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Multiplicity of Solutions of the Weighted p-Laplacian with Isolated Singularity and Diffusion Suppressed by Convection*

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Abstract In this paper, the authors study the multiplicity of solutions to the weighted p-Laplacian with isolated singularity and diffusion suppressed by convection

$$-\mathrm{div}(|x|^{\alpha}|\nabla u|^{p-2}\nabla u)+\lambda\frac{1}{|x|^{\beta}}|\nabla u|^{p-2}\nabla u\cdot x=|x|^{\gamma}g(|x|)\quad\text{in } B\setminus\{0\}$$

subject to nonlinear Robin boundary value condition

$$|x|^{\alpha}|\nabla u|^{p-2}\nabla u\cdot\vec{n}=A-\rho u\quad\text{on }\partial B,$$

where $\lambda>0$, $B\subset\mathbb{R}^N(N\geq 2)$ is the unit ball centered at the origin, $\alpha>0$, $p>1$, $\beta\in\mathbb{R}$, $\gamma>-N$, $g\in C([0,1])$ with $g(0)>0$, $A\in\mathbb{R}$, $\rho>0$ and $\vec{n}$ is the unit outward normal. The same problem with diffusion promoted by convection, namely $\lambda\leq 0$, has already been discussed by the last two authors (Song-Yin (2012)), where the existence, nonexistence and classification of singularities for solutions are presented. Completely different from [Song, H. J. and Yin, J. X., Removable isolated singularities of solutions to the weighted p-Laplacian with singular convection, *Math. Meth. Appl. Sci.*, **35**, 2012, 1089–1100], in the present case $\lambda>0$, namely the diffusion is suppressed by the convection, non-singular solutions are not only existent but also may be infinite which vary according only to the values of solutions at the isolated singular point. At the same time, the singular solutions may exist only if the diffusion dominates the convection.

Keywords Weighted p-Laplacian, Multiplicity of solutions, Isolated singularities, Convection
2000 MR Subject Classification 35J66, 35J75

1 Introduction

This is a continuation of the work [1] on the study of isolated singularities of solutions to the following nonlinear Robin boundary value problem:

$$-\mathrm{div}(|x|^{\alpha}|\nabla u|^{p-2}\nabla u)+\lambda\frac{1}{|x|^{\beta}}|\nabla u|^{p-2}\nabla u\cdot x=|x|^{\gamma}g(|x|)\quad\text{in } B\setminus\{0\},\tag{1.1}$$

$$|x|^{\alpha}|\nabla u|^{p-2}\nabla u\cdot\vec{n}=A-\rho u\quad\text{on }\partial B,\tag{1.2}$$

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where $B \subset \mathbb{R}^N (N \geq 2)$ is the unit ball centered at the origin, $\alpha > 0$, $p > 1$, $\gamma > -N$, λ, β, $A \in \mathbb{R}$, $\rho > 0$, $g(r)$ is a continuous function defined on [0,1] with $g(0) > 0$, and $\vec{n}$ denotes the unit outward normal to the boundary ∂B. We have classified the isolated singularities for the problem when $\lambda \leq 0$ in [1]. The essential point to set $\lambda \leq 0$ in [1] is due to the fact that the convection may exhibit (sometimes very strong) degeneracy at the isolated singular point, being consistent with the diffusion, namely the convection promotes the diffusion, which ensures the uniqueness of solutions. When $\lambda > 0$, the convection is with (sometimes very strong) singularity, being opposite from that of the diffusion, namely the convection suppresses the diffusion; this causes many difficulties for the classification of the isolated singularities and the proofs of the nonexistence of solutions.

The study of isolated singularities for quasilinear elliptic equations was initiated by Serrin in [2–3], where the growth of lower-order terms is at most that of the principal part. Since then, great attention has been paid to the study of isolated singularities of various equations; see [4–5] for the fractional Laplacian, [6] for nonhomogeneous divergence-form operators, [7] for nonlinear equations with singular potentials, [8–9] for equations with nonlinearities depending on the gradient, [10–11] for the weighted p-Laplacian and so on. However, as far as we know, there are only a few papers concerning isolated singularities for equations involving convection explicitly. An elaborate and rarely known result was obtained in 1995 by Guedda and Kirane [12] for positive solutions to the equation

$$-\Delta u + \frac{1}{2}\nabla u \cdot x + \frac{u}{q-1} - u^q = 0 \quad \text{in } B \setminus \{0\},$$

where $N > 2$ and $1 < q < \frac{N+2}{N-2}$.

In this paper, we aim at studying the multiplicity of solutions of problem (1.1)–(1.2) with an isolated singularity when $\lambda > 0$. To describe the singularities of solutions, let us begin by giving the definitions of non-singular solutions and singular solutions to (1.1)–(1.2). As in [1], we denote by ∇z and $\nabla^* z$ the generalized gradients of a function z in B and $B \setminus \{0\}$, respectively. The generalized derivatives of z in other domains will still be denoted by ∇z when no confusion arises. Besides, unless otherwise stated, B_δ always denotes the open ball of radius δ centered at the origin.

Definition 1.1 *A function $u(x)$ is called a generalized solution of problem* (1.1)–(1.2)*, if $u \in C(\overline{B}\setminus\{0\}) \cap L^\infty(B) \cap W^{1,p}(B \setminus \overline{B}_\delta)$ for any $0 < \delta < 1$, $|x|^\alpha |\nabla^* u|^p$, $\frac{1}{|x|^{\beta-1}}|\nabla^* u|^{p-1} \in L^1(B)$, and u satisfies*

$$\begin{aligned} &\int_B |x|^\alpha |\nabla^* u|^{p-2}\nabla^* u \cdot \nabla\varphi \mathrm{d}x + \lambda \int_B \frac{1}{|x|^\beta}\varphi |\nabla^* u|^{p-2}\nabla^* u \cdot x \mathrm{d}x \\ &= \int_{\partial B}(A - \rho u)\varphi \mathrm{d}S + \int_B |x|^\gamma g(|x|)\varphi \mathrm{d}x \end{aligned} \tag{1.3}$$

for any $\varphi \in C^\infty(\overline{B})$ which vanishes in a certain neighbourhood of the origin.

Definition 1.2 *A function $u(x)$ is called a generalized solution of* (1.1) *in the whole ball B with the boundary value condition* (1.2)*, if $u \in C(\overline{B}) \cap W^{1,p}(B \setminus \overline{B}_\delta)$ for any $0 < \delta < 1$, $|x|^\alpha |\nabla u|^p$, $\frac{1}{|x|^{\beta-1}}|\nabla u|^{p-1} \in L^1(B)$, and u satisfies*

$$\int_B |x|^\alpha |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi \mathrm{d}x + \lambda \int_B \frac{1}{|x|^\beta}\varphi |\nabla u|^{p-2}\nabla u \cdot x \mathrm{d}x$$

$$= \int_{\partial B} (A - \rho u)\varphi \mathrm{d}S + \int_B |x|^\gamma g(|x|)\varphi \mathrm{d}x \tag{1.4}$$

for any $\varphi \in C^\infty(\overline{B})$.

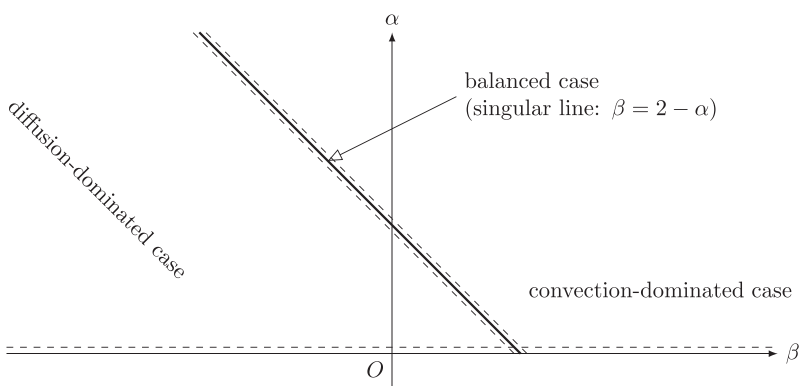
Definition 1.3 *Assume that u is a generalized solution of problem* (1.1)–(1.2). *We say u is a non-singular solution of* (1.1)–(1.2) *or has a removable singularity at the origin if u further solves* (1.1) *in the whole ball B; otherwise u is a singular solution or has a non-removable singularity at the origin.*

It should be pointed out that in some cases, for clarification of the discussion of the removability of singularities, we shall supplement a reasonable auxiliary condition at the origin, namely,

$$\lim_{x\to 0} u(x) = \theta, \tag{1.5}$$

where θ is a real number.

Remark 1.1 Under the assumptions, we find that being the same as $\lambda \le 0$ in [1], $\beta = 2-\alpha$ is a singular line for which $\beta\alpha$-plane is divided into three parts: $\beta = 2-\alpha$ is a balanced case, the diffusion dominates the convection if $\beta < 2-\alpha$, and the convection dominates the diffusion if $\beta > 2-\alpha$. Since the classification in $\lambda > 0$ is the same as that in $\lambda \le 0$, we show these three cases for $\lambda \in \mathbb{R}$ more clearly with the following diagram:



Now we state the main results of this paper. We consider the balanced case $\beta = 2-\alpha$ first. Recall that when $\lambda \le 0$, there may exist infinitely many singular solutions but one non-singular solution at most. However, when $\lambda > 0$, not only singular solution does not exist, but also there may exist infinitely many non-singular solutions.

Theorem 1.1 *Assume* $\beta = 2-\alpha$.

(i) *If*

$$\alpha \ge p + \gamma,$$

then problem (1.1)–(1.2) *has no solution.*

(ii) *If*

$$\min\{p - N + \lambda, p + \gamma\} \le \alpha < p + \gamma, \tag{1.6}$$

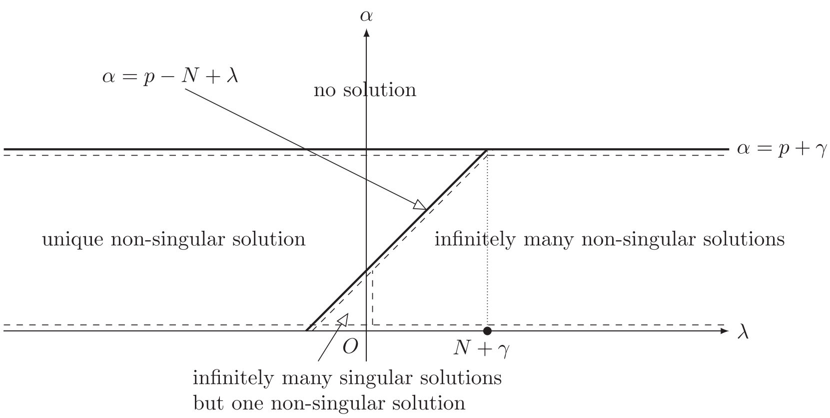
then (1.1)–(1.2) *has a unique solution* u *possessing the limit* $\lim_{x\to 0} u(x)$, *which is further non-singular.*

(iii) *If*

$$0 < \alpha < \min\{p - N + \lambda, p + \gamma\}, \tag{1.7}$$

then for any $\theta \in \mathbb{R}$, (1.1), (1.2) *and* (1.5) *has a unique solution, which also satisfies the equation* (1.1) *in the whole ball* B. *That is to say, problem* (1.1)–(1.2) *admits infinitely many non-singular solutions.*

Remark 1.2 When p, γ, N have different values, the figures are slightly different, but we prefer to show the case $p > N$. Being same as the previous remark, we show it for $\lambda \in \mathbb{R}$:



From the figure, we see that when $0 < \lambda < N + \gamma$, the first critical exponent for α: $\min\{p - N + \lambda, p + \gamma\} = p - N + \lambda$, does not change (see [1]), and as the value of λ increases, the length of the interval in which (1.1)–(1.2) has infinitely many non-singular solutions is lengthened, attains the maximum at $\lambda = N + \gamma$ and keeps unchanged when $\lambda > N + \gamma$; correspondingly, the length of the interval in which (1.1)–(1.2) has a unique non-singular solution is shortened and becomes zero when $\lambda \ge N + \gamma$. Besides, for $0 < \alpha < \min\{p - N + \lambda, p + \gamma\}$, infinitely many solutions to (1.1)–(1.2) all have removable singularities at the origin here, but only one solution possesses removable singularities in the case $\lambda \le 0$.

In fact, if $p \le N$, the singular solution in the case $\lambda \le 0$ does not exist, and if $p \le -\gamma$, there is no solution with $\lambda \in \mathbb{R}$.

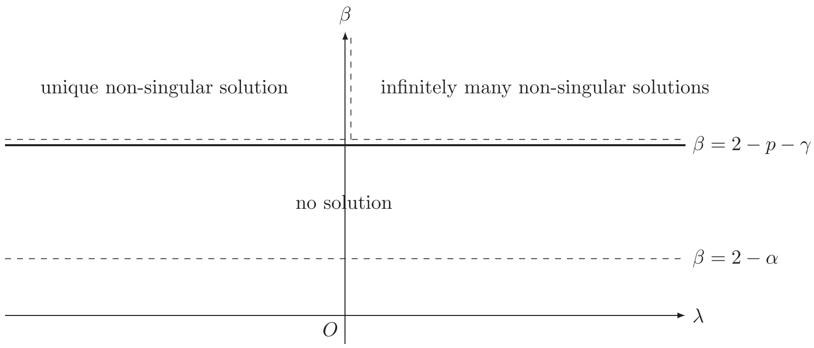
Next, we consider the case $\beta > 2 - \alpha$. We find that when $\beta > 2 - \alpha$ and $\beta > 2 - p - \gamma$, the result that there are infinitely many non-singular solutions is different from that in the case $\lambda \le 0$, where problem (1.1)–(1.2) has a unique non-singular solution.

Theorem 1.2 *Suppose* $\beta > 2 - \alpha$.

(i) *If* $\beta \le 2 - p - \gamma$, *then problem* (1.1)–(1.2) *has no solution.*

(ii) *If $\beta > 2-p-\gamma$, then for any $\theta \in \mathbb{R}$,* (1.1), (1.2) *and* (1.5) *has a unique solution, which further satisfies* (1.1) *in the whole ball B. In other words, problem* (1.1)–(1.2) *has infinitely many non-singular solutions.*

Remark 1.3 The figure as $\alpha > p+\gamma$ is slightly different from the figure as $\alpha \le p+\gamma$. Again, we show the case $\alpha > p+\gamma$ for $\lambda \in \mathbb{R}$ here:



From the figure, we see that even if p and γ have different values, there exist infinitely many non-singular solutions if $\beta > \max\{2-p-\gamma, 2-\alpha\}$, which is different from the case $\beta = 2-\alpha$ where there is no solution if $p \le -\gamma$.

Finally, we consider the case $\beta < 2-\alpha$, the result is exactly the same as that in the case $\lambda \le 0$.

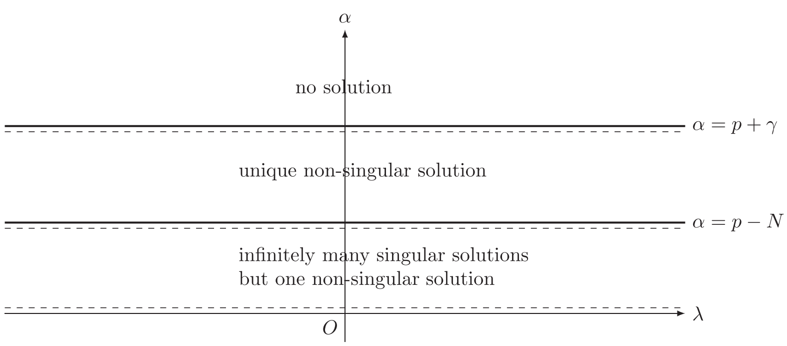
Theorem 1.3 *Let $\beta < 2-\alpha$.*

(i) *If $\alpha \ge p+\gamma$, then problem* (1.1)–(1.2) *has no solution.*

(ii) *If $p-N \le \alpha < p+\gamma$, then* (1.1)–(1.2) *has a unique solution, which is non-singular.*

(iii) *If $0 < \alpha < p-N$, then for any $\theta \in \mathbb{R}$,* (1.1)–(1.2) *has a unique solution such that* (1.5) *holds, among which one and only one is non-singular.*

Remark 1.4 Although the result is the same as that in the case $\lambda \le 0$, we show it for $\lambda \in \mathbb{R}$:



In fact, the figure as $p \le N$ is slightly different from that as $p > N$ we have shown. If $p \le N$, the singular solution will not exist.

The rest of this paper is organized as follows. In Section 2, we introduce several related notations and present some auxiliary lemmas. Subsequently, we carry out the proofs of Theorem 1.1 in Section 3 and Theorem 1.2 in Section 4; since Theorem 1.3 can be proven by using a procedure quite similar to that in [1], for brevity we omit the details.

2 Preliminaries

In this section, we first recall some notations and properties of weighted Sobolev spaces, and then establish the uniqueness of solutions and a comparison principle for an auxiliary problem, which will play a crucial role in obtaining the asymptotic properties of solutions near the isolated singular point $x = 0$.

Following [13], let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $M \subset \partial\Omega$, $1 < p < \infty$, $\sigma \in \mathbb{R}$ and $d_M(x) = \operatorname{dist}(x, M) = \inf\limits_{y\in M} |x - y|$ for $x \in \Omega$. Define

$$W^{1,p}(\Omega; d_M, \sigma) = \Big\{ f = f(x) : \int_\Omega |f(x)|^p d_M^\sigma(x)\mathrm{d}x,\ \int_\Omega |\nabla f(x)|^p d_M^\sigma(x)\mathrm{d}x < \infty \Big\}.$$

Then the weighted Sobolev space $W^{1,p}(\Omega; d_M, \sigma)$ is a Banach space when equipped with the norm

$$\|f\|_{1,p;d_M,\sigma} = \Big(\int_\Omega |f(x)|^p d_M^\sigma(x)\mathrm{d}x + \int_\Omega |\nabla f(x)|^p d_M^\sigma(x)\mathrm{d}x \Big)^{\frac{1}{p}}. \tag{2.1}$$

Denote by $W_M^{1,p}(\Omega; d_M, \sigma)$ the closure of the set

$$C_M^\infty(\Omega) = \Big\{ f \in C^\infty(\overline{\Omega}) : \operatorname{supp} f \cap \overline{M} = \emptyset \Big\} \tag{2.2}$$

in the norm (2.1), and

$$H^{1,p}(\Omega; d_M, \sigma) = \Big\{ f = f(x) : \int_\Omega |f(x)|^p d_M^{\sigma-p}(x)\mathrm{d}x,\ \int_\Omega |\nabla f(x)|^p d_M^\sigma(x)\mathrm{d}x < \infty \Big\},$$

which is also a Banach space when provided with the norm

$$\|f\|_{(H),1,p;d_M,\sigma} = \Big(\int_\Omega |f(x)|^p d_M^{\sigma-p}(x)\mathrm{d}x + \int_\Omega |\nabla f(x)|^p d_M^\sigma(x)\mathrm{d}x \Big)^{\frac{1}{p}}.$$

For the weighted Sobolev spaces above, the following embedding result holds (see [1, 13–14]).

Lemma 2.1 *$H^{1,p}(\Omega; d_M, \sigma) \hookrightarrow W_M^{1,p}(\Omega; d_M, \sigma)$ for all $\sigma \in \mathbb{R}$.*

Next, we state some Hardy type inequalities without proof; see [1] for details.

Lemma 2.2 (i) *Let $\sigma < p - N$ and $f(x)$ be a continuous function defined on $\overline{B}$ satisfying $f(0) = 0$ and*

$$\int_{B\setminus\{0\}} |\nabla^* f(x)|^p |x|^\sigma \mathrm{d}x < \infty.$$

Then the following inequality holds:

$$\int_{B\setminus\{0\}} |f(x)|^p|x|^{\sigma-p}\mathrm{d}x \le \Big(\frac{p}{p-N-\sigma}\Big)^p \int_{B\setminus\{0\}} |\nabla^* f(x)|^p|x|^\sigma \mathrm{d}x.$$

(ii) *Assume that* $h\in C(\overline{B\setminus B_\delta})\cap W^{1,p}(B\setminus \overline{B}_\delta)$ *for some* $0<\delta<1$ *and* $h=0$ *on* ∂B_δ. *Then the inequality*

$$\int_{B\setminus\overline{B}_\delta} |h(x)|^p(|x|-\delta)^{-p}\mathrm{d}x \le \Big(\frac{p}{p-1}\frac{1}{\delta^{N-1}}\Big)^p \int_{B\setminus\overline{B}_\delta} |\nabla h(x)|^p\mathrm{d}x$$

holds.

Remark 2.1 Under the assumptions of Lemma 2.2, $f\in H^{1,p}(B\setminus\{0\};|x|,\sigma)$ and $h\in H^{1,p}(B\setminus\overline{B}_\delta;|x|-\delta,0)$. By Lemma 2.1, we further have $f\in W^{1,p}_{\{0\}}(B\setminus\{0\};|x|,\sigma)$ and $h\in W^{1,p}_{\partial B_\delta}(B\setminus\overline{B}_\delta;|x|-\delta,0)$.

Before proceeding further, we present a remark on Definition 1.1 including the notations introduced above.

Remark 2.2 Assume that u is a generalized solution of (1.1)–(1.2). Then we see that for any $\varphi\in C^\infty_{\{0\}}(B\setminus\{0\})$,

$$\int_B |x|^{\alpha-\lambda}|\nabla^*u|^{p-2}\nabla^*u\cdot\nabla\varphi\mathrm{d}x = \int_{\partial B}(A-\rho u)\varphi\mathrm{d}S + \int_B |x|^{\gamma-\lambda}g(|x|)\varphi\mathrm{d}x$$

if $\beta=2-\alpha$, and

$$\begin{aligned}&\int_B \mathrm{e}^{-\lambda\frac{|x|^{2-\alpha-\beta}}{2-\alpha-\beta}}|x|^\alpha|\nabla^*u|^{p-2}\nabla^*u\cdot\nabla\varphi\mathrm{d}x\\ &= \mathrm{e}^{-\frac{\lambda}{2-\alpha-\beta}}\int_{\partial B}(A-\rho u)\varphi\mathrm{d}S + \int_B \mathrm{e}^{-\lambda\frac{|x|^{2-\alpha-\beta}}{2-\alpha-\beta}}|x|^\gamma g(|x|)\varphi\mathrm{d}x\end{aligned}$$

if $\beta\ne 2-\alpha$.

The rest of this section is devoted to the discussion of an auxiliary problem for which the uniqueness of solutions and a comparison principle will be derived.

Given $0<\delta<1$ and $u_0\in C(\overline{B}\setminus\{0\})$, consider the problem

$$-\operatorname{div}(|x|^\alpha|\nabla v_\delta|^{p-2}\nabla v_\delta)+\lambda\frac{1}{|x|^\beta}|\nabla v_\delta|^{p-2}\nabla v_\delta\cdot x = |x|^\gamma g(|x|) \quad \text{in } B\setminus\overline{B}_\delta, \tag{2.3}$$

$$|x|^\alpha|\nabla v_\delta|^{p-2}\nabla v_\delta\cdot\vec{n} = A-\rho v_\delta \quad \text{on } \partial B, \tag{2.4}$$

$$v_\delta = u_0 \quad \text{on } \partial B_\delta. \tag{2.5}$$

We say a function $v_\delta(x)$ is a generalized solution of problem (2.3)–(2.5), provided that $v_\delta\in C(\overline{B\setminus B_\delta})\cap W^{1,p}(B\setminus\overline{B}_\delta)$, v_δ satisfies

$$\begin{aligned}&\int_{B\setminus\overline{B}_\delta} |x|^\alpha|\nabla v_\delta|^{p-2}\nabla v_\delta\cdot\nabla\varphi\mathrm{d}x + \lambda\int_{B\setminus\overline{B}_\delta}\frac{1}{|x|^\beta}\varphi|\nabla v_\delta|^{p-2}\nabla v_\delta\cdot x\mathrm{d}x\\ &= \int_{\partial B}(A-\rho v_\delta)\varphi\mathrm{d}S + \int_{B\setminus\overline{B}_\delta}|x|^\gamma g(|x|)\varphi\mathrm{d}x\end{aligned}$$

for any $\varphi\in C^\infty_{\partial B_\delta}(B\setminus\overline{B}_\delta)$, and (2.5) holds in a point-wise sense.

Remark 2.3 Similarly, we have for every $\varphi \in C^\infty_{\partial B_\delta}(B \setminus \overline{B}_\delta)$,

$$\int_{B\setminus\overline{B}_\delta} |x|^{\alpha-\lambda}|\nabla v_\delta|^{p-2}\nabla v_\delta \cdot \nabla\varphi \mathrm{d}x = \int_{\partial B}(A-\rho v_\delta)\varphi \mathrm{d}S + \int_{B\setminus\overline{B}_\delta} |x|^{\gamma-\lambda}g(|x|)\varphi \mathrm{d}x \tag{2.6}$$

if $\beta = 2-\alpha$, and

$$\begin{aligned}&\int_{B\setminus\overline{B}_\delta} \mathrm{e}^{-\lambda\frac{|x|^{2-\alpha-\beta}}{2-\alpha-\beta}}|x|^{\alpha}|\nabla v_\delta|^{p-2}\nabla v_\delta \cdot \nabla\varphi \mathrm{d}x \\ &= \mathrm{e}^{-\frac{\lambda}{2-\alpha-\beta}}\int_{\partial B}(A-\rho v_\delta)\varphi \mathrm{d}S + \int_{B\setminus\overline{B}_\delta} \mathrm{e}^{-\lambda\frac{|x|^{2-\alpha-\beta}}{2-\alpha-\beta}}|x|^{\gamma}g(|x|)\varphi \mathrm{d}x \end{aligned}\tag{2.7}$$

if $\beta \neq 2-\alpha$. Using the denseness of $C^\infty_{\partial B_\delta}(B \setminus \overline{B}_\delta)$ in the space $W^{1,p}_{\partial B_\delta}(B \setminus \overline{B}_\delta; |x|-\delta, 0)$ and the trace theorem for the standard Sobolev space $W^{1,p}(B \setminus \overline{B}_\delta)$, one can deduce that the integral equalities (2.6) and (2.7) further hold for any $\varphi \in W^{1,p}_{\partial B_\delta}(B \setminus \overline{B}_\delta; |x|-\delta, 0) \cap C(\overline{B \setminus B_\delta})$.

Lemma 2.3 *Problem* (2.3)–(2.5) *admits at most one generalized solution.*

Proof Suppose that v_δ and ω_δ are two generalized solutions of (2.3)–(2.5) and write $z = v_\delta - \omega_\delta$. Then

$$z \in C(\overline{B \setminus B_\delta}) \cap W^{1,p}(B \setminus \overline{B}_\delta) \quad \text{and} \quad z(x) = 0 \text{ on } \partial B_\delta.$$

By Remark 2.1, $z \in W^{1,p}_{\partial B_\delta}(B \setminus \overline{B}_\delta; |x|-\delta, 0)$. Thus, it follows from Remark 2.3 that

$$\int_{B\setminus\overline{B}_\delta} |x|^{\alpha-\lambda}(|\nabla v_\delta|^{p-2}\nabla v_\delta - |\nabla\omega_\delta|^{p-2}\nabla\omega_\delta)\cdot(\nabla v_\delta - \nabla\omega_\delta)\mathrm{d}x = -\rho\int_{\partial B}(v_\delta-\omega_\delta)^2\mathrm{d}S$$

if $\beta = 2-\alpha$, and

$$\begin{aligned}&\int_{B\setminus\overline{B}_\delta} \mathrm{e}^{-\lambda\frac{|x|^{2-\alpha-\beta}}{2-\alpha-\beta}}|x|^{\alpha}(|\nabla v_\delta|^{p-2}\nabla v_\delta - |\nabla\omega_\delta|^{p-2}\nabla\omega_\delta)\cdot(\nabla v_\delta - \nabla\omega_\delta)\mathrm{d}x \\ &= -\rho\mathrm{e}^{-\frac{\lambda}{2-\alpha-\beta}}\int_{\partial B}(v_\delta-\omega_\delta)^2\mathrm{d}S\end{aligned}$$

if $\beta \neq 2-\alpha$. Noticing that $(|\nabla v_\delta|^{p-2}\nabla v_\delta - |\nabla\omega_\delta|^{p-2}\nabla\omega_\delta)\cdot(\nabla v_\delta - \nabla\omega_\delta) \geq 0$ and $\rho > 0$, we therefore conclude

$$\nabla v_\delta = \nabla\omega_\delta \quad \text{a.e. in } B\setminus\overline{B}_\delta \quad \text{and} \quad v_\delta = \omega_\delta \quad \text{on } \partial B,$$

which implies $v_\delta = \omega_\delta$. The proof is complete.

Lemma 2.4 *Let v^1_δ, v^2_δ be respectively the generalized solutions of* (2.3)–(2.4) *with inner boundary data u_{01}, u_{02}. If $u_{01} \leq u_{02}$ on ∂B_δ, then it follows that $v^1_\delta \leq v^2_\delta$ on $\overline{B \setminus B_\delta}$.*

Proof Set $z = (v^1_\delta - v^2_\delta)^+ = \max\{v^1_\delta - v^2_\delta, 0\}$. Then $z \in W^{1,p}_{\partial B_\delta}(B \setminus \overline{B}_\delta; |x|-\delta, 0)$, for which Remark 2.3 implies that

$$\int_{B\setminus\overline{B}_\delta} |x|^{\alpha-\lambda}(|\nabla v^1_\delta|^{p-2}\nabla v^1_\delta - |\nabla v^2_\delta|^{p-2}\nabla v^2_\delta)\cdot\nabla(v^1_\delta - v^2_\delta)^+\mathrm{d}x = -\rho\int_{\partial B}(v^1_\delta - v^2_\delta)(v^1_\delta - v^2_\delta)^+\mathrm{d}S$$

if $\beta = 2-\alpha$, and

$$\begin{aligned}&\int_{B\setminus\overline{B}_\delta} \mathrm{e}^{-\lambda\frac{|x|^{2-\alpha-\beta}}{2-\alpha-\beta}}|x|^{\alpha}(|\nabla v^1_\delta|^{p-2}\nabla v^1_\delta - |\nabla v^2_\delta|^{p-2}\nabla v^2_\delta)\cdot\nabla(v^1_\delta - v^2_\delta)^+\mathrm{d}x \\ &= -\rho\mathrm{e}^{-\frac{\lambda}{2-\alpha-\beta}}\int_{\partial B}(v^1_\delta - v^2_\delta)(v^1_\delta - v^2_\delta)^+\mathrm{d}S\end{aligned}$$

if $\beta \neq 2-\alpha$. Consequently, $v^1_\delta \leq v^2_\delta$ on $\overline{B \setminus B_\delta}$. The proof is complete.

3 Proof of Theorem 1.1

In this section, we study the case where $\beta = 2-\alpha$ and (1.1) becomes

$$-\mathrm{div}(|x|^\alpha |\nabla u|^{p-2}\nabla u) + \lambda \frac{1}{|x|^{2-\alpha}}|\nabla u|^{p-2}\nabla u \cdot x = |x|^\gamma g(|x|) \quad \text{in } B\setminus\{0\}. \tag{3.1}$$

We shall start by considering radially symmetric solutions of (3.1) and (1.2), and then give the proof of Theorem 1.1.

Let $u(r)$ $(r=|x|)$ be a radially symmetric solution of (3.1), (1.2) and (1.5). By a simple calculation, we obtain

$$(r^{N-1+\alpha-\lambda}\phi_p(u_r))_r = -r^{N-1-\lambda+\gamma}g(r), \quad 0<r<1, \tag{3.2}$$

$$\phi_p(u_r(1)) = A - \rho u(1), \tag{3.3}$$

$$\lim_{r\to 0^+} u(r) = \theta, \tag{3.4}$$

where $\phi_p(s) = |s|^{p-2}s$. We call $u(r)$ a solution of (3.2)–(3.3), if $u \in C^1((0,1]) \cap L^\infty((0,1))$, $r^{N-1+\alpha-\lambda}$ $\phi_p(u_r) \in C^1((0,1))$, and (3.2)–(3.3) hold in a point-wise sense. Then we have the following proposition.

Proposition 3.1 *Let $\alpha_* = \min\{p-N+\lambda, p+\gamma\}$. If $\alpha \ge p+\gamma$, then problem* (3.2)–(3.3) *has no solution*; *if $\alpha_* \le \alpha < p+\gamma$, then* (3.2)–(3.3) *admits a unique solution u, which satisfies that the limit $\lim\limits_{r\to 0^+} u(r)$ exists*; *if $0<\alpha<\alpha_*$, then for any $\theta\in\mathbb{R}$, problem* (3.2)–(3.4) *admits a unique solution, that is to say,* (3.2)–(3.3) *has an infinite number of solutions in $C([0,1])$.*

Proof Assume that u solves (3.2)–(3.3). Integrating (3.2) yields

$$r^{N-1+\alpha-\lambda}\phi_p(u_r) = A - \rho u(1) + \int_r^1 s^{N-1-\lambda+\gamma}g(s)\mathrm{d}s. \tag{3.5}$$

In the sequel we distinguish three cases: $0<\lambda<N+\gamma$, $\lambda = N+\gamma$ and $\lambda > N+\gamma$.

Case I $0<\lambda<N+\gamma$. We now obtain from (3.5) that

$$\lim_{r\to 0^+} r^{N-1+\alpha-\lambda}\phi_p(u_r) = A - \rho u(1) + \int_0^1 s^{N-1-\lambda+\gamma}g(s)\mathrm{d}s. \tag{3.6}$$

If $\alpha \ge p-N+\lambda$, then (3.6), combined with the fact $u\in L^\infty((0,1))$, implies

$$\lim_{r\to 0^+} r^{N-1+\alpha-\lambda}\phi_p(u_r) = 0.$$

As a result,

$$u(1) = l^* \triangleq \frac{A + \int_0^1 s^{N-1-\lambda+\gamma}g(s)\mathrm{d}s}{\rho}, \tag{3.7}$$

and by (3.5),

$$u(r) = l^* + \int_r^1 \phi_q\Big(\frac{1}{\tau^{N-1+\alpha-\lambda}}\int_0^\tau s^{N-1-\lambda+\gamma}g(s)\mathrm{d}s\Big)\mathrm{d}\tau, \tag{3.8}$$

where $q = \frac{p}{p-1}$. Using $g(0) > 0$, we compute

$$u'(r) \sim \frac{1}{r^{\frac{\alpha-1-\gamma}{p-1}}} \quad (r \to 0^+). \tag{3.9}$$

Here and below, by $f \sim h$ $(r \to 0^+)$ we mean that

$$\lim_{r\to 0^+} \frac{f(r)}{h(r)} = C$$

for some nonzero constant C. Hence, we derive from (3.8) and (3.9) that if $\alpha \geq p+\gamma$, (3.2)–(3.3) admits no solution; if $p-N+\lambda \leq \alpha < p+\gamma$, the function u defined by (3.8) is exactly a solution and the unique solution of (3.2)–(3.3), which satisfies that $\lim\limits_{r\to 0^+} u(r)$ exists.

If $0 < \alpha < p - N + \lambda$, given $\theta \in \mathbb{R}$, solving (3.2)–(3.4) yields

$$u(r) = \theta + \int_0^r \phi_q\Big(\frac{1}{\tau^{N-1+\alpha-\lambda}}\Big[A - \rho u(1) + \int_\tau^1 s^{N-1-\lambda+\gamma} g(s)\mathrm{d}s\Big]\Big)\mathrm{d}\tau.$$

In particular,

$$u(1) = \theta + \int_0^1 \phi_q\Big(\frac{1}{\tau^{N-1+\alpha-\lambda}}\Big[A - \rho u(1) + \int_\tau^1 s^{N-1-\lambda+\gamma} g(s)\mathrm{d}s\Big]\Big)\mathrm{d}\tau,$$

that is, $u(r)$ is a solution of (3.2)–(3.4) if and only if $u(1)$ is a fixed point of the function:

$$\Psi_\theta(l) = \theta + \int_0^1 \phi_q\Big(\frac{1}{\tau^{N-1+\alpha-\lambda}}\Big[A - \rho l + \int_\tau^1 s^{N-1-\lambda+\gamma} g(s)\mathrm{d}s\Big]\Big)\mathrm{d}\tau. \tag{3.10}$$

In fact, it is not difficult to verify that $\Psi_\theta(l)$ is well defined for all $l \in \mathbb{R}$, continuous and strictly decreasing, and

$$\lim_{l\to-\infty} \Psi_\theta(l) = +\infty, \quad \lim_{l\to+\infty} \Psi_\theta(l) = -\infty.$$

Thus, $\Psi_\theta(l)$ has one and only one fixed point, denoted by l_θ, and accordingly, problem (3.2)–(3.4) admits a unique solution:

$$u(r) = \theta + \int_0^r \phi_q\Big(\frac{1}{\tau^{N-1+\alpha-\lambda}}\Big[A - \rho l_\theta + \int_\tau^1 s^{N-1-\lambda+\gamma} g(s)\mathrm{d}s\Big]\Big)\mathrm{d}\tau, \quad r \in [0,1]. \tag{3.11}$$

Case II $\lambda = N + \gamma$. In this case, (3.5) becomes

$$r^{\alpha-1-\gamma}\phi_p(u_r) = A - \rho u(1) + \int_r^1 \frac{g(s)}{s}\mathrm{d}s,$$

and we apply $g(0) > 0$ again to get

$$u'(r) \sim \frac{1}{r^{\frac{\alpha-1-\gamma}{p-1}}}\Big(\ln\frac{1}{r}\Big)^{\frac{1}{p-1}} \quad (r \to 0^+). \tag{3.12}$$

This together with $u \in L^\infty((0,1))$ suggests that (3.2)–(3.3) has no solution provided that $\alpha \geq p+\gamma$. While if $0 < \alpha < p+\gamma$, then we can similarly deduce that for each $\theta \in \mathbb{R}$, the function $u(r)$ given by (3.11) uniquely solves (3.2)–(3.4), where l_θ is still the unique fixed point of (3.10).

Case III $\lambda > N+\gamma$. In this case, there holds

$$u'(r) \sim \frac{1}{r^{\frac{\alpha-1-\gamma}{p-1}}} \quad (r \to 0^+), \tag{3.13}$$

and so if $\alpha \geq p+\gamma$, then (3.2)–(3.3) admits no solution. Analogously, if $0 < \alpha < p+\gamma$, for any $\theta \in \mathbb{R}$, (3.2)–(3.4) has a unique solution, given by (3.11). The proof is complete.

We next turn to studying the generalized solutions of problem (3.1), (1.2). Assume that u is a generalized solution of (3.1), (1.2), and denote

$$\begin{aligned} &m(r) = \min_{x \in \partial B_r} u(x), \quad M(r) = \max_{x \in \partial B_r} u(x), \quad 0 < r < 1, \\ &m = \inf_{x \in B \setminus \{0\}} u(x), \quad M = \sup_{x \in B \setminus \{0\}} u(x), \quad U = \frac{1}{|\partial B|} \int_{\partial B} u(x) \mathrm{d}S. \end{aligned} \tag{3.14}$$

Fixing $0 < \delta < 1$, consider the problems

$$\begin{cases} -\mathrm{div}(|x|^\alpha |\nabla v_\delta|^{p-2} \nabla v_\delta) + \lambda \dfrac{1}{|x|^{2-\alpha}} |\nabla v_\delta|^{p-2} \nabla v_\delta \cdot x = |x|^\gamma g(|x|) \quad \text{in } B \setminus \overline{B}_\delta, \\ |x|^\alpha |\nabla v_\delta|^{p-2} \nabla v_\delta \cdot \vec{n} = A - \rho v_\delta \quad \text{on } \partial B, \\ v_\delta = m(\delta) \quad \text{on } \partial B_\delta, \end{cases}$$

and

$$\begin{cases} -\mathrm{div}(|x|^\alpha |\nabla \omega_\delta|^{p-2} \nabla \omega_\delta) + \lambda \dfrac{1}{|x|^{2-\alpha}} |\nabla \omega_\delta|^{p-2} \nabla \omega_\delta \cdot x = |x|^\gamma g(|x|) \quad \text{in } B \setminus \overline{B}_\delta, \\ |x|^\alpha |\nabla \omega_\delta|^{p-2} \nabla \omega_\delta \cdot \vec{n} = A - \rho \omega_\delta \quad \text{on } \partial B, \\ \omega_\delta = M(\delta) \quad \text{on } \partial B_\delta. \end{cases}$$

In view of Lemma 2.3, we compute

$$v_\delta(x) = m(\delta) + \int_\delta^{|x|} \phi_q \Big(\frac{1}{\tau^{N-1+\alpha-\lambda}} \Big[A - \rho l^\delta_{m(\delta)} + \int_\tau^1 s^{N-1-\lambda+\gamma} g(s) \mathrm{d}s \Big] \Big) \mathrm{d}\tau, \tag{3.15}$$

and

$$\omega_\delta(x) = M(\delta) + \int_\delta^{|x|} \phi_q \Big(\frac{1}{\tau^{N-1+\alpha-\lambda}} \Big[A - \rho l^\delta_{M(\delta)} + \int_\tau^1 s^{N-1-\lambda+\gamma} g(s) \mathrm{d}s \Big] \Big) \mathrm{d}\tau, \tag{3.16}$$

where l^δ_σ represents the unique fixed point of the function

$$\Phi^\delta_\sigma(l) = \sigma + \int_\delta^1 \phi_q \Big(\frac{1}{\tau^{N-1+\alpha-\lambda}} \Big[A - \rho l + \int_\tau^1 s^{N-1-\lambda+\gamma} g(s) \mathrm{d}s \Big] \Big) \mathrm{d}\tau.$$

According to Lemma 2.4,

$$v_\delta(x) \leq u(x) \leq \omega_\delta(x) \quad \text{for all } x \in \overline{B \setminus B_\delta} \tag{3.17}$$

and particularly,

$$l^\delta_{m(\delta)} \leq U \leq l^\delta_{M(\delta)}, \tag{3.18}$$

because $v_\delta(x) \equiv l^\delta_{m(\delta)}$ and $\omega_\delta(x) \equiv l^\delta_{M(\delta)}$ on ∂B. Combining (3.15)–(3.18), we find that for every $0<\delta<1$ and every $x \in \overline{B \setminus B_\delta}$,

$$\begin{aligned} m(\delta) &+ \int_\delta^{|x|} \phi_q\Big(\frac{1}{\tau^{N-1+\alpha-\lambda}}\Big[A-\rho U+\int_\tau^1 s^{N-1-\lambda+\gamma} g(s)\mathrm{d}s\Big]\Big)\mathrm{d}\tau \le u(x) \\ &\le M(\delta) + \int_\delta^{|x|} \phi_q\Big(\frac{1}{\tau^{N-1+\alpha-\lambda}}\Big[A-\rho U+\int_\tau^1 s^{N-1-\lambda+\gamma} g(s)\mathrm{d}s\Big]\Big)\mathrm{d}\tau, \end{aligned} \tag{3.19}$$

which implies

$$\Big|\int_\delta^1 \phi_q\Big(\frac{1}{\tau^{N-1+\alpha-\lambda}}\Big[A-\rho U+\int_\tau^1 s^{N-1-\lambda+\gamma} g(s)\mathrm{d}s\Big]\Big)\mathrm{d}\tau\Big| \le M-m \tag{3.20}$$

for any $0<\delta<1$.

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1 We prove (i) by contradiction and suppose that $u(x)$ is a generalized solution to (3.1), (1.2). When $0<\lambda<N+\gamma$, there holds

$$\lim_{\tau\to 0^+}\int_\tau^1 s^{N-1-\lambda+\gamma} g(s)\mathrm{d}s = \int_0^1 s^{N-1-\lambda+\gamma} g(s)\mathrm{d}s,$$

and if we denote

$$\Delta = A-\rho U+\int_0^1 s^{N-1-\lambda+\gamma} g(s)\mathrm{d}s,$$

then based on $\alpha \ge p+\gamma$ and $g(0)>0$, a short calculation gives

$$\lim_{\delta\to 0^+}\int_\delta^1 \phi_q\Big(\frac{1}{\tau^{N-1+\alpha-\lambda}}\Big[A-\rho U+\int_\tau^1 s^{N-1-\lambda+\gamma} g(s)\mathrm{d}s\Big]\Big)\mathrm{d}\tau = \begin{cases} +\infty & \text{if } \Delta>0, \\ -\infty & \text{if } \Delta \le 0, \end{cases}$$

which contradicts (3.20). When $\lambda \ge N+\gamma$,

$$\lim_{\tau\to 0^+}\int_\tau^1 s^{N-1-\lambda+\gamma} g(s)\mathrm{d}s = +\infty,$$

and by $\alpha \ge p+\gamma$ one can obtain

$$\lim_{\delta\to 0^+}\int_\delta^1 \phi_q\Big(\frac{1}{\tau^{N-1+\alpha-\lambda}}\Big[A-\rho U+\int_\tau^1 s^{N-1-\lambda+\gamma} g(s)\mathrm{d}s\Big]\Big)\mathrm{d}\tau = +\infty,$$

leading to a contradiction again. The assertion (i) is thus proven.

We now proceed to the proof of the assertion (ii). Notice that the hypothesis (1.6) is equivalent to

$$p-N+\lambda \le \alpha < p+\gamma \quad \text{and} \quad 0<\lambda<N+\gamma. \tag{3.21}$$

Assume that u is a generalized solution of (3.1) and (1.2) such that $\lim\limits_{x\to 0} u(x)$ exists. Then a combination of (3.20) and (3.21) gives $U=l^*$; see (3.7). Thus, (3.19) reduces to

$$m(\delta) - \int_\delta^{|x|} \phi_q\Big(\frac{1}{\tau^{N-1+\alpha-\lambda}}\int_0^\tau s^{N-1-\lambda+\gamma} g(s)\mathrm{d}s\Big)\mathrm{d}\tau \le u(x)$$

$$\leq M(\delta) - \int_{\delta}^{|x|} \phi_q\Big(\frac{1}{\tau^{N-1+\alpha-\lambda}} \int_0^{\tau} s^{N-1-\lambda+\gamma} g(s)\mathrm{d}s\Big)\mathrm{d}\tau. \tag{3.22}$$

Fixing $x \in \overline{B} \setminus \{0\}$ and sending $\delta \to 0$ in (3.22), we apply (3.21) to obtain

$$u(x) = \lim_{y\to 0} u(y) - \int_0^{|x|} \phi_q\Big(\frac{1}{\tau^{N-1+\alpha-\lambda}} \int_0^{\tau} s^{N-1-\lambda+\gamma} g(s)\mathrm{d}s\Big)\mathrm{d}\tau,$$

which says that u must be radially symmetric about the origin. Since $u = l^*$ on ∂B,

$$\lim_{x\to 0} u(x) = l^* + \int_0^1 \phi_q\Big(\frac{1}{\tau^{N-1+\alpha-\lambda}} \int_0^{\tau} s^{N-1-\lambda+\gamma} g(s)\mathrm{d}s\Big)\mathrm{d}\tau.$$

Hence, it is concluded that u must have the following form

$$u(x) = l^* + \int_{|x|}^1 \phi_q\Big(\frac{1}{\tau^{N-1+\alpha-\lambda}} \int_0^{\tau} s^{N-1-\lambda+\gamma} g(s)\mathrm{d}s\Big)\mathrm{d}\tau \quad \text{for } x \in \overline{B} \setminus \{0\}. \tag{3.23}$$

It is not difficult to verify that the function u defined by (3.23) solves problem (3.1) and (1.2). We next show that (3.23) further satisfies (3.1) in the whole ball B. It follows from (3.9) that $\nabla u = \nabla^* u = u'(r)\frac{x}{r}$, where $r = |x|$, and

$$\lim_{r\to 0} r^{N-1+\alpha} \phi_p(u_r) = 0. \tag{3.24}$$

Let $\xi_\varepsilon(s) \in C^\infty(\mathbb{R})$ be such that

$$\xi_\varepsilon(s) = 0 \quad \text{for } s \leq \varepsilon, \quad \xi_\varepsilon(s) = 1 \quad \text{for } s \geq 2\varepsilon \quad \text{and} \quad 0 \leq \xi'_\varepsilon(s) \leq \frac{C}{\varepsilon}, \tag{3.25}$$

where $0 < \varepsilon \ll 1$. Then for any $\varphi \in C^\infty(\overline{B})$, plugging $\varphi(x)\xi_\varepsilon(|x|)$ into (1.3) yields

$$\begin{aligned} &\int_B |x|^\alpha |\nabla u|^{p-2} \nabla u \cdot \nabla(\varphi\xi_\varepsilon)\mathrm{d}x + \lambda \int_B \frac{1}{|x|^{2-\alpha}} \varphi\xi_\varepsilon |\nabla u|^{p-2}\nabla u \cdot x\mathrm{d}x \\ &= \int_{\partial B} (A - \rho u)\varphi \mathrm{d}S + \int_B |x|^\gamma g(|x|)\varphi\xi_\varepsilon \mathrm{d}x. \end{aligned} \tag{3.26}$$

Since sending $\varepsilon \to 0$ in (3.26) gives

$$\begin{aligned} &\int_B |x|^\alpha |\nabla u|^{p-2} \nabla u \cdot \nabla\varphi \mathrm{d}x + |\partial B|\varphi(0) \lim_{r\to 0} r^{N-1+\alpha}\phi_p(u_r) + \lambda \int_B \frac{1}{|x|^{2-\alpha}} \varphi |\nabla u|^{p-2}\nabla u \cdot x\mathrm{d}x \\ &= \int_{\partial B} (A - \rho u)\varphi \mathrm{d}S + \int_B |x|^\gamma g(|x|)\varphi \mathrm{d}x, \end{aligned}$$

where $|\partial B|$ denotes the surface area of the unit ball, we derive (1.4) using (3.24) and complete the proof of (ii).

To obtain (iii), given $\theta \in \mathbb{R}$, we consider the function

$$u(x) = \theta + \int_0^{|x|} \phi_q\Big(\frac{1}{\tau^{N-1+\alpha-\lambda}} \Big[A - \rho l_\theta + \int_\tau^1 s^{N-1-\lambda+\gamma} g(s)\mathrm{d}s\Big]\Big)\mathrm{d}\tau, \tag{3.27}$$

where l_θ is the unique fixed point of $\Psi_\theta(l)$ defined by (3.10). Proceeding as above, one can conclude that the expression (3.27) is a solution and the only solution to (3.1) and (1.2) with the inner boundary condition $\lim_{x\to 0} u(x) = \theta$, and fulfills (3.1) in the whole ball B for every $\theta \in \mathbb{R}$. The proof is complete.

4 Proof of Theorem 1.2

In this section, we study the case where $\beta > 2-\alpha$ and carry out the proof of Theorem 1.2.

Using a similar procedure as that in Section 3, we begin by analyzing radially symmetric solutions. Let $u(r)$ be a radially symmetric solution to (1.1), (1.2) and (1.5). Then a short calculation yields

$$\Big(\mathrm{e}^{-\lambda\frac{r^{2-\alpha-\beta}}{2-\alpha-\beta}} r^{N-1+\alpha}\phi_p(u_r)\Big)_r + \mathrm{e}^{-\lambda\frac{r^{2-\alpha-\beta}}{2-\alpha-\beta}} r^{N-1+\gamma} g(r) = 0, \quad 0<r<1, \tag{4.1}$$

$$\phi_p(u_r(1)) = A - \rho u(1), \tag{4.2}$$

$$\lim_{r\to 0^+} u(r) = \theta. \tag{4.3}$$

The solutions of (4.1)–(4.2) are defined similarly, for which the following result holds.

Proposition 4.1 *Assume that $\beta > 2-\alpha$. If $\beta \le 2-p-\gamma$, problem (4.1)–(4.2) admits no solution, while if $\beta > 2-p-\gamma$, for any given $\theta\in\mathbb{R}$, (4.1)–(4.3) admits a unique solution.*

Proof Assume that u is a solution of (4.1)–(4.2). Integrating (4.1) from r to 1 leads to

$$\begin{aligned} u'(r) = \phi_q\Big(&\frac{\mathrm{e}^{\lambda r^{2-\alpha-\beta}}}{2-\alpha-\beta}\frac{1}{r^{N-1+\alpha}}\Big[\mathrm{e}^{-\frac{\lambda}{2-\alpha-\beta}}(A-\rho u(1)) \\ &+ \int_r^1 \mathrm{e}^{-\lambda\frac{t^{2-\alpha-\beta}}{2-\alpha-\beta}} t^{N-1+\gamma} g(t)\mathrm{d}t\Big]\Big). \end{aligned} \tag{4.4}$$

Observing that for any $l\in\mathbb{R}$,

$$\lim_{\tau\to 0^+}\frac{\mathrm{e}^{-\frac{\lambda}{2-\alpha-\beta}}(A-\rho l) + \int_\tau^1 \mathrm{e}^{-\lambda\frac{t^{2-\alpha-\beta}}{2-\alpha-\beta}} t^{N-1+\gamma} g(t)\mathrm{d}t}{\mathrm{e}^{-\lambda\frac{\tau^{2-\alpha-\beta}}{2-\alpha-\beta}}\tau^{N+\gamma+\beta-(2-\alpha)}} = \frac{g(0)}{\lambda} > 0, \tag{4.5}$$

we derive from (4.4) that

$$u'(r) \sim \frac{1}{r^{\frac{1-\beta-\gamma}{p-1}}} \quad (r\to 0^+). \tag{4.6}$$

By (4.6) and $u\in L^\infty((0,1))$, we arrive at that (4.1)–(4.2) has no solution provided that $\beta \le 2-p-\gamma$. When $\beta > 2-p-\gamma$, for any $\theta\in\mathbb{R}$, we consider the function

$$\begin{aligned} u(r) = \theta + \int_0^r \phi_q\Big(&\mathrm{e}^{\frac{\lambda\tau^{2-\alpha-\beta}}{2-\alpha-\beta}}\frac{1}{\tau^{N-1+\alpha}}\Big[\mathrm{e}^{-\frac{\lambda}{2-\alpha-\beta}}(A-\rho l_\theta^+) \\ &+ \int_\tau^1 \mathrm{e}^{-\lambda\frac{t^{2-\alpha-\beta}}{2-\alpha-\beta}} t^{N-1+\gamma} g(t)\mathrm{d}t\Big]\Big)\mathrm{d}\tau, \end{aligned} \tag{4.7}$$

where l_θ^+ is the unique fixed point of

$$\begin{aligned} \Psi_\theta^+(l) = \theta + \int_0^1 \phi_q\Big(&\mathrm{e}^{\frac{\lambda\tau^{2-\alpha-\beta}}{2-\alpha-\beta}}\frac{1}{\tau^{N-1+\alpha}}\Big[\mathrm{e}^{-\frac{\lambda}{2-\alpha-\beta}}(A-\rho l) \\ &+ \int_\tau^1 \mathrm{e}^{-\lambda\frac{t^{2-\alpha-\beta}}{2-\alpha-\beta}} t^{N-1+\gamma} g(t)\mathrm{d}t\Big]\Big)\mathrm{d}\tau. \end{aligned}$$

Using (4.5), it is not difficult to see that the function u defined by (4.7) is well-defined for each $0<r\le 1$, and uniquely solves (4.1)–(4.3). The proof is complete.

For generalized solutions u of (1.1), (1.2), an argument similar to that leading to (3.19) shows that

$$\begin{aligned}
&m(\delta)+\int_\delta^{|x|}\phi_q\Big(\mathrm{e}^{\lambda\frac{\tau^{2-\alpha-\beta}}{2-\alpha-\beta}}\frac{1}{\tau^{N-1+\alpha}}\Big[\mathrm{e}^{-\frac{\lambda}{2-\alpha-\beta}}(A-\rho U)\\
&\quad+\int_\tau^1\mathrm{e}^{-\lambda\frac{t^{2-\alpha-\beta}}{2-\alpha-\beta}}t^{N-1+\gamma}g(t)\mathrm{d}t\Big]\Big)\mathrm{d}\tau\\
&\le u(x)\le M(\delta)+\int_\delta^{|x|}\phi_q\Big(\mathrm{e}^{\lambda\frac{\tau^{2-\alpha-\beta}}{2-\alpha-\beta}}\frac{1}{\tau^{N-1+\alpha}}\Big[\mathrm{e}^{-\frac{\lambda}{2-\alpha-\beta}}(A-\rho U)\\
&\quad+\int_\tau^1\mathrm{e}^{-\lambda\frac{t^{2-\alpha-\beta}}{2-\alpha-\beta}}t^{N-1+\gamma}g(t)\mathrm{d}t\Big]\Big)\mathrm{d}\tau
\end{aligned}\tag{4.8}$$

for all $0<\delta<1$ and all $x\in\overline{B\setminus B_\delta}$.

Proof of Theorem 1.2 It is seen from (4.8) that

$$\Big|\int_\delta^1\phi_q\Big(\mathrm{e}^{\lambda\frac{\tau^{2-\alpha-\beta}}{2-\alpha-\beta}}\frac{1}{\tau^{N-1+\alpha}}\Big[\mathrm{e}^{-\frac{\lambda}{2-\alpha-\beta}}(A-\rho U)+\int_\tau^1\mathrm{e}^{-\lambda\frac{t^{2-\alpha-\beta}}{2-\alpha-\beta}}t^{N-1+\gamma}g(t)\mathrm{d}t\Big]\Big)\mathrm{d}\tau\Big|\le M-m$$

for any $0<\delta<1$, which combined with (4.5) implies that problem (1.1)–(1.2) has no generalized solution provided that $\beta\le 2-p-\gamma$; the assertion (i) is thus proven.

Let $\beta>2-p-\gamma$ and θ be any given real number. We first suppose that u is a generalized solution to (1.1), (1.2) and (1.5). Then by (4.5) and (4.8), we find

$$\begin{aligned}
u(x)=\theta&+\int_0^{|x|}\phi_q\Big(\mathrm{e}^{\lambda\frac{\tau^{2-\alpha-\beta}}{2-\alpha-\beta}}\frac{1}{\tau^{N-1+\alpha}}\Big[\mathrm{e}^{-\frac{\lambda}{2-\alpha-\beta}}(A-\rho U)\\
&+\int_\tau^1\mathrm{e}^{-\lambda\frac{t^{2-\alpha-\beta}}{2-\alpha-\beta}}t^{N-1+\gamma}g(t)\mathrm{d}t\Big]\Big)\mathrm{d}\tau,
\end{aligned}\tag{4.9}$$

and according to (3.14),

$$\begin{aligned}
U=\theta&+\int_0^1\phi_q\Big(\mathrm{e}^{\lambda\frac{\tau^{2-\alpha-\beta}}{2-\alpha-\beta}}\frac{1}{\tau^{N-1+\alpha}}\Big[\mathrm{e}^{-\frac{\lambda}{2-\alpha-\beta}}(A-\rho U)\\
&+\int_\tau^1\mathrm{e}^{-\lambda\frac{t^{2-\alpha-\beta}}{2-\alpha-\beta}}t^{N-1+\gamma}g(t)\mathrm{d}t\Big]\Big)\mathrm{d}\tau.
\end{aligned}\tag{4.10}$$

Since (4.10) has a unique root, the uniqueness of generalized solutions is derived. Next, using (4.6) and $\beta>\max\{2-\alpha,2-p-\gamma\}$, one can see that (4.9) with U satisfying (4.10) solves (1.1), (1.2) and (1.5). Finally, the removability of singularities follows by a similar argument as that in the proof of (ii) of Theorem 1.1. The proof is complete.

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