD^ju}{v^2}\Big)_k\\&=-\Big(\frac{2u_k^iu^j}{v^2}-\sum_{m=1}^n\frac{2u^mu_{mk}u^iu^j}{v^4}\Big),\end{aligned}$$

where $v=\sqrt{1+|Du|^2}=\sqrt{1+u_1^2}$.

Assume that u_1 is big enough such that u_1, ω_1, ω^1, $|D\omega|$ and v are equivalent with each other at (x_0,t_0). Otherwise, the conclusion of Lemma 2.2 is proved. It is also noticeable that $|\omega_i|\leq c_2$, $i=2,\cdots,n$ for some nonnegative constant c_2. Here, in the proof, c_2 is denoted to

[4] This change can always be found. In fact, one can firstly rotate τ_i, $i=1,2,\cdots,n$, such that the gradient vector Du lies in the same or the opposite direction with τ_1. Denote the hyperplane, which is orthogonal with τ_1, by Π. Then rotate $\tau_2,\tau_3,\cdots,\tau_n$ in Π, corresponding to an orthogonal matrix, such that the real symmetric matrices $(u_{ij})_{2\leq i,j\leq n}$, $(\sigma_{ij})_{2\leq i,j\leq n}$ change into diagonal matrices.

be a nonnegative constant which may change in different places but has nothing to do with T'. Since $(\beta_{ij})_{n\times n} \geq k_0(\delta_{ij})_{n\times n}$, one can easily get

$$\begin{aligned} I_2 &= \sum_{i,j=1}^{n} (\zeta g^{ij}\beta_{ij} - \zeta^2 g^{ij}\beta_i\beta_j) \\ &\geq \zeta\Big[\sum_{i=2}^{n}\sigma^{ii}k_0 + \frac{k_0}{v^2}\Big] - \zeta^2\Big(\frac{\beta_1^2}{v^2} + \sum_{i=2}^{n}\sigma^{ii}\beta_i^2\Big). \end{aligned} \tag{2.5}$$

Set $J := \sum_{i,j=1}^{n} g^{ij}|D\omega|_{ij}^2 - |D\omega|_t^2$, one has

$$\begin{aligned} J &= \sum_{i,j=1}^{n} g^{ij}|D\omega|_{ij}^2 - |D\omega|_t^2 \\ &= \sum_{i,j,k=1}^{n} g^{ij}(2\omega_j^k\omega_{ki} + 2\omega^k\omega_{ikj}) - 2\sum_{k=1}^{n}\omega^k\omega_{tk} \\ &= \sum_{i,j,k,l=1}^{n} g^{ij}[2\omega_j^k\omega_{ki} + 2\omega^k(u_{ijk} + R_{ikj}^l u_l - G_{ikj})] - 2\sum_{k=1}^{n}\omega^k\omega_{tk} \\ &= \sum_{i,j,k=1}^{n} g^{ij}[2\omega_j^k\omega_{ki} + 2\omega^k(u_{ijk} + R_{ikj}^1 u_1 - G_{ikj})] - 2\sum_{k=1}^{n}\omega^k\omega_{tk} \\ &= 2\sum_{k=1}^{n}\omega^k\Big[\sum_{i,j=1}^{n} g^{ij}(u_{ijk} + R_{ikj}^1 u_1 - G_{ikj}) - \omega_{tk}\Big] + 2\sum_{i,j,k=1}^{n} g^{ij}\omega_{ki}\omega_j^k \\ &= 2\sum_{k=1}^{n}\omega^k\Big[\sum_{i,j=1}^{n} g^{ij}(u_{ijk} + R_{ikj}^1 u_1 - G_{ijk}) - u_{tk}\Big] + 2\sum_{i,j,k=1}^{n} g^{ij}\omega_{ki}\omega_j^k \\ &= -2\sum_{i,j,k=1}^{n} g^{ij}\omega^k G_{ijk} + 2\sum_{i,j,k=1}^{n} g^{ij}\omega^k R_{ikj}^1 u_1 - 2\sum_{i,j,k=1}^{n}\omega^k(g^{ij})_k u_{ij} + 2\sum_{i,j,k=1}^{n} g^{ij}\omega_{ki}\omega_j^k \\ &\triangleq J_1 + J_2 + J_3 + J_4, \end{aligned}$$

where R_{ikj}^l, $1 \leq i,j,k,l \leq n$, are coefficients of the curvature tensor on M^n. It is obvious that

$$\begin{aligned} J_1 &= -2\sum_{i,j,k=1}^{n} g^{ij}\omega^k G_{ijk} \\ &\geq -c_2 v. \end{aligned} \tag{2.6}$$

Next we deal with J_2, J_3. In fact, using the nonnegativity of the Ricci curvature on M^n, we have

$$\begin{aligned} J_2 &= 2\sum_{i,j,k=1}^{n} g^{ij}\omega^k R_{ikj}^1 u_1 \\ &= 2\sum_{i=2}^{n}\sigma^{ii}R_{i1i}^1 v^2 + 2\sum_{i,k=2}^{n}\sigma^{ii}\omega^k R_{iki}^1 v + 2R_{111}^1 + 2\sum_{k=2}^{n}\frac{\omega^k R_{1k1}^1}{v} \\ &\geq c_2 v^2 + O(1)v + 2R_{111}^1 \end{aligned}$$

$$\geq c_2 v^2 + O(1)v \tag{2.7}$$

and

$$\begin{aligned}
J_3 &= -2\sum_{i,j,k=1}^{n} \omega^k (g^{ij})_k u_{ij} \\
&= \sum_{i,j,k=1}^{n} 4\omega^k \frac{u_k^i u^j u_{ij}}{v^2} - \sum_{m,i,j,k=1}^{n} 4\omega^k \frac{u^m u^i u^j u_{mk} u_{ij}}{v^4} \\
&= \sum_{i,l,k=1}^{n} 4\omega^k \frac{\sigma^{il} u_1 u_{lk} u_{1i}}{v^2} - \sum_{k=1}^{n} 4\omega^k \frac{u_1^3 u_{1k} u_{11}}{v^4} \\
&= 4\frac{\omega^1 (u_{11})^2 u_1}{v^4} + 4\sum_{i=2}^{n} \frac{\omega^i u_1 u_{11} u_{1i}}{v^4} + 4\sum_{i=2}^{n} \frac{\omega^1 u_1 \sigma^{ii} (u_{1i})^2}{v^2} + 4\sum_{i=2}^{n} \frac{\omega^i \sigma^{ii} u_{1i} u_{ii} u_1}{v^2} \\
&\triangleq J_{31} + J_{32} + J_{33} + J_{34}.
\end{aligned}$$

By (2.2)–(2.3), for $2 \leq i \leq n$, we have

$$u_{1i} - G_{1i} = \frac{-\zeta\beta_i |D\omega|^2 - 2\omega^i u_{ii} + 2\sum\limits_{k=2}^{n} \omega^k G_{ik}}{2\omega^1} \tag{2.8}$$

and

$$\sum_{i=2}^{n} \frac{2\omega^i (u_{1i} - G_{1i})}{|D\omega|^2} = -\zeta\beta_1 - \frac{2\omega^1 (u_{11} - G_{11})}{|D\omega|^2}. \tag{2.9}$$

By (2.8), for $2 \leq i \leq n$, it follows that

$$\begin{aligned}
u_{1i} &= \frac{-\zeta\beta_i |D\omega|^2 - 2\omega^i u_{ii} + 2\sum\limits_{k=2}^{n} \omega^k G_{ik}}{2\omega^1} + G_{1i} \\
&= -\frac{1}{2}\zeta\beta_i v - \frac{\omega^i u_{ii}}{\omega^1} + O(1).
\end{aligned} \tag{2.10}$$

By (2.9), we have

$$\begin{aligned}
\sum_{i=2}^{n} \frac{2\omega^i (u_{1i} - G_{1i})}{|D\omega|^2} &= \sum_{i=2}^{n} \frac{2\omega^i}{|D\omega|^2} \Big(\frac{-\zeta\beta_i |D\omega|^2 - 2\omega^i u_{ii} + 2\sum\limits_{k=2}^{n} \omega^k G_{ik}}{2\omega^1} \Big) \\
&= \sum_{i=2}^{n} O\Big(\frac{|\zeta\beta_i|}{v}\Big) - \sum_{i=2}^{n} \frac{2(\omega^i)^2 u_{ii}}{|D\omega|^2 \omega^1} + \sum_{i,k=2}^{n} \frac{2\omega^i \omega^k G_{ki}}{|D\omega|^2 \omega^1}.
\end{aligned} \tag{2.11}$$

By (2.10)–(2.11), we have

$$\begin{aligned}
-\zeta\beta_1 - \frac{2\omega^1 (u_{11} - G_{11})}{|D\omega|^2} &= \sum_{i=2}^{n} O\Big(\frac{|\zeta\beta_i|}{v}\Big) - \sum_{i=2}^{n} \frac{2(\omega^i)^2 u_{ii}}{|D\omega|^2 \omega^1} + \sum_{i,k=2}^{n} \frac{2\omega^i \omega^k G_{ki}}{|D\omega|^2 \omega^1}, \\
-\zeta\beta_1 - \frac{2v (u_{11} - G_{11})}{v^2} &= O\Big(\frac{1}{v}\Big) + O\Big(\frac{1}{v^3}\Big) u_{ii} + O\Big(\frac{1}{v^3}\Big).
\end{aligned}$$

So

$$u_{11}=-\frac{1}{2}\zeta\beta_1 v+\sum_{i=2}^{n}O\Big(\frac{1}{v^2}\Big)u_{ii}+O(1). \tag{2.12}$$

Now, we deal with $J_{31}, J_{32}, J_{33}, J_{34}$, respectively. It is obvious that

$$J_{31}+J_{33}\geq 0. \tag{2.13}$$

For the term J_{32},

$$\begin{aligned}
J_{32}&=4\sum_{i=2}^{n}\frac{\omega^i u_1 u_{11}u_{1i}}{v^4}\\
&=4\sum_{i=2}^{n}\frac{\omega^i u_1}{v^4}\Big(-\frac{1}{2}\zeta\beta_1 v+\sum_{i=2}^{n}O\Big(\frac{1}{v^2}\Big)u_{ii}+O(1)\Big)\cdot\Big(-\frac{1}{2}\zeta\beta_i v-\frac{\omega^i u_{ii}}{\omega^1}+O(1)\Big)\\
&=O\Big(\frac{\zeta^2|\beta_i\beta_1|}{v}\Big)+O\Big(\frac{1}{v^2}\Big)+\sum_{i=2}^{n}\Big[O\Big(\frac{|\zeta\beta_1|}{v^2}\Big)+O\Big(\frac{|\zeta\beta_i|}{v^4}\Big)\Big]u_{ii}+\sum_{i=2}^{n}O\Big(\frac{1}{v^6}\Big)u_{ii}^2.
\end{aligned} \tag{2.14}$$

Besides, we have

$$\begin{aligned}
J_{34}&=4\sum_{i=2}^{n}\frac{\omega^i\sigma^{ii}u_{1i}u_{ii}u_1}{v^2}\\
&=4\sum_{i=2}^{n}\frac{\omega^i u_1\sigma^{ii}u_{ii}}{v^2}\Big(-\frac{1}{2}\zeta\beta_i v-\frac{\omega^i u_{ii}}{\omega^1}+O(1)\Big)\\
&=\sum_{i=2}^{n}O\Big(\frac{1}{v^2}\Big)u_{ii}^2+\sum_{i=2}^{n}O(|\zeta\beta_i|)u_{ii}.
\end{aligned} \tag{2.15}$$

By (2.13)–(2.15), it follows that

$$\begin{aligned}
J_3\geq{}&\sum_{i=2}^{n}\Big[O\Big(\frac{1}{v^6}\Big)+O\Big(\frac{1}{v^2}\Big)\Big]u_{ii}^2+\sum_{i=2}^{n}\Big[O\Big(\frac{|\zeta\beta_1|}{v^2}\Big)+O\Big(\frac{|\zeta\beta_i|}{v^4}\Big)+O(|\zeta\beta_i|)\Big]u_{ii}\\
&+O\Big(\frac{\zeta^2|\beta_i\beta_1|}{v}\Big)+O\Big(\frac{1}{v^2}\Big).
\end{aligned} \tag{2.16}$$

Then, for J_4, we have

$$\begin{aligned}
J_4&=2\sum_{i,j,k=1}^{n}g^{ij}\omega_{ki}\omega_j^k\\
&=2\sum_{l,i,j,k=1}^{n}g^{ij}\sigma^{kl}\omega_{ki}\omega_{lj}\\
&=2\sum_{i=1}^{n}\frac{\sigma^{ii}}{v^2}(\omega_{1i})^2+2\sum_{i=2}^{n}\sigma^{ii}(\omega_{1i})^2+2\sum_{i=2}^{n}(\sigma^{ii})^2(\omega_{ii})^2\\
&\geq\sum_{i=2}^{n}\Big(1+\frac{1}{v^2}\Big)\sigma^{ii}u_{1i}^2+\sum_{i=2}^{n}(\sigma^{ii})^2u_{ii}^2-c_2.
\end{aligned} \tag{2.17}$$

By (2.6)–(2.7), (2.16) and (2.18), we write all the terms containing u_{ii} in J as below

$$\sum_{i=2}^{n}\Big[O\Big(\frac{1}{v^6}\Big)+O\Big(\frac{1}{v^2}\Big)+(\sigma^{ii})^2\Big]u_{ii}^2+\sum_{i=2}^{n}\Big[O(|\zeta\beta_i|)+O\Big(\frac{|\zeta\beta_i|}{v^4}\Big)+O\Big(\frac{|\zeta\beta_1|}{v^2}\Big)\Big]u_{ii}$$
$$\geq -\sum_{i=2}^{n}\frac{O(|\zeta\beta_i|^2)}{(\sigma^{ii})^2},$$

where the inequality holds since $ax^2+bx\geq -\frac{b^2}{4a}$ for $a>0$. Therefore, we can obtain

$$\begin{aligned}J&=J_1+J_2+J_3+J_4\\&\geq -\sum_{i=2}^{n}\frac{O(|\zeta\beta_i|^2)}{(\sigma^{ii})^2}-c_2v+O(1)v+c_2v^2.\end{aligned}\tag{2.18}$$

Hence,

$$\begin{aligned}I_1&=\frac{J}{|D\omega|^2}\\&\geq -\frac{\sum\limits_{i=2}^{n}\frac{O(|\zeta\beta_i|^2)}{(\sigma^{ii})^2}+c_2v-O(1)v-c_2v^2}{|D\omega|^2}.\end{aligned}\tag{2.19}$$

By (2.4)–(2.5) and (2.19), at the maximum point (x_0,t_0), we can get

$$\begin{aligned}0&\geq\sum_{i,j=1}^{n}g^{ij}\Phi_{ij}-\Phi_t\\&\geq -\frac{\sum\limits_{i=2}^{n}\frac{O(|\zeta\beta_i|^2)}{(\sigma^{ii})^2}+c_2v-O(1)v-c_2v^2}{|D\omega|^2}\\&\quad+\zeta\Big[\sum_{i=2}^{n}\sigma^{ii}k_0+\frac{k_0}{v^2}\Big]-\zeta^2\Big(\frac{\beta_1^2}{v^2}+\sum_{i=2}^{n}\sigma^{ii}\beta_i^2\Big)\\&\geq\zeta\Big[\sum_{i=2}^{n}\sigma^{ii}k_0+\frac{k_0}{v^2}\Big]-\zeta^2\Big(\frac{\beta_1^2}{v^2}+\sum_{i=2}^{n}\sigma^{ii}\beta_i^2\Big)\end{aligned}$$

Let $\lambda=\min(\sigma^{ii})$, $\Lambda=\max(\sigma^{ii})$, $i\geq 2$. Taking $0<\zeta<\min\big\{\frac{\lambda(n-1)k_0}{\Lambda},2\kappa_1\big\}$, we can obtain

$$v(x_0,t_0)\leq c_3,$$

where c_3 is independent of T'. Then the conclusion of Lemma 2.2 follows immediately.

By Lemmas 2.1–2.2, together with the Schauder estimate for parabolic PDEs (i.e., one can control $C^{2,\alpha}$ by C^{α}, and then, by iterating, the regularity can be improved), we can get uniform estimates in any C^k-norm for the derivatives of u, and locally (in time) uniform bounds for the C^0-norm, which leads to the long-time existence, with uniform bounds on all higher derivatives of u, to the IBVP ($\sharp$). This finishes the proof of (1) of Theorem 1.1.

3 Asymptotic Behavior

In order to study the asymptotic behavior of the solution to the IVBP ($\sharp$), we need the following two conclusions.

Lemma 3.1 *Let Ω be a strictly convex bounded domain in M^n, $n \geq 2$, and $\partial\Omega \in C^3$. Assume that $\varepsilon > 0$, the Ricci curvature of M^n is nonnegative, ϕ is a function defined on $\overline{\Omega}$, and there exists a positive constant $L > 0$ such that*

$$|\phi|_{C^3(\overline{\Omega})} \leq L.$$

Let $u \in C^2(\overline{\Omega}) \cap C^3(\Omega)$ be a solution to the following BVP:

$$\begin{cases} \varepsilon u = \sum\limits_{i=1}^{n} \Big(\sigma^{ij} - \dfrac{D^i u D^j u}{1+|Du|^2}\Big) D_i D_j u & \text{in } \Omega, \\ D_{\vec{\nu}} u = \phi(x) & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Then there exists a constant $c_4 := c_4(n, \Omega, L) > 0$ such that

$$\sup_{\overline{\Omega}} |Du| \leq c_4.$$

Proof Let $\Phi(x) = \log |D\omega|^2 + \zeta\beta$, where $\omega = u + \phi(x)\beta$, and ζ will be determined later. Denote by $G = -\phi(x)\beta$.

If one chooses $0 < \zeta < 2\kappa_1$, using an almost same procedure as that in (2.1), it is easy to show that the maximum of Φ can only be achieved in the interior of Ω.

Assume that $\Phi(x)$ attains its maximum at $x_0 \in \Omega$, then we have at this point that

$$\Phi_i(x_0) = \frac{|D\omega|_i^2}{|D\omega|^2} + \zeta\beta_i = 0$$

and

$$\begin{aligned} 0 \geq \Phi_{ij}(x_0) &= \frac{|D\omega|_{ij}^2}{|D\omega|^2} - \frac{|D\omega|_i^2 |D\omega|_j^2}{|D\omega|^4} + \zeta\beta_{ij} \\ &= \frac{|D\omega|_{ij}^2}{|D\omega|^2} + \zeta\beta_{ij} - \zeta^2 \beta_i \beta_j. \end{aligned}$$

It follows that

$$\begin{aligned} 0 &\geq \sum_{i,j=1}^{n} g^{ij} \Phi_{ij} \\ &= \sum_{i,j=1}^{n} g^{ij} \frac{|D\omega|_{ij}^2}{|D\omega|^2} + \sum_{i,j=1}^{n} \zeta g^{ij} \beta_{ij} - \sum_{i,j=1}^{n} \zeta^2 g^{ij} \beta_i \beta_j \\ &\triangleq I_1 + I_2, \end{aligned} \tag{3.2}$$

where

$$I_1 = \sum_{i,j=1}^{n} g^{ij} \frac{|D\omega|_{ij}^2}{|D\omega|^2}$$

and

$$I_2 = \sum_{i,j=1}^{n} \zeta g^{ij} \beta_{ij} - \sum_{i,j=1}^{n} \zeta^2 g^{ij} \beta_i \beta_j.$$

As in Lemma 2.2, one can choose suitable local coordinates around x_0 such that $|Du| = u_1$, $(u_{ij})_{2\le i,j\le n}$ is diagonal, and $(\sigma_{ij})_{2\le i,j\le n}$ is diagonal. Similarly, for the term I_2, at x_0, we have

$$I_2 \ge \zeta\Big[\sum_{i=2}^{n}\sigma^{ii}k_0 + \frac{k_0}{v^2}\Big] - \zeta^2\Big(\frac{\beta_1^2}{v^2} + \sum_{i=2}^{n}\sigma^{ii}\beta_i^2\Big).$$

Set $J := \sum\limits_{i,j=1}^{n} g^{ij}|D\omega|^2_{ij}$. By direct calculation, one has

$$\begin{aligned}
J &= \sum_{i,j=1}^{n} g^{ij}|D\omega|^2_{ij}\\
&= \sum_{i,j,k=1}^{n} g^{ij}(2\omega_j^k\omega_{ki} + 2\omega^k\omega_{ikj})\\
&= \sum_{i,j,k,l=1}^{n} g^{ij}[2\omega_j^k\omega_{ki} + 2\omega^k(u_{ijk} + R^l_{ikj}u_l - G_{ikj})]\\
&= 2\sum_{i,j,k=1}^{n} g^{ij}\omega^k(u_{ijk} + R^1_{ikj}u_1 - G_{ikj}) + 2\sum_{i,j,k=1}^{n} g^{ij}\omega_j^k\omega_{ki}\\
&= -2\sum_{i,j,k=1}^{n} g^{ij}\omega^k G_{ijk} + 2\sum_{i,j,k=1}^{n} g^{ij}\omega^k R^1_{ikj}u_1 - 2\sum_{i,j,k=1}^{n}\omega^k(g^{ij})_k u_{ij}\\
&\quad + 2\sum_{k=1}^{n}\omega^k(\varepsilon u_k) + 2\sum_{i,j,k=1}^{n} g^{ij}\omega_{ki}\omega_j^k\\
&\triangleq J_1 + J_2 + J_3 + J_4 + J_5.
\end{aligned}$$

Without loss of generality, one may assume that u_1 is big enough, then

$$J_4 = 2\varepsilon v^2 \ge 0.$$

Otherwise, the conclusion of Lemma 3.1 follows.

As in Lemma 2.2, the other terms J_1, J_2, J_3, J_5 can be controlled similarly. In (3.2), taking $0 < \zeta < \min\big\{\frac{\lambda(n-1)k_0}{\Lambda}, 2\kappa_1\big\}$, we can obtain

$$v(x_0) \le c_5$$

for some positive constant $c_5 := c_5(n, \Omega, L)$, which is independent of ε. Then the assertion of Lemma 3.1 follows.

Theorem 3.1 *Let Ω be a strictly convex bounded domain in M^n with C^3 boundary $\partial\Omega$, $n \ge 2$. Assume that the Ricci curvature of M^n is nonnegative. For $\phi(x) \in C^3(\overline{\Omega})$, there exists a unique $\lambda \in \mathbb{R}$ and $\omega \in C^{2,\alpha}(\overline{\Omega})$ solving the* BVP (‡). *Moreover, the solution ω is unique up to a constant.*

Proof We use a similar method to that of the proof of [15, Theorem 1.2].

For each fixed $\varepsilon > 0$, we firstly show the existence of the solution to the BVP (3.1). Based on the C^1-estimate (see Lemma 3.1), the only obstacle is to derive a priori C^0-estimate for the solution $u_\varepsilon(x)$ to the BVP (3.1).

Let f be a smooth function on $\overline{\Omega}$ satisfying $D_{\vec{\nu}} f < -\sup_{\overline{\Omega}} |\phi(x)|$. Let ρ be a point where $f - u_\varepsilon$ achieves its minimum. Denote by T the tangent vector to $\partial\Omega$. If $\rho \in \partial\Omega$, then $D_T f(\rho) = D_T u_\varepsilon(\rho)$ and $D_{\vec{\nu}} f(\rho) \geq D_{\vec{\nu}} u_\varepsilon(\rho) = \phi(\rho)$, which is contradict with the choice of f. So, $\rho \in \Omega$, and then $Df(\rho) = Du_\varepsilon(\rho)$ and $D^2 f(\rho) \geq D^2 u_\varepsilon(\rho)$. This gives the existence of a constant $c_6 := c_6(f)$ such that

$$c_6 \geq \sum_{i,j=1}^{n} g^{ij}(Df) f_{ij}(\rho) \geq \sum_{i,j=1}^{n} g^{ij}(Du_\varepsilon)(u_\varepsilon)_{ij}(\rho) = \varepsilon u_\varepsilon(\rho)$$

Together with the fact $f(x) - u_\varepsilon(x) \geq f(\rho) - u_\varepsilon(\rho)$ for $x \in \Omega$, we have

$$\varepsilon u_\varepsilon(x) \leq \varepsilon f(x) - \varepsilon f(\rho) + c_6.$$

Similarly, one can get a lower bound for $\varepsilon u_\varepsilon(x)$. Therefore, $\sup_{\overline{\Omega}} |\varepsilon u_\varepsilon| \leq c_7$ holds for some nonnegative constant c_7. By the standard theory of second-order elliptic PDEs,[5] one can get the existence of the solution to the BVP (3.1).

Set $\omega_\varepsilon := u_\varepsilon - \frac{1}{|\Omega|}\int_\Omega u_\varepsilon \mathrm{d}x$. It is easy to check that ω_ε satisfies

$$\begin{cases} \displaystyle\sum_{i,j=1}^{n} \Big(\sigma^{ij} - \frac{(\omega_\varepsilon)^i(\omega_\varepsilon)^j}{1+|D\omega_\varepsilon|^2}\Big)(\omega_\varepsilon)_{ij} = \varepsilon\omega_\varepsilon + \varepsilon\frac{1}{|\Omega|}\int_\Omega u_\varepsilon \mathrm{d}x & \text{in } \Omega, \\ D_{\vec{\nu}}\omega_\varepsilon = (\omega_\varepsilon)_{\vec{\nu}} = \phi(x) & \text{on } \partial\Omega. \end{cases}$$

By

$$\sup_{\overline{\Omega}} |D\omega_\varepsilon| = \sup_{\overline{\Omega}} |Du_\varepsilon| \leq c_4$$

(see Lemma 3.1) and the fact that ω_ε has at least one zero point, we have $|\omega_\varepsilon| \leq c_8$ for some nonnegative constant $c_8 := c_8(c_4, c_7)$, which also gives the boundedness of $\frac{1}{|\Omega|}\int_\Omega(\varepsilon u_\varepsilon)\mathrm{d}x$. By the Schauder theory for second-order elliptic PDEs, one has $|\omega_\varepsilon|_{C^{2,\alpha}(\overline{\Omega})} \leq c_9$ for some nonnegative constant $c_9 := c_9(c_8)$. Taking $\varepsilon \to 0$, we have $\omega_\varepsilon \to \omega$ and $\varepsilon\omega_\varepsilon + \varepsilon\frac{1}{|\Omega|}\int_\Omega u_\varepsilon \mathrm{d}x \to \lambda$, where (λ, ω) solves the BVP (‡).

Assume that there exist two pairs (λ_1, u_1) and (λ_2, u_2) solving the BVP (‡). Without loss of generality, we may assume that $\lambda_1 \leq \lambda_2$. Let $\omega = u_1 - u_2$, and by the linearization process for the quasilinear elliptic PDEs, it is clear that ω satisfies

$$\begin{cases} \displaystyle\sum_{i,j=1}^{n} \widetilde{g}^{ij}\omega_{ij} + \sum_{i=1}^{n} b_i\omega_i = \lambda_1 - \lambda_2 \leq 0 & \text{in } \Omega, \\ D_{\vec{\nu}}\omega = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.3}$$

where $\widetilde{g}^{ij} = g^{ij}(Du_1)$ and $b_i = \sum_{k,l=1}^{n} (u_2)_{kl}\int_0^1 g^{kl}_{,p_i}(\eta Du_1 + (1-\eta)Du_2)\mathrm{d}\eta$. By Hopf's lemma, ω must be a constant, which gives the uniqueness (up to a constant) of the solution to the BVP (‡). Consequently, we have $\lambda_1 = \lambda_2$. This completes the proof of Theorem 3.1.

[5] In fact, here we can use two steps based on the standard theory of second-order elliptic PDEs to get the existence.

Step 1: Denote by $(\star)_i$ a family of boundary value problems indexed by a parameter i in a bounded closed interval, say [0,1]. Moreover, when $i = 1$, $(\star)_1$ is exactly the target equation (i.e., BVP (3.1)), and when $i = 0$, $(\star)_0$ is a BVP which can be solved.

Step 2: Show that the set $\mathcal{A}$ of all $i \in [0, 1]$ for which BVPs $(\star)_i$ can be solved is not only open but also closed, which means $\mathcal{A}$ should be the whole segment $0 \leq i \leq 1$. Therefore, the target equation can be solved.

Let

$$\widetilde{\omega}(x,t) := \omega + \lambda t, \tag{3.4}$$

where (λ, ω) is the solution to the BVP (‡). It is easy to check that $\widetilde{\omega}$ solves the following IBVP

$$\begin{cases} u_t = \sum\limits_{i,j=1}^{n} \Big(\sigma^{ij} - \dfrac{D^i u D^j u}{1+|Du|^2}\Big) D_i D_j u & \text{on } \Omega \times (0,\infty), \\ D_{\vec{\nu}} u = \phi(x) & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = \omega(x) & \text{on } \Omega. \end{cases} \tag{3.5}$$

As mentioned at the end of Section 2, by Lemmas 2.1–2.2, the Schauder theory for parabolic PDEs, one can obtain the long-time existence for the IBVP (♯), i.e., $T = \infty$.

Corollary 3.1 *For a solution $u = u(x,t)$ to the* IBVP (♯)*, there exists a positive constant c_{10}, independent of t, such that*

$$|u(x,t) - \lambda t| \le c_{10}.$$

Proof Set $z(x,t) := u(x,t) - \widetilde{\omega}(x,t)$. By the linearization process, it is easy to check that $z(x,t)$ satisfies

$$\begin{cases} z_t = \sum\limits_{i,j=1}^{n} \widetilde{g}^{ij} z_{ij} + \sum\limits_{i=1}^{n} b_i z_i & \text{in } \Omega \times (0,\infty), \\ D_{\vec{\nu}} z = 0 & \text{on } \partial\Omega \times (0,\infty), \\ z(x,0) = u_0(x) - \omega(x) & \text{on } \Omega, \end{cases}$$

where $\widetilde{g}^{ij} = g^{ij}(Du)$ and $b_i = \sum\limits_{k,l=1}^{n} (\widetilde{\omega})_{kl} \int_0^1 g^{kl}_{,p_i}(\eta Du + (1-\eta) D\widetilde{\omega}) \mathrm{d}\eta$. By the maximum principle of second-order parabolic PDEs, we know that z attains its maximum and minimum on $\Omega \times \{0\}$. Hence, one has

$$\sup_{\Omega \times (0,\infty)} |u - \lambda t| \le \sup_{\Omega} |\omega| + \sup_{\Omega} |u_0 - \omega|,$$

which implies the conclusion of Corollary 3.1.

Lemma 3.2 *Let u_1 and u_2 be any two solutions to the* IBVP (♯) *with initial data $u_{0,1}$ and $u_{0,2}$ respectively. Let $u = u_1 - u_2$, then u converges to a constant function as $t \to \infty$. In particular, the limit of any solution to the* IBVP (♯) *is $\widetilde{\omega}$ up to a constant.*

Proof We use a similar method to that of the proof of [15, Lemma 2.5].

As shown in Corollary 3.1, it is easy to know that u satisfies

$$\begin{cases} z_t = \sum\limits_{i,j=1}^{n} \widetilde{g}^{ij} z_{ij} + \sum\limits_{i}^{n} b_i z_i & \text{in } \Omega \times (0,\infty), \\ z_\nu = 0 & \text{on } \partial\Omega \times (0,\infty), \\ z(x,0) = u_{0,1}(x) - u_{0,2}(x) & \text{on } \Omega, \end{cases} \tag{3.6}$$

where $\widetilde{g}^{ij} = g^{ij}(Du_1)$ and $b_i = \sum\limits_{k,l=1}^{n} (u_2)_{kl} \int_0^1 g^{kl}_{,p_i}(\eta Du_1 + (1-\eta) Du_2) \mathrm{d}\eta$. Set

$$\operatorname{osc}(u)(t) = \max_{\Omega} u(x,t) - \min_{\Omega} u(x,t).$$

By the strong maximum principle of second-order parabolic PDEs and Hopf's lemma, one knows that $\operatorname{osc}(u)(t)$ is a strictly decreasing function unless u is a constant.

Now, we claim that

$$\lim_{t\to\infty}\operatorname{osc}(u)(t)=0.$$

Otherwise, one has $\lim\limits_{t\to\infty}\operatorname{osc}(u)(t)=\chi$ for some $\chi>0$. In fact, given a sequence $t_n\to+\infty$, define

$$u_{1,n}(\cdot,t):=u_1(\cdot,t+t_n)-\lambda t_n$$

and

$$u_{2,n}(\cdot,t):=u_2(\cdot,t+t_n)-\lambda t_n$$

By Corollary 3.1, for $i=1,2$, we have $|u_{i,n}-\lambda t|\le c_{10}$. By Lemmas 2.1–2.2 and the Schauder theory of second-order parabolic PDEs, it follows that for any k, $u_{1,n}(\cdot,t)$ and $u_{2,n}(\cdot,t)$ are locally (in time) C^k uniformly bounded with respect to n. Therefore, there exists a subsequence (still denoted by t_n) such that $u_{1,n}(\cdot,t)$ and $u_{2,n}(\cdot,t)$ converge locally uniformly in any C^k to $u_1^*(\cdot,t)$ and $u_2^*(\cdot,t)$, respectively, i.e.,

$$u_1^*(\cdot,t)=\lim_{n\to\infty}u_{1,n}(\cdot,t),\quad u_2^*(\cdot,t)=\lim_{n\to\infty}u_{2,n}(\cdot,t).$$

Set $u^*:=u_1^*-u_2^*$, and then we have

$$\begin{aligned}\operatorname{osc}(u^*)(t)&=\operatorname{osc}(u_1^*-u_2^*)\\&=\lim_{n\to\infty}\operatorname{osc}(u_1(x,t+t_n)-\lambda t_n-u_2(x,t+t_n)+\lambda t_n)\\&=\lim_{n\to\infty}\operatorname{osc}(u_1(x,t+t_n)-u_2(x,t+t_n))\\&=\lim_{n\to\infty}\operatorname{osc}(u)(t+t_n)\\&=\chi.\end{aligned}\tag{3.7}$$

The second equality in (3.7) holds since $u_{1,n}(\cdot,t)$ and $u_{1,n}(\cdot,t)$ are uniformly convergent.

Besides, it is easy to check that u^* satisfies

$$\begin{cases}z_t=\sum\limits_{i,j=1}^n\widetilde{g}^{ij}z_{ij}+\sum\limits_{i=1}^n b_iz_i & \text{in } \Omega\times(-\infty,\infty),\\ D_{\vec{\nu}}z=0 & \text{on } \partial\Omega\times(-\infty,\infty),\end{cases}$$

where $\widetilde{g}^{ij}=g^{ij}(Du_1^*)$ and $b_i=\sum\limits_{k,l=1}^n(u_2^*)_{kl}\int_0^1 g^{kl}_{,p_i}(\eta Du_1^*+(1-\eta)Du_2^*)\mathrm{d}\eta$. By the strong maximum principle of second-order parabolic PDEs and Hopf's lemma, we know u^* is a constant. This is contradict with $\operatorname{osc}(u^*)(t)\equiv\chi$. Our claim follows. So, one has $\lim\limits_{t\to\infty}\max\limits_{\Omega}u=\lim\limits_{t\to\infty}\min\limits_{\Omega}u=c_{11}$ for some constant c_{11}, which implies $\lim\limits_{t\to\infty}|u-c_{11}|=0$. This finishes the proof of Lemma 3.2.

Clearly, by Corollary 3.1 and Lemma 3.2, we know that the limit of any solution to the IBVP (♯) is $\widetilde{w}=w+\lambda t$ up to a constant, where (λ,ω) is the solution to the BVP (‡). This completes the proof of (2) of Theorem 1.1.

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