Chin. Ann. Math. Ser. B
43(5), 2022, 685–718
DOI: 10.1007/s11401-022-0353-2

Chinese Annals of Mathematics, Series B

Positive Solutions for Asymptotically Linear Cone-Degenerate Elliptic Equations*

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Abstract In this paper, the authors study the asymptotically linear elliptic equation on manifold with conical singularities

$$-\Delta_{\mathbb{B}}u + \lambda u = a(z)f(u), \quad u \geq 0 \text{ in } \mathbb{R}^N_+,$$

where $N = n+1 \geq 3$, $\lambda > 0$, $z = \big(t, x_1, \cdots, x_n\big)$, and $\Delta_{\mathbb{B}} = (t\partial_t)^2 + \partial_{x_1}^2 + \cdots + \partial_{x_n}^2$. Combining properties of cone-degenerate operator, the Pohozaev manifold and qualitative properties of the ground state solution for the limit equation, we obtain a positive solution under some suitable conditions on a and f.

Keywords Asymptotically linear, Pohozaev identity, Cone degenerate elliptic operators
2000 MR Subject Classification 35J20, 58J05

1 Introduction

In this paper, we consider the following degenerate equation

$$-\Delta_{\mathbb{B}}u + \lambda u = a(z)f(u), \quad u \geq 0 \text{ in } \mathbb{R}^N_+ \tag{1.1}$$

for $N = n+1 \geq 3$, $\lambda > 0$ and $z = (t, x) \in \mathbb{R}^N_+$. The operator $\Delta_{\mathbb{B}}$ is defined by $(t\partial_t)^2 + \partial_{x_1}^2 + \cdots + \partial_{x_n}^2$, which is an elliptic operator with totally characteristic degeneracy on the boundary $t = 0$ (we also call it Fuchsian type Laplace operator), and the corresponding gradient operator is denoted by $\nabla_{\mathbb{B}} = (t\partial_t, \partial_{x_1}, \cdots, \partial_{x_n})$.

The analysis on manifolds with conical singularities and the properties of elliptic operators are intensively studied. Based on Schulze's cone algebra (see [22]), Schrohe and Seiler [21] introduced the so-called L_p-theory for the cone Sobolev spaces. Recently, Chen, Liu and Wei [7] established the so-called cone Sobolev inequality and Poincaré inequality for the weighted Sobolev spaces. Such kind of inequalities are fundamental to prove the existence of the solutions for nonlinear problems with totally characteristic degeneracy. First, by using these inequalities and the variational method they got the existence theorem for a class of semilinear totally

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*This work was supported by the National Natural Science Foundation of China (Nos. 11631011, 11601402, 12171183, 11831009, 12071364, 11871387).

characteristic elliptic equations with subcritical cone Sobolev exponents in [7]. Then, they studied equations with critical cone Sobolev exponents in [6]. At last, they obtained multiple solutions for equations with subcritical or critical cone Sobolev exponents in [8]. For more results on totally characteristic elliptic equations, one can refer to [5, 9, 17]. All the results of above concerned the equations with super-linear term.

In this paper, we will assume the following conditions on f:

(f1) $f \in C(\mathbb{R}, \mathbb{R}^+)$, $f(s) = 0$ for $s \leq 0$ and $\lim\limits_{s\to 0^+} \frac{f(s)}{s} = 0$;

(f2) $\lim\limits_{s\to+\infty} \frac{f(s)}{s} = 1$;

(f3) set $F(s) = \int_0^s f(\tau)\mathrm{d}\tau$ and $Q(s) = \frac{1}{2}f(s)s - F(s)$. There exists $D \geq 1$ such that

$$0 < Q(s) \leq DQ(\tau) \quad \text{for all } 0 < s \leq \tau, \qquad \lim_{s\to+\infty} Q(s) = +\infty.$$

And the function $a\colon \mathbb{R}^N_+ \to \mathbb{R}$ satisfies:

(A1) $a(z) \in C^2(\mathbb{R}^N_+, \mathbb{R}^+)$ with $\inf_{\mathbb{R}^N_+} a(z) > 0$;

(A2) $\liminf\limits_{|z|\to+\infty} a(z) = a_\infty > \lambda$;

(A3) $(\ln t)t\partial_t a(t,x) + \sum\limits_{i=1}^{n} x_i\partial_{x_i} a(t,x) \geq 0$ for all $(t,x) \in \mathbb{R}^N_+$, with strict inequality holding on a set of positive measure;

(A4) $a(t,x) + \frac{1}{N}\big[(\ln t)t\partial_t a(t,x) + \sum\limits_{i=1}^{n} x_i\partial_{x_i} a(t,x)\big] < a_\infty$ for all $(t,x) \in \mathbb{R}^N_+$;

(A5) $(\ln t)t\partial_t a(t,x) + \sum\limits_{i=1}^{n} x_i\partial_{x_i} a(t,x) + \frac{1}{N}(\ln t, x)\cdot H_a(t,x)\cdot(\ln t, x) \geq 0$ for all $(t,x) \in \mathbb{R}^N_+$,

where $(\ln t, x)\cdot H_a(t,x)\cdot(\ln t,x) := (\ln t)^2[t\partial_t a + t^2\partial_t^2 a] + 2\sum\limits_{i=1}^{n} x_i(\ln t)t\partial^2_{tx_i} a + \sum\limits_{i,j=1}^{n} x_i x_j \partial^2_{x_i x_j} a$ is the value of Hessian matrix $H_a(t,x)$ of function a, in the sense of measure $\frac{\mathrm{d}t}{t}\mathrm{d}x$, applying at the vector $(\ln t, x)$.

Note that, the condition (f2) means that the nonlinear term is asymptotically linear at infinity. This model with standard version comes from nonlinear optics, see [24–25]. However, when we consider a similar nonlinear elliptic problem on manifold with conical singularities, then near the conic point, we can use the cylindrical coordinates transformation to make the model to be problem (1.1) (see [22]). Since the nonlinear term is not homogeneous and is asymptotically linear at infinity, not all functions $u \in \mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}^N_+)\backslash\{0\}$ can be projected on the Nehari manifold (see [11]) and the method in [12, 20] also fails, in which they exploited the fact that, under suitable hypotheses including that the nonlinear term of the elliptic equation is homogeneous and super-quadratic at infinity, the mountain pass min-max level of the energy functional associated with the equations is equal to the minimum of the energy functional restricted to the Nehari manifold. Fortunately, all functions on an open subset of $\mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}^N_+)\backslash\{0\}$ can be projected on the Pohozaev manifold associated with the equation, and we can restrict on this manifold to find the critical points. This is inspired by [15]. At the same time, we should replace the Palais-Smale condition by the Cerami condition (see [4, 10]):

(Ce) the functional I satisfies the Cerami condition if, for any sequence $\{u_j\} \subset \mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}^N_+)$ such that $\{I(u_j)\}$ is bounded and $\|I'(u_j)\|(1+\|u_j\|) \to 0$, there exists a convergent subsequence.

Consider the energy functional $I\colon \mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}^N_+) \to \mathbb{R}$,

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N_+} |\nabla_{\mathbb{B}} u|^2 \frac{\mathrm{d}t}{t}\mathrm{d}x - \int_{\mathbb{R}^N_+} \Big[a(z)F(u) - \frac{1}{2}\lambda u^2\Big]\frac{\mathrm{d}t}{t}\mathrm{d}x,$$

naturally associated with problem (1.1). Then we have following nonexistence result.

Theorem 1.1 *Assume that* (A1)–(A5) *and* (f1)–(f3) *hold, then*

$$p := \inf_{u\in\mathcal{P}} I(u) = c := \min_{\gamma\in\Gamma} \max_{\tau\in[0,1]} I(\gamma(\tau)) \tag{1.2}$$

is not a critical level of I and the infimum above is not achieved. Here, $\mathcal{P}$ is Pohozaev manifold associated with problem (1.1) *which will be defined bellow in* (2.7) *and*

$$\Gamma := \left\{\gamma \in C([0,1]; \mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}_+^N)) \mid \gamma(0) = 0, I(\gamma(1)) < 0\right\}.$$

Remark 1.1 Theorem 1.1 shows that the mountain pass value is not the critical level of $I(u)$. This means that to find the critical point of $I(u)$, we can not use the idea by the classical mountain pass lemma under the conditions on a and f in Theorem 1.1. Moreover, one can only except the existence of solutions with higher level energy.

Consider also the limiting problem corresponding with problem (1.1),

$$-\Delta_{\mathbb{B}} u + \lambda u = a_\infty f(u) \quad \text{in } \mathbb{R}_+^N, \tag{1.3}$$

and its associated energy functional

$$I_\infty(u) = \frac{1}{2}\int_{\mathbb{R}_+^N} |\nabla_{\mathbb{B}} u|^2 \frac{\mathrm{d}t}{t}\mathrm{d}x - \int_{\mathbb{R}_+^N} \left[a_\infty F(u) - \frac{1}{2}\lambda u^2\right]\frac{\mathrm{d}t}{t}\mathrm{d}x.$$

In Section 3 we will show that problem (1.3) has exact one positive solution which is "radial" and a least energy solution under some conditions. Then, we have following existence result. Suppose

(f4) $f \in C^1(\mathbb{R}, \mathbb{R}^+) \cap \mathrm{Lip}\ (\mathbb{R}, \mathbb{R}^+)$.

Theorem 1.2 *Assume that* (A1)–(A5) *and* (f1)–(f4) *hold. Then problem* (1.1) *admits a positive solution $u \in \mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}_+^N)$.*

Remark 1.2 To prove Theorem 1.2, the main idea is to use linking argument together with barycenter functional restricted to Pohozaev manifold $\mathcal{P}$. A crucial step is to construct Cerami sequence and we give a clear exposition.

Remark 1.3 Conditions (A2), (A3) and (A4) imply that

$$(\ln t)\cdot t\partial_t a(t,x) + \sum_{i=1}^n x_i\partial_{x_i} a(t,x) \to 0 \quad \text{if } |z| \to \infty. \tag{1.4}$$

Conditions (f1) and (f2) show that, given $\varepsilon > 0$ and $2 \le p \le 2^*$, there exists a positive constant $C = C(\varepsilon, p)$ such that for all $s \in \mathbb{R}$,

$$|F(s)| \le \frac{\varepsilon}{2}|s|^2 + C|s|^p. \tag{1.5}$$

Remark 1.4 An example of function f satisfying conditions (f1)–(f4) is

$$f(s) = \begin{cases} \dfrac{s^3}{1+s^2}, & s \ge 0, \\ 0, & s \le 0. \end{cases} \tag{1.6}$$

One can verify that

$$a(z) = a_\infty - \frac{1}{\sqrt{(\ln t)^2 + |x|^2 + k}} \quad \text{with } k > \frac{1}{a_\infty} \text{ and } a_\infty > \lambda$$

satisfies previous assumptions (A1)–(A5).

This paper is organized as follows. In Section 2, we introduce the cone Sobolev spaces and corresponding properties. At the same time, we establish the distance and Pohozaev identity on cone. In Section 3, we study some properties for solutions of limiting problem, in particular, the least energy solution is considered. Then, we give the nonexistence result in Section 4. At last, we prove the existence of a positive solution for problem (1.1) in Section 5.

2 Preliminaries

2.1 Cone Sobolev spaces and inequalities on $\mathbb{R}_+^N$

Definition 2.1 *For $(t,x) \in \mathbb{R}_+^N$, $\gamma \in \mathbb{R}$ and $1 \le p < +\infty$, we say that $u(t,x) \in L_p^\gamma\big(\mathbb{R}_+^N, \frac{dt}{t}dx\big)$ if*

$$\|u\|_{L_p^\gamma} = \Big(\int_{\mathbb{R}_+^N} t^N |t^{-\gamma} u(t,x)|^p \frac{dt}{t} dx\Big)^{\frac{1}{p}} < +\infty.$$

The definition of the weighted Sobolev space for $1 \le p < +\infty$ is as follows.

Definition 2.2 *For $m \in \mathbb{N}$ and $\gamma \in \mathbb{R}$, the space*

$$\mathcal{H}_p^{m,\gamma}(\mathbb{R}_+^N) = \Big\{u \in \mathcal{D}'(\mathbb{R}_+^N) :\ (t\partial_t)^k \partial_x^\alpha u \in L_p^\gamma\Big(\mathbb{R}_+^N, \frac{dt}{t}dx\Big)\Big\}$$

for arbitrary $k \in \mathbb{N}$ and multi-index $\alpha \in \mathbb{N}^n$ with $k + |\alpha| \le m$.

It is easy to see that $\mathcal{H}_p^{m,\gamma}\big(\mathbb{R}_+^N, \frac{dt}{t}dx\big)$ is a Banach space with norm

$$\|u\|_{\mathcal{H}_p^{m,\gamma}(\mathbb{R}_+^N)} = \sum_{k+|\alpha| \le m} \Big(\int_{\mathbb{R}_+^N} t^N |t^{-\gamma} (t\partial_t)^k \partial_x^\alpha u|^p \frac{dt}{t} dx\Big)^{\frac{1}{p}}.$$

Proposition 2.1 (Cone Sobolev inequality, see [7]) *Assume that $1 \le p < N$, $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$ and $\gamma \in \mathbb{R}$. The following estimate*

$$\|u\|_{L_{p^*}^{\gamma^*}(\mathbb{R}_+^N)} \le c_1 \|(t\partial_t)u\|_{L_p^\gamma(\mathbb{R}_+^N)} + (c_1 + \alpha c_2) \sum_{i=1}^n \|\partial_{x_i} u\|_{L_p^\gamma(\mathbb{R}_+^N)} + c_2 \|u\|_{L_p^\gamma(\mathbb{R}_+^N)}$$

holds for all $u(t,x) \in C_0^\infty(\mathbb{R}_+^N)$, where

$$\gamma^* = \gamma - 1, \quad c_1 = \frac{\alpha}{N}, \quad c_2 = \frac{1}{N}\Big|\frac{(N-1)(N-p\gamma)}{N-p}\Big|^{\frac{1}{N}} \quad \text{for } \alpha = \frac{(N-1)p}{N-p}.$$

Moreover, if $u(t,x) \in \mathcal{H}_p^{1,\gamma}(\mathbb{R}_+^N)$, we have

$$\|u\|_{L_{p^*}^{\gamma^*}(\mathbb{R}_+^N)} \le c\|u\|_{\mathcal{H}_p^{1,\gamma}(\mathbb{R}_+^N)},$$

where the constant $c = c_1 + \alpha c_2$.

In what follows, we denote $\|u\| := \big(\int_{\mathbb{R}_+^N} |\nabla_{\mathbb{B}} u|^2 \frac{dt}{t} dx + \lambda \int_{\mathbb{R}_+^N} u^2 \frac{dt}{t} dx\big)^{\frac{1}{2}}$ the norm in $\mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}_+^N)$.

2.2 Distance on cone

Since the distance on cone is $\mathrm{d}s^2 = \frac{1}{t^2}(\mathrm{d}t)^2 + \sum\limits_{i=1}^{n}(\mathrm{d}x_i)^2$, we obtain the distance between point $z = \big(t, x_1, \cdots, x_n\big)$ and $z_0 = \big(t_0, x_1^0, \cdots, x_n^0\big)$ on cone is

$$d(z, z_0) = \sqrt{(\ln t - \ln t_0)^2 + \sum_{i=1}^{n}(x_i - x_i^0)^2}. \tag{2.1}$$

For simplicity, we denote the points $z = (t, x)$, $w = (s, y)$ in $\mathbb{R}_+^N$ and $|z| = \sqrt{(\ln t)^2 + \sum\limits_{i=1}^{n} x_i^2}$. For functional $g = g(z, u) = g(t, x, u)$ with $t \in \mathbb{R}_+, x \in \mathbb{R}^n$ and $u(t, x) \in \mathbb{R}$, we denote

$$\partial_0 g = \partial_t g(t, x, u), \quad \partial_i g = \partial_{x_i} g(t, x, u), \quad \partial_{N+1} g = \partial_u g(t, x, u).$$

We introduce the open "ball" in $\mathbb{R}_+^N$ in the sense of measure $\frac{\mathrm{d}t}{t}\mathrm{d}x$ with center $w = (s, y)$ and radius r as follows:

$$\Omega_r(s, y) := \Big\{(t, x) \in \mathbb{R}_+^N;\ \Big(\ln\frac{t}{s}\Big)^2 + \sum_{i=1}^{n}(x_i - y_i)^2 < r^2\Big\}.$$

We say $u = u(z)$ is "radially symmetric" about $w = (s, y)$, if $u(z_1) = u(z_2)$ as $d(z_1, w) = d(z_2, w)$.

For $\mu > 0$ and $u \in \mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}_+^N)$, we introduce a scaled function $u_\mu(t, x) := u\big(t^{\frac{1}{\mu}}, \frac{x}{\mu}\big)$. And for the point $(s, y) \in \mathbb{R}_+^N$, let $T_{sy}u(t, x) := u\big(\frac{t}{s}, x - y\big)$ denote the translation function.

Remark 2.1 By (2.1), we have

$$d\Big(\Big(t^{\frac{1}{\mu}}, \frac{x}{\mu}\Big), (1, 0)\Big) = \sqrt{(\ln t^{\frac{1}{\mu}})^2 + \Big(\frac{|x|}{\mu}\Big)^2} = \frac{1}{\mu}\sqrt{(\ln t)^2 + |x|^2} = \frac{1}{\mu} d((t, x), (1, 0))$$

and

$$d\Big(\Big(\frac{t_0}{s}, x^0 - y\Big), \Big(\frac{t}{s}, x - y\Big)\Big) = \sqrt{(\ln t - \ln t_0)^2 + |x^0 - x|^2} = d((t_0, x^0), (t, x)).$$

Therefore, the introduction of scaled function u_μ and translation function $T_{sy}u$ has meaning.

2.3 Pohozaev identity and manifold

In this section, we deduce the Pohozaev identity on cone and introduce the corresponding manifold. The original work is Pohozaev [19].

Proposition 2.2 *Let $u \in \mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}_+^N)\backslash\{0\}$ be a solution of (1.1), then u satisfies*

$$\begin{aligned} \frac{N-2}{2}\int_{\mathbb{R}_+^N} |\nabla_{\mathbb{B}} u|^2 \frac{\mathrm{d}t}{t}\mathrm{d}x = N\int_{\mathbb{R}_+^N} G(z, u)\frac{\mathrm{d}t}{t}\mathrm{d}x + \int_{\mathbb{R}_+^N} \Big[(\ln t)t\partial_t a(t, x) \\ + \sum_{i=1}^{n} x_i \partial_{x_i} a(t, x)\Big] F(u)\frac{\mathrm{d}t}{t}\mathrm{d}x, \end{aligned} \tag{2.2}$$

where $G(z, u) = a(z)F(u) - \frac{1}{2}\lambda u^2$.

Proof We write $-\Delta_{\mathbb{B}} u = a(z)f(u) - \lambda u =: g(z,u)$ and introduce the transform $T(s)u(t,x) := u\big(t^{\frac{1}{s}}, \frac{x}{s}\big)$, then $T(1) = id$ and

$$\frac{\partial}{\partial s}\Big|_{s=1} T(s) = -(\ln t, x)\cdot \nabla_{\mathbb{B}}.$$

Set $\varphi \in \mathcal{D}(\mathbb{R})$ satisfy $0 \le \varphi \le 1, \varphi(r) = 1$ if $r \le 1$, and $\varphi(r) = 0$ if $r \ge 2$. Define

$$\varphi_k(t,x) = \varphi\Big(\frac{(\ln t)^2 + |x|^2}{k^2}\Big),$$

then there exists a constant $c \ge 0$ such that for all integer k, we have

$$\varphi_k \le c, \quad |(\ln t, x)| \cdot |\nabla_{\mathbb{B}} \varphi_k(t,x)| \le c. \tag{2.3}$$

To obtain the Pohozaev identity, we multiple $-\Delta_{\mathbb{B}} u = g(z,u)$ by $\varphi_k(t,x)((\ln t, x)\cdot \nabla_{\mathbb{B}} u)$ and get

$$\Delta_{\mathbb{B}} u \ \varphi_k(t,x)((\ln t,x)\cdot\nabla_{\mathbb{B}} u) + g(z,u)\varphi_k(t,x)((\ln t,x)\cdot\nabla_{\mathbb{B}} u) = 0.$$

On one hand,

$$\begin{aligned}
&\Delta_{\mathbb{B}} u \ \varphi_k(t,x)((\ln t,x)\cdot\nabla_{\mathbb{B}} u)\\
&= \nabla_{\mathbb{B}}\cdot(\nabla_{\mathbb{B}} u \ \varphi_k(t,x)(\ln t,x)\cdot\nabla_{\mathbb{B}} u) - (\nabla_{\mathbb{B}} u\cdot\nabla_{\mathbb{B}}\varphi_k)((\ln t,x)\cdot\nabla_{\mathbb{B}} u)\\
&\quad - \varphi_k\{\nabla_{\mathbb{B}}[(\ln t,x)\cdot\nabla_{\mathbb{B}} u]\cdot\nabla_{\mathbb{B}} u\}\\
&= \nabla_{\mathbb{B}}\cdot(\varphi_k\nabla_{\mathbb{B}} u \ (\ln t,x)\cdot\nabla_{\mathbb{B}} u) - (\nabla_{\mathbb{B}} u\cdot\nabla_{\mathbb{B}}\varphi_k)((\ln t,x)\cdot\nabla_{\mathbb{B}} u)\\
&\quad - \varphi_k\Big\{|\nabla_{\mathbb{B}} u|^2 + (\ln t,x)\cdot\nabla_{\mathbb{B}}\frac{|\nabla_{\mathbb{B}} u|^2}{2}\Big\}\\
&= \nabla_{\mathbb{B}}\cdot(\varphi_k\nabla_{\mathbb{B}} u \ (\ln t,x)\cdot\nabla_{\mathbb{B}} u) - (\nabla_{\mathbb{B}} u\cdot\nabla_{\mathbb{B}}\varphi_k)((\ln t,x)\cdot\nabla_{\mathbb{B}} u)\\
&\quad - \Big\{\varphi_k|\nabla_{\mathbb{B}} u|^2 + \nabla_{\mathbb{B}}\cdot\Big(\varphi_k(t,x)(\ln t,x)\frac{|\nabla_{\mathbb{B}} u|^2}{2}\Big) - \nabla_{\mathbb{B}}\varphi_k\cdot(\ln t,x)\frac{|\nabla_{\mathbb{B}} u|^2}{2} - N\varphi_k\frac{|\nabla_{\mathbb{B}} u|^2}{2}\Big\}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\nabla_{\mathbb{B}}\cdot[G(z,u)\varphi_k(t,x)(\ln t,x)] &= [t\partial_0 G + tg(z,u)\partial_t u]\varphi_k \ln t + Gt\partial_t\varphi_k \ \ln t + G\varphi_k\\
&\quad + \sum_{i=1}^n[\partial_i G\varphi_k x_i + g(z,u)\partial_{x_i} u \ \varphi_k x_i + G\partial_{x_i}\varphi_k x_i] + G\varphi_k(N-1)\\
&= g(z,u)\varphi_k(t,x)((\ln t,x)\cdot\nabla_{\mathbb{B}} u) + \Big[(\ln t)\cdot t\partial_0 G + \sum_{i=1}^n x_i\partial_i G\Big]\varphi_k\\
&\quad + G\nabla_{\mathbb{B}}\varphi_k\cdot(\ln t,x) + NG\varphi_k.
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
&\nabla_{\mathbb{B}}\cdot(\nabla_{\mathbb{B}} u \ \varphi_k(t,x)(\ln t,x)\cdot\nabla_{\mathbb{B}} u) - (\nabla_{\mathbb{B}} u\cdot\nabla_{\mathbb{B}}\varphi_k)((\ln t,x)\cdot\nabla_{\mathbb{B}} u) - \varphi_k|\nabla_{\mathbb{B}} u|^2\\
&- \nabla_{\mathbb{B}}\cdot\Big(\varphi_k(t,x)(\ln t,x)\frac{|\nabla_{\mathbb{B}} u|^2}{2}\Big) + \nabla_{\mathbb{B}}\varphi_k\cdot(\ln t,x)\frac{|\nabla_{\mathbb{B}} u|^2}{2} + N\varphi_k\frac{|\nabla_{\mathbb{B}} u|^2}{2}\\
&+ \nabla_{\mathbb{B}}\cdot[G\varphi_k(t,x)(\ln t,x)] - \Big[(\ln t)t\partial_0 G + \sum_{i=1}^n x_i\partial_i G\Big]\varphi_k - G\nabla_{\mathbb{B}}\varphi_k\cdot(\ln t,x) - NG(z,u)\varphi_k = 0.
\end{aligned}$$

Integrating on $\Omega_r(1,0)$ with measure $\frac{\mathrm{d}t}{t}\mathrm{d}x$, we obtain

$$\int_{\partial\Omega_r(1,0)} \Big[\nabla_{\mathbb{B}} u\ (\ln t, x)\cdot \nabla_{\mathbb{B}} u - (\ln t, x)\frac{|\nabla_{\mathbb{B}} u|^2}{2} + G(z,u)(\ln t, x)\Big]\varphi_k \cdot \nu \mathrm{d}S$$
$$= \int_{\Omega_r(1,0)} \Big\{\Big[NG(z,u) - \frac{N-2}{2}|\nabla_{\mathbb{B}} u|^2 + (\ln t)t\partial_0 G + \sum_{i=1}^n x_i\partial_i G\Big]\varphi_k$$
$$+ (\nabla_{\mathbb{B}} u\cdot \nabla_{\mathbb{B}}\varphi_k)\ (\ln t, x)\cdot \nabla_{\mathbb{B}} u - \nabla_{\mathbb{B}}\varphi_k\cdot(\ln t, x)\frac{|\nabla_{\mathbb{B}} u|^2}{2} + G(z,u)\nabla_{\mathbb{B}}\varphi_k\cdot(\ln t, x)\Big\}\frac{\mathrm{d}t}{t}\mathrm{d}x,$$

where $\nu = \frac{(\ln t, x)}{r}$ is the unit outward normal vector of $\partial\Omega_r(1,0)$.

Hence (2.3) and the Lebesgue dominated convergence theorem imply that

$$\int_{\partial\Omega_r(1,0)} \Big[\nabla_{\mathbb{B}} u\ (\ln t, x)\cdot \nabla_{\mathbb{B}} u - (\ln t, x)\frac{|\nabla_{\mathbb{B}} u|^2}{2} + G(z,u)(\ln t, x)\Big]\cdot \nu \mathrm{d}S$$
$$= \int_{\Omega_r(1,0)} \Big[NG(z,u) - \frac{N-2}{2}|\nabla_{\mathbb{B}} u|^2 + (\ln t)t\partial_0 G + \sum_{i=1}^n x_i\partial_i G\Big]\frac{\mathrm{d}t}{t}\mathrm{d}x. \tag{2.4}$$

Since

$$\int_{\mathbb{R}_+^{n+1}} \Big[|G(z,u)| + \frac{1}{2}|\nabla_{\mathbb{B}} u|^2\Big]\frac{\mathrm{d}t}{t}\mathrm{d}x$$
$$= \int_0^\infty \Big\{\int_{\partial\Omega_r(1,0)} \Big[|G(z,u)| + \frac{1}{2}|\nabla_{\mathbb{B}} u|^2\Big]\mathrm{d}S\Big\}\mathrm{d}R < +\infty, \tag{2.5}$$

then there exists a sequence $R_n \to +\infty$ such that as $n \to +\infty$,

$$R_n \int_{\partial\Omega_r(1,0)} \Big[|G(z,u)| + \frac{1}{2}|\nabla_{\mathbb{B}} u|^2\Big]\mathrm{d}S \to 0. \tag{2.6}$$

If this is false and

$$\varliminf_{R\to+\infty} R\int_{\partial\Omega_r(1,0)} \Big[|G(z,u)| + \frac{1}{2}|\nabla_{\mathbb{B}} u|^2\Big]\mathrm{d}S = \alpha > 0,$$

then $|G(z,u)| + \frac{1}{2}|\nabla_{\mathbb{B}} u|^2 \notin L_1^N\big(\mathbb{R}_+^N, \frac{\mathrm{d}t}{t}\mathrm{d}x\big)$, which contradicts (2.5).

Thus, combining (2.4) and (2.6) we get

$$\frac{N-2}{2}\int_{\mathbb{R}_+^N} |\nabla_{\mathbb{B}} u|^2 \frac{\mathrm{d}t}{t}\mathrm{d}x = \int_{\mathbb{R}_+^N} \Big[NG(z,u) + (\ln t)t\partial_0 G(z,u) + \sum_{i=1}^n x_i\partial_i G(z,u)\Big]\frac{\mathrm{d}t}{t}\mathrm{d}x,$$

which means (2.2).

Now we define the Pohozaev manifold associated with (1.1) by

$$\mathcal{P} := \{u \in \mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}_+^N)\backslash\{0\};\quad u \text{ satisfies } (2.2)\}. \tag{2.7}$$

In the following discussion, we denote $\frac{\mathrm{d}t}{t}\mathrm{d}x$ by $\mathrm{d}\sigma$.

Lemma 2.1 *Let functional* $J\colon \mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}_+^N) \to \mathbb{R}$ *be defined by*

$$J(u) = \frac{N-2}{2}\int_{\mathbb{R}_+^N} |\nabla_{\mathbb{B}} u|^2 \mathrm{d}\sigma - N\int_{\mathbb{R}_+^N} G(z,u)\mathrm{d}\sigma - \int_{\mathbb{R}_+^N} \Big[(\ln t)t\partial_t a(t,x) + \sum_{i=1}^n x_i\partial_{x_i} a(t,x)\Big]F(u)\mathrm{d}\sigma.$$

Then, it holds that

(1) $\{u \equiv 0\}$ *is an isolated point of* $J^{-1}(\{0\})$;

(2) $\mathcal{P} = \{u \in \mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}_+^N)\backslash\{0\}\colon J(u) = 0\}$ *is a closed set*;

(3) $\mathcal{P}$ *is a* C^1 *manifold*;

(4) *there exists* $\alpha > 0$ *such that* $\|u\| > \alpha$ *for all* $u \in \mathcal{P}$.

Proof (1) By condition (A4), we have

$$\begin{aligned} J(u) &= \frac{N-2}{2}\int_{\mathbb{R}_+^N} |\nabla_{\mathbb{B}} u|^2 \mathrm{d}\sigma - N\int_{\mathbb{R}_+^N}\Big(\Big[a(t,x) \\ &\quad + \frac{(\ln t)t\partial_t a(t,x) + \sum\limits_{i=1}^n x_i\partial_{x_i}a(t,x)}{N}\Big]F(u) - \frac{1}{2}\lambda u^2\Big)\mathrm{d}\sigma \\ &> \frac{N-2}{2}\int_{\mathbb{R}_+^N} |\nabla_{\mathbb{B}} u|^2 \mathrm{d}\sigma - N\int_{\mathbb{R}_+^N}\Big[a_\infty F(u) - \frac{1}{2}\lambda u^2\Big]\mathrm{d}\sigma \\ &\geq \frac{N-2}{2}\|u\|^2 - N\int_{\mathbb{R}_+^N} a_\infty F(u)\mathrm{d}\sigma. \end{aligned}$$

Then the Cone Sobolev embedding and condition (1.5) imply that

$$\begin{aligned} J(u) &> \frac{N-2}{2}\|u\|^2 - \frac{\varepsilon N a_\infty}{2\lambda}\int_{\mathbb{R}_+^N}\lambda|u|^2\mathrm{d}\sigma - CNa_\infty\int_{\mathbb{R}_+^N}|u|^p\mathrm{d}\sigma \\ &\geq \frac{1}{2}\Big(N-2-\frac{\varepsilon N a_\infty}{\lambda}\Big)\|u\|^2 - CNa_\infty\|u\|^p. \end{aligned}$$

If we take $\varepsilon > 0$ small enough and $0 < \rho < 1$ such that

$$(N-2)\lambda - \varepsilon N a_\infty > 0, \quad \rho^p < \frac{1}{4CNa_\infty}\Big(N-2-\frac{\varepsilon N a_\infty}{\lambda}\Big)\rho^2,$$

then if $\|u\| = \rho$, we have

$$J(u) > \frac{1}{4}\Big(N-2-\frac{\varepsilon N a_\infty}{\lambda}\Big)\rho^2 > 0.$$

And $J(u) > 0$ if $0 < \|u\| < \rho$.

(2) Since $J(u)$ is a C^1 functional, thus $\mathcal{P}\cup\{0\} = J^{-1}(\{0\})$ is a closed subset. Moreover, $\{u \equiv 0\}$ is an isolated point of $J^{-1}(\{0\})$ and then $\mathcal{P}$ is a closed set.

(3) Considering the derivative of J at u, we have

$$\langle J'(u), u\rangle = (N-2)\int_{\mathbb{R}_+^N}|\nabla_{\mathbb{B}}u|^2\mathrm{d}\sigma - N\int_{\mathbb{R}_+^N}\Big(\Big[a + \frac{(\ln t)t\partial_t a + \sum\limits_{i=1}^n x_i\partial_{x_i}a}{N}\Big]f(u)u - \lambda u^2\Big)\mathrm{d}\sigma.$$

Since $u \in \mathcal{P}$, if follows that

$$\begin{aligned} \langle J'(u), u\rangle &= 2N\int_{\mathbb{R}_+^N}\Big[a(t,x) + \frac{(\ln t)t\partial_t a(t,x) + \sum\limits_{i=1}^n x_i\partial_{x_i}a(t,x)}{N}\Big]F(u)\mathrm{d}\sigma \\ &\quad - N\int_{\mathbb{R}_+^N}\Big[a(t,x) + \frac{(\ln t)t\partial_t a(t,x) + \sum\limits_{i=1}^n x_i\partial_{x_i}a(t,x)}{N}\Big]f(u)u\mathrm{d}\sigma \end{aligned}$$

$$= 2N \int_{\mathbb{R}^N_+} \Big[a(t,x) + \frac{(\ln t)t\partial_t a(t,x) + \sum\limits_{i=1}^{n} x_i \partial_{x_i} a(t,x)}{N}\Big] \cdot \Big[F(u) - \frac{1}{2} f(u)u\Big] \mathrm{d}\sigma < 0,$$

where we used (A1), (A3) and (f3).

Therefore, if $u \in \mathcal{P}$ then $\langle J'(u), u\rangle < 0$. Thus by implicit function theorem, we know that $\mathcal{P}$ is a C^1 manifold.

(4) Since 0 is an isolated point of $J^{-1}\big(\{0\}\big)$, there must be a "ball" $\|u\| \le \alpha$ which does not intersect $\mathcal{P}$ and the assertion is proved.

3 Energy Levels for Limiting Problem

In this section, we study some crucial properties of solutions for the limiting problem

$$-\Delta_{\mathbb{B}} u + \lambda u = a_\infty f(u) \quad \text{in } \mathbb{R}^N_+, \tag{3.1}$$

where $\lambda > 0$ and $a_\infty > \lambda$.

Let $I_\infty \colon \mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}^N_+) \to \mathbb{R}$ be the energy functional corresponding to (3.1), namely

$$I_\infty(u) = \frac{1}{2} \int_{\mathbb{R}^N_+} |\nabla_{\mathbb{B}} u|^2 \mathrm{d}\sigma - \int_{\mathbb{R}^N_+} G_\infty(u) \mathrm{d}\sigma \quad \text{with } G_\infty(u) = a_\infty F(u) - \frac{1}{2}\lambda u^2.$$

We say that a solution Φ of (3.1) is a least energy solution to (3.1) if

$$I_\infty(\Phi) = m, \quad m := \inf\{I_\infty(u) \colon u \in \mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}^N_+)\backslash\{0\} \text{ is a solution of (3.1)}\}.$$

The Pohozaev identity corresponding to (3.1) can be stated as

$$\frac{N-2}{2} \int_{\mathbb{R}^N_+} |\nabla_{\mathbb{B}} u|^2 \mathrm{d}\sigma = N \int_{\mathbb{R}^N_+} G_\infty(u) \mathrm{d}\sigma,$$

and we introduce the manifold

$$\mathcal{P}_\infty := \{u \in \mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}^N_+)\backslash\{0\}, J_\infty(u) = 0\},$$

where $J_\infty(u) = \frac{N-2}{2} \int_{\mathbb{R}^N_+} |\nabla_{\mathbb{B}} u|^2 \mathrm{d}\sigma - N \int_{\mathbb{R}^N_+} G_\infty(u) \mathrm{d}\sigma$. Consider the set of paths

$$\Gamma_\infty := \{\gamma \in C([0,1]; \mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}^N_+)) \mid \gamma(0) = 0, I_\infty(\gamma(1)) < 0\},$$

and define the min-max mountain pass level

$$c_\infty := \min_{\gamma \in \Gamma_\infty} \max_{\tau \in [0,1]} I_\infty(\gamma(\tau)).$$

Then, we have following property, which is important for the proof of nonexistence.

Proposition 3.1 *It holds $m = c_\infty$.*

In order to prove this result, based on a key observation, we deduce the following property.

Proposition 3.2 *Suppose that $a_\infty > \lambda > 0$ and* (f1), (f2) *hold. Then there exists a nontrivial "radial" least energy solution u for problem* (3.1) *such that*

$$I_\infty(u) = m = \inf_{v \in \mathcal{P}_\infty} I_\infty(v).$$

Proof We divide the proof into three steps.

Step 1 The set $\left\{u \in \mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}_+^N), \int_{\mathbb{R}_+^N} G_\infty(u)\mathrm{d}\sigma = 1\right\}$ is not empty.

Since $a_\infty > \lambda$ and $\lim\limits_{s\to+\infty} \frac{f(s)}{s} = 1$, there exists $\zeta > 0$ such that

$$G_\infty(\zeta) = a_\infty F(\zeta) - \frac{1}{2}\lambda\zeta^2 > 0.$$

Now for $R > 1$, we define

$$u_R(z) = \begin{cases} \zeta & \text{for } |z| \le R, \\ \zeta(R+1-r) & \text{for } r = |z| \in [R, R+1], \\ 0 & \text{for } |z| \ge R+1. \end{cases}$$

Then, $u_R(z) \in \mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}_+^N)$ and

$$\int_{\mathbb{R}_+^N} G_\infty(u_R)\mathrm{d}\sigma \ge G_\infty(\zeta)\big|\Omega_R\big|_{\mathbb{B}} - \big|\Omega_{R+1}\backslash\Omega_R\big|_{\mathbb{B}} \max_{s\in[0,\zeta]} |G_\infty(s)|,$$

where $|\Omega_R|_{\mathbb{B}}$ is the volume of Ω_R in the sense of measure $\frac{\mathrm{d}t}{t}\mathrm{d}x$. Therefore, there exist constants $C, C' > 0$ such that

$$\int_{\mathbb{R}_+^N} G_\infty(u_R)\mathrm{d}\sigma \ge CR^N - C'R^{N-1}.$$

For R large enough, this shows that $\int_{\mathbb{R}_+^N} G_\infty(u_R)\mathrm{d}\sigma > 0$.

Since $\int_{\mathbb{R}_+^N} G_\infty((u_R)_\mu)\mathrm{d}\sigma = \mu^N \int_{\mathbb{R}_+^N} G_\infty(u_R)\mathrm{d}\sigma$, there exists a proper constant $\mu > 0$ such that $\int_{\mathbb{R}_+^N} G_\infty((u_R)_\mu)\mathrm{d}\sigma = 1$.

Step 2 For any $u(t,x) \in \mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}_+^N)$, there exists $v(t,x) := u(\mathrm{e}^t, x) \in H^1(\mathbb{R}^N)$. And for any $v(t,x) \in H^1(\mathbb{R}^N)$, there exists $u(t,x) := v(\ln t, x) \in \mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}_+^N)$. At the same time

$$\begin{aligned}\int_{\mathbb{R}_+^N} \Big[(t\partial_t u(t,x))^2 + \sum_{i=1}^n (\partial_{x_i} u(t,x))^2\Big]\frac{\mathrm{d}t}{t}\mathrm{d}x &= \int_{\mathbb{R}_+^N} \Big[(t\partial_t v(\ln t,x))^2 + \sum_{i=1}^n (\partial_{x_i} v(\ln t,x))^2\Big]\frac{\mathrm{d}t}{t}\mathrm{d}x \\ &= \int_{\mathbb{R}^N} \Big[(\partial_t v(t,x))^2 + \sum_{i=1}^n (\partial_{x_i} v(t,x))^2\Big]\mathrm{d}t\mathrm{d}x\end{aligned}$$

and

$$\int_{\mathbb{R}_+^N} \Big[a_\infty F(u) - \frac{1}{2}\lambda u^2\Big]\frac{\mathrm{d}t}{t}\mathrm{d}x = \int_{\mathbb{R}^N} \Big[a_\infty F(v) - \frac{1}{2}\lambda v^2\Big]\mathrm{d}t\mathrm{d}x.$$

Thus, problem

$$\min\Big\{\int_{\mathbb{R}_+^N} |\nabla_{\mathbb{B}} u|^2 \mathrm{d}\sigma;\ \ u \in \mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}_+^N), \int_{\mathbb{R}_+^N} G_\infty(u)\mathrm{d}\sigma = 1\Big\} \tag{3.2}$$

is equivalent to the problem

$$\min\Big\{\int_{\mathbb{R}^N} |\nabla v|^2 \mathrm{d}t\mathrm{d}x;\ \ v \in H^1(\mathbb{R}^N), \int_{\mathbb{R}^N} G_\infty(v)\mathrm{d}t\mathrm{d}x = 1\Big\}. \tag{3.3}$$

From Berestycki and Lions's paper [2], we know that there exists a positive, spherically symmetric solution $\overline{v} \in H^1(\mathbb{R}^N)$ for problem (3.3). By (2.1), if $\overline{v}(t,x)$ is symmetric with $(0,0)$, then $\overline{u}(t,x) := \overline{v}(\ln t, x)$ will be symmetric with $(1,0)$. Thus, we have a positive, "radially symmetric" solution $\overline{u} \in \mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}_+^N)$ for problem (3.2). What's more, there exists a Lagrange multiple θ such that

$$-\Delta_{\mathbb{B}}\overline{u} = \theta(a_\infty f(\overline{u}) - \lambda\overline{u}).$$

If $\theta = 0$, we have $\overline{u} = 0$, which is impossible. Let us show that $\theta > 0$. Suppose by contradiction that $\theta < 0$. Observe that $a_\infty f(\overline{u}) - \lambda\overline{u} \neq 0$, since $a_\infty f(s) - \lambda s \neq 0$ for $s > 0$ small, $a_\infty f(\overline{u}) - \lambda\overline{u} = 0$ gives $\overline{u} \equiv 0$ or $f(\overline{u}) = \big(\frac{\lambda}{a_\infty}\big)\overline{u} \neq 0$, both cases contradict $\int_{\mathbb{R}_+^N} G_\infty(\overline{u})\mathrm{d}\sigma = 1$. Consider a function $\varphi \in \mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}_+^N)$ such that $\langle a_\infty f(\overline{u}) - \lambda\overline{u}, \varphi\rangle > 0$. Since

$$\int_{\mathbb{R}_+^N} G_\infty(\overline{u} + \varepsilon\varphi)\mathrm{d}\sigma \simeq \int_{\mathbb{R}_+^N} G_\infty(\overline{u})\mathrm{d}\sigma + \varepsilon\big\langle a_\infty f(\overline{u}) - \lambda\overline{u}, \varphi\big\rangle$$

and

$$\int_{\mathbb{R}_+^N} |\nabla_{\mathbb{B}}(\overline{u} + \varepsilon\varphi)|^2\mathrm{d}\sigma \simeq \int_{\mathbb{R}_+^N} |\nabla_{\mathbb{B}}\overline{u}|^2\mathrm{d}\sigma + 2\varepsilon\theta\big\langle a_\infty f(\overline{u}) - \lambda\overline{u}, \varphi\big\rangle \quad \text{for } \varepsilon \to 0 \ \text{ and } \ \theta < 0,$$

one can find $\varepsilon > 0$ small enough so that $\overline{\varphi} = \overline{u} + \varepsilon\varphi$ satisfying

$$\int_{\mathbb{R}_+^N} G_\infty(\overline{\varphi})\mathrm{d}\sigma > \int_{\mathbb{R}_+^N} G_\infty(\overline{u})\mathrm{d}\sigma = 1 \quad \text{and} \quad \int_{\mathbb{R}_+^N} |\nabla_{\mathbb{B}}\overline{\varphi}|^2\mathrm{d}\sigma < \int_{\mathbb{R}_+^N} |\nabla_{\mathbb{B}}\overline{u}|^2\mathrm{d}\sigma.$$

Again by a scaled change, there exists a $0 < \mu < 1$ such that

$$\int_{\mathbb{R}_+^N} G_\infty(\overline{\varphi})\mathrm{d}\sigma = 1 \quad \text{and} \quad \int_{\mathbb{R}_+^N} |\nabla_{\mathbb{B}}\overline{\varphi}|^2\mathrm{d}\sigma < \int_{\mathbb{R}_+^N} |\nabla_{\mathbb{B}}\overline{u}|^2\mathrm{d}\sigma,$$

which is impossible. Hence $\theta > 0$.

Therefore, $\overline{u}$ satisfies, at least in $\mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}_+^N)$, the following equation with $\theta > 0$,

$$-\Delta_{\mathbb{B}}\overline{u} = \theta(a_\infty f(\overline{u}) - \lambda\overline{u}),$$

and so $\overline{u}_{\sqrt{\theta}} = \overline{u}\big(t^{\frac{1}{\sqrt{\theta}}}, \frac{x}{\sqrt{\theta}}\big)$ is a solution of problem (3.1).

Step 3 We prove that $u = \overline{u}_{\sqrt{\theta}}$ is a least energy solution for problem (3.1). Note that

$$\int_{\mathbb{R}_+^N} G_\infty(\overline{u})\mathrm{d}\sigma = 1, \quad \int_{\mathbb{R}_+^N} |\nabla_{\mathbb{B}}u|^2\mathrm{d}\sigma = \frac{2N}{N-2}\int_{\mathbb{R}_+^N} G_\infty(u)\mathrm{d}\sigma$$

and

$$\int_{\mathbb{R}_+^N} |\nabla_{\mathbb{B}}\overline{u}|^2\mathrm{d}\sigma = \min\Big\{\int_{\mathbb{R}_+^N} |\nabla u|^2\mathrm{d}\sigma;\ u \in \mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}_+^N), \int_{\mathbb{R}_+^N} G_\infty(u)\mathrm{d}\sigma = 1\Big\}.$$

Since

$$\int_{\mathbb{R}_+^N} |\nabla_{\mathbb{B}}u|^2\mathrm{d}\sigma = \theta^{\frac{N-2}{2}}\int_{\mathbb{R}_+^N} |\nabla_{\mathbb{B}}\overline{u}|^2\mathrm{d}\sigma \quad \text{and} \quad \int_{\mathbb{R}_+^N} G_\infty(u)\mathrm{d}\sigma = \theta^{\frac{N}{2}}\int_{\mathbb{R}_+^N} G_\infty(\overline{u})\mathrm{d}\sigma = \theta^{\frac{N}{2}},$$

we have

$$\theta = \frac{N-2}{2N}\int_{\mathbb{R}_+^N} |\nabla_{\mathbb{B}}\overline{u}|^2\mathrm{d}\sigma.$$

Moreover,

$$I_\infty(u)=\frac{1}{N}\int_{\mathbb{R}^N_+}|\nabla_{\mathbb{B}}u|^2\mathrm{d}\sigma=\frac{1}{N}\Big(\frac{N-2}{2N}\Big)^{\frac{N-2}{2}}\Big(\int_{\mathbb{R}^N_+}|\nabla_{\mathbb{B}}\overline{u}|^2\mathrm{d}\sigma\Big)^{\frac{N}{2}}.$$

On the other hand, let v be another solution of (3.1). Then

$$\int_{\mathbb{R}^N_+}|\nabla_{\mathbb{B}}v|^2\mathrm{d}\sigma=\frac{2N}{N-2}\int_{\mathbb{R}^N_+}G_\infty(v)\mathrm{d}\sigma.$$

Let $\mu>0$ satisfying $\int_{\mathbb{R}^N_+}G_\infty(v_\mu)\mathrm{d}\sigma=1$, then

$$\mu=\Big(\int_{\mathbb{R}^N_+}G_\infty(v)\mathrm{d}\sigma\Big)^{-\frac{1}{N}}=\Big(\frac{N-2}{2N}\int_{\mathbb{R}^N_+}|\nabla_{\mathbb{B}}v|^2\mathrm{d}\sigma\Big)^{-\frac{1}{N}}.$$

Therefore, we get

$$I_\infty(v)=\frac{1}{N}\int_{\mathbb{R}^N_+}|\nabla_{\mathbb{B}}v|^2\mathrm{d}\sigma=\frac{1}{N}\Big(\frac{N-2}{2N}\Big)^{\frac{N-2}{2}}\Big(\int_{\mathbb{R}^N_+}|\nabla_{\mathbb{B}}v_\mu|^2\mathrm{d}\sigma\Big)^{\frac{N}{2}}.$$

Since $\overline{u}$ solves problem (3.2) and $\int_{\mathbb{R}^N_+}G_\infty(v_\mu)\mathrm{d}\sigma=1$, we obtain

$$\int_{\mathbb{R}^N_+}|\nabla_{\mathbb{B}}\overline{u}|^2\mathrm{d}\sigma\le\int_{\mathbb{R}^N_+}|\nabla_{\mathbb{B}}v_\mu|^2\mathrm{d}\sigma.$$

Thus

$$I_\infty(u)\le I_\infty(v)\quad\text{for all solution } v \text{ of problem (3.1), i.e., } I_\infty(u)=m.$$

What's more,

$$\begin{aligned}I_\infty(u)&=\min_{\int_{\mathbb{R}^N_+}G_\infty(\varphi)\mathrm{d}\sigma=1}\frac{1}{N}\Big(\frac{N-2}{2N}\Big)^{\frac{N-2}{2}}\Big(\int_{\mathbb{R}^N_+}|\nabla_{\mathbb{B}}\varphi|^2\mathrm{d}\sigma\Big)^{\frac{N}{2}}\\&=\min_{J_\infty(\varphi_{\sqrt{\theta_\varphi}})=0}\frac{1}{N}\Big(\frac{N-2}{2N}\Big)^{\frac{N-2}{2}}\Big(\int_{\mathbb{R}^N_+}|\nabla_{\mathbb{B}}\varphi|^2\mathrm{d}\sigma\Big)^{\frac{N}{2}}=\min_{J_\infty(v)=0}I_\infty(v)=\min_{v\in\mathcal{P}_\infty}I_\infty(v).\end{aligned}$$

Lemma 3.1 *For all $\gamma\in\Gamma_\infty$, we have $\gamma([0,1])\cap\mathcal{P}_\infty\neq\emptyset$.*

Proof Similar to the proof of Lemma 2.1, we know that there exists $\rho>0$ such that

$$J_\infty(u)>0\quad\text{if } 0<\|u\|<\rho.$$

Observe that

$$J_\infty(u)=NI_\infty(u)-\int_{\mathbb{R}^N_+}|\nabla_{\mathbb{B}}u|^2\mathrm{d}\sigma.$$

For any $\gamma\in\Gamma_\infty$, it holds

$$J_\infty(\gamma(0))=0\quad\text{and}\quad J_\infty(\gamma(1))\le NI_\infty(\gamma(1))<0.$$

Thus, there exists $\tau\in(0,1)$ such that

$$\|\gamma(\tau)\|\ge\rho\quad\text{and}\quad J_\infty(\gamma(\tau))=0.$$

This means $\gamma(\tau)\in\gamma([0,1])\cap\mathcal{P}_\infty$.

Lemma 3.2 *Let Φ be a least energy solution to problem* (3.1)*. Then there exists $\gamma \in \Gamma_\infty$ such that*

$$\Phi \in \gamma([0,1]) \quad \textit{and} \quad \max_{\tau \in [0,1]} I_\infty(\gamma(\tau)) = I_\infty(\Phi) = m.$$

Proof Let Φ be a least energy solution of (3.1) and $\int_{\mathbb{R}_+^N} |\nabla_{\mathbb{B}}\Phi|^2 \mathrm{d}\sigma = \frac{2N}{N-2} \int_{\mathbb{R}_+^N} G_\infty(\Phi)\mathrm{d}\sigma$. We can define a continuous path $\gamma_1(\tau)(t,x) = \Phi\big(t^{\frac{1}{\tau}}, \frac{x}{\tau}\big)$ for $\tau > 0$ and $\gamma_1(0) = 0$. Then $I_\infty(\gamma_1(0)) = 0$ and

$$I_\infty(\gamma_1(\tau)) = \frac{\tau^{N-2}}{2} \int_{\mathbb{R}_+^N} |\nabla_{\mathbb{B}}\Phi|^2 \mathrm{d}\sigma - \tau^N \int_{\mathbb{R}_+^N} G_\infty(\Phi)\mathrm{d}\sigma \quad \text{for } \tau > 0.$$

In particular, $\gamma_1(1) = \Phi$. By taking derivative, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau} I_\infty(\gamma_1(\tau)) &= \frac{N-2}{2}\tau^{N-3} \int_{\mathbb{R}_+^N} |\nabla_{\mathbb{B}}\Phi|^2 \mathrm{d}\sigma - N\tau^{N-1} \int_{\mathbb{R}_+^N} G_\infty(\Phi)\mathrm{d}\sigma \\ &= \frac{N-2}{2}\tau^{N-3}(1-\tau^2) \int_{\mathbb{R}_+^N} |\nabla_{\mathbb{B}}\Phi|^2 \mathrm{d}\sigma.\end{aligned}$$

Thus $I_\infty(\gamma_1(1)) = \max\limits_{\tau \in [0,1]} I_\infty(\gamma_1(\tau))$.

Since $\int_{\mathbb{R}_+^N} G_\infty(\Phi)\mathrm{d}\sigma > 0$, we can take $L > 1$ large enough such that $I_\infty(\gamma_1(L)) < 0$. Let $\gamma(\tau) = \gamma_1(\tau L)$, then we have $\gamma \in \Gamma_\infty$ and the result follows.

Proof of Proposition 3.1 Combining Proposition 3.2 and Lemma 3.1, we get $m \le c_\infty$. Considering the path $\gamma \in \Gamma_\infty$ provided by Lemma 3.2, we have

$$\max_{\tau \in [0,1]} I_\infty(\gamma(\tau)) = m.$$

Taking the infimum over Γ_∞, we obtain

$$\inf_{\gamma \in \Gamma_\infty} \max_{\tau \in [0,1]} I_\infty(\gamma(\tau)) \le m.$$

Therefore, $c_\infty \le m$. And we get the assertion.

Put $l = \inf\{u \in (0,\infty) \mid G_\infty(u) > 0\}$. Then we have following uniqueness result.

Proposition 3.3 *Suppose*

$$\frac{-\lambda u + a_\infty f(u)}{u - l} \text{ is non-increasing on the subset of } (l,\infty) \text{ where } -\lambda u + a_\infty f(u) > 0. \tag{3.4}$$

Then problem (3.1) *has at most one positive solution.*

In fact, if f is defined in (1.6), then condition (3.4) is satisfied. Therefore, in this case, together with the result in Proposition 3.2, we know that problem (3.1) has exact one positive solution and this solution is radial. We will give the proof in Appendix.

4 Nonexistence Result

We begin by presenting the main relations between Pohozaev manifold $\mathcal{P}$ associated with non-autonomous problem (1.1) and Pohozaev manifold $\mathcal{P}_\infty$ associated with limiting problem (3.1). Note that condition (A3), (A4) imply that

$$I_\infty(u) < I(u) \quad \text{for all } u \in \mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}_+^N) \backslash \{0\}.$$

We will show in this section that

$$p := \inf_{u\in\mathcal{P}} I(u) = c_\infty$$

and this level is not a critical level for functional I.

Lemma 4.1 *If $\int_{\mathbb{R}^N_+} G_\infty(u)\mathrm{d}\sigma > 0$, then there exist unique $\mu_1, \mu_2 > 0$ such that*

$$u_{\mu_1} \in \mathcal{P}_\infty \quad \text{and} \quad u_{\mu_2} \in \mathcal{P}.$$

Proof First, we consider the case $\mathcal{P}_\infty$. Let $\varphi\colon (0,\infty) \to \mathbb{R}$ defined by

$$\varphi(\mu) = I_\infty(u_\mu) = I_\infty\Big(u\Big(t^{\frac{1}{\mu}}, \frac{x}{\mu}\Big)\Big) = \frac{1}{2}\mu^{N-2}\int_{\mathbb{R}^N_+} |\nabla_{\mathbb{B}} u|^2 \mathrm{d}\sigma - \mu^N \int_{\mathbb{R}^N_+} G_\infty(u)\mathrm{d}\sigma.$$

Taking derivative of φ, we get

$$\varphi'(\mu) = \frac{N-2}{2}\mu^{N-3}\int_{\mathbb{R}^N_+} |\nabla_{\mathbb{B}} u|^2 \mathrm{d}\sigma - N\mu^{N-1}\int_{\mathbb{R}^N_+} G_\infty(u)\mathrm{d}\sigma = \frac{J_\infty(u_\mu)}{\mu}.$$

Thus, $\varphi'(\mu) = 0$ if and only if either $\mu = 0$ or $\mu = \mu_1 = \big(\frac{(N-2)\int_{\mathbb{R}^N_+} |\nabla_{\mathbb{B}} u|^2 \mathrm{d}\sigma}{2N\int_{\mathbb{R}^N_+} G_\infty(u)\mathrm{d}\sigma}\big)^{\frac{1}{2}} > 0$.

By the formula of $\varphi'(\mu)$, we know $u_\mu \in \mathcal{P}_\infty$ if and only if $\varphi'(\mu) = 0$ for some $\mu > 0$ and then we have $u_{\mu_1} \in \mathcal{P}_\infty$. Observe that φ is positive for $\mu > 0$ small enough while is negative for $\mu > 0$ large, thus the unique critical point of φ is a global maximum point for φ.

Now, we turn to the case $\mathcal{P}$. First, we define the function

$$\begin{aligned}\Psi(\mu) :&= I(u_\mu) = \frac{1}{2}\mu^{N-2}\int_{\mathbb{R}^N_+} |\nabla_{\mathbb{B}} u|^2 \mathrm{d}\sigma - \int_{\mathbb{R}^N_+} G(t, x, u_\mu)\mathrm{d}\sigma \\ &= \frac{1}{2}\mu^{N-2}\int_{\mathbb{R}^N_+} |\nabla_{\mathbb{B}} u|^2 \mathrm{d}\sigma - \mu^N \int_{\mathbb{R}^N_+} \Big[a(t^\mu, \mu x)F(u) - \lambda\frac{u^2}{2}\Big]\mathrm{d}\sigma.\end{aligned}$$

Taking derivative of $\Psi(\mu)$ and recalling that $N \geq 3$, we get

$$\begin{aligned}\Psi'(\mu) = {} & \frac{N-2}{2}\mu^{N-3}\int_{\mathbb{R}^N_+} |\nabla_{\mathbb{B}} u|^2 \mathrm{d}\sigma - N\mu^{N-1}\int_{\mathbb{R}^N_+} \Big[a(t^\mu, \mu x)F(u) - \lambda\frac{u^2}{2}\Big]\mathrm{d}\sigma \\ & - \mu^N \int_{\mathbb{R}^N_+} \Big[t^\mu \ln t\partial_0 a(t^\mu, \mu x) + \sum_{i=1}^n x_i\partial_i a(t^\mu, \mu x)\Big]F(u)\mathrm{d}\sigma = \frac{J(u_\mu)}{\mu}.\end{aligned}$$

Thus we find that

$$u_\mu \in \mathcal{P} \quad \text{if and only if} \quad \Psi'(\mu) = 0 \ \ \text{for some} \ \ \mu > 0.$$

Note that, by condition (A2) and the Lebesgue dominated convergence theorem, we have

$$\lim_{\mu\to\infty}\int_{\mathbb{R}^N_+} \Big[a(t^\mu, \mu x)F(u) - \lambda\frac{u^2}{2}\Big]\mathrm{d}\sigma = \int_{\mathbb{R}^N_+} \Big[a_\infty F(u) - \lambda\frac{u^2}{2}\Big]\mathrm{d}\sigma = \int_{\mathbb{R}^N_+} G_\infty(u)\mathrm{d}\sigma. \tag{4.1}$$

Also using (1.4) and again Lebesgue dominated convergence theorem it follows that

$$\lim_{\mu\to\infty}\int_{\mathbb{R}^N_+} \Big[t^\mu \ln t^\mu \partial_0 a(t^\mu, \mu x) + \sum_{i=1}^n \mu x_i\partial_i a(t^\mu, \mu x)\Big]F(u)\mathrm{d}\sigma = 0. \tag{4.2}$$

Therefore, if $\mu > 0$ large enough, then

$$\Psi'(\mu) = \mu^{N-3}\Big\{\frac{N-2}{2}\int_{\mathbb{R}_+^N} |\nabla_{\mathbb{B}} u|^2 \mathrm{d}\sigma - N\mu^2\Big(\int_{\mathbb{R}_+^N} G_\infty(u)\mathrm{d}\sigma + o_\mu(1)\Big)\Big\}.$$

Since $\int_{\mathbb{R}_+^N} G_\infty(u)\mathrm{d}\sigma > 0$, it follows that $\Psi'(\mu) < 0$ for $\mu > 0$ large enough.

On the other hand, for $\mu > 0$ small enough, from conditions (A1), (A3), (A4) and (1.5), we have

$$0 < a(t,x) + \frac{1}{N}\Big[(\ln t)t\partial_t a(t,x) + \sum_{i=1}^n x_i \partial_{x_i} a(t,x)\Big] < a_\infty$$

and

$$\begin{aligned}-\frac{\lambda}{2}\int_{\mathbb{R}_+^N} u^2 \mathrm{d}\sigma &\le \int_{\mathbb{R}_+^N} \Big\{\Big[a(t^\mu,\mu x) + \frac{t^\mu \ln t^\mu \partial_0 a(t^\mu,\mu x) + \sum_{i=1}^n \mu x_i \partial_i a(t^\mu,\mu x)}{N}\Big]F(u) - \lambda\frac{u^2}{2}\Big\}\mathrm{d}\sigma \\ &< \int_{\mathbb{R}_+^N} G_\infty(u)\mathrm{d}\sigma < \frac{Ca_\infty}{2}\int_{\mathbb{R}_+^N} u^2\mathrm{d}\sigma,\end{aligned}$$

where C is a positive constant independent of μ.

Then, taking $\mu > 0$ small enough, we have $\Psi'(\mu) > 0$. Since $\Psi'(\mu)$ is continuous, there exists $\mu_2 = \mu_2(u) > 0$ such that $\Psi'(\mu_2) = 0$. This means that $u_{\mu_2} \in \mathcal{P}$.

To show the uniqueness of μ_2, $\Psi'(\mu) = 0$ implies that

$$\frac{N-2}{2}\int_{\mathbb{R}_+^N} |\nabla_{\mathbb{B}} u|^2\mathrm{d}\sigma = N\mu^2\int_{\mathbb{R}_+^N}\Big\{\Big[a + \frac{t^\mu \ln t^\mu \partial_0 a(t^\mu,\mu x) + \sum_{i=1}^n \mu x_i \partial_i a(t^\mu,\mu x)}{N}\Big]F(u) - \lambda\frac{u^2}{2}\Big\}\mathrm{d}\sigma.$$

Denoting

$$\psi(\mu) := \int_{\mathbb{R}_+^N}\Big\{\Big[a(t^\mu,\mu x) + \frac{t^\mu \ln t^\mu \partial_0 a(t^\mu,\mu x) + \sum_{i=1}^n \mu x_i \partial_i a(t^\mu,\mu x)}{N}\Big]F(u) - \lambda\frac{u^2}{2}\Big\}\mathrm{d}\sigma,$$

then we have

$$\begin{aligned}\psi'(\mu) = \frac{1}{\mu}\int_{\mathbb{R}_+^N}&\Big\{t^\mu \ln t^\mu \partial_0 a(t^\mu,\mu x) + \sum_{i=1}^n \mu x_i\partial_i a(t^\mu,\mu x) + \frac{1}{N}\Big[(\ln t^\mu)^2 t^\mu \partial_0 a(t^\mu,\mu x) \\ &+ (\ln t^\mu)^2 t^{2\mu}\partial_0^2 a(t^\mu,\mu x) + 2\sum_{i=1}^n t^\mu \ln t^\mu \mu x_i \partial_{0i}^2 a(t^\mu,\mu x) + \sum_{i,j=1}^n \mu^2 x_i x_j \partial_{ij}^2 a(t^\mu,\mu x)\Big] \\ &+ \frac{1}{N}\Big[t^\mu \ln t^\mu \partial_0 a(t^\mu,\mu x) + \sum_{i=1}^n \mu x_i \partial_i a(t^\mu,\mu x)\Big]\Big\}F(u)\mathrm{d}\sigma.\end{aligned}$$

Conditions (A3) and (A5) tell us that $\psi'(\mu) > 0$. Therefore, $\psi(\mu)$ is an increasing function of μ and there exists a unique $\mu > 0$ such that

$$\frac{N-2}{2}\int_{\mathbb{R}_+^N} |\nabla_{\mathbb{B}} u|^2 \mathrm{d}\sigma = N\mu^2\psi(\mu).$$

The uniqueness of μ_2 is verified.

Remark 4.1 Note that the hypothesis (A5) was used in the previous lemma only to show the uniqueness of μ_2.

Lemma 4.2 *Let* $\mathcal{O} := \big\{u \in \mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}_+^N); \int_{\mathbb{R}_+^N} G_\infty(u)\mathrm{d}\sigma > 0\big\}$ *be an open subset of* $\mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}_+^N)$. *The map* $\mu_2\colon \mathcal{O} \to \mathbb{R}^+$ *defined by* $u \mapsto \mu_2(u)$, *such that* $u_{\mu_2(u)} \in \mathcal{P}$, *is continuous.*

Proof Consider the sequence $\{u_j\} \subset \mathcal{O}$ such that $u_j \to u \in \mathcal{O}$. We will show that $\mu_2(u_j) \to \mu_2(u)$.

First, $\{\mu_2(u_j)\}$ is bounded. Indeed, recall the expression $\Psi'(\mu) = 0$ in the proof of the previous lemma applied to u_j and $\mu_2(u_j)$:

$$\begin{aligned}\frac{N-2}{2}\int_{\mathbb{R}_+^N}|\nabla_{\mathbb{B}}u_j|^2\mathrm{d}\sigma &= N(\mu_2(u_j))^2\int_{\mathbb{R}_+^N}\Big[a(t^{\mu_2(u_j)},\mu_2(u_j)x)F(u)-\lambda\frac{u^2}{2}\Big]\mathrm{d}\sigma\\ &\quad+(\mu_2(u_j))^2\int_{\mathbb{R}_+^N}\Big[t^{\mu_2(u_j)}\ln t^{\mu_2(u_j)}\partial_0 a(t^{\mu_2(u_j)},\mu_2(u_j)x)\\ &\quad+\sum_{i=1}^n \mu_2(u_j)x_i\partial_i a(t^{\mu_2(u_j)},\mu_2(u_j)x)\Big]F(u)\mathrm{d}\sigma.\end{aligned}$$

Since $\mu_2(u_j) > 0$ for all $j \in \mathbb{N}$, suppose by contradiction that $\mu_2(u_j) \to +\infty$ as $j \to \infty$. Then, by (4.1)–(4.2) and Lebesgue dominated convergence theorem we get that the right hand side of above equality goes to infinity while the left hand side tends to $\frac{N-2}{2}\int_{\mathbb{R}_+^N}|\nabla_{\mathbb{B}}u|^2\mathrm{d}\sigma < \infty$, which is a contradiction. Therefore, we know that $\{\mu_2(u_j)\}$ is bounded and has a convergence subsequence, say $\mu_2(u_j) \to \overline{\mu}_2$. In turn, by Lebesgue dominated convergence theorem, as $j \to \infty$, we obtain

$$\int_{\mathbb{R}_+^N} a(t^{\mu_2(u_j)},\mu_2(u_j)x)F(u)\mathrm{d}\sigma \to \int_{\mathbb{R}_+^N} a(t^{\overline{\mu}_2},\overline{\mu}_2 x)F(u)\mathrm{d}\sigma$$

and

$$\begin{aligned}&\int_{\mathbb{R}_+^N}\Big[t^{\mu_2(u_j)}\ln t^{\mu_2(u_j)}\partial_0 a(t^{\mu_2(u_j)},\mu_2(u_j)x)+\sum_{i=1}^n \mu_2(u_j)x_i\partial_i a(t^{\mu_2(u_j)},\mu_2(u_j)x)\Big]F(u)\mathrm{d}\sigma\\ &\to\int_{\mathbb{R}_+^N}\Big[t^{\overline{\mu}_2}\ln t^{\overline{\mu}_2}\partial_0 a(t^{\overline{\mu}_2},\overline{\mu}_2 x)+\sum_{i=1}^n \overline{\mu}_2 x_i\partial_i a(t^{\overline{\mu}_2},\overline{\mu}_2 x)\Big]F(u)\mathrm{d}\sigma.\end{aligned}$$

Since $u_j \to u \in \mathcal{O}$, we get

$$\begin{aligned}\frac{N-2}{2}\int_{\mathbb{R}_+^N}|\nabla_{\mathbb{B}}u|^2\mathrm{d}\sigma &= N(\overline{\mu}_2)^2\int_{\mathbb{R}_+^N}\Big[a(t^{\overline{\mu}_2},\overline{\mu}_2 x)F(u)-\lambda\frac{u^2}{2}\Big]\mathrm{d}\sigma\\ &\quad+(\overline{\mu}_2)^2\int_{\mathbb{R}_+^N}\Big[t^{\overline{\mu}_2}\ln t^{\overline{\mu}_2}\partial_0 a(t^{\overline{\mu}_2},\overline{\mu}_2 x)+\sum_{i=1}^n \overline{\mu}_2 x_i\partial_i a(t^{\overline{\mu}_2},\overline{\mu}_2 x)\Big]F(u)\mathrm{d}\sigma.\end{aligned}$$

This means that $u_{\overline{\mu}_2} \in \mathcal{P}$, and by the uniqueness of the projection in $\mathcal{P}$ we get $\overline{\mu}_2 = \mu_2(u)$.

Lemma 4.3 *For* $u \in \mathcal{P}_\infty$, *there exists a unique* $\mu > 1$ *such that* $u_\mu \in \mathcal{P}$.

Proof Let $u \in \mathcal{P}_\infty$, then $\int_{\mathbb{R}_+^N} G_\infty(u)\mathrm{d}\sigma > 0$ and Lemma 4.1 assert the existence of a unique $\mu > 0$ such that $u_\mu \in \mathcal{P}$. Moreover, we have

$$\frac{N-2}{2}\int_{\mathbb{R}_+^N}|\nabla_{\mathbb{B}}u|^2\mathrm{d}\sigma$$

$$= N\mu^2 \int_{\mathbb{R}_+^N} \Big\{\Big[a + \frac{t^\mu \ln t^\mu \partial_0 a(t^\mu, \mu x) + \sum\limits_{i=1}^n \mu x_i \partial_i a(t^\mu, \mu x)}{N}\Big] F(u) - \lambda \frac{u^2}{2}\Big\} \mathrm{d}\sigma.$$

Using the hypotheses (A4), we have

$$\frac{N-2}{2} \int_{\mathbb{R}_+^N} |\nabla_{\mathbb{B}} u|^2 \mathrm{d}\sigma < N\mu^2 \int_{\mathbb{R}_+^N} \Big[a_\infty F(u) - \lambda \frac{u^2}{2}\Big] \mathrm{d}\sigma = N\mu^2 \int_{\mathbb{R}_+^N} G_\infty(u) \mathrm{d}\sigma.$$

But by $u \in \mathcal{P}_\infty$, we know $\frac{N-2}{2} \int_{\mathbb{R}_+^N} |\nabla_{\mathbb{B}} u|^2 \mathrm{d}\sigma = N \int_{\mathbb{R}_+^N} G_\infty(u) \mathrm{d}\sigma$. Hence we find $\mu > 1$.

Lemma 4.4 *For $u \in \mathcal{P}$, there exists a unique $0 < \mu < 1$ such that $u_\mu \in \mathcal{P}_\infty$.*

Proof First, we verify that if $u \in \mathcal{P}$, then $\int_{\mathbb{R}_+^N} G_\infty(u) \mathrm{d}\sigma > 0$. In fact, by $u \in \mathcal{P}$ and condition (A4), u satisfies

$$\begin{aligned} \frac{N-2}{2} \int_{\mathbb{R}_+^N} |\nabla_{\mathbb{B}} u|^2 \mathrm{d}\sigma &= N \int_{\mathbb{R}_+^N} \Big\{\Big[a(t,x) + \frac{(\ln t) t \partial_t a(t,x) + \sum\limits_{i=1}^n x_i \partial_{x_i} a(t,x)}{N}\Big] F(u) - \frac{1}{2}\lambda u^2\Big\} \mathrm{d}\sigma \\ &< N \int_{\mathbb{R}_+^N} \Big[a_\infty F(u) - \frac{1}{2}\lambda u^2\Big] \mathrm{d}\sigma = N \int_{\mathbb{R}_+^N} G_\infty(u) \mathrm{d}\sigma. \end{aligned}$$

Since $u \neq 0$ and $u \in \mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}_+^N)$, we get $\int_{\mathbb{R}_+^N} |\nabla_{\mathbb{B}} u|^2 \mathrm{d}\sigma > 0$ and hence $\int_{\mathbb{R}_+^N} G_\infty(u) \mathrm{d}\sigma > 0$.

Therefore, there exists a unique $\mu > 0$ such that $u_\mu \in \mathcal{P}_\infty$. Notice that

$$\frac{N-2}{2N} \int_{\mathbb{R}_+^N} |\nabla_{\mathbb{B}} u|^2 \mathrm{d}\sigma < \int_{\mathbb{R}_+^N} G_\infty(u) \mathrm{d}\sigma,$$

and $u_\mu \in \mathcal{P}_\infty$, then we have

$$\mu^2 = \frac{(N-2) \int_{\mathbb{R}_+^N} |\nabla_{\mathbb{B}} u|^2 \mathrm{d}\sigma}{2N \int_{\mathbb{R}_+^N} G_\infty(u) \mathrm{d}\sigma} < 1.$$

Thus we have $\mu < 1$.

Remark 4.2 As a result, the function $u \in \mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}_+^N) \backslash \{0\}$ can be projected to $\mathcal{P}$ and $\mathcal{P}_\infty$ if and only if $\int_{\mathbb{R}_+^N} G_\infty(u) \mathrm{d}\sigma > 0$.

Lemma 4.5 *If $u \in \mathcal{P}_\infty$, then $T_{sy} u(t,x) = u\big(\frac{t}{s}, x - y\big) \in \mathcal{P}_\infty$ for all $w = (s,y) \in \mathbb{R}_+^N$. Moreover, there exists $\mu_w > 1$ such that*

$$(T_{sy} u)_{\mu_w} \in \mathcal{P} \quad \textit{and} \quad \lim_{|w| \to \infty} \mu_w = 1.$$

Proof For $u \in \mathcal{P}_\infty$, since

$$\begin{aligned} J_\infty(T_{sy} u) = \frac{N-2}{2} &\int_{\mathbb{R}_+^N} \Big[\Big(t \partial_t u\Big(\frac{t}{s}, x - y\Big)\Big)^2 + \sum_{i=1}^n \Big(\partial_{x_i} u\Big(\frac{t}{s}, x - y\Big)\Big)^2\Big] \frac{\mathrm{d}t}{t} \mathrm{d}x \\ &- N \int_{\mathbb{R}_+^N} \Big[a_\infty F\Big(u\Big(\frac{t}{s}, x - y\Big)\Big) - \frac{1}{2}\lambda u^2\Big(\frac{t}{s}, x - y\Big)\Big] \frac{\mathrm{d}t}{t} \mathrm{d}x \end{aligned}$$

$$= \frac{N-2}{2}\int_{\mathbb{R}_+^N} |\nabla_{\mathbb{B}} u|^2 \frac{\mathrm{d}t}{t}\mathrm{d}x - N\int_{\mathbb{R}_+^N} \Big[a_\infty F(u) - \frac{1}{2}\lambda u^2\Big]\frac{\mathrm{d}t}{t}\mathrm{d}x = J_\infty(u),$$

we know $T_{sy}u(t,x) \in \mathcal{P}_\infty$ for all $w=(s,y)\in\mathbb{R}_+^N$. What's more, from Lemma 4.3, there exists $\mu_w > 1$ such that $(T_{sy}u)_{\mu_w}\in\mathcal{P}$.

Suppose by contradiction that there exists a sequence $w_j\in\mathbb{R}_+^N$ such that $|w_j|\to+\infty$ and $\mu_{w_j}\to A>1$ or $+\infty$. Now for $(T_{sy}u)_{\mu_w}\in\mathcal{P}$, we have

$$\begin{aligned}\frac{N-2}{2}\int_{\mathbb{R}_+^N}|\nabla_{\mathbb{B}}u|^2\mathrm{d}\sigma &= N\mu_w^2\int_{\mathbb{R}_+^N}\Big\{a(st^{\mu_w}, y+\mu_w x)\\ &\quad + \frac{1}{N}\Big[st^{\mu_w}\ln(st^{\mu_w})\partial_0 a(st^{\mu_w}, y+\mu_w x)\\ &\quad + \sum_{i=1}^n (y_i+\mu x_i)\partial_i a(st^{\mu_w}, y+\mu_w x)\Big]F(u) - \lambda\frac{u^2}{2}\Big\}\mathrm{d}\sigma\\ &= N\mu_w^2\int_{\mathbb{R}_+^N}\Big[K(st^{\mu_w}, y+\mu_w x)F(u) - \lambda\frac{u^2}{2}\Big]\mathrm{d}\sigma, \end{aligned}\tag{4.3}$$

where

$$K(st^{\mu_w}, y+\mu_w x) = a + \frac{st^{\mu_w}\ln(st^{\mu_w})\partial_0 a + \sum\limits_{i=1}^n (y_i+\mu_w x_i)\partial_i a}{N}\bigg|_{(st^{\mu_w}, y+\mu_w x)}.$$

From condition (A4) and (1.5) we have

$$\begin{aligned}0 &\le K(st^{\mu_w}, y+\mu_w x)F(u(t,x)) - \lambda\frac{u^2(t,x)}{2}\\ &< a_\infty F(u(t,x)) - \lambda\frac{u^2(t,x)}{2} \le Cu^2(t,x) \quad \text{for a.e. } (t,x)\in\mathbb{R}_+^N.\end{aligned}$$

Applying (4.3) to $(T_{sy}u)_{\mu_{w_j}}$, if $\mu_{w_j}\to+\infty$, we get that the right hand side goes to infinity while the left hand side is a constant $\frac{N-2}{2}\int\limits_{\mathbb{R}_+^N}|\nabla_{\mathbb{B}}u|^2\mathrm{d}\sigma < +\infty$. This is a contradiction. If $\mu_{w_j}\to A>1$, then

$$\lim_{|w_j|\to+\infty}\int_{\mathbb{R}_+^N}\Big[K(st^{\mu_w}, y+\mu_w x)F(u) - \lambda\frac{u^2}{2}\Big]\mathrm{d}\sigma = \int_{\mathbb{R}_+^N} G_\infty(u)\mathrm{d}\sigma.$$

This means that the right hand side goes to $NA^2\int_{\mathbb{R}_+^N} G_\infty(u)\mathrm{d}\sigma$ while the left hand side is a constant $\frac{N-2}{2}\int_{\mathbb{R}_+^N}|\nabla_{\mathbb{B}}u|^2\mathrm{d}\sigma<+\infty$. But $A>1$ and $u\in\mathcal{P}_\infty$, we get also a contradiction.

Under the assumption of Lemma 4.5, we have following lemma.

Lemma 4.6 $\sup\limits_{w\in\mathbb{R}_+^N}\mu_w = \overline{\mu} < +\infty$ *and* $\overline{\mu}>1$.

Proof Lemma 4.5 tells us that there exists $R>0$ such that $|\mu_w|<2$ for $|w|>R$. We show that there exists $M>0$ such that

$$\sup\big\{\mu_w; |w|\le R\big\}\le M.$$

Suppose that there exists a subsequence $\{w_j\} \subset \mathbb{R}^N_+$ with $|w_j| \le R$ such that $\mu_{w_j} \to +\infty$ as $j \to +\infty$. As in previous lemma, but now with $\mu_{w_j} \to +\infty$, we have

$$\lim_{\mu_{w_j}\to+\infty} \int_{\mathbb{R}^N_+} \Big[K(st^{\mu_w}, y+\mu_w x)F(u) - \lambda\frac{u^2}{2}\Big]\mathrm{d}\sigma = \int_{\mathbb{R}^N_+} G_\infty(u)\mathrm{d}\sigma$$

and then

$$\frac{N-2}{2}\int_{\mathbb{R}^N_+} |\nabla_{\mathbb{B}} u|^2 \mathrm{d}\sigma = N\mu_{w_j}^2\Big[\int_{\mathbb{R}^N_+} G_\infty(u)\mathrm{d}\sigma + o_{w_j}(1)\Big].$$

But $\mu_{w_j} \to +\infty$ and the left hand side is a fixed number, this is absurd. Thus $\sup\limits_{w\in\mathbb{R}^N_+} \mu_w < +\infty$.

Lemma 4.7 *It holds* $p = \inf\limits_{u\in\mathcal{P}} I(u) = c_\infty$.

Proof Let $\Phi \in \mathcal{H}^{1,\frac{N}{2}}_2(\mathbb{R}^N_+)$ be a ground state solution of limiting problem (3.1), $\Phi \in \mathcal{P}_\infty$ and $I_\infty(\Phi) = c_\infty$. Given any $w = (s, y) \in \mathbb{R}^N_+$, from previous discussions, we know that $T_{sy}\Phi \in \mathcal{P}_\infty$ and $I_\infty(T_{sy}\Phi) = c_\infty$. From Lemma 4.5, there exists a $\mu_w > 1$ such that $\widetilde{\Phi}_w := (T_{sy}\Phi)_{\mu_w} \in \mathcal{P}$. Thus, we get

$$\begin{aligned}
&|I(\widetilde{\Phi}_w) - c_\infty| = |I(\widetilde{\Phi}_w) - I_\infty(T_{sy}\Phi)| \\
={}& \Big|\frac{1}{2}\int_{\mathbb{R}^N_+} |\nabla_{\mathbb{B}}\widetilde{\Phi}_w|^2\mathrm{d}\sigma - \int_{\mathbb{R}^N_+} G(t,x,\widetilde{\Phi}_w)\mathrm{d}\sigma - \frac{1}{2}\int_{\mathbb{R}^N_+} |\nabla_{\mathbb{B}} T_{sy}\Phi|^2\mathrm{d}\sigma + \int_{\mathbb{R}^N_+} G_\infty(T_{sy}\Phi)\mathrm{d}\sigma\Big| \\
={}& \Big|\frac{\mu_w^{N-2}-1}{2}\int_{\mathbb{R}^N_+} |\nabla_{\mathbb{B}}\Phi|^2\mathrm{d}\sigma - \int_{\mathbb{R}^N_+} [\mu_w^N a(st^{\mu_w}, y+\mu_w x) - a_\infty]F(\Phi)\mathrm{d}\sigma \\
&+ (\mu_w^N - 1)\frac{\lambda}{2}\int_{\mathbb{R}^N_+} \Phi^2\mathrm{d}\sigma\Big| \\
\le{}& \frac{|\mu_w^{N-2}-1|}{2}\int_{\mathbb{R}^N_+} |\nabla_{\mathbb{B}}\Phi|^2\mathrm{d}\sigma + \int_{\mathbb{R}^N_+} |\mu_w^N a(st^{\mu_w}, y+\mu_w x) - a_\infty|\cdot F(\Phi)\mathrm{d}\sigma \\
&+ |\mu_w^N - 1|\frac{\lambda}{2}\int_{\mathbb{R}^N_+} \Phi^2\mathrm{d}\sigma.
\end{aligned}$$

Since $\mu_w \to 1$ and $a(st, y+x) \to a_\infty$ as $|w| \to \infty$, we obtain

$$|I(\widetilde{\Phi}_w) - c_\infty| = o_w(1) + \int_{\mathbb{R}^N_+} |a(st, y+x) - a_\infty|\cdot F(\Phi)\mathrm{d}\sigma + o_w(1) = o_w(1),$$

and $\lim\limits_{|w|\to\infty} I(\widetilde{\Phi}_w) = c_\infty$. Therefore, $p = \inf\limits_{u\in\mathcal{P}} I(u) \le c_\infty$.

On the other hand, for $v \in \mathcal{P}$, then from Lemma 4.4 we get $0 < \mu < 1$ such that $v_\mu \in \mathcal{P}_\infty$. At the same time, using $N \ge 3$, (A3) and Proposition 3.1, we have

$$\begin{aligned}
I(v) &= \frac{1}{N}\int_{\mathbb{R}^N_+} |\nabla_{\mathbb{B}} v|^2\mathrm{d}\sigma + \frac{1}{N}\int_{\mathbb{R}^N_+} \Big[(\ln t)t\partial_t a(t,x) + \sum_{i=1}^n x_i\partial_{x_i} a(t,x)\Big]F(v)\mathrm{d}\sigma \\
&\ge \frac{1}{N}\int_{\mathbb{R}^N_+} |\nabla_{\mathbb{B}} v|^2\mathrm{d}\sigma > \frac{\mu^{N-2}}{N}\int_{\mathbb{R}^N_+} |\nabla_{\mathbb{B}} v|^2\mathrm{d}\sigma = I_\infty(v_\mu) \ge \inf_{u\in\mathcal{P}_\infty} I_\infty(u) = m = c_\infty.
\end{aligned}$$

Therefore we find $p \ge c_\infty$. And then we get the assertion.

Lemma 4.8 $\mathcal{P}$ *is a natural constraint of problem* (1.1).

Proof Let $u \in \mathcal{P}$ be a critical point of I, restricted to the manifold $\mathcal{P}$. We obtain that there exists a Lagrange multiple θ such that

$$I'(u) + \theta J'(u) = 0.$$

We will show that $\theta = 0$. Apply the linear functional above at any $v \in \mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}_+^N)$,

$$\begin{aligned}0 = &\int_{\mathbb{R}_+^N} \nabla_{\mathbb{B}} u \cdot \nabla_{\mathbb{B}} v \mathrm{d}\sigma - \int_{\mathbb{R}_+^N} [a(t,x)f(u)v - \lambda uv] \mathrm{d}\sigma + \theta\Big[(N-2)\int_{\mathbb{R}_+^N} \nabla_{\mathbb{B}} u \cdot \nabla_{\mathbb{B}} v \mathrm{d}\sigma \\ &- \int_{\mathbb{R}_+^N} \Big[Na(t,x) + (\ln t)t\partial_t a(t,x) + \sum_{i=1}^n x_i \partial_{x_i} a(t,x)\Big] f(u)v - \lambda uv \mathrm{d}\sigma\Big],\end{aligned}$$

so that u satisfies

$$-(1+\theta(N-2))\triangle_{\mathbb{B}} u + \lambda(1+\theta N)u = \Big[(1+\theta N)a(t,x) + \theta\Big((\ln t)t\partial_t a(t,x) + \sum_{i=1}^n x_i \partial_{x_i} a(t,x)\Big)\Big] f(u).$$

The solution of this equation satisfies a Pohozaev manifold identity $J^*(u) = 0$, i.e.,

$$\begin{aligned}&(1+\theta(N-2))\frac{N-2}{2}\int_{\mathbb{R}_+^N} |\nabla_{\mathbb{B}} u|^2 \mathrm{d}\sigma \\ &= \int_{\mathbb{R}_+^N} \Big[NG^*(z,u) + (\ln t)t\partial_0 G^*(z,u) + \sum_{i=1}^n x_i \partial_i G^*(z,u)\Big] \mathrm{d}\sigma,\end{aligned}$$

where

$$G^*(z,u) = \Big[(1+\theta N)a(t,x) + \theta((\ln t)t\partial_0 a(t,x) + \sum_{i=1}^n x_i \partial_i a(t,x))\Big] F(u) - \frac{(1+\theta N)}{2}\lambda u^2,$$

and then

$$\begin{aligned}(\ln t)t\partial_0 G^*(z,u) + \sum_{i=1}^n x_i \partial_i G^*(z,u) &= \Big\{(1+\theta(N+1))\Big[(\ln t)t\partial_0 a(t,x) + \sum_{i=1}^n x_i \partial_i a(t,x)\Big] \\ + \theta\Big[(\ln t)^2(t\partial_0 a(t,x) + t^2\partial_0^2 a(t,x)) &+ 2\sum_{i=1}^n x_i (\ln t)t\partial_{0i}^2 a(t,x) + \sum_{i,j=1}^n x_i x_j \partial_{ij}^2 a(t,x)\Big]\Big\} F(u).\end{aligned}$$

Thus, we have

$$\begin{aligned}&(1+\theta(N-2))\frac{N-2}{2}\int_{\mathbb{R}_+^N} |\nabla_{\mathbb{B}} u|^2 \mathrm{d}\sigma \\ &= N(1+\theta N)\int_{\mathbb{R}_+^N} \Big\{\Big[a + \frac{(\ln t)t\partial_0 a + \sum_{i=1}^n x_i \partial_i a}{N}\Big] F(u) - \frac{1}{2}\lambda u^2\Big\} \mathrm{d}\sigma \\ &+ (N+1)\theta\int_{\mathbb{R}_+^N} \Big\{(\ln t)t\partial_0 a + \sum_{i=1}^n x_i \partial_i a \\ &+ \frac{1}{N+1}\Big[(\ln t)^2 t\partial_0 a + (\ln t)^2 t^2 \partial_0^2 a + 2\sum_{i=1}^n x_i (\ln t)t\partial_{0i}^2 a + \sum_{i,j=1}^n x_i x_j \partial_{ij}^2 a\Big]\Big\} F(u) \mathrm{d}\sigma.\end{aligned}$$

Since $u \in \mathcal{P}$, substituting (2.2) in the above equation, we get

$$\begin{aligned}&-\theta(N-2)\int_{\mathbb{R}_+^N}|\nabla_{\mathbb{B}}u|^2\mathrm{d}\sigma\\&=(N+1)\theta\int_{\mathbb{R}_+^N}\Big\{(\ln t)t\partial_0 a+\sum_{i=1}^n x_i\partial_i a+\frac{1}{N+1}\Big[(\ln t)^2 t\partial_0 a+(\ln t)^2t^2\partial_0^2 a\\&\quad+2\sum_{i=1}^n x_i(\ln t)t\partial_{0i}^2 a+\sum_{i,j=1}^n x_ix_j\partial_{ij}^2 a\Big]\Big\}F(u)\mathrm{d}\sigma.\end{aligned}$$

By the assumption (A5), if $\theta > 0$, then the right hand side of above equality is nonnegative as

$$\begin{aligned}&(\ln t)t\partial_0 a+\sum_{i=1}^n x_i\partial_i a+\frac{1}{N+1}\Big[(\ln t)^2 t\partial_0 a+(\ln t)^2t^2\partial_0^2 a+2\sum_{i=1}^n x_i(\ln t)t\partial_{0i}^2 a+\sum_{i,j=1}^n x_ix_j\partial_{ij}^2 a\Big]\\&\geq\frac{N}{N+1}\Big((\ln t)t\partial_0 a+\sum_{i=1}^n x_i\partial_i a+\frac{1}{N}\Big[(\ln t)^2 t\partial_0 a+(\ln t)^2t^2\partial_0^2 a\\&\quad+2\sum_{i=1}^n x_i(\ln t)t\partial_{0i}^2 a+\sum_{i,j=1}^n x_ix_j\partial_{ij}^2 a\Big]\Big),\end{aligned}$$

while the left hand side is negative. If $\theta < 0$, we can also get a contradiction. Therefore $\theta = 0$, and $I'(u) = 0$ which means that u is a critical point of I in $\mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}_+^N)$.

Proof of Theorem 1.1 The fact $p = c$ will be proved in Lemma 5.3. Suppose by contradiction that there exists a critical point $v \in \mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}_+^N)$ of I at level c. Then, particularly, $v \in \mathcal{P}$ and $I(v) = c = p$. Let $0 < \mu < 1$ satisfies $v_\mu \in \mathcal{P}_\infty$. One has,

$$\begin{aligned}p=I(v)&=\frac{1}{N}\int_{\mathbb{R}_+^N}|\nabla_{\mathbb{B}}v|^2\mathrm{d}\sigma+\frac{1}{N}\int_{\mathbb{R}_+^N}\Big[(\ln t)t\partial_t a(t,x)+\sum_{i=1}^n x_i\partial_{x_i}a(t,x)\Big]F(v)\mathrm{d}\sigma\\&\geq\frac{1}{N}\int_{\mathbb{R}_+^N}|\nabla_{\mathbb{B}}v|^2\mathrm{d}\sigma>\frac{\mu^{N-2}}{N}\int_{\mathbb{R}_+^N}|\nabla_{\mathbb{B}}v|^2\mathrm{d}\sigma=I_\infty(v_\mu)\geq\inf_{u\in\mathcal{P}_\infty}I_\infty(u)=m=c_\infty,\end{aligned}$$

i.e., $p > c_\infty$, which is contradict to Lemma 4.7.

What's more, the infimum p is not achieved. Otherwise, if there exists $u \in \mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}_+^N)$ such that $I(u) = p$ and $I'|_{\mathcal{P}}(u) = 0$, then by Lemma 4.8 we know $I'(u) = 0$ in $\mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}_+^N)$, contradicting to the first part of this proof. Therefore, we get the assertion in Theorem 1.1.

5 Existence of a Positive Solution

In this section, we will prove that problem (1.1) has a positive solution. Note that p is not critical level for functional I and we should search for solutions in higher level energy. We will use linking argument together with barycenter functional restricted to Pohozaev manifold $\mathcal{P}$.

We begin by showing that the min-max level of the mountain pass theorem for the functionals I and I_∞ are equality.

Lemma 5.1 *Functional I satisfies the geometrical properties of the mountain pass theorem.*

Proof First, it is clearly that $I(0) = 0$. Then similarly to the proof of Lemma 2.1, we know that there exists $\rho > 0$ such that $I(u) > 0$ if $0 < \|u\| < \rho$. At last, let u be a least energy solution to problem (3.1), then Lemma 3.2 tells us that there exists a $\gamma \in \Gamma_\infty$ such that $\gamma(\tau) = u\big(t^{\frac{1}{L\tau}}, \frac{x}{L\tau}\big)$ for $\tau > 0$ and $L > 0$ sufficiently large. Taking $\gamma_w(\tau) = T_{sy}\gamma(\tau) = u\big(\big(\frac{t}{s}\big)^{\frac{1}{L\tau}}, \frac{x-y}{L\tau}\big)$ and by condition (A2), we obtain

$$\begin{aligned} I(\gamma_w(1)) &= \frac{L^{N-2}}{2}\int_{\mathbb{R}_+^N} |\nabla_{\mathbb{B}} u|^2 \mathrm{d}\sigma - L^N \int_{\mathbb{R}_+^N} \Big[a(st^L, y+Lx)F(u) - \frac{1}{2}\lambda u^2\Big]\mathrm{d}\sigma \\ &= I_\infty(\gamma(1)) + L^N \int_{\mathbb{R}_+^N} [a_\infty - a(st^L, y+Lx)]F(u)\mathrm{d}\sigma \\ &= I_\infty(\gamma(1)) + o_w(1) < 0 \quad \text{for } |w| \text{ large,} \end{aligned}$$

since $I_\infty(\gamma(1)) < 0$. We deduce the assertion.

Lemma 5.2 *It holds $c = c_\infty$.*

Proof For $\gamma \in \Gamma$, $I(\gamma(1)) < 0$. Since $I_\infty(u) \le I(u)$, we get $I_\infty(\gamma(1)) < 0$. Thus $\Gamma \subset \Gamma_\infty$ and

$$c_\infty = \min_{\gamma\in\Gamma_\infty}\max_{\tau\in[0,1]} I_\infty(\gamma(\tau)) \le \min_{\gamma\in\Gamma}\max_{\tau\in[0,1]} I_\infty(\gamma(\tau)) \le \min_{\gamma\in\Gamma}\max_{\tau\in[0,1]} I(\gamma(\tau)) = c,$$

which means that $c_\infty \le c$.

On the other hand, for any $\varepsilon > 0$, let $\gamma \in \Gamma_\infty$ such that $I_\infty(\gamma(\tau)) \le c_\infty + \varepsilon$ for all $\tau \in [0,1]$. Choosing $w = (s, y) \in \mathbb{R}_+^N$ and translating $T_{sy}\gamma(\tau)$, for w large enough, we have $T_{sy} \circ \gamma \in \Gamma$ (see the proof of Lemma 5.1). Let $\tau_0 \in [0,1]$ such that $I(T_{sy}\gamma(\tau))$ taking its maximum at τ_0, then

$$c_\infty + \varepsilon \ge I_\infty(\gamma(\tau_0)) = \lim_{|w|\to\infty} I(\gamma(\tau_0)) = \max_{\tau\in[0,1]} I(T_{sy}\gamma(\tau)) \ge \inf_{\gamma\in\Gamma}\max_{\tau\in[0,1]} I(\gamma(\tau)) = c.$$

Since ε is arbitrary, we get $c_\infty \ge c$ and the equality follows.

As a result, Lemmas 4.7 and 5.2 imply the following lemma.

Lemma 5.3 *It holds $p = c$, where p and c are defined in* (1.2).

Recall that a sequence $\{u_j\}$ is said to be Cerami sequence for functional I at level $d \in \mathbb{R}$, denoted by $(Ce)_d$, if $I(u_j) \to d$ and $\|I'(u_j)\|(1 + \|u_j\|) \to 0$. Applying the concentration-compactness Lemma of Lions (see [23]), we show that, for $d > 0$, any $(Ce)_d$ sequence is bounded, up to a subsequence.

Lemma 5.4 *If $\{u_j\}$ is a $(Ce)_d$ sequence for functional I, then it has a bounded subsequence.*

Proof Suppose by contradiction that $\|u_j\| \to \infty$. Define $\widetilde{u}_j := \frac{u_j}{\|u_j\|}$, then $\{\widetilde{u}_j\}$ is a bounded sequence and $\|\widetilde{u}_j\| = 1$. Therefore, $\widetilde{u}_j \to \widetilde{u}$, up to a subsequence, and one of the following two cases occurs:

$$\text{Case 1}: \quad \exists\, R > 0 \quad \text{s.t.} \quad \limsup_{j\to\infty} \sup_{(s,y)\in\mathbb{R}_+^N} \int_{\Omega_R(s,y)} |\widetilde{u}_j|^2 \mathrm{d}\sigma > 0;$$

$$\text{Case 2}: \quad \forall\, R > 0 \qquad \limsup_{j\to\infty} \sup_{(s,y)\in\mathbb{R}_+^N} \int_{\Omega_R(s,y)} |\widetilde{u}_j|^2 \mathrm{d}\sigma = 0.$$

If Case 2 holds, taking $L > 2\sqrt{dD}$, with D from the condition (f3), then we get

$$I\Big(\frac{L}{\|u_j\|}u_j\Big) = \frac{1}{2}L^2 - \int_{\mathbb{R}_+^N} a(t,x)F\Big(\frac{L}{\|u_j\|}u_j\Big)\mathrm{d}\sigma.$$

For any $\varepsilon > 0$, (f1) and (f2) give

$$\int_{\mathbb{R}_+^N} a(t,x)F\Big(\frac{L}{\|u_j\|}u_j\Big)\mathrm{d}\sigma < a_\infty \int_{\mathbb{R}_+^N} \frac{\varepsilon}{2}L^2\widetilde{u}_j^2\mathrm{d}\sigma + C_\varepsilon \int_{\mathbb{R}_+^N} |\widetilde{u}_j|^p\mathrm{d}\sigma \le \frac{\varepsilon a_\infty}{2\lambda}L^2 + o_j(1),$$

where $\int_{\mathbb{R}_+^N} |\widetilde{u}_j|^p\mathrm{d}\sigma \to 0$ by a variant of Lion's lemma. Take $\varepsilon = \frac{\lambda}{2a_\infty}$, and then we obtain

$$I\Big(\frac{L}{\|u_j\|}u_j\Big) \ge \frac{L^2}{4} - o_j(1).$$

Since $\|u_j\| \to \infty$, we have $\frac{L}{\|u_j\|} \in (0,1)$ for $j > 0$ sufficiently large and

$$\max_{\tau\in[0,1]} I(\tau u_j) \ge I\Big(\frac{L}{\|u_j\|}u_j\Big) \ge \frac{L^2}{4} - o_j(1).$$

Let τ_j satisfy $I(\tau_j u_j) = \max\limits_{\tau\in[0,1]} I(\tau u_j)$. Thus

$$I(\tau_j u_j) \ge \frac{L^2}{4} - o_j(1). \tag{5.1}$$

On the other hand, since $\tau_j \le 1$, using (f3) we have

$$\begin{aligned} I(\tau_j u_j) &= I(\tau_j u_j) - \frac{1}{2}I'(\tau_j u_j)(\tau_j u_j) = \int_{\mathbb{R}_+^N} a(z)\Big[\frac{1}{2}f(\tau_j u_j)(\tau_j u_j) - F(\tau_j u_j)\Big]\mathrm{d}\sigma \\ &\le D\int_{\mathbb{R}_+^N} a(z)\Big[\frac{1}{2}f(u_j)(u_j) - F(u_j)\Big]\mathrm{d}\sigma \\ &= D\Big[I(u_j) - \frac{1}{2}I'(u_j)(u_j)\Big] = Dd + o_j(1). \end{aligned} \tag{5.2}$$

Combining (5.1) and (5.2), we obtain

$$\frac{L^2}{4} - o_j(1) \le I(\tau_j u_j) \le Dd + o_j(1).$$

But $L > 2\sqrt{dD}$, we get a contradiction.

If Case 1 occurs, i.e., there exists $R > 0$ such that $\limsup\limits_{j\to\infty} \sup\limits_{(s,y)\in\mathbb{R}_+^N} \int_{\Omega_R(s,y)} |\widetilde{u}_j|^2\mathrm{d}\sigma = \alpha > 0$. If $\{w_j = (s_j, y_j)\}$ is a sequence such that $|w_j| \to +\infty$ and $\int_{\Omega_R(s_j,y_j)} |\widetilde{u}_j|^2\mathrm{d}\sigma > \frac{\alpha}{2}$. Recalling that $T_{s_j y_j}\widetilde{u}_j(\cdot) \rightharpoonup \overline{u}(\cdot)$, we get

$$\int_{\Omega_R(1,0)} |\widetilde{u}_j(ts_j, x+y_j)|^2\mathrm{d}\sigma > \frac{\alpha}{2} \quad \text{and} \quad \int_{\Omega_R(1,0)} |\overline{u}(t,x)|^2\mathrm{d}\sigma > \frac{\alpha}{2},$$

which means $\overline{u} \ne 0$. Then, there exists $\Omega \subset \Omega_R(1,0)$ with $|\Omega|_{\mathbb{B}} > 0$, such that

$$0 \ne \overline{u}(t,x) = \lim_{j\to+\infty} \widetilde{u}_j(ts_j, x+y_j) = \lim_{j\to+\infty} \frac{u_j(ts_j, x+y_j)}{\|u_j\|}, \quad \text{a.e. } (t,x) \in \Omega. \tag{5.3}$$

Since $\|u_j\| \to \infty$, this implies $u_j(ts_j, x+y_j) \to \infty$. We claim that, $u_j(ts_j, x+y_j) \to +\infty$ for $(t,x) \in \Omega$. In fact, let $v_j(t,x) = \widetilde{u}_j(ts_j, x+y_j)$ and for a sequence $\zeta_j \to 0$ in the dual space of $\mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}_+^N)$ as $j \to \infty$, we get

$$-\Delta_{\mathbb{B}} v_j + \lambda v_j = \frac{a(ts_j, x+y_j)}{\|u_j\|} f(\|u_j\| v_j) + \frac{\zeta_j}{\|u_j\|}.$$

Testing this equation by v_j^- (the negative part of v_j) and taking into account that

$$\int_{\mathbb{R}_+^N} \frac{a(ts_j, x+y_j)}{\|u_j\|} f(\|u_j\|v_j)v_j^- \mathrm{d}\sigma = 0, \quad \frac{\langle \zeta_j, v_j^- \rangle}{\|u_j\|} = \frac{\langle \zeta_j, u_j^-(\cdot s_j, \cdot + y_j)\rangle}{\|u_j\|} = o_j(1),$$

we obtain $\int_{\mathbb{R}_+^N} |\nabla_{\mathbb{B}} v_j^-|\mathrm{d}\sigma + \int_{\mathbb{R}_+^N} \lambda |v_j^-|^2 \mathrm{d}\sigma = \|v_j^-\|^2 = o_j(1)$ as $j \to \infty$. In particular, by the cone Sobolev embedding, we have $\|v_j^-\|_{L_p^{\frac{N}{p}}} = o_j(1)$ as $j \to \infty$ for all $2 \le p \le 2^*$. Note that $v_j = \widetilde{u}_j(ts_j, x+y_j) \to \overline{u}$ in $L_p^{\frac{N}{p}}$, then we have $v_j^- \to \overline{u}_j^-$ in $L_p^{\frac{N}{p}}$. And then $\overline{u}_j^- = 0$ on Ω, which means $\overline{u} > 0$ on Ω. In turn, by (5.3), we get the claim. Therefore, by condition (f3), Fatou lemma and (A1) with $\widetilde{\alpha} = \inf_{\mathbb{R}_+^N} a > 0$, we have

$$\begin{aligned}
&\liminf_{j\to\infty} \int_{\mathbb{R}_+^N} a(t,x)\Big[\frac{1}{2} f(u_j)(u_j) - F(u_j)\Big]\mathrm{d}\sigma \\
&= \liminf_{j\to\infty} \int_{\mathbb{R}_+^N} a(ts_j, x+y_j)\Big[\frac{1}{2} f(u_j(ts_j, x+y_j))u_j(ts_j, x+y_j) - F(u_j(ts_j, x+y_j))\Big]\mathrm{d}\sigma \\
&\ge \widetilde{\alpha} \liminf_{j\to\infty} \int_{\Omega} \Big[\frac{1}{2} f(u_j(ts_j, x+y_j))u_j(ts_j, x+y_j) - F(u_j(ts_j, x+y_j))\Big]\mathrm{d}\sigma \\
&\ge \widetilde{\alpha} \int_{\Omega} \liminf_{j\to\infty} \Big[\frac{1}{2} f(u_j(ts_j, x+y_j))u_j(ts_j, x+y_j) - F(u_j(ts_j, x+y_j))\Big]\mathrm{d}\sigma = +\infty.
\end{aligned}$$

On the other hand, $|I'(u_j)(u_j)| \le \|I'(u_j)\| \cdot \|u_j\| \to 0$ as $j \to \infty$. Thus,

$$\int_{\mathbb{R}_+^N} a(t,x)\Big(\frac{1}{2} f(u_j)(u_j) - F(u_j)\Big)\mathrm{d}\sigma = I(u_j) - \frac{1}{2} I'(u_j)(u_j) = d + o_j(1),$$

which gives a contradiction. If $\{w_j\}$ is bounded, say $|w_j| < \widetilde{R}$ for some $\widetilde{R}$, then we can get

$$\frac{\alpha}{2} \le \int_{\Omega_R(s_j, y_j)} |\widetilde{u}_j|^2 \mathrm{d}\sigma \le \int_{\Omega_{2\widetilde{R}}(1,0)} |\widetilde{u}_j|^2 \mathrm{d}\sigma.$$

Since $\widetilde{u}_j \to \widetilde{u}$ in $L_2^{\frac{N}{2}}(\Omega_{2\widetilde{R}}(1,0))$, it follows that

$$\frac{\alpha}{2} \le \int_{\Omega_{2\widetilde{R}}(1,0)} |\widetilde{u}|^2 \mathrm{d}\sigma.$$

Similar to the previous case, there exists $\Omega \subset \Omega_{2\widetilde{R}}(1,0)$ of positive measure such that (5.3) holds. Then the argument follows as above for the case where $\{w_j\}$ is unbounded and we can also get a contradiction. Therefore, neither Case 1 or Case 2 can occur and we have the assertion.

Now, we show the existence of a Cerami sequence for functional I at level c. We need following Ghoussoub-Preiss theorem. One can refer to [13, Theorem 6] (see also [14]).

Proposition 5.1 *Let X be a Banach space and $\Phi\colon X \to \mathbb{R}$ be a continuous, Gâteaux-differentiable functional, such that $\Phi'\colon X \to X$ is continuous from the norm of X to the weak $*$ topology of its dual space X^*. Take two points z_0, z_1 in X and consider the set $\widetilde{\Gamma}$ of all continuous paths from z_0 to z_1:* $\widetilde{\Gamma} = \big\{\gamma \in C^0([0,1]; X) \mid \gamma(0) = z_0, \gamma(1) = z_1\big\}$.

Define

$$\alpha = \inf_{\gamma \in \widetilde{\Gamma}} \max_{\tau \in [0,1]} \Phi(\gamma(\tau)).$$

Assume there exists a closed subset Y of X such that $Y \cap \Phi_\alpha$ separates z_0 and z_1. Then there exists a sequence $\{z_j\} \subset X$ such that, as $j \to \infty$,

(1) $\delta(z_j, Y) \to 0$;

(2) $\Phi(z_j) \to \alpha$;

(3) $(1 + \|z_j\|)\|\Phi'(z_j)\| \to 0$.

Here, $\Phi_\alpha = \{z \in X \mid \Phi(z) \geq \alpha\}$ and the geodesic distance $\delta(z_1, z_2)$ between z_1 and z_2 in X is

$$\delta(z_1, z_2) = \inf\Big\{ \int_0^1 \frac{\|\gamma'(\tau)\|}{1 + \|\gamma(\tau)\|} \mathrm{d}\tau \;\Big|\; \gamma \in C^1([0,1]; X), \gamma(0) = z_1, \gamma(1) = z_2 \Big\}.$$

Lemma 5.5 *Let c be min-max mountain pass level for functional I, then there exists a $(Ce)_c$ sequence $\{u_j\}$ in $\mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}_+^N)$.*

Proof We apply the above Proposition 5.1 with $X = \mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}_+^N)$ and $\Phi = I$. Consider $z_0 = 0$ and $z_1 \in \mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}_+^N)$ such that $I(z_1) < 0$. The existence of z_1 is guaranteed by the mountain pass geometry of I. Then $c = \inf\limits_{\gamma \in \widetilde{\Gamma}} \max\limits_{\tau \in [0,1]} I(\gamma(\tau))$ and the closed subset $\mathcal{P} \cap \{I(u) \geq c\} = \mathcal{P}$ separates z_0 and z_1. In fact, $z_0 = 0 \notin \mathcal{P}$ and $z_1 \notin \mathcal{P}$, since $J(z_1) < NI(z_1) < 0$. Moreover, there exists $\rho > 0$ such that $J(u) > 0$ for $0 < \|u\| < \rho$. We have $\mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}_+^N) \backslash \mathcal{P} = \{0\} \cup \{J > 0\} \cup \{J < 0\}$. The "ball" $\Omega_\rho(z_0)$ is in a connected component of $\{0\} \cup \{J > 0\}$ and z_1 belongs to a connected component of $\{J(u) < 0\}$. Therefore, we get a sequence $\{u_j\} \subset \mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}_+^N)$ such that

$$\delta(u_j, \mathcal{P}) \to 0, \quad I(u_j) \to c, \quad (1 + \|u_j\|)\|I'(u_j)\| \to 0.$$

Lemma 5.6 *Let $\{u_j\} \subset \mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}_+^N)$ be a bounded sequence such that*

$$I(u_j) \to d \quad \textit{and} \quad \|I'(u_j)\|(1 + \|u_j\|) \to 0.$$

Then, replacing $\{u_j\}$ by a subsequence if necessary, there exist a solution $\overline{u}$ of (1.1)*, a number $k \in \mathbb{N} \cup \{0\}$, k functions $u^1, u^2, \cdots, u^k$ and k sequences of points $\{w_j^i\} \subset \mathbb{R}_+^N$, $1 \leq i \leq k$, satisfying*:

(1) $u_j \to \overline{u}$ *in* $\mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}_+^N)$;

or

(2) u^i *are nontrivial solutions of limiting problem* (1.3);

(3) $|w_j^i| \to \infty$ *and* $|w_j^i - w_j^l| \to \infty$ *for* $i \neq l$;

(4) $u_j(t,x) - \sum\limits_{i=1}^k u^i\big(\frac{t}{s_j^i}, x - y_j^i\big) \to \overline{u}(t,x)$;

(5) $I(u_j) \to I(\overline{u}) + \sum\limits_{i=1}^k I_\infty(u^i)$.

Remark 5.1 Nowadays, the proof of this lemma is standard and is a version of the concentration-compactness lemma of Lions [16, 23]. The main ingredients are Lions lemma and Brezis-Lieb lemma (see [3]). One can also refer to [8]. In fact, the solutions $u^i \in \mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}_+^N)$ can be chosen as positive and "radially symmetric" about some point.

Corollary 5.1 *If $I(u_j) \to c_\infty$ and $\|I'(u_j)\|(1+\|u_j\|) \to 0$, then either $\{u_j\}$ is relatively compact or Lemma* 5.6 *holds with $k=1$ and $\overline{u}=0$.*

Lemma 5.7 *Suppose that*

$$\text{the limiting problem (1.3) admits a unique positive "radial" solution,} \tag{5.4}$$

then I satisfies condition $(Ce)_d$ at level $d \in (c_\infty, 2c_\infty)$.

Proof Let $\{u_j\} \subset \mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}_+^N)$ satisfy $I(u_j) \to d$ and $\|I'(u_j)\|(1+\|u_j\|) \to 0$ as $j \to \infty$. By Lemma 5.4, $\{u_j\}$ is a bounded sequence. Applying Lemma 5.6, up to a subsequence, we get

$$u_j(t,x) - \sum_{i=1}^{k} u^i\Big(\frac{t}{s_j^i}, x-y_j^i\Big) \to \overline{u}(t,x) \quad \text{in } \mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}_+^N), \quad I(u_j) \to I(\overline{u}) + \sum_{i=1}^{k} I_\infty(u^i).$$

Since $d < 2c_\infty$, we have $k < 2$. If $k = 1$, we get

• If $\overline{u} \neq 0$, then $I(\overline{u}) \geq p = c_\infty$ and $I(u_j) \geq 2c_\infty$. This is impossible.

• If $\overline{u} = 0$, then $I(u_j) \to I_\infty(u^1)$. Under (5.4), we know $I_\infty(u^1) = c_\infty$, against $d > c_\infty$.

Therefore, we have $k = 0$ and $u_j \to \overline{u}$.

Remark 5.2 Note that, if f is defined as (1.6), then problem (1.3) has exact on positive solution. Therefore, condition (5.4) is satisfied. We will give the proof in Appendix.

Lemma 5.8 *If $I(u_j) \to d > 0$ and $\{u_j\} \in \mathcal{P}$, then $\{u_j\}$ is bounded.*

Proof $I(u_j) \to d > 0$ implies that $\{I(u_j)\}$ is bounded in $\mathbb{R}$. If $\{u_j\} \in \mathcal{P}$, by condition (A3), we know

$$d + 1 \geq I(u_j) \geq \frac{1}{N} \int_{\mathbb{R}_+^N} |\nabla_{\mathbb{B}} u_j|^2 \mathrm{d}\sigma,$$

i.e., $\int_{\mathbb{R}_+^N} |\nabla_{\mathbb{B}} u_j|^2 \mathrm{d}\sigma$ is bounded. By the cone Sobolev inequality, $\int_{\mathbb{R}_+^N} |u_j|^{2^*} \mathrm{d}\sigma$ is also bounded. Applying (1.5) with $\varepsilon\|a\|_{L^\infty} < \lambda$, we get

$$\int_{\mathbb{R}_+^N} a(z)F(u_j)\mathrm{d}\sigma \leq \frac{\varepsilon\|a\|_{L^\infty}}{2} \int_{\mathbb{R}_+^N} |u_j|^2 \mathrm{d}\sigma + C(\varepsilon) \int_{\mathbb{R}_+^N} |u_j|^{2^*} \mathrm{d}\sigma.$$

Inserting this to the expression of I, we have

$$d + 1 \geq I(u_j) \geq \frac{1}{2} \int_{\mathbb{R}_+^N} |\nabla_{\mathbb{B}} u_j|^2 \mathrm{d}\sigma + \frac{1}{2}(\lambda - \varepsilon\|a\|_{L^\infty}) \int_{\mathbb{R}_+^N} |u_j|^2 \mathrm{d}\sigma - C(\varepsilon) \int_{\mathbb{R}_+^N} |u_j|^{2^*} \mathrm{d}\sigma,$$

thus $\int_{\mathbb{R}_+^N} |u_j|^2 \mathrm{d}\sigma$ is bounded. Therefore, $\{u_j\}$ is bounded in $\mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}_+^N)$.

Next, we introduce the barycenter function, which is crucial for proving the existence of solution for problem (1.1). For $u \in \mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}_+^N) \backslash \{0\}$, set

$$\eta(u)(t,x) := \frac{1}{|\Omega_1|} \int_{\Omega_1(t,x)} |u(s,y)| \frac{\mathrm{d}s}{s} \mathrm{d}y,$$

then $\eta(u)$ is a continuous function and a.e. finite. Let

$$\widehat{u}(t,x) := \Big[\eta(u)(t,x) - \frac{1}{2} \max \eta(u)\Big]^+.$$

It follows that $\widehat{u} \in C_0(\mathbb{R}_+^N)$. Now, we can define the barycenter of u.

Definition 5.1 *For $u \in \mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}_+^N)\backslash\{0\}$, define the barycenter of u by*

$$\beta(u) := \frac{1}{\|\widehat{u}\|_{L_1^N}} \int_{\mathbb{R}_+^N} (\ln t, x)\widehat{u}(t,x)\frac{\mathrm{d}t}{t}\mathrm{d}x \in \mathbb{R}^N.$$

Since $\widehat{u}$ has compact support, $\beta(u)$ is well defined. Furthermore, we have following properties.

Lemma 5.9 (1) *β is a continuous functional in $\mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}_+^N)\backslash\{0\}$;*
(2) *if u is "radially symmetric", then $\beta(u) = 0$;*
(3) *given $w = (s, y) \in \mathbb{R}_+^N$, then $\beta(T_{sy}u) = \beta(u) + (\ln s, y)$.*

Proof We prove (3). Set $v(t,x) = T_{sy}u = u\big(\frac{t}{s}, x-y\big)$, then

$$\eta(v)(t,x) = \frac{1}{|\Omega_1|}\int_{\Omega_1(t,x)} \Big|u\Big(\frac{\tau}{s}, \xi - y\Big)\Big|\frac{\mathrm{d}\tau}{\tau}\mathrm{d}\xi = \frac{1}{|\Omega_1|}\int_{\Omega_1(\frac{t}{s},x-y)} |u(\tau,\xi)|\frac{\mathrm{d}\tau}{\tau}\mathrm{d}\xi.$$

Let $\alpha = \max \eta(v)$, then

$$\widehat{v}(t,x) = \Big[\frac{1}{|\Omega_1|}\int_{\Omega_1(\frac{t}{s},x-y)} |u(\tau,\xi)|\frac{\mathrm{d}\tau}{\tau}\mathrm{d}\xi - \frac{1}{2}\alpha\Big]^+.$$

Therefore, we get

$$\begin{aligned}
\beta(v) &= \frac{1}{\|\widehat{u}\|_{L_1^N}} \int_{\mathbb{R}_+^N} (\ln t, x)\Big[\frac{1}{|\Omega_1|}\int_{\Omega_1(\frac{t}{s},x-y)} |u(\tau,\xi)|\frac{\mathrm{d}\tau}{\tau}\mathrm{d}\xi - \frac{1}{2}\alpha\Big]^+\frac{\mathrm{d}t}{t}\mathrm{d}x \\
&= \frac{1}{\|\widehat{u}\|_{L_1^N}} \int_{\mathbb{R}_+^N} (\ln t + \ln s, x+y)\Big[\frac{1}{|\Omega_1|}\int_{\Omega_1(t,x)} |u(\tau,\xi)|\frac{\mathrm{d}\tau}{\tau}\mathrm{d}\xi - \frac{1}{2}\alpha\Big]^+\frac{\mathrm{d}t}{t}\mathrm{d}x \\
&= \beta(u) + (\ln s, y)\frac{1}{\|\widehat{u}\|_{L_1^N}} \int_{\mathbb{R}_+^N} \Big[\frac{1}{|\Omega_1|}\int_{\Omega_1(t,x)} |u(\tau,\xi)|\frac{\mathrm{d}\tau}{\tau}\mathrm{d}\xi - \frac{1}{2}\alpha\Big]^+\frac{\mathrm{d}t}{t}\mathrm{d}x \\
&= \beta(u) + (\ln s, y)\frac{1}{\|\widehat{u}\|_{L_1^N}} \int_{\mathbb{R}_+^N} \widehat{u}(t,x)\frac{\mathrm{d}t}{t}\mathrm{d}x = \beta(u) + (\ln s, y).
\end{aligned}$$

Now, we define

$$b := \inf\big\{I(u)\colon u \in \mathcal{P} \text{ and } \beta(u) = 0\big\}.$$

It is clear that $b \geq c_\infty$. Moreover, the following results hold.

Lemma 5.10 *Suppose $\{u_j\}, \{v_j\} \subset \mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}_+^N)$ satisfying $\|u_j - v_j\| \to 0$ and $I'(v_j) \to 0$ as $j \to \infty$. Then, $I'(u_j) \to 0$ as $j \to \infty$.*

Proof By assumption (f4), $f \in C^1(\mathbb{R}, \mathbb{R}^+) \cap \mathrm{Lip}\,(\mathbb{R}, \mathbb{R}^+)$. Then for every $u, v, \varphi \in \mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}_+^N)$,

$$I''(u)(v,\varphi) = \int_{\mathbb{R}_+^N} \nabla_{\mathbb{B}} v \cdot \nabla_{\mathbb{B}} \varphi \mathrm{d}\sigma + \int_{\mathbb{R}_+^N} \lambda v\varphi \mathrm{d}\sigma - \int_{\mathbb{R}_+^N} a(z)f'(u)v\varphi\mathrm{d}\sigma. \tag{5.5}$$

By the mean value theorem, there exists $\xi \in (0,1)$ such that

$$I'(v)(\varphi) - I'(u)(\varphi) = I''(u + \xi(v-u))(\varphi, v-u).$$

Thus, taking into account that $|f'(u_j+\xi_j(v_j-u_j))|\le C$ a.e. and by assumption (f1), for any $j\ge 1$ we find $\xi_j\in(0,1)$ such that

$$\begin{aligned}&I'(v_j)(\varphi)-I'(u_j)(\varphi)=I''(u_j+\xi_j(v_j-u_j))(\varphi,v_j-u_j)\\&=\int_{\mathbb{R}^N_+}\nabla_{\mathbb{B}}(v_j-u_j)\cdot\nabla_{\mathbb{B}}\varphi\mathrm{d}\sigma+\int_{\mathbb{R}^N_+}\lambda(v_j-u_j)\varphi\mathrm{d}\sigma-\int_{\mathbb{R}^N_+}a(z)f'(u_j+\xi_j(v_j-u_j))(v_j-u_j)\varphi\mathrm{d}\sigma\\&\le C\|\varphi\|\cdot\|v_j-u_j\|+Ca_\infty\int_{\mathbb{R}^N_+}|v_j-u_j|\cdot|\varphi|\mathrm{d}\sigma\le C\|\varphi\|\cdot\|v_j-u_j\|.\end{aligned}$$

Take the supremum over $\varphi\in\mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}^N_+)$ with $\|\varphi\|\le 1$, then we obtain as $j\to\infty$,

$$\|I'(v_j)-I'(u_j)\|\le C\|v_j-u_j\|=o_j(1).$$

Therefore, $I'(u_j)\to 0$ as $j\to\infty$.

Lemma 5.11 *It holds $b>c_\infty$.*

Proof Suppose by contradiction that $b=c_\infty$. Then there exists a sequence $\{u_j\}$ with $u_j\in\mathcal{P}$ and $\beta(u_j)=0$ such that $I(u_j)\to b$. From Lemma 5.8, $\{u_j\}$ is bounded. Since $c=c_\infty$ and $c=p$, so $p=b$, which implies $\{u_j\}$ is a minimizing sequence for I on $\mathcal{P}$. By Ekeland variational principle, there exists another sequence $\{\widetilde{u}_j\}\subset\mathcal{P}$ such that, as $j\to\infty$,

$$I(\widetilde{u}_j)\to p,\quad I'|_{\mathcal{P}}(\widetilde{u}_j)\to 0,\quad \|\widetilde{u}_j-u_j\|\to 0.$$

Let us now deduce that $I'(\widetilde{u}_j)\to 0$ as $j\to\infty$. Otherwise, there exists $\alpha>0$ and a subsequence $\{\widetilde{u}_{j_k}\}$ such that

$$\|I'(\widetilde{u}_{j_k})\|\ge\alpha\quad\text{for all }\ k\ge 1\ \text{ large.}$$

Arguing as in the proof of Lemma 5.10, there exists a positive constant C such that

$$|I'(\widetilde{u}_{j_k})(\varphi)-I'(v)(\varphi)|\le C\|\varphi\|\,\|\widetilde{u}_{j_k}-v\|\quad\text{for all } k\ge 1 \text{ and any } v,\varphi\in\mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}^N_+).$$

Taking the supremum over $\|\varphi\|\le 1$, we obtain $\|I'(\widetilde{u}_{j_k})-I'(v)\|\le C\|\widetilde{u}_{j_k}-v\|$ for all $k\ge 1$ and any $v\in\mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}^N_+)$. Therefore, if $\|\widetilde{u}_{j_k}-v\|<\frac{\widetilde{\delta}}{C}:=2\delta$, we have $\|I'(\widetilde{u}_{j_k})-I'(v)\|\le\widetilde{\delta}$ for any $v\in\mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}^N_+)$ and all $k\ge 1$. This means, $\alpha-\widetilde{\delta}<\|I'(\widetilde{u}_{j_k})\|-\widetilde{\delta}<\|I'(v)\|$ for all $k\ge 1$ large. For $\widetilde{\delta}\in(0,\alpha)$ and $\widetilde{\alpha}:=\alpha-\widetilde{\delta}>0$, we obtain

$$\forall\ v\in\mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}^N_+):\ v\in\Omega_{2\delta}(\widetilde{u}_{j_k})\ \Rightarrow\ \|I'(v)\|>\widetilde{\alpha}.$$

Set $\varepsilon:=\min\left\{\frac{p}{2},\frac{\widetilde{\alpha}\delta}{8}\right\}$ and $Q:=\{\widetilde{u}_{j_k}\}$. Then, by a virtue of the Deformation Lemma, we get a deformation ψ: $[0,1]\times\mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}^N_+)\to\mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}^N_+)$ at level p such that

$$\psi(1,I^{p+\varepsilon}\cap Q)\subset I^{p-\varepsilon},\quad I\big(\psi(1,u)\big)\le I(u)\quad\text{for all } u\in\mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}^N_+).\tag{5.6}$$

Note that, for each $k\ge 1$, by (A4) we get

$$\begin{aligned}\int_{\mathbb{R}^N_+}G_\infty(\widetilde{u}_{j_k})\mathrm{d}\sigma&\ge\int_{\mathbb{R}^N_+}\Big[\Big(a(t,x)+\frac{1}{N}\Big[\ln t\ t\partial_t a(t,x)+\sum_{i=1}^n x_i\partial_{x_i}a(t,x)\Big]\Big)F(\widetilde{u}_{j_k})-\frac{\lambda}{2}\widetilde{u}_{j_k}^2\Big]\mathrm{d}\sigma\\&=\frac{N-2}{2N}\int_{\mathbb{R}^N_+}|\nabla_{\mathbb{B}}\widetilde{u}_{j_k}|^2\mathrm{d}\sigma>0.\end{aligned}$$

Then there exists a unique $\tau > 0$ such that $\widetilde{u}_{j_k}\big(t^{\frac{1}{\tau}}, \frac{x}{\tau}\big) \in \mathcal{P}$. Since $\widetilde{u}_{j_k} \in \mathcal{P}$, for k large enough,

$$\max_{\tau>0} I\Big(\widetilde{u}_{j_k}\Big(t^{\frac{1}{\tau}}, \frac{x}{\tau}\Big)\Big) = I(\widetilde{u}_{j_k}) < p + \varepsilon.$$

Therefore, from (5.6) we obtain

$$\max_{\tau>0} I\Big(\psi\Big(1, \widetilde{u}_{j_k}\Big(t^{\frac{1}{\tau}}, \frac{x}{\tau}\Big)\Big)\Big) < p - \varepsilon.$$

On the other hand, for k and L fixed large, $\gamma(\tau) = \psi\big(1, \widetilde{u}_{j_k}\big(t^{\frac{1}{L\tau}}, \frac{x}{L\tau}\big)\big)$ for $\tau > 0$ and $\gamma(0) = 0$ is a path in Γ since

$$\begin{aligned} I(\gamma(1)) &= I\Big(\psi\Big(1, \widetilde{u}_{j_k}\Big(t^{\frac{1}{L}}, \frac{x}{L}\Big)\Big)\Big) \le I\Big(\widetilde{u}_{j_k}\Big(t^{\frac{1}{L}}, \frac{x}{L}\Big)\Big) \\ &= \frac{1}{2}L^{N-2}\int_{\mathbb{R}^N_+} |\nabla_{\mathbb{B}}\widetilde{u}_{j_k}|^2 \mathrm{d}\sigma - L^N \int_{\mathbb{R}^N_+} \Big[a(t^L, Lx)F(\widetilde{u}_{j_k}) - \lambda\frac{\widetilde{u}_{j_k}^2}{2}\Big]\mathrm{d}\sigma \\ &= \frac{1}{2}L^{N-2}\int_{\mathbb{R}^N_+} |\nabla_{\mathbb{B}}\widetilde{u}_{j_k}|^2 \mathrm{d}\sigma - L^N\Big(\int_{\mathbb{R}^N_+} G_\infty(\widetilde{u}_{j_k})\mathrm{d}\sigma + o_L(1)\Big) < 0 \quad \text{for } L \to \infty. \end{aligned}$$

Hence, we get

$$c \le \max_{\tau\in[0,1]} I\Big(\psi\Big(1, \widetilde{u}_{j_k}\Big(t^{\frac{1}{(L\tau)}}, \frac{x}{L\tau}\Big)\Big)\Big) = \max_{\tau>0} I\Big(\psi\Big(1, \widetilde{u}_{j_k}\Big(t^{\frac{1}{\tau}}, \frac{x}{\tau}\Big)\Big)\Big) < p - \varepsilon < p,$$

which contradicts with the fact $p = c$. Thus, we know $I'(\widetilde{u}_j) \to 0$ as $j \to \infty$. Since $\|\widetilde{u}_j - u_j\| \to 0$, by Lemma 5.10, we obtain $I'(u_j) \to 0$ as $j \to \infty$. As a result, $\{u_j\}$ satisfies the assumptions of Corollary 5.1 and since $p = c_\infty$ is not attained, then Lemma 5.6 holds with $k = 1$ and $\overline{u} = 0$. This implies

$$u_j(t, x) = u^1\Big(\frac{t}{s_j}, x - y_j\Big) + o_j(1) \quad \text{as } j \to \infty,$$

where $w_j = (s_j, y_j) \in \mathbb{R}^N_+$, $|w_j| \to \infty$ and u^1 is a solution of limiting problem (1.3). Making a translation, we get

$$u_j(ts_j, x + y_j) = u^1(t, x) + o_j(1).$$

Calculating the barycenter functional on both sides, then we have

$$\beta(u_j(ts_j, x + y_j)) = \beta(u_j) - w_j = -w_j \quad \text{and} \quad \beta(u^1(t, x) + o_j(1)) \to \beta(u^1),$$

since $\beta(\cdot)$ is a continuous functional. On one side, $\beta(u^1)$ is a fixed real value and, on the other side $|w_j| \to \infty$, so we arrive a contradiction. Thus, we get $b > c_\infty$.

Consider the positive, "radially symmetric", ground state solution $\Phi \in \mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}^N_+)$ of limiting problem (3.1). We define the operator $\Pi \colon \mathbb{R}^N_+ \to \mathcal{P}$ by

$$\Pi[s, y](t, x) := \Phi\Big(\Big(\frac{t}{s}\Big)^{\frac{1}{\mu_w}}, \frac{x - y}{\mu_w}\Big),$$

where $\mu_w > 1$ projects $T_{sy}\Phi$ onto $\mathcal{P}$. Then, Π is a continuous function of $w = (s, y)$ since μ_w is unique and $\mu_w(T_{sy}\Phi)$ is continuous. Moreover, we have following properties.

Lemma 5.12 $\beta(\Pi[s, y](t, x)) = (\ln s, y)$ *for all* $w = (s, y) \in \mathbb{R}^N_+$.

Proof Let $v(t,x)=\Phi\big(\big(\frac{t}{s}\big)^{\frac{1}{\mu_w}},\frac{x-y}{\mu_w}\big)$, then

$$\begin{aligned}\eta(v)(t,x)&=\frac{1}{|\Omega_1|}\int_{\Omega_1(t,x)}|v(\tau,\xi)|\frac{\mathrm{d}\tau}{\tau}\mathrm{d}\xi=\frac{1}{|\Omega_1|}\int_{\Omega_1(\frac{t}{s},x-y)}\Big|\Phi\Big(\tau^{\frac{1}{\mu_w}},\frac{\xi}{\mu_w}\Big)\Big|\frac{\mathrm{d}\tau}{\tau}\mathrm{d}\xi\\&=\eta(\Phi_{\mu_w})\Big(\frac{t}{s},x-y\Big).\end{aligned}$$

Here, $\Phi_\mu(t,x)=\Phi(t^{\frac{1}{\mu}},\frac{x}{\mu})$. By the fact that $\widehat{v}(t,x)=\widehat{\Phi_{\mu_w}}(\frac{t}{s},x-y)$ and $\|\widehat{v}\|_{L_1^N}=\|\widehat{\Phi_{\mu_w}}\|_{L_1^N}$, then

$$\begin{aligned}\beta(v)&=\frac{1}{\|\widehat{v}\|_{L_1^N}}\int_{\mathbb{R}_+^N}(\ln t,x)\widehat{\Phi_{\mu_w}}\Big(\frac{t}{s},x-y\Big)\frac{\mathrm{d}t}{t}\mathrm{d}x=\frac{1}{\|\widehat{v}\|_{L_1^N}}\int_{\mathbb{R}_+^N}(\ln t+\ln s,x+y)\widehat{\Phi_{\mu_w}}(t,x)\frac{\mathrm{d}t}{t}\mathrm{d}x\\&=\frac{1}{\|\widehat{v}\|_{L_1^N}}\int_{\mathbb{R}_+^N}(\ln t,x)\widehat{\Phi_{\mu_w}}(t,x)\frac{\mathrm{d}t}{t}\mathrm{d}x+\frac{1}{\|\widehat{v}\|_{L_1^N}}\int_{\mathbb{R}_+^N}(\ln s,y)\widehat{\Phi_{\mu_w}}(t,x)\frac{\mathrm{d}t}{t}\mathrm{d}x\\&=\beta(\Phi_{\mu_w})+(\ln s,y)\frac{1}{\|\widehat{v}\|_{L_1^N}}\int_{\mathbb{R}_+^N}\widehat{v}(st,x+y)\frac{\mathrm{d}t}{t}\mathrm{d}x=0+(\ln s,y)=(\ln s,y),\end{aligned}$$

since Φ is “radially symmetric” and so Φ_{μ_w} is “radially symmetric”.

Lemma 5.13 *$I(\Pi[s,y])\searrow c_\infty$, if $|w|\to+\infty$.*

Proof In fact, $\Pi[s,y]\in\mathcal{P}$ and on $\mathcal{P}$ the functional I can be written as, with $v(t,x)=\Phi\big(\big(\frac{t}{s}\big)^{\frac{1}{\mu_w}},\frac{x-y}{\mu_w}\big)$,

$$I(\Pi[s,y])=\frac{1}{N}\int_{\mathbb{R}_+^N}|\nabla_{\mathbb{B}}v|^2\mathrm{d}\sigma+\frac{1}{N}\int_{\mathbb{R}_+^N}\Big[\ln t\; t\partial_t a(t,x)+\sum_{i=1}^n x_i\partial_{x_i}a(t,x)\Big]F(t,x,v)\mathrm{d}\sigma.$$

Moreover, since $\Phi\in\mathcal{P}_\infty$, we have $I_\infty(\Phi)=\frac{1}{N}\int_{\mathbb{R}_+^N}|\nabla_{\mathbb{B}}\Phi|^2\mathrm{d}\sigma$. Thus, we obtain

$$\begin{aligned}I(\Pi[s,y])&=\frac{\mu_w^{N-2}}{N}\int_{\mathbb{R}_+^N}|\nabla_{\mathbb{B}}\Phi|^2\mathrm{d}\sigma+\frac{\mu_w^N}{N}\int_{\mathbb{R}_+^N}\Big[st^{\mu_w}\ln st^{\mu_w}\partial_0 a(st^{\mu_w},y+\mu_w x)\\&\quad+\sum_{i=1}^n(y_i+\mu_w x_i)\partial_i a(st^{\mu_w},y+\mu_w x)\Big]F(t,x,\Phi)\mathrm{d}\sigma\\&=\mu_w^{N-2}I_\infty(\Phi)+\frac{\mu_w^N}{N}\int_{\mathbb{R}_+^N}\Big[st^{\mu_w}\ln st^{\mu_w}\partial_0 a(st^{\mu_w},y+\mu_w x)\\&\quad+\sum_{i=1}^n(y_i+\mu_w x_i)\partial_i a(st^{\mu_w},y+\mu_w x)\Big]F(t,x,\Phi)\mathrm{d}\sigma>c_\infty.\end{aligned}$$

Applying Lebesgue dominated convergence theorem, (1.4) and $\mu_w\to1$ if $|w|\to+\infty$, we have

$$\lim_{|w|\to+\infty}\int_{\mathbb{R}_+^N}\Big[st^{\mu_w}\ln st^{\mu_w}\partial_0 a(st^{\mu_w},y+\mu_w x)+\sum_{i=1}^n(y_i+\mu_w x_i)\partial_i a(st^{\mu_w},y+\mu_w x)\Big]F(t,x,\Phi)\mathrm{d}\sigma=0.$$

Therefore, we get $I(\Pi[s,y])\searrow c_\infty$, if $|w|\to+\infty$.

Denote $\overline{\mu}=\sup\limits_{w=(s,y)\in\mathbb{R}_+^N}\mu_w$, then we have following lemma.

Lemma 5.14 *Let C be the constant such that $|F(s)| \le Cs^2$. Assume that* (1.3) *admits a unique positive solution which is "radially symmetric" about some point and*

$$\|a_\infty - a\|_{L^\infty} < \frac{c_\infty}{C\overline{\mu}^N \|\Phi\|^2_{L_2^{\frac{N}{2}}}}. \tag{5.7}$$

Then, $I(\Pi[s,y]) < 2c_\infty$ *for any* $(s,y) \in \mathbb{R}_+^N$.

Proof Note that I_∞ is invariant under the translation T_{sy}, and the maximum of $\tau \to I_\infty\big(\Phi\big(t^{\frac{1}{\tau}}, \frac{x}{\tau}\big)\big)$ is attained at $\tau = 1$. Since $\mu_w > 1$, we get

$$\begin{aligned}
I(\Pi[s,y]) &= I_\infty(\Pi[s,y]) + I(\Pi[s,y]) - I_\infty(\Pi[s,y]) \\
&\le I_\infty(\Phi) + \int_{\mathbb{R}_+^N} (a_\infty - a(t,x))F(t,x,\Pi[s,y])\mathrm{d}\sigma \\
&< c_\infty + \frac{c_\infty}{\overline{\mu}^N \|\Phi\|^2_{L_2^{\frac{N}{2}}} C} \int_{\mathbb{R}_+^N} C\Phi^2\Big(\Big(\frac{t}{s}\Big)^{\frac{1}{\mu_w}}, \frac{x-y}{\mu_w}\Big)\mathrm{d}\sigma \\
&= c_\infty + \frac{c_\infty \mu_w^N}{\overline{\mu}^N \|\Phi\|^2_{L_2^{\frac{N}{2}}}} \int_{\mathbb{R}_+^N} \Phi^2 \mathrm{d}\sigma \le 2c_\infty.
\end{aligned}$$

We will need a version of linking theorem with Cerami condition by Bartolo, Benci and Fortunato in [1].

Definition 5.2 *Let S be a closed subset of Banach space X and Q be a subset manifold of X with relative boundary ∂Q. We say that S and ∂Q link if*:

(1) $S \cap \partial Q = \emptyset$;

(2) *for any* $h \in C^0(X,X)$ *such that* $h\big|_{\partial Q} = id$, $h(Q) \cap S \neq \emptyset$.

Moreover, if S and Q are as above and B is a subset of $C^0(X,X)$, then S and ∂Q link with respect to B if (1) *and* (2) *hold for any* $h \in B$.

Proposition 5.2 (Linking) *Suppose that $I \in C^1(X,\mathbb{R})$ is a functional satisfying* (*Ce*) *condition. Consider a closed subset $S \subset X$ and a submanifold $Q \subset X$ with relative boundary ∂Q such that*

(1) *S links ∂Q*;

(2) $e = \inf\limits_{u\in S} I(u) > \sup\limits_{u\in\partial Q} I(u) = e_0$;

(3) $\sup\limits_{u\in Q} I(u) < +\infty$.

If $B = \{h \in C^0(X,X);\ h|_{\partial Q} = id\}$, *then* $\widetilde{e} = \inf\limits_{h\in B} \sup\limits_{u\in Q} I(h(u)) \ge e$ *is a critical value of I.*

Now, we are ready to prove the main existence result (Theorem 1.2).

Proof of Theorem 1.2 Since $b > c_\infty$ from Lemma 5.11 and $I(\Pi[s,y]) \searrow c_\infty$ if $|w| \to +\infty$ from Lemma 5.13, there exists $\rho > 0$ such that

$$c_\infty < \max_{|w|=\rho} I(\Pi[s,y]) < b. \tag{5.8}$$

In order to apply the linking theorem, we take

$$Q := \Pi(\overline{\Omega_\rho(1,0)}) \quad \text{and} \quad S := \big\{u \in \mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}_+^N)\colon u \in \mathcal{P}, \beta(u) = 0\big\}.$$

Then we can deduce that ∂Q and S link with respect to $B = \{h \in C(Q, \mathcal{P})\colon h|_{\partial Q} = id\}$. In fact, since $\beta(\Pi[s,y](t,x)) = (\ln s, y)$, from Lemma 5.12, we have $\partial Q \cap S = \emptyset$, because if $u \in S$ then $\beta(u) = 0$ and if $u \in \partial Q$, i.e., $u = \Pi[s,y]$ for some $w = (s,y) \in \mathbb{R}^N_+$ with $|w| = \rho$ then $\beta(u) = (\ln s, y) \neq 0$. On the other hand, given $h \in B$, let $\chi\colon \overline{\Omega_\rho(1,0)} \to \mathbb{R}^N$ be defined by $\chi(s,y) = \beta \circ h \circ \Pi[s,y]$. Then by composition, χ is continuous. Moreover, for any $|w| = \rho$, we get $h \circ \Pi[s,y] = \Pi[s,y]$ as $h\big|_{\partial Q} = id$ and from Lemma 5.12, $\chi(s,y) = (\ln s, y)$. Since $\|\chi(s,y)\|^2_{\mathbb{R}^N} = (\ln s)^2 + |y|^2 = |(s,y)|^2$, by the fixed point theorem, there exists $\widetilde{w} \in \overline{\Omega_\rho(1,0)}$ such that $\chi(\widetilde{s}, \widetilde{y}) = 0$, which implies $h(\Pi[\widetilde{s}, \widetilde{y}]) \in S$. Thus, $h(Q) \cap S \neq \emptyset$ and ∂Q and S link.

From the definition of b, Q and the inequality (5.8), we can write

$$b = \inf_S I > \max_{\partial Q} I.$$

Now, define

$$d = \inf_{h \in B} \max_{u \in Q} I\big(h(u)\big).$$

Then $d \geq b$. Indeed, since ∂Q and S link, if $h \in B$ then there exists $\varphi \in S$ such that $\varphi \in h(Q)$, i.e., $\varphi = h(v)$ for some $v \in Q$. Therefore,

$$\max_{u \in Q} I\big(h(u)\big) \geq I\big(h(v)\big) = I(\varphi) \geq \inf_{u \in S} I(u) = b,$$

which means $d \geq b$. In particular, $d > c_\infty$. Moreover, if $h = id$, then from Lemma 5.14, we obtain

$$\inf_{h \in B} \max_{u \in Q} I(h(u)) \leq \max_{u \in Q} I(u) < 2c_\infty.$$

This means $c_\infty < d < 2c_\infty$ under condition (5.7). Thus, from Lemma 5.7, functional I satisfies (Ce) condition at level d. By Proposition 5.2 (linking theorem), d is a critical value of I. Therefore, there exists a positive solution for problem (1.1). Theorem 1.2 is proved.

6 Appendix

Consider the limiting problem

$$-\Delta_{\mathbb{B}} u + \lambda u = a_\infty f(u) \quad \text{in } \mathbb{R}^N_+, \tag{6.1}$$

where $a_\infty > \lambda > 0$. The purpose of this section is to prove that problem (6.1) can have at most one positive solution in $\mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}^N_+)$.

First, we recall a well know result from Peletier and Serrin in [18]. For the classical elliptic problem

$$\Delta u + g(u) = 0 \quad \text{in } \mathbb{R}^n, \quad u(x) \to 0 \quad \text{as } x \to \infty, \tag{6.2}$$

in which $n > 1$, $x = (x_1, \cdots, x_n)$, and g satisfies the following hypotheses:

(H1) g is defined and locally Lipschitz continuous on $(0, \infty)$.

(H2) $\lim\limits_{u \to 0} \frac{g(u)}{u} = -m$, where m is a positive constant.

(H3) $\int_0^\delta g(u)\mathrm{d}u > 0$ for some positive constant δ.

(H4) $\frac{g(u)}{u-l}$ is non-increasing on the subset of (l, ∞) where $g(u) > 0$.

Here $l = \inf\big\{u \in (0,\infty) : \int_0^u g(s)\mathrm{d}s > 0\big\}$. Then problem (6.2) has at most one solution. One can refer to [18, Theorem 2] for details.

Now, we give an example which satisfies above conditions.

Lemma 6.1 *Let f be defined as in* (1.6) *and $\lambda < a_\infty \le \frac{4}{3}\lambda$, then $g(u) = -\lambda u + a_\infty f(u)$ satisfies conditions* (H1)–(H4).

Proof Since

$$g'(u) = -\lambda + a_\infty \frac{3u^2 + u^4}{(1+u^2)^2}, \quad g'(0) = -\lambda, \quad \lim_{u\to\infty} g'(u) = a_\infty - \lambda > 0,$$

we have $g'(u)$ is bounded on $(0,\infty)$. This means that condition (H1) is satisfied.

It is clear that condition (H2) is satisfied with $m = \lambda$.

If $g(u) > 0$, then $u > \sqrt{\frac{\lambda}{a_\infty - \lambda}}$. On the other hand, $g''(u) = a_\infty \frac{2u(3-u^2)}{(1+u^2)^3}$, which means that $g'(u)$ is decreasing on $(\sqrt{3},\infty)$. Hence $g'(u)$ is positive and decreasing on $(\sqrt{3},\infty)$. This means that $g(u)$ is increasing at least on $(\sqrt{3},\infty)$. Since $\lim\limits_{u\to\infty} g(u) = +\infty$, condition (H3) is satisfied with $\delta > \sqrt{\frac{\lambda}{a_\infty - \lambda}}$.

Let $h(u) = g'(u)(u-l) - g(u)$. Then

$$h(l) = -g(l) < 0 \quad \text{since} \;\; l > \sqrt{\frac{\lambda}{a_\infty - \lambda}}, \quad \text{and} \quad \lim_{u\to\infty} h(u) = l(\lambda - a_\infty) < 0.$$

If $\lambda < a_\infty \le \frac{4}{3}\lambda$, then $\sqrt{\frac{\lambda}{a_\infty-\lambda}} \ge \sqrt{3}$ and $l > \sqrt{\frac{\lambda}{a_\infty-\lambda}} \ge \sqrt{3}$, then $h'(u) = g''(u)(u-l) < 0$ on (l,∞). Hence $h(u) < 0$ on (l,∞). This means that

$$\Big[\frac{g(u)}{u-l}\Big]' = \frac{g'(u)(u-l) - g(u)}{(u-l)^2} = \frac{h(u)}{(u-l)^2} < 0 \quad \text{on } (l,\infty),$$

and condition (H4) is satisfied.

Next, if f and a satisfy the conditions (f1)–(f4), (A1)–(A5) and (3.4), then $g(u) = -\lambda u + a_\infty f(u)$ will satisfy conditions (H1)–(H4) with $m = \lambda$. Therefore, problem

$$\Delta u(t,x) - \lambda u(t,x) + a_\infty f(u) = 0 \quad \text{in } \mathbb{R}^N, \quad u(t,x) \to 0 \quad \text{as } (t,x) \to \infty \tag{6.3}$$

has at most one positive solution, where $N = n+1$.

Since for any $u(t,x) \in \mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}_+^N)$ satisfying problem (6.1), there exists $v(t,x) := u(\mathrm{e}^t, x) \in H^1(\mathbb{R}^N)$ satisfying problem (6.3). And for any $v(t,x) \in H^1(\mathbb{R}^N)$ satisfying problem (6.3), there exists $u(t,x) := v(\ln t, x) \in \mathcal{H}_2^{1,\frac{N}{2}}(\mathbb{R}_+^N)$ satisfying problem (6.1). Then, we know problem (6.1) has at most on positive solution.

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