

Uniqueness of Solution to Systems of Elliptic Operators and Application to Asymptotic Synchronization of Linear Dissipative Systems II: Case of Multiple Feedback Dampings*

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Abstract In this paper, the authors consider the asymptotic synchronization of a linear dissipative system with multiple feedback dampings. They first show that under the observability of a scalar equation, Kalman's rank condition is sufficient for the uniqueness of solution to a complex system of elliptic equations with mixed observations. The authors then establish a general theory on the asymptotic stability and the asymptotic synchronization for the corresponding evolutionary system subjected to mixed dampings of various natures. Some classic models are presented to illustrate the field of applications of the abstract theory.

Keywords Kalman rank condition, Uniqueness, Asymptotic synchronization,
Kelvin-Voigt damping

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1 Introduction

Synchronization is a widespread natural phenomenon. It was first observed by Huygens [11] in 1665. The theoretical research on synchronization from the mathematical point of view dates back to Wiener in 1950s in [43] (Chapter 10). The previous study focused on the systems described by ordinary differential equations. Since 2012, Li and Rao started the research on the exact boundary synchronization for a coupled system of wave equations (see [18, 20–23, 26]), later the approximate synchronization has been carried out for a coupled system of wave equations with various boundary controls (see [19, 25, 27, 30]). The most part of their results was recently collected in the monograph [28]. Consequently, this kind of study of synchronization becomes a part of research in control theory. The optimal control for the exact synchronization of parabolic system was recently investigated in [42]. We quote [1, 6] for the synchronization of distributed parameter systems on networks.

By duality, the approximate boundary controllability of a coupled system of wave equations can be transformed to the uniqueness of solution to the corresponding adjoint system. Since

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the adjoint system is constituted of many wave equations of the same type and observed by an incomplete system of observations, it is not a standard uniqueness of continuation, and only Kalman's rank condition is not sufficient for the uniqueness. In order to obtain the uniqueness of solution to this complex system, our basic idea is to combine the uniform observability of a scalar equation and the algebraic structure of the coupling matrices, namely, Kalman's rank condition. The first attempt for realizing this idea was carried out in [24–25] for a system of wave equations with Dirichlet boundary conditions by incomplete Neumann observations. Later, this idea was used in [19, 30] for Neumann and Robin conditions, and further developed in [29] for an elliptic system with Neumann boundary conditions observed by incomplete Dirichlet observations. We quote [34] for a close work on the observability of heat equations by internal observations.

The goal of the present paper is to generalize the results in [29] from the special case of one sole damping to the general case of several dampings with different natures.

Let H and V be two separated Hilbert spaces such that $V \subset H$ with dense and compact imbedding.

Let L be the duality operator from V onto the dual space V' , such that

$$\langle L\phi, \psi \rangle_{V', V} = (\phi, \psi)_V, \quad \forall \phi, \psi \in V. \quad (1.1)$$

By Riesz-Fréchet's representation theorem, L is an isomorphism from V onto V' . Moreover, taking H as the pivot space, for all $\psi \in V$ and $\phi \in H$, we have

$$\langle \phi, \psi \rangle_{V', V} = (\phi, \psi)_H. \quad (1.2)$$

Let g_s ($1 \leq s \leq M$) be linear compact operators from V into V' , such that

$$\langle g_s \phi, \psi \rangle_{V', V} = \langle g_s \psi, \phi \rangle_{V', V}, \quad \langle g_s \phi, \phi \rangle_{V', V} \geq 0 \quad (1.3)$$

and

$$\langle g_s \phi, \phi \rangle_{V', V} = 0 \quad \text{if and only if} \quad g_s \phi = 0. \quad (1.4)$$

Denote by $\mathcal{V}$ and $\mathcal{H}$ the product spaces:

$$\mathcal{V} = V^N, \quad \mathcal{H} = H^N. \quad (1.5)$$

For $U = (u^{(1)}, \dots, u^{(N)})^T$, let the vector operators $\mathcal{L}$, respectively, $\mathcal{G}_s$ be defined by

$$\mathcal{L}U = \begin{pmatrix} Lu^{(1)} \\ \vdots \\ Lu^{(N)} \end{pmatrix}, \quad \mathcal{G}_s U = \begin{pmatrix} g_s u^{(1)} \\ \vdots \\ g_s u^{(N)} \end{pmatrix}, \quad 1 \leq s \leq M. \quad (1.6)$$

Let A and D_s ($1 \leq s \leq M$) be symmetric and positive semi-definite matrices. Consider the following second order evolution system with several dampings of different natures.

$$U'' + \mathcal{L}U + AU + \sum_{s=1}^M D_s \mathcal{G}_s U' = 0. \quad (1.7)$$

It is easy to show that (1.7) generates a semi-group of contractions with compact resolvent in the space $\mathcal{V} \times \mathcal{H}$.

The case $M = 1$:

$$U'' + \mathcal{L}U + AU + D_1 \mathcal{G}_1 U' = 0 \quad (1.8)$$

was studied in [29], and we showed that Kalman rank condition

$$\text{rank}(D_1, AD_1, \dots, A^{N-1}D_1) = N \quad (1.9)$$

is necessary for the asymptotic stability of system (1.8). Moreover, under suitable conditions on the pair of operators (L, g_1) , Kalman rank condition (1.9) is also sufficient for the asymptotic stability of system (1.8) (see [29, Theorem 3.4]). In [31], we carried out a complete study on the uniform synchronization of system (1.8). In particular, we justified the necessity of diverse conditions of compatibility on the matrices A and D_1 . Moreover, in [32] we considered a coupled system of wave equations in a rectangular domain, which does not satisfy the usual multiplier geometrical condition.

The aim of the present work is to investigate the asymptotic stability of system (1.7) under the common action of M feedback dampings $D_1 \mathcal{G}_1 U', \dots, D_M \mathcal{G}_M U'$. In Proposition 2.2 below, we will show that Kalman rank condition

$$\text{rank}(D, AD, \dots, A^{N-1}D) = N \quad (1.10)$$

with the composite matrix by blocks:

$$D = (D_1, D_2, \dots, D_M) \quad (1.11)$$

is still necessary for the asymptotic stability of system (1.7). Moreover, under suitable conditions on the matrix A and on the pairs (L, g_s) for $1 \leq s \leq M$, we will show in Theorem 3.2 that Kalman rank condition (1.10) is still sufficient for the asymptotic stability of system (1.7). The involved dampings in system (1.7) can be of different types, for example, boundary damping, locally distributed viscous dampings, locally distributed Kelvin-Voigt damping or bending moment damping etc. Therefore, it provides a rich freedom for the choice of feedback controls in applications. This is the main advantage of the approach.

The materials in the paper are organized as follows. In §2, we first formulate the problem in the framework of semi-groups. Then by the classic method of frequency domain, we reduce the asymptotic stability to the uniqueness of solution to an over-determined elliptic system. In §3, under the assumptions that A is closed to a scalar matrix and L can be uniformly observed by the operator g_s for $1 \leq s \leq M$, we establish the corresponding uniqueness theorem. We study the corresponding asymptotic synchronization in §4. In order to illustrate the abstract result, we give some examples of applications in §5.

2 Setting of Problem

In this section, we will characterize the asymptotic stability of system (1.7) by the method of frequency domain. We first make some necessary arrangement.

Since for $1 \leq s \leq M$, D_s are symmetric and positive semi-definite matrices, by conditions (1.3)–(1.4), it is easy to check that

$$\langle D_s \mathcal{G}_s U, V \rangle = \langle D_s \mathcal{G}_s V, U \rangle, \quad \langle D_s \mathcal{G}_s U, U \rangle \geq 0 \quad (2.1)$$

and

$$\langle D_s \mathcal{G}_s U, U \rangle = 0 \quad \text{if and only if} \quad D_s \mathcal{G}_s U = 0. \quad (2.2)$$

Clearly, by (2.1)–(2.2), we have

$$\sum_{s=1}^M \langle D_s \mathcal{G}_s U, U \rangle \geq 0 \quad (2.3)$$

and

$$\sum_{s=1}^M \langle D_s \mathcal{G}_s U, U \rangle = 0 \quad \text{if and only if} \quad D_s \mathcal{G}_s U = 0, \quad 1 \leq s \leq M. \quad (2.4)$$

Then, defining the linear operator $\mathcal{A}$ by

$$\mathcal{A}(U, \widehat{U}) = \left(\widehat{U}, -\mathcal{L}U - AU - \sum_{s=1}^M D_s \mathcal{G}_s \widehat{U} \right) \quad (2.5)$$

with the domain

$$D(\mathcal{A}) = \left\{ (U, \widehat{U}) \in \mathcal{V} \times \mathcal{V} : \mathcal{L}U + AU + \sum_{s=1}^M D_s \mathcal{G}_s \widehat{U} \in \mathcal{V} \right\}, \quad (2.6)$$

we transform (1.7) into an abstract formulation as follows:

$$(U, \widehat{U})' = \mathcal{A}(U, \widehat{U}). \quad (2.7)$$

It was shown in [29, Proposition 3.1] that operator $\mathcal{A}$ generates a semi-group of contractions with compact resolvent in the space $\mathcal{V} \times \mathcal{H}$.

We recall the following generalized rank condition of Kalman type, which will play an important role in the study of uniqueness.

Proposition 2.1 (see [25, Lemma 2.1]) *Let $d \geq 0$ be an integer. The Kalman rank condition*

$$\text{rank}(D, AD, \dots, A^{N-1}D) = N - d \quad (2.8)$$

holds if and only if d is the largest dimension of the subspaces which are invariant for A^T and contained in $\text{Ker}(D^T)$.

Proposition 2.2 *If system (1.7) is asymptotically stable, then we necessarily have Kalman rank condition (1.10) with D given by (1.11).*

Proof If (1.10) fails, by Proposition 2.1, there exists a unit vector $E \in \mathbb{R}^N$ and a real number a such that

$$A^T E = aE, \quad D^T E = 0. \quad (2.9)$$

Noting that A and D_s with $1 \leq s \leq M$ are symmetric, we get

$$AE = aE, \quad D_s E = 0, \quad s = 1, \dots, M. \quad (2.10)$$

Then, applying E^T to (1.7) and setting $u = E^T U$ we get

$$u'' + Lu + au = 0, \quad (2.11)$$

which is conservative, therefore, unstable.

Theorem 2.1 *System (1.7) is asymptotically stable if and only if for any given $\beta \in \mathbb{R}$, the over-determined system of the state variable $\Phi = (\phi^{(1)}, \dots, \phi^{(N)})^T$:*

$$\mathcal{L}\Phi + A\Phi = \beta^2 \Phi \quad (2.12)$$

associated with the conditions

$$D_s \mathcal{G}_s \Phi = 0, \quad 1 \leq s \leq M \quad (2.13)$$

has only the trivial solution.

Proof Noting that $\mathcal{A}^{-1}$ is compact in the space $\mathcal{V} \times \mathcal{H}$, by the classic theory of semi-groups (see [3, 4, 37]), the dissipative system (1.7) is asymptotically stable if and only if $\mathcal{A}$ has no pure imaginary eigenvalues. Indeed, assume that $\mathcal{A}$ has a pure imaginary eigenvalue, namely, there exist $\beta \in \mathbb{R}$ and a non-trivial $(\Phi, \Psi) \in \mathcal{V} \times \mathcal{H}$, such that

$$\mathcal{A}(\Phi, \Psi) = i\beta(\Phi, \Psi), \quad (2.14)$$

namely,

$$\Psi = i\beta\Phi, \quad -\mathcal{L}\Phi - A\Phi - \sum_{s=1}^M D_s \mathcal{G}_s \Psi = i\beta\Psi. \quad (2.15)$$

Inserting the first equation into the second one, we get

$$\mathcal{L}\Phi + A\Phi + i\beta \sum_{s=1}^M D_s \mathcal{G}_s \Phi = \beta^2 \Phi. \quad (2.16)$$

Since $\mathcal{L} + A$ is symmetric and coercive, we have $\beta \neq 0$. Then, noting that $\mathcal{L}$ and $D_s \mathcal{G}_s$ ($1 \leq s \leq M$) are symmetric, we deduce that (2.16) is equivalent to the system

$$\mathcal{L}\Phi + A\Phi = \beta^2 \Phi \quad \text{and} \quad \sum_{s=1}^M D_s \mathcal{G}_s \Phi = 0. \quad (2.17)$$

Using (2.4), the second condition in (2.17) implies condition (2.13), then it gives a contradiction. The proof is complete.

3 Uniqueness Theorem Under Kalman Rank Condition

In this section, we will show both the necessity and the sufficiency of Kalman rank condition (1.10) for the uniqueness of solution to the over-determined system (2.12)–(2.13).

Proposition 3.1 *Assume that the over-determined system (2.12)–(2.13) has only the trivial solution. Then the pair (A, D) necessarily satisfies Kalman rank condition (1.10) with D given by (1.11).*

Proof This is a direct consequence of Proposition 2.2 and Theorem 2.1. However, we prefer to give a direct proof here.

Otherwise, let a and E be chosen as in (2.10). Let $v \in V$ be a non-zero element and $\lambda \in \mathbb{R}^+$ be large enough, such that $Lv = \lambda v$ and $\lambda + a > 0$.

Defining

$$\beta^2 = \lambda + a \quad \text{and} \quad \Phi = vE, \quad (3.1)$$

we have

$$\mathcal{L}\Phi + A\Phi = LvE + vAE = (\lambda + a)vE = \beta^2\Phi. \quad (3.2)$$

So, Φ is a solution to (2.12). Moreover, noting that G_s is of diagonal form, we check easily that Φ satisfies the dissipation condition (2.13):

$$D_s G_s \Phi = g_s v D_s E = 0, \quad s = 1, \dots, M. \quad (3.3)$$

Thus, we get a contradiction.

Now we make some preparation for the proof of sufficiency. Since Kalman rank condition (1.10) is stable under invertible linear transformation, without loss of generality, the symmetric matrix A can be written as

$$A = \text{diag}(\overbrace{\lambda_1, \dots, \lambda_1}^{\sigma_1}, \dots, \overbrace{\lambda_m, \dots, \lambda_m}^{\sigma_m}),$$

where $\lambda_k \geq 0$ are eigenvalues of A with multiplicity σ_k ($k = 1, \dots, m$).

Accordingly, let

$$\mu_0 = 0 : \mu_k = \mu_{k-1} + \sigma_k, \quad k = 1, \dots, m. \quad (3.4)$$

For any given p with $1 \leq s \leq M$, we write

$$D_s = (d_1^{(s)}, \dots, d_{\mu_1}^{(s)}, \dots, d_{\mu_{m-1}+1}^{(s)}, \dots, d_{\mu_m}^{(s)}), \quad (3.5)$$

where the vector $d_i^{(s)} \in \mathbb{R}^N$ denotes the i -th column of the matrix D_s .

Let

$$D^T = \begin{pmatrix} D_1 \\ \vdots \\ D_M \end{pmatrix} = (d_1, \dots, d_{\mu_1}, \dots, d_{\mu_{m-1}+1}, \dots, d_{\mu_m}), \quad (3.6)$$

where the vector $d_i \in \mathbb{R}^{MN}$ is composed of the i -th column of the matrix D_s with $1 \leq s \leq M$:

$$d_i = \begin{pmatrix} d_i^{(1)} \\ \vdots \\ d_i^{(M)} \end{pmatrix}. \quad (3.7)$$

Proposition 3.2 *For any given integer k with $1 \leq k \leq m$, the vectors $d_{\mu_{k-1}+1}, \dots, d_{\mu_k}$ of the composite matrix D^T are linearly independent.*

Proof Denote by ε_i the canonical basis vectors in $\mathbb{R}^N$. Since $D^T \varepsilon_i = d_i$ and the subspace $\text{Span}\{\varepsilon_{\mu_{k-1}+1}, \dots, \varepsilon_{\mu_k}\}$ is invariant for A , by Proposition 2.1, Kalman rank condition (1.9) implies that

$$\sum_{i=\mu_{k-1}+1}^{\mu_k} \alpha_i d_i = 0 \quad (3.8)$$

if and only if $\alpha_{\mu_{k-1}+1} = \dots = \alpha_{\mu_k} = 0$. Therefore, the column vectors $d_{\mu_{k-1}+1}, \dots, d_{\mu_k}$ are linearly independent.

Definition 3.1 *For any given s with $1 \leq s \leq M$, the operator L is g_s -observable, if there exists a constant $c_1 > 0$, independent of $\beta \in \mathbb{R}$ and $f \in H$, such that the estimate*

$$\|\phi_s\|_H \leq c_1 \|f\|_H \quad (3.9)$$

holds for any given solution ϕ_s to the over-determined scalar problem

$$\beta^2 \phi_s - L\phi_s = f \quad \text{with} \quad g_s \phi_s = 0. \quad (3.10)$$

By the continuous embedding $H \subset V$, there exists a constant $c_2 > 0$, such that

$$\|\phi\|_H \leq c_2 \|\phi\|_V, \quad \forall \phi \in V. \quad (3.11)$$

Theorem 3.1 *Assume that*

(a) *there exists $a \in \mathbb{R}$, such that the following ε -closing condition*

$$\|A - aI\|_2 \leq \varepsilon \quad (3.12)$$

holds with $\varepsilon < \frac{1}{c}$, where $c = \max(c_1, c_2^2)$;

(b) *the pair (A, D) satisfies Kalman rank condition (1.10) with D given by (1.11);*

(c) *the operator L is g_s -observable for $1 \leq s \leq M$.*

Then, the over-determined system (2.12)–(2.13) has only the trivial solution.

Proof Applying D_s to (2.12) and noting $W = D_s \Phi$, we get

$$(\beta^2 - a)W - LW = D_s A \Phi - aW. \quad (3.13)$$

Setting

$$W = (w_j), \quad D_s A \Phi - aW = (f_j), \quad D_s = (d_{ij}^{(s)}) \quad (3.14)$$

for $1 \leq i, j \leq N$, we have

$$w_j = \sum_{i=1}^N d_{ij}^{(s)} \phi_i = \sum_{k=1}^m \sum_{i=\mu_{k-1}+1}^{\mu_k} d_{ij}^{(s)} \phi_i \quad (3.15)$$

and

$$f_j = \sum_{k=1}^m (\lambda_k - a) \sum_{i=\mu_{k-1}+1}^{\mu_k} d_{ij}^{(s)} \phi_i. \quad (3.16)$$

On the other hand, noting that $\mathcal{G}_s$ is diagonal, condition (2.13) leads to

$$\mathcal{G}_s W = \mathcal{G}_s D_s \Phi = D_s \mathcal{G}_s \Phi = 0. \quad (3.17)$$

Then, taking the j -th component of (3.13) and (3.17), we get

$$(\beta^2 - a)w_j - Lw_j = f_j \quad (3.18)$$

with the additional condition

$$g_s w_j = 0. \quad (3.19)$$

If $\beta^2 - a \leq 0$, multiplying (3.18) by w_j , we get

$$-(\beta^2 - a)\|w_j\|_H^2 + \|w_j\|_V^2 = -(f_j, w_j)_H \leq \|f_j\|_H \|w_j\|_H. \quad (3.20)$$

Then, noting (3.11), we have

$$\|w_j\|_H \leq c\|f_j\|_H. \quad (3.21)$$

If $\beta^2 - a > 0$, the observability of (L, g_s) implies again (3.21).

On the other hand, noting that L is self-adjoint, we have

$$(\phi_i, \phi_j)_H = 0, \quad \mu_{k-1} + 1 \leq i \leq \mu_k, \quad \mu_{l-1} + 1 \leq j \leq \mu_l, \quad k \neq l. \quad (3.22)$$

Then it follows from (3.16) that

$$\begin{aligned} \|f_j\|_H^2 &\leq \sup_{1 \leq k \leq m} |a - \lambda_k|^2 \sum_{k=1}^m \left\| \sum_{i=\mu_{k-1}+1}^{\mu_k} d_{ij}^{(s)} \phi_i \right\|_H^2 \\ &= \sup_{1 \leq k \leq m} |a - \lambda_k|^2 \|w_j\|_H^2, \quad j = 1, \dots, N. \end{aligned} \quad (3.23)$$

Hence, noting the ε -closing condition (3.12) and (3.21), we get

$$\|f_j\|_H \leq \sup_{1 \leq k \leq m} |a - \lambda_k| \|w_j\|_H \leq \varepsilon \|w_j\|_H \leq c\varepsilon \|f_j\|_H. \quad (3.24)$$

Then, it follows from (3.24) that $f_j = 0$ and then $w_j = 0$ for $j = 1, \dots, N$, provided that $\varepsilon c < 1$. Thus we get

$$\sum_{i=\mu_{k-1}+1}^{\mu_k} d_{ij}^{(s)} \phi_i = 0, \quad 1 \leq j \leq N, \quad (3.25)$$

namely

$$\sum_{i=\mu_{k-1}+1}^{\mu_k} d_i^{(s)} \phi_i = 0, \quad 1 \leq k \leq m, \quad (3.26)$$

where $d_i^{(s)}$ is the i -th column vector of the matrix D_s .

Noting (3.7), we arrange (3.26) by blocks into the following expression

$$\sum_{i=\mu_{k-1}+1}^{\mu_k} \begin{pmatrix} d_i^{(s)} \\ \vdots \\ d_i^{(M)} \end{pmatrix} \phi_i = \sum_{i=\mu_{k-1}+1}^{\mu_k} d_i \phi_i = 0, \quad 1 \leq k \leq m. \quad (3.27)$$

By Proposition 3.2, the column vectors $d_{\mu_{k-1}+1}, \dots, d_{\mu_k}$ of D^T are linearly independent, then we get

$$\phi_i = 0, \quad \mu_{k-1} + 1 \leq i \leq \mu_k, \quad 1 \leq k \leq m, \quad (3.28)$$

namely, $\Phi \equiv 0$. The proof is thus complete.

Theorem 3.1 can be read as “under suitable conditions, the observability of the scalar equation implies the stability of the whole system”. By this way, we provide a simple and efficient approach to solve a seemingly difficult problem of asymptotic stability of a complex system.

As a direct consequence of Theorems 2.1 and 3.1, we have the following important result.

Theorem 3.2 *Under the same assumptions as those in Theorem 3.1, system (1.7) is asymptotically stable.*

4 Asymptotic Synchronization by Groups

By Proposition 3.1, when the pair (A, D) does not satisfy Kalman rank condition (1.10), system (1.7) is not asymptotically stable. Instead of stability, we consider the asymptotic synchronization by groups.

Let $p \geq 1$ be an integer such that

$$0 = n_0 < n_1 < \dots < n_s = N \quad (4.1)$$

with $n_r - n_{r-1} \geq 2$ for $r = 1, \dots, p$. We re-arrange the components of the state variable U into p groups

$$(u^{(1)}, \dots, u^{(n_1)}), (u^{(n_1+1)}, \dots, u^{(n_2)}), \dots, (u^{(n_{p-1}+1)}, \dots, u^{(n_p)}). \quad (4.2)$$

Let S_r be a full row-rank matrix of order $(n_r - n_{r-1} - 1) \times (n_r - n_{r-1})$:

$$S_r = \begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \end{pmatrix}, \quad 1 \leq r \leq p. \quad (4.3)$$

Define the $(N - p) \times N$ matrix C_p of synchronization by p -groups as

$$C_p = \begin{pmatrix} S_1 & & & \\ & S_2 & & \\ & & \ddots & \\ & & & S_p \end{pmatrix}. \quad (4.4)$$

Let

$$e_r = (0, \dots, 0, \overset{(n_{r-1}+1)}{1}, \dots, \overset{(n_r)}{1}, 0, \dots, 0)^T, \quad 1 \leq r \leq p. \quad (4.5)$$

Then

$$\text{Ker}(C_p) = \text{Span}\{e_1, \dots, e_p\}. \quad (4.6)$$

Definition 4.1 System (1.7) is asymptotically synchronizable by p -groups, if for any given initial data $(U_0, U_1) \in \mathcal{V} \times \mathcal{H}$, the corresponding solution U satisfies

$$(u^{(k)} - u^{(l)}, u^{(k)'} - u^{(l)'}) \rightarrow (0, 0) \quad \text{in } V \times H \quad (4.7)$$

as $t \rightarrow +\infty$ for all $n_{r-1} + 1 \leq k, l \leq n_r$ and $1 \leq r \leq p$, or equivalently

$$C_p(U, U') \rightarrow (0, 0) \quad \text{in } (V \times H)^{N-p} \quad \text{as } t \rightarrow +\infty. \quad (4.8)$$

Let us recall some known results.

If system (1.1) is asymptotically synchronizable by p -groups, by [29, Theorem 4.7], we have

$$\text{rank}(D, AD, \dots, A^{N-1}D) \geq N - p. \quad (4.9)$$

Moreover, if system (1.1) is asymptotically synchronizable by p -groups under the minimum rank condition

$$\text{rank}(D, AD, \dots, A^{N-1}D) = N - p, \quad (4.10)$$

by [29, Theorem 4.8], A satisfies the condition of C_p -compatibility:

$$A\text{Ker}(C_p) \subseteq \text{Ker}(C_p) \quad (4.11)$$

and D satisfies the condition of strong C_p -compatibility:

$$\text{Ker}(C_p) \subseteq \text{Ker}(D_s), \quad 1 \leq s \leq M. \quad (4.12)$$

In this situation, by [29, Proposition 4.2], there exist a symmetric and positive semi-definite matrix $\bar{A}$ of order $(N - p)$ and a symmetric and positive semi-definite matrices $\bar{D}_s$ ($1 \leq s \leq M$) of order $(N - p)$, such that

$$(C_p C_p^T)^{-\frac{1}{2}} C_p A = \bar{A} (C_p C_p^T)^{-\frac{1}{2}} C_p \quad (4.13)$$

and

$$(C_p C_p^T)^{-\frac{1}{2}} C_p D_s = \bar{D}_s (C_p C_p^T)^{-\frac{1}{2}} C_p, \quad 1 \leq s \leq M. \quad (4.14)$$

Applying $(C_p C_p^T)^{-\frac{1}{2}} C_p$ to system (1.7) and setting $W = (C_p C_p^T)^{-\frac{1}{2}} C_p U$, we get the following reduced system

$$W'' + \mathcal{L}W + \overline{A}W + \sum_{s=1}^M \overline{D}_s \mathcal{G}_s W' = 0. \quad (4.15)$$

Obviously, the asymptotic synchronization by p -groups of system (1.7) is equivalent to the asymptotic stability of the reduced system (4.15).

Since the reduced matrices $\overline{A}$ and $\overline{D}_s$ ($1 \leq s \leq M$) are still symmetric and positive semi-definite, the asymptotic stability of the reduced system (1.7) can be treated by Theorem 3.2. More precisely, let

$$\overline{D} = (\overline{D}_1, \overline{D}_2, \dots, \overline{D}_M). \quad (4.16)$$

Since $\overline{A}$ is of order $(N - p)$, by Proposition 2.2, the following rank condition on the reduced matrices $\overline{A}$ and $\overline{D}$:

$$\text{rank}(\overline{D}, \overline{A}\overline{D}, \dots, \overline{A}^{N-p-1}\overline{D}) = N - p \quad (4.17)$$

is required for the asymptotic stability of the reduced system (4.15).

We first present a basic relation between the rank of the original matrices and the reduced ones.

Proposition 4.1 *Let A satisfy (4.11), respectively, D satisfy (4.12). We have*

$$\text{rank}(\overline{D}, \overline{A}\overline{D}, \dots, \overline{A}^{N-p-1}\overline{D}) = \text{rank}(D, AD, \dots, A^{N-1}D). \quad (4.18)$$

Proof First, it follows from (4.14) that

$$\overline{D}_s = (C_p C_p^T)^{-\frac{1}{2}} C_p D_s C_p^T (C_p C_p^T)^{-\frac{1}{2}}, \quad 1 \leq s \leq M.$$

Then, using (4.16), we have

$$\overline{D} = (C_p C_p^T)^{-\frac{1}{2}} C_p D \{C_p^T\}_M \{(C_p C_p^T)^{-\frac{1}{2}}\}_M,$$

where

$$\{C_p^T\}_M = \begin{pmatrix} C_p^T & & & \\ & C_p^T & & \\ & & \ddots & \\ & & & C_p^T \end{pmatrix}_M \quad (4.19)$$

is a diagonal matrix of M blocks of C_p^T , respectively,

$$\{(C_p C_p^T)^{-\frac{1}{2}}\}_M = \begin{pmatrix} (C_p C_p^T)^{-\frac{1}{2}} & & & \\ & (C_p C_p^T)^{-\frac{1}{2}} & & \\ & & \ddots & \\ & & & (C_p C_p^T)^{-\frac{1}{2}} \end{pmatrix}_M \quad (4.20)$$

is a diagonal matrix of M blocks of $(C_p C_p^T)^{-\frac{1}{2}}$.

Noting (4.13) and (4.19), we have

$$\begin{aligned}\overline{A} \overline{D} &= \overline{A} (C_p C_p^T)^{-\frac{1}{2}} C_p D \{C_p^T\}_M \{(C_p C_p^T)^{-\frac{1}{2}}\}_M \\ &= (C_p C_p^T)^{-\frac{1}{2}} C_p A D \{C_p^T\}_M \{(C_p C_p^T)^{-\frac{1}{2}}\}_M.\end{aligned}\quad (4.21)$$

Successively, we have

$$\begin{aligned}\overline{A}^2 \overline{D} &= \overline{A} (\overline{A} \overline{D}) \\ &= \overline{A} (C_p C_p^T)^{-\frac{1}{2}} C_p A D \{C_p^T\}_M \{(C_p C_p^T)^{-\frac{1}{2}}\}_M \\ &= (C_p C_p^T)^{-\frac{1}{2}} C_p A^2 D \{C_p^T\}_M \{(C_p C_p^T)^{-\frac{1}{2}}\}_M, \dots\end{aligned}\quad (4.22)$$

Thus, we have

$$\begin{aligned}&(\overline{D}, \overline{A} \overline{D}, \dots, \overline{A}^{N-1} \overline{D}) \\ &= (C_p C_p^T)^{-\frac{1}{2}} C_p (D, AD, \dots, A^{N-1} D) \{\{C_p^T\}_M\}_N \{\{(C_p C_p^T)^{-\frac{1}{2}}\}_M\}_N,\end{aligned}\quad (4.23)$$

where

$$\{\{C_p^T\}_M\}_N = \begin{pmatrix} \{C_p^T\}_M & & & \\ & \{C_p^T\}_M & & \\ & & \ddots & \\ & & & \{C_p^T\}_M \end{pmatrix}_N \quad (4.24)$$

is a diagonal matrix of N blocks of $\{C_p^T\}_M$, similarly, $\{\{(C_p C_p^T)^{-\frac{1}{2}}\}_M\}_N$ is a diagonal matrix of N blocks of $\{(C_p C_p^T)^{-\frac{1}{2}}\}_M$.

Since $(C_p C_p^T)^{-\frac{1}{2}}$ and $\{\{(C_p C_p^T)^{-\frac{1}{2}}\}_M\}_N$ are invertible, by Cayley-Hamilton's theorem, it follows from (4.23) that

$$\begin{aligned}&\text{rank}(\overline{D}, \overline{A} \overline{D}, \dots, \overline{A}^{N-p-1} \overline{D}) \\ &= \text{rank}(\overline{D}, \overline{A} \overline{D}, \dots, \overline{A}^{N-1} \overline{D}) \\ &= \text{rank} C_p (D, AD, \dots, A^{N-1} D) \{\{C_p^T\}_M\}_N.\end{aligned}\quad (4.25)$$

On the other hand, since A is symmetric, by the condition of C_p -compatibility (4.11), we have $A \text{Im}(C_p^T) \subseteq \text{Im}(C_p^T)$. On the other hand, the condition of strong C_p -compatibility (4.12) implies that $\text{Im}(D) \subseteq \text{Im}(C_p^T)$. Then, we can successively get

$$\text{Im}(AD) = A \text{Im}(D) \subseteq A \text{Im}(C_p^T) \subseteq \text{Im}(C_p^T), \dots \quad (4.26)$$

It follows that

$$\text{Im}(D, AD, \dots, A^{N-1} D) \subseteq \text{Im}(C_p^T). \quad (4.27)$$

Then, we get

$$\text{Ker}(C_p) \cap \text{Im}(D, AD, \dots, A^{N-1} D) \subseteq \text{Ker}(C_p) \cap \text{Im}(C_p^T) = \{0\}. \quad (4.28)$$

By [28, Proposition 2.7], we get

$$\begin{aligned} & \text{rank} C_p(D, AD \cdots, A^{N-1}D) \{\{C_p^T\}_M\}_N \\ &= \text{rank}(D, AD \cdots, A^{N-1}D) \{\{C_p^T\}_M\}_N. \end{aligned} \quad (4.29)$$

Now, consider the transposition of the matrix in the right-hand side of (4.29) :

$$\{\{C_p\}_M\}_N \begin{pmatrix} D^T \\ D^T A \\ \vdots \\ D^T A^{N-1} \end{pmatrix}. \quad (4.30)$$

First, we have

$$\text{Ker}(\{\{C_p\}_M\}_N) = (\text{Ker}(C_p))^{MN}. \quad (4.31)$$

Next, by the condition of strong C_p -compatibility (4.12), we have $\text{Im}(D_s) \subseteq \text{Im}(C_p^T)$ for $1 \leq s \leq M$, namely,

$$\text{Im}(D^T) = \text{Im} \begin{pmatrix} D_1 \\ D_2 \\ \vdots \\ D_M \end{pmatrix} \subseteq (\text{Im}(C_p^T))^M. \quad (4.32)$$

Then we get

$$\text{Im} \begin{pmatrix} D^T \\ D^T A \\ \vdots \\ D^T A^{N-1} \end{pmatrix} \subseteq \text{Im} \begin{pmatrix} D^T \\ D^T \\ \vdots \\ D^T \end{pmatrix} \subseteq (\text{Im}(C_p^T))^{MN}. \quad (4.33)$$

It follows that

$$\begin{aligned} & \text{Ker}(\{\{C_p\}_M\}_N) \cap \text{Im} \begin{pmatrix} D^T \\ D^T A \\ \vdots \\ D^T A^{N-1} \end{pmatrix} \\ & \subseteq (\text{Ker}(C_p))^{MN} \cap (\text{Im}(C_p^T))^{MN} \\ & = (\text{Ker}(C_p))^{MN} \cap ((\text{Ker}(C_p))^\perp)^{MN} \\ & \subseteq (\text{Ker}(C_p))^{MN} \cap ((\text{Ker}(C_p))^{MN})^\perp = \{0\}. \end{aligned} \quad (4.34)$$

Once again, by [28, Proposition 2.7], we get

$$\text{rank} \{\{C_p\}_M\}_N \begin{pmatrix} D^T \\ D^T A \\ \vdots \\ D^T A^{N-1} \end{pmatrix} = \text{rank} \begin{pmatrix} D^T \\ D^T A \\ \vdots \\ D^T A^{N-1} \end{pmatrix}, \quad (4.35)$$

namely,

$$\begin{aligned} & \text{rank}(D, AD, \dots, A^{N-1}D)\{\{C_p^T\}_M\}_N \\ &= \text{rank}(D, AD, \dots, A^{N-1}D). \end{aligned} \quad (4.36)$$

Finally, combining (4.25), (4.29) and (4.36), we get the desired rank relation (4.18). The proof is achieved.

As a direct application of Theorem 3.2, we have the following result.

Theorem 4.1 *Let A satisfy (4.11), respectively, D satisfy (4.12). Assume that*

- (a) *the ε -closing condition (3.12) holds,*
- (b) *the pair (A, D) satisfies the rank condition (4.10),*
- (c) *the operator L is g_s -observable for $1 \leq s \leq M$.*

Then, system (1.7) is asymptotically synchronizable by p -groups. Moreover, for any given initial data $(U_0, U_1) \in \mathcal{V} \times \mathcal{H}$, there exist linearly independent functions $u_1, \dots, u_p$ such that

$$(u^{(k)}(t) - u_r(t), (u^{(k)})'(t) - u'_r(t)) \rightarrow (0, 0) \quad \text{in } V \times H \quad (4.37)$$

as $t \rightarrow +\infty$ for all $n_{r-1} + 1 \leq k \leq n_r$ and $1 \leq r \leq p$.

Proof First, by [28, Proposition 2.21], the spectrum of $\overline{A}$ is a part of that of A , so, the ε -closing condition (3.12) still holds for the reduced matrix $\overline{A}$. On the other hand, combining the rank condition (4.10) and the rank relation (4.18), we get

$$\text{rank}(\overline{D}, \overline{A}\overline{D}, \dots, \overline{A}^{N-p-1}\overline{D}) = N - p. \quad (4.38)$$

We can thus apply Theorem 3.2 to the reduce system (4.15) for obtaining the approximate stability.

Moreover, for any given initial data $(U_0, U_1) \in (V \times H)^N$, let U be the corresponding solution to system (1.7). Let $u_r = \frac{\langle U, e_r \rangle}{\|e_r\|}$ for $r = 1, \dots, p$. Then, projecting to $\text{Ker}(C_p)$ and to $\text{Im}(C_p^T)$, respectively, we get

$$U = \sum_{r=1}^p \frac{u_r e_r}{\|e_r\|} + C_p^T (C_p C_p^T)^{-1} C_p U. \quad (4.39)$$

Moreover, by (4.8), we get

$$\begin{aligned} & \left(U - \sum_{r=1}^p \frac{u_r e_r}{\|e_r\|}, U' - \sum_{r=1}^p \frac{u'_r e_r}{\|e_r\|} \right) \\ &= C_p^T (C_p C_p^T)^{-1} (C_p U, C_p U') \rightarrow (0, 0) \quad \text{in } \mathcal{V} \times \mathcal{H} \quad \text{as } t \rightarrow +\infty. \end{aligned} \quad (4.40)$$

Noting (4.5), we see that (4.40) exactly means (4.37).

Now we will precisely show the dynamics of the functions $u_1, \dots, u_p$. Since A is symmetric, noting (4.11), there exist some real numbers α_{rl} with $\alpha_{rl} = \alpha_{lr}$, such that

$$Ae_r = \sum_{l=1}^p \alpha_{rl} \frac{\|e_r\|}{\|e_l\|} e_l, \quad r = 1, \dots, p. \quad (4.41)$$

Moreover, by (4.12) we have

$$D_s e_r = 0, \quad r = 1, \dots, p; \quad s = 1, \dots, M. \quad (4.42)$$

Then, applying e_r to (1.7), we get

$$u_r'' + L u_r + \sum_{l=1}^p \alpha_{rl} u_l = 0 \quad (4.43)$$

associated with the initial data

$$t = 0 : \quad u_r = \frac{(U_0, e_r)}{\|e_r\|}, \quad u_r' = \frac{(U_1, e_r)}{\|e_r\|}. \quad (4.44)$$

The proof is complete.

Remark 4.1 The convergence (4.7) is called the asymptotic synchronization by p -groups in the consensus sense, while the convergence (4.37) is in the pinning sense. $(u_1, \dots, u_p)^T$ is called the asymptotically synchronizable state by p -groups. Theorem 4.1 indicates that the two notions are simply the same. Moreover, since the functions $u_1, \dots, u_p$ are linearly independent, there does not exist an extended matrix $\tilde{C}_q$ ($q < p$) such that

$$\tilde{C}_q(U(t), U'(t)) \rightarrow (0, 0) \quad \text{in } (V \times H)^{N-q} \quad \text{as } t \rightarrow +\infty. \quad (4.45)$$

Therefore, unlike the case of approximate boundary synchronization by p -groups (see Chapter 11 in [28]), there is no possibility to get any induced synchronization in the present situation.

5 Applications

In this section, we denote by $\Omega \subset \mathbb{R}^n$ a bounded domain with smooth boundary Γ and by $\omega \subset \Omega$ a neighbourhood of the boundary Γ .

Let a and b be given smooth and positive functions in Ω such that

$$a(x) \geq a_0 > 0, \quad b(x) \geq b_0 > 0, \quad \forall x \in \omega. \quad (5.1)$$

The coupling matrix A , as well as the damping matrices $D_1, D_2, \dots$, appearing in diverse models, are assumed to be symmetric and positive semi-definite.

5.1 Wave equations with mixed dampings

Consider the following system of wave equations with boundary viscous damping and locally distributed viscous and Kelvin-Voigt dampings (see [36])

$$\begin{cases} U'' - \Delta U + AU + aD_1 U' - D_2 \operatorname{div}(b \nabla U') = 0 & \text{in } \Omega, \\ \partial_\nu U + D_3 U' = 0 & \text{on } \Gamma, \end{cases} \quad (5.2)$$

where ∂_ν denotes the outward normal derivative on the boundary.

Let

$$H = L^2(\Omega), \quad V = H^1(\Omega). \quad (5.3)$$

Multiplying system (5.2) by a test function $\Phi \in (H^1(\Omega))^N$ and integrating by parts, we get the variational formulation:

$$\begin{aligned} & \int_{\Omega} (U'', \Phi) dx + \int_{\Omega} (\nabla U, \nabla \Phi) dx + \int_{\Omega} (AU, \Phi) dx \\ & + \int_{\Omega} (aD_1 U', \Phi) dx + \int_{\Omega} (bD_2 \nabla U', \nabla \Phi) dx + \int_{\Gamma} (D_3 U', \Phi) d\Gamma = 0, \end{aligned} \quad (5.4)$$

where $(\cdot, \cdot)$ denotes the inner product in $\mathbb{R}^N$ or in $\mathbb{M}^N(\mathbb{R})$.

Let L be defined by

$$\langle Lu, \phi \rangle = \int_{\Omega} \nabla u \cdot \nabla \phi dx, \quad (5.5)$$

respectively, g_1, g_2 and g_3 be defined by

$$\begin{cases} \langle g_1 u, \phi \rangle = \int_{\Omega} au\phi dx, \\ \langle g_2 u, \phi \rangle = \int_{\Omega} b\nabla u \cdot \nabla \phi dx, \\ \langle g_3 u, \phi \rangle = \int_{\Gamma} u\phi d\Gamma. \end{cases} \quad (5.6)$$

Setting $\mathcal{L}, \mathcal{G}_1, \mathcal{G}_2$ and $\mathcal{G}_3$ as in (1.6), the variational equation (5.4) can be rewritten as

$$U'' + \mathcal{L}U + AU + D_1 \mathcal{G}_1 U' + D_2 \mathcal{G}_2 U' + D_3 \mathcal{G}_3 U' = 0. \quad (5.7)$$

Obviously, the operators L, g_1, g_2 and g_3 satisfy conditions (1.1)–(1.4). Then, system (5.7) generates a semi-group of contractions with compact resolvent in the space $(H^1(\Omega) \times L^2(\Omega))^N$.

Theorem 5.1 *Let A satisfy (4.11), respectively, D satisfy (4.12). Assume furthermore that A satisfies (3.12) and the pair (A, D) satisfies (4.10) with $D = (D_1, D_2, D_3)$. Then system (5.2) is asymptotically synchronizable by p -groups in the space $(H^1(\Omega) \times L^2(\Omega))^N$.*

Proof By Theorem 4.1, it is sufficient to show that there exists $c > 0$ independent of $\beta \in \mathbb{R}$ and $f \in L^2(\Omega)$, such that the following uniform observability inequality

$$\int_{\Omega} |\phi|^2 dx \leq c \int_{\Omega} |f|^2 dx \quad (5.8)$$

holds for any given solution $\phi \in H^1(\Omega)$ to the over-determined system

$$\begin{cases} \beta^2 \phi + \Delta \phi = f & \text{in } \Omega, \\ \partial_{\nu} \phi = 0 & \text{on } \Gamma \end{cases} \quad (5.9)$$

associated with each of the following conditions

$$\begin{cases} \phi = 0 & \text{in } \Gamma, \\ a\phi = 0 & \text{in } \Omega, \\ \operatorname{div}(b\nabla \phi) = 0 & \text{in } \Omega. \end{cases} \quad (5.10)$$

Let $m = x - x_0$. Recall the following formula

$$2 \int_{\Omega} \phi(m \cdot \nabla \phi) dx = -n \int_{\Omega} |\phi|^2 dx + \int_{\Gamma} (m \cdot \nu) |\phi|^2 d\Gamma \quad (5.11)$$

for all $\phi \in H^1(\Omega)$, and Rellich's identity (see [33])

$$\begin{aligned} & 2 \int_{\Omega} \Delta \phi(m \cdot \nabla \phi) dx \\ &= (n-2) \int_{\Omega} |\nabla \phi|^2 dx + \int_{\Gamma} (2\partial_{\nu} \phi(m \cdot \nabla \phi) - (m \cdot \nu) |\nabla \phi|^2) d\Gamma \end{aligned} \quad (5.12)$$

for all $\phi \in H^2(\Omega)$.

Case 1 $\phi = 0$ on Γ . Multiplying the equation in (5.9) by $2m \cdot \nabla \phi + (n-1)\phi$, we get

$$\begin{aligned} & \beta^2 \int_{\Omega} \phi(2m \cdot \nabla \phi + (n-1)\phi) dx + \int_{\Omega} \Delta \phi(2m \cdot \nabla \phi + (n-1)\phi) dx \\ &= \int_{\Omega} f(2m \cdot \nabla \phi + (n-1)\phi) dx. \end{aligned} \quad (5.13)$$

Since $\phi \in H_0^2(\Omega)$, applying formula (5.11) to the first term on the left-hand side of (5.13) gives

$$\beta^2 \int_{\Omega} \phi(2m \cdot \nabla \phi + (n-1)\phi) dx = -\beta^2 \int_{\Omega} |\phi|^2 dx. \quad (5.14)$$

Similarly, applying Rellich's identity (5.12) to the second term on the left-hand side of (5.13) gives

$$\int_{\Omega} \Delta \phi(2m \cdot \nabla \phi + (n-1)\phi) dx = - \int_{\Omega} |\nabla \phi|^2 dx. \quad (5.15)$$

Inserting (5.14) and (5.15) into (5.13) gives

$$\int_{\Omega} (|\beta \phi|^2 + |\nabla \phi|^2) dx = - \int_{\Omega} f(2m \cdot \nabla \phi + (n-1)\phi) dx. \quad (5.16)$$

For any given $\varepsilon > 0$, by Cauchy-Schwarz's inequality, there exists a positive constant C_{ε} such that

$$\int_{\Omega} (|\beta \phi|^2 + |\nabla \phi|^2) dx \leq C_{\varepsilon} \int_{\Omega} |f|^2 dx + \varepsilon \int_{\Omega} |\phi|^2 dx. \quad (5.17)$$

Since $\phi \in H_0^2(\Omega)$, by Poincaré's inequality, for $\varepsilon > 0$ small enough, we can find a positive constant c such that

$$\int_{\Omega} (|\phi|^2 + |\nabla \phi|^2) dx \leq c \int_{\Omega} |f|^2 dx. \quad (5.18)$$

This is a stronger version of (5.8).

Case 2 $a\phi = 0$ in Ω . Using (5.1), we get $\phi = 0$ on Γ , then, we return to Case 1.

Case 3 $\operatorname{div}(b\nabla \phi) = 0$ in Ω . Integrating by parts, we get

$$\int_{\Omega} \operatorname{div}(b\nabla \phi) \phi dx = \int_{\Omega} b |\nabla \phi|^2 dx = 0. \quad (5.19)$$

Since $b \geq 0$, we get $b\nabla \phi = 0$ in Ω . Noting (5.1), it follows that $\nabla \phi = 0$ in ω . Then by the homogeneous boundary condition on Γ , we get $\phi = 0$ in ω , in particular, $\phi = 0$ on Γ , then we return to Case 1.

Remark 5.1 In fact, the uniform estimate (5.8) is based on the uniform stability of equation (5.2) for a scalar equation (i.e. for $N = 1$), which was abundantly studied by different approaches in literatures. We only quote [13, 16–17] and the references therein for boundary feedback. The uniform decay was first established by multipliers in [10] as ω is a neighbourhood of the boundary. The explicit decay rate was given in [41] under suitable geometric condition. Later, the result was generalized in [44] to semi-linear case. When Ω is a compact Riemann manifold without boundary and ω satisfies the geometric optic condition, the uniform stability was established by a micro-local approach in [39]. Moreover, the volume of the damping region ω can be sufficiently small in [5] etc.

5.2 Kirchhoff plate equations with locally distributed Kelvin-Voigt dampings

Consider the following system of Kirchhoff plate equations with locally distributed viscous and Kelvin-Voigt dampings (see [8, 14–15]) for more precise description):

$$\begin{cases} U'' + \Delta^2 U + AU + aD_1U' + D_2\Delta(b\Delta U') = 0 & \text{in } \Omega, \\ U = \partial_\nu U = 0 & \text{on } \Gamma. \end{cases} \quad (5.20)$$

Let

$$H = L^2(\Omega), \quad V = H_0^2(\Omega). \quad (5.21)$$

Multiplying system (5.20) by a test function $\Phi \in (H_0^2(\Omega))^N$ and integrating by parts, we get the following variational formulation:

$$\begin{aligned} & \int_{\Omega} (U'', \Phi) dx + \int_{\Omega} (\nabla U, \nabla \Phi) dx + \int_{\Omega} (AU, \Phi) dx \\ & + \int_{\Omega} (aD_1U', \Phi) dx + \int_{\Omega} (bD_2\Delta U', \Delta \Phi) dx = 0, \end{aligned} \quad (5.22)$$

where $(\cdot, \cdot)$ denotes the inner product in $\mathbb{R}^N$ or in $\mathbb{M}^N(\mathbb{R})$.

Let L be defined by

$$\langle Lu, \phi \rangle = \int_{\Omega} \nabla u \cdot \nabla \phi dx, \quad (5.23)$$

respectively, g_1, g_2 be defined by

$$\langle g_1 u, \phi \rangle = \int_{\Omega} au \phi dx, \quad \langle g_2 u, \phi \rangle = \int_{\Omega} b \Delta u \Delta \phi dx. \quad (5.24)$$

Setting $\mathcal{L}$, $\mathcal{G}_1$ and $\mathcal{G}_2$ as in (1.6), the variational equation (5.22) can be rewritten as

$$U'' + \mathcal{L}U + AU + D_1\mathcal{G}_1U' + D_2\mathcal{G}_2U' = 0. \quad (5.25)$$

Theorem 5.2 *Let A satisfy (4.11), respectively, D satisfy (4.12). Assume that A satisfies (3.12) and the pair (A, D) satisfies (4.10) with $D = (D_1, D_2)$. Then system (5.20) is asymptotically synchronizable by p -groups in the space $(H_0^2(\Omega) \times L^2(\Omega))^N$.*

Proof By Theorem 4.1, it is sufficient to show that there exists $c > 0$ independent of $\beta \in \mathbb{R}$ and $f \in L^2(\Omega)$, such that the the following uniform observability inequality

$$\int_{\Omega} |\phi|^2 dx \leq c \int_{\Omega} |f|^2 dx \quad (5.26)$$

holds for any solution ϕ to the system

$$\begin{cases} -\beta^2 \phi + \Delta^2 \phi = f & \text{in } \Omega, \\ \phi = \partial_{\nu} \phi = 0 & \text{on } \Gamma \end{cases} \quad (5.27)$$

associated with each of the conditions

$$\begin{cases} a\phi = 0 & \text{in } \Omega, \\ \Delta(b\Delta\phi) = 0 & \text{in } \Omega. \end{cases} \quad (5.28)$$

Let us recall a formula of integration by parts (see [33]):

$$2 \int_{\Omega} (\Delta^2 \phi)(m \cdot \nabla \phi) dx = (4 - n) \int_{\Omega} |\Delta \phi|^2 dx, \quad \phi \in H_0^4(\Omega). \quad (5.29)$$

Case 1 $a\phi = 0$ in Ω . By (5.1), we get $\phi = 0$ in ω . Multiplying the equation in (5.27) by $2m \cdot \nabla \phi + (n - 2)\phi$, we have

$$\begin{aligned} & -\beta^2 \int_{\Omega} \phi(2m \cdot \nabla \phi + (n - 2)\phi) dx + \int_{\Omega} \Delta^2(2m \cdot \nabla \phi + (n - 2)\phi) dx \\ &= - \int_{\Omega} f(2m \cdot \nabla \phi + (n - 2)\phi) dx. \end{aligned} \quad (5.30)$$

Since $\phi \in H_0^4(\Omega)$, applying formula (5.11) to the first term on the left-hand side of (5.30) gives

$$-\beta^2 \int_{\Omega} \phi(2m \cdot \nabla \phi + (n - 2)\phi) dx = 2\beta^2 \int_{\Omega} |\phi|^2 dx. \quad (5.31)$$

Applying formular (5.29) to the second term on the left-hand side of (5.30) gives

$$\int_{\Omega} \Delta^2 \phi(2m \cdot \nabla \phi + (n - 2)\phi) dx = 2 \int_{\Omega} |\Delta \phi|^2 dx. \quad (5.32)$$

Inserting (5.31) and (5.32) into (5.30) gives

$$2 \int_{\Omega} |\beta \phi|^2 dx + 2 \int_{\Omega} |\Delta \phi|^2 dx = \int_{\Omega} f(2m \cdot \nabla \phi + (n - 2)\phi) dx. \quad (5.33)$$

For any given $\varepsilon > 0$, by Cauchy-Schwarz's inequality, there exists a constant $C_{\varepsilon} > 0$ such that

$$\int_{\Omega} (|\beta \phi|^2 + |\Delta \phi|^2) dx \leq C_{\varepsilon} \int_{\Omega} |f|^2 dx + \varepsilon \int_{\Omega} (|\nabla \phi|^2 + |\phi|^2) dx. \quad (5.34)$$

Since $-\Delta$ is an isomorphism from $H_0^1(\Omega) \cap H^2(\Omega)$ onto $L^2(\Omega)$, for $\varepsilon > 0$ small enough, we can find a constant $c > 0$, such that

$$\|\phi\|_{H_0^2(\Omega)}^2 \sim \int_{\Omega} |\Delta \phi|^2 dx \leq c \int_{\Omega} |f|^2 dx, \quad (5.35)$$

which is a stronger version of (5.26).

Case 2 $\Delta(b\Delta\phi) = 0$ in Ω . Integrating by parts, we get

$$\int_{\Omega} \Delta(b\Delta\phi)\phi dx = \int_{\Omega} b|\Delta\phi|^2 dx = 0. \quad (5.36)$$

Since $b \geq 0$, we get $b\Delta\phi = 0$ in Ω . Using (5.1), it follows that $\Delta\phi = 0$ in ω . Then, noting the homogeneous boundary condition on Γ , Carleman uniqueness theorem (see [7, 40]) implies that $\phi = 0$ in ω , then we return to Case 1.

5.3 Euler-Bernoulli beam equations with mixed dampings

In the two previous subsections, we have considered the case of mixed dampings for wave equations and of locally distributed dampings for plate equations. However, when ω is not a neighbourhood of Γ , the situation is technically complicated! As a beginning, we will consider a system of beam equations. There are many to pursue... In particular, the discussion below can also be carried out for many other situations, such as Timoshenko beam [2, 12], Bresse beam [35] etc.

Let a, b be smooth and positive functions in $(0, 1)$ such that

$$a(x) \geq a_0 > 0, \quad b(x) \geq b_0 > 0, \quad 0 < \alpha^- < x < \alpha^+ < 1. \quad (5.37)$$

Consider the following system of Euler-Bernoulli beam equations with locally distributed and boundary dampings:

$$\begin{cases} U'' + U_{xxxx} + AU + aD_3U' - D_4(bU'_x)_x = 0 & \text{in } (0, 1), \\ U(0) = U_x(0) = 0, \\ U_{xxx}(1) = D_1U'(1), \\ U_{xx}(1) = -D_2U'_x(1). \end{cases} \quad (5.38)$$

Let

$$H = L^2(0, 1), \quad V = \{u \in H^2(0, 1) : u(0) = u_x(0) = 0\}. \quad (5.39)$$

Multiplying system (5.38) by $\Phi \in V^N$ and integrating by parts, we get

$$\begin{aligned} & \int_0^1 (U'', \Phi) dx + \int_0^1 (U_{xx}, \Phi_{xx}) dx + \int_0^1 (AU, \Phi) dx + (D_1U'(1), \Phi(1)) \\ & + (D_2U'_x(1), \Phi_x(1)) + \int_0^1 (aD_3U', \Phi) dx + \int_0^1 (bD_4U'_x, \Phi_x) dx = 0, \end{aligned} \quad (5.40)$$

where $(\cdot, \cdot)$ denotes the inner product in $\mathbb{R}^N$.

Let L be defined by

$$\langle Lu, \phi \rangle = \int_{\Omega} u_{xx} \phi_{xx} dx, \quad (5.41)$$

respectively, g_1, g_2, g_3 and g_4 be defined by

$$g_1 u = u(1), \quad g_2 u = u_x(1) \quad (5.42)$$

and

$$\langle g_3 u, \phi \rangle = \int_0^1 a u \phi dx, \quad \langle g_4 u, \phi \rangle = \int_0^1 b u_x \phi_x dx. \quad (5.43)$$

Setting $\mathcal{L}$, $\mathcal{G}_1$ and $\mathcal{G}_2$ as in (1.6), the variational equation (5.40) can be rewritten as

$$U'' + \mathcal{L}U + AU + D_1 \mathcal{G}_1 U' + D_2 \mathcal{G}_2 U' + D_3 \mathcal{G}_3 U' + D_4 \mathcal{G}_4 U' = 0. \quad (5.44)$$

Obviously, the operators L and g_1, g_2, g_3 and g_4 satisfy well conditions (1.1)–(1.4). Then, system (5.44) generates a semi-group of contractions with compact resolvent on the space $(V \times H)^N$.

Theorem 5.3 *Let A satisfy (4.11), respectively, D satisfy (4.12). Assume that A satisfies (3.12) and the pair (A, D) satisfies (4.10) with $D = (D_1, D_2, D_3, D_4)$. Then system (5.38) is asymptotically synchronizable by p -groups in the space $(V \times H)^N$.*

Proof By Theorem 4.1, it is sufficient to show that there exists a positive constant c , independent of $\beta \in \mathbb{R}$ and $f \in L^2(0, 1)$, such that the following uniform observability inequality

$$\int_0^1 |\phi|^2 dx \leq c \int_0^1 |f|^2 dx \quad (5.45)$$

holds for any solution to the system

$$\begin{cases} \beta^2 \phi - \phi_{xxxx} = f, & 0 < x < 1, \\ \phi(0) = \phi_x(0) = 0, \\ \phi_{xx}(1) = \phi_{xxx}(1) = 0 \end{cases} \quad (5.46)$$

associated with each of the conditions

$$\begin{cases} \phi(1) = 0, \\ \phi_x(1) = 0, \\ a\phi = 0, & 0 < x < 1, \\ (b\phi_x)_x = 0, & 0 < x < 1. \end{cases} \quad (5.47)$$

Case 1 $\phi(1) = 0$. Multiplying the equation in (5.46) by $2x\phi_x$ and integrating by parts, we get

$$\begin{aligned} & [x(\beta\phi)^2]_0^1 - 2[x\phi_x\phi_{xxx}]_0^1 + [x(\phi_{xx})^2]_0^1 + [\phi_x\phi_{xx}]_0^1 \\ &= \int_0^1 |\beta\phi|^2 dx + 3 \int_0^1 |\phi_{xx}|^2 dx + \int_0^1 2xf\phi_x dx. \end{aligned} \quad (5.48)$$

Using the boundary conditions in (5.46), it follows that

$$\int_0^1 (|\beta\phi|^2 dx + 3|\phi_{xx}|^2 dx) = - \int_0^1 2xf\phi_x dx. \quad (5.49)$$

Noting the boundary conditions $\phi(0) = \phi_x(0) = 0$, by Poincaré's inequality

$$\int_0^1 |\phi_{xx}|^2 dx \geq c \|\phi\|_{H^2(0,1)}^2 \quad (5.50)$$

and Cauchy-Schwarz' inequality, we get

$$\int_0^1 (|\beta\phi|^2 + |\phi_{xx}|^2) dx \leq c \int_0^1 |f|^2 dx. \quad (5.51)$$

Here and hereafter, c will denote a positive constant. It follows that

$$\|\phi\|_{H^2(0,1)} \leq c \|f\|_{L^2(0,1)}, \quad (5.52)$$

which is a stronger version of (5.45).

Case 2 $\phi_x(1) = 0$. Multiplying the equation in (5.46) by $2x\phi_{xxx}$ and integrating by parts, we get

$$\begin{aligned} & 2\beta^2[\phi x\phi_{xx}]_0^1 - \beta^2[x\phi_x^2]_0^1 - 2\beta^2[\phi\phi_x]_0^1 - [x\phi_{xxx}^2]_0^1 \\ &= -3\beta^2 \int_0^1 \phi_x^2 dx - \int_0^1 \phi_{xxx}^2 dx + \int_0^1 2xf\phi_{xxx} dx. \end{aligned} \quad (5.53)$$

Using the boundary conditions in (5.46), it follows that

$$\int_0^1 (3|\beta\phi_x|^2 + |\phi_{xxx}|^2) dx = \int_0^1 2xf\phi_{xxx} dx. \quad (5.54)$$

By Cauchy-Schwarz' inequality, we have

$$\int_0^1 (|\beta\phi_x|^2 + |\phi_{xxx}|^2) dx \leq c \int_0^1 |f|^2 dx. \quad (5.55)$$

Noting the boundary conditions $\phi(0) = \phi_x(0) = \phi_{xx}(1) = 0$, by Poincaré's inequality

$$\int_0^1 |\phi_{xxx}|^2 dx \geq c \|\phi\|_{H^3(0,1)}^2, \quad (5.56)$$

we get a stronger version of (5.45):

$$\|\phi\|_{H^3(0,1)} \leq c \|f\|_{L^2(0,1)}. \quad (5.57)$$

Case 3 $a\phi = 0$ for $0 < x < 1$. The condition implies that $\phi \equiv 0$ for $\alpha^- < x < \alpha^+$, then, in particular, $\phi(\alpha^-) = \phi_x(\alpha^-) = \phi_{xx}(\alpha^-) = 0$. Applying (5.51) on the interval $(0, \alpha^-)$, we have

$$\int_0^{\alpha^-} (|\beta\phi|^2 + |\phi_{xx}|^2) dx \leq \int_0^{\alpha^-} |f|^2 dx. \quad (5.58)$$

On the interval $(\alpha^+, 1)$, we have $\phi(\alpha^+) = \phi_x(\alpha^+) = \phi_{xx}(\alpha^+) = 0$. Multiplying the equation in (5.46) by $2(x-1)\phi_x$ and applying (5.48) on the interval $(\alpha^+, 1)$, we get

$$\int_{\alpha^+}^1 (|\beta\phi|^2 + 3|\phi_{xx}|^2) dx = - \int_{\alpha^+}^1 2xf\phi_x dx. \quad (5.59)$$

By Cauchy-Schwarz' inequality, we have

$$\int_{\alpha^+}^1 (|\beta\phi|^2 + |\phi_{xx}|^2) dx \leq c \int_{\alpha^+}^1 |f|^2 dx. \quad (5.60)$$

Since $\phi \equiv 0$ on the interval $[\alpha^-, \alpha^+]$, combining (5.58) and (5.60), we get

$$\int_0^1 (|\beta\phi|^2 + |\phi_{xx}|^2) dx \leq c \int_0^1 |f|^2 dx. \quad (5.61)$$

which, together with (5.50), yields a stronger version of (5.45).

Case 4 $(b\phi_x)_x = 0$ for $0 < x < 1$. Then $b\phi_x$ is a constant for $0 < x < 1$. Since $\phi_x(0) = 0$, we have $b\phi_x = 0$.

In particular, $\phi_x(\alpha^-) = \phi_{xx}(\alpha^-) = \phi_{xxx}(\alpha^-) = 0$. Applying (5.55) on the interval $(0, \alpha^-)$, we have

$$\int_0^{\alpha^-} (|\beta\phi_x|^2 + |\phi_{xxx}|^2) dx \leq c \int_0^{\alpha^-} |f|^2 dx. \quad (5.62)$$

On the interval $(\alpha^+, 1)$, we have $\phi_x(\alpha^+) = \phi_{xx}(\alpha^+) = \phi_{xxx}(\alpha^+) = 0$. Multiplying the equation in (5.46) by $2x\phi_{xxx}$ and using (5.53) on the interval $(\alpha^+, 1)$, we get

$$\begin{aligned} & 2\beta^2[x\phi\phi_{xx}]_{\alpha^+}^1 - \beta^2[x\phi_x^2]_{\alpha^+}^1 - 2\beta^2[\phi\phi_x]_{\alpha^+}^1 - [x\phi_{xxx}^2]_{\alpha^+}^1 \\ &= - \int_{\alpha^+}^1 \phi_{xxx}^2 dx - 3\beta^2 \int_{\alpha^+}^1 \phi_x^2 dx + \int_{\alpha^+}^1 2xf\phi_{xxx} dx. \end{aligned} \quad (5.63)$$

Using the boundary conditions on $x = \alpha^+$ and on $x = 1$, we get

$$\begin{aligned} & \int_{\alpha^+}^1 (3|\beta\phi_x|^2 + |\phi_{xxx}|^2) dx \\ &= \int_{\alpha^+}^1 2xf\phi_x dx + \beta^2\phi_x^2(\alpha^+) + 2\beta^2\phi(\alpha^+)\phi_x(\alpha^+). \end{aligned} \quad (5.64)$$

Multiplying the equation in (5.46) by ϕ_{xxx} and integrating by parts on the interval $(\alpha^+, 1)$, we get

$$\beta^2[\phi\phi_{xx}]_{\alpha^+}^1 - \beta^2[\phi_x^2]_{\alpha^+}^1 = \int_{\alpha^+}^1 f\phi_{xxx} dx. \quad (5.65)$$

Using the boundary conditions on $x = \alpha^+$ and on $x = 1$, we get

$$\beta^2\phi_x^2(1) = \int_{\alpha^+}^1 f\phi_{xxx} dx. \quad (5.66)$$

Inserting (5.66) into (5.64), we get

$$\int_{\alpha^+}^1 (|\beta\phi_x|^2 + |\phi_{xxx}|^2) dx \leq c \int_{\alpha^+}^1 |f|^2 dx + |\beta\phi(\alpha^+)|^2. \quad (5.67)$$

Using (5.62), we get

$$|\beta\phi(\alpha^+)|^2 = |\beta\phi(\alpha^-)|^2 \leq c\|\beta\phi\|_{H^1(0, \alpha^-)}^2 \leq c \int_0^{\alpha^-} |f|^2 dx. \quad (5.68)$$

Inserting (5.68) into (5.67), we get

$$\int_{\alpha^+}^1 (|\beta\phi_x|^2 + |\phi_{xxx}|^2) dx \leq c \int_0^1 |f|^2 dx. \quad (5.69)$$

Since $\phi_x \equiv 0$ on the interval (α^-, α^+) , combining (5.62) and (5.69), we get

$$\int_0^1 (|\beta\phi_x|^2 + |\phi_{xxx}|^2) dx \leq c \int_0^1 |f|^2 dx, \quad (5.70)$$

which, together with (5.56), yields a stronger version of (5.45).

Remark 5.2 Roughly speaking, we can stabilize the beam system (5.38) by using several dampings of different types. This is the great advantage of the method. However, there are many variances, for example, the supports of the damping coefficients a and b in (5.37) can be different. In particular, the support of b can be a neighbour of the ends of the interval $[0, 1]$. We can also add the Kelvin-Voigt damping $D_5(cU'_{xx})_{xx}$.

Remark 5.3 For all the models considered here, the observability inequality is obtained by the multiplier method under the geometrical multiplier condition. It is stronger than the required observability inequality, for example, (5.18) is much stronger than (5.8) etc. We hope that this regularity should be served to establish a polynomial decay rate for the smooth initial data:

$$\|C_p(U(t), U'(t))\|_{(H^1(\Omega) \times L^2(\Omega))^{N-p}} = O((1+t)^{-\delta}), \quad (5.71)$$

where the constant $\delta > 0$ is independent of the initial data. We refer to [9, 38] for the recent progress on the polynomial stability of indirectly damped wave equations.

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