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Boundary Hölder Estimates for a Class of Degenerate Elliptic Equations in Piecewise Smooth Domains*

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Abstract In this paper, the authors will apply De Giorgi-Nash-Moser iteration to establish boundary Hölder estimates for a class of degenerate elliptic equations in piecewise C^2-smooth domains.

Keywords Degenerate elliptic, De Giorgi-Nash-Moser iteration, Boundary Hölder regularity
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1 Introduction

The present paper is interested in the boundary Hölder regularity for some class of degenerate elliptic equations in a general piecewise C^2-smooth domain. Consider

$$\partial_i(a_{ij}\partial_j u) + b_i\partial_i u = f \quad \text{in } \Omega = B_1(0)\bigcap_{\sigma=n+1}^{n+m}\{g^\sigma > 0\} \subset \mathbb{R}^{n+m} \tag{1.1}$$

for some $g^\sigma \in C^2(\overline{B}_1(0)), \sigma = n+1, \cdots, n+m$. Here we always use $i, j, k, \cdots$ to denote the indices $1, 2, 3, \cdots, n+m$ and $\sigma, \tau, \cdots$ to denote the indices $n+1, \cdots, n+m$. Assume that

$$g^\sigma(0) = 0, \quad \det(\partial_\tau g^\sigma)(0) \neq 0. \tag{1.2}$$

For simplicity, we also denote $g^i \triangleq x_i,\ i = 1, \cdots, n$ and with $G_{ij} = \partial_{x_i} g^j$ where $i, j = 1, 2, \cdots, n+m$, (G^{ij}) is the inverse of the matrix (G_{ij}). We also assume

$$C_*\begin{pmatrix} I_n & 0 \\ 0 & M_m \end{pmatrix} \geq (a_{ij}\partial_i g^k \partial_j g^l)_{k,l=1,\cdots,n+m} \geq \frac{1}{C_*}\begin{pmatrix} I_n & 0 \\ 0 & M_m \end{pmatrix} \tag{1.3}$$

for some positive constant $C_* > 1$. Here I_n is the $n \times n$ unit matrix and

$$M_m = \begin{pmatrix} g^{n+1} & 0 & \cdots & 0 \\ 0 & g^{n+2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots. & g^{n+m} \end{pmatrix}. \tag{1.4}$$

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Moreover, $b_i\partial_i g^\sigma = b_{\sigma 1} + b_{\sigma 2}$ and

$$\begin{aligned}&\sum_{i,j=1}^{n+m}|a_{ij}| + \sum_{i=1}^{n+m}|b_i| + \sum_{\sigma=n+1}^{n+m}\Big(|b_{\sigma 1}| + \Big|\sum_{i=1}^{n+m} G^{i\sigma}\partial_{x_i} b_{\sigma 1}\Big|\Big) \le C_*,\\ &b_{\sigma 1}\ge 0 \quad \text{on } g^\sigma = 0, \quad |b_{\sigma 2}| \le C_*\sqrt{g^\sigma}, \quad \sigma = n+1,\cdots,n+m.\end{aligned} \tag{1.5}$$

We have the following result.

Theorem 1.1 *Let* (1.2)–(1.5) *be fulfilled. Suppose that* $u\in C^2(\Omega)\cap L^\infty(\Omega)$ *solves* (1.1) *with* $f\in L^q(\Omega)$ *for some* $q > \frac{n+2m}{2}$. *Then* $u \in C^\alpha(\mathcal{N}(0)\cap\overline{\Omega})$ *for a neighbourhood* $\mathcal{N}(0)$ *of the origin and some constant* $\alpha\in(0,1)$ *depending only on* C_*, q *and* $|g^\sigma|_{C^2(\overline{\Omega})}$, $\sigma = n+1,\cdots,n+m$.

We are going to reduce the proof for Theorem 1.1 to a model problem by the following transformation.

The Coordinate Transformation Set

$$\mathcal{T}: x\to y = (y_1,\cdots,y_{n+m}) = (x_1,\cdots,x_n,g^{n+1},\cdots,g^{n+m})$$

and $\widetilde{a}_{ij} = a_{kl}\partial_k g^i\partial_l g^j$. A simple calculation yields that

$$\partial_{y_i}(\widetilde{a}_{ij}\partial_{y_j}u) + \widetilde{b}_i\partial_{y_i}u = f \quad \text{in } \mathcal{N}(0)\cap\mathcal{T}(\Omega) \tag{1.6}$$

for some neighbourhood $\mathcal{N}(0)$ of the origin $\mathcal{T}(0)$, where

$$\widetilde{b}_i = b_k\frac{\partial g^i}{\partial x_k} - a_{kj}\frac{\partial^2 g^l}{\partial x_k\partial x_h}\frac{\partial x_h}{\partial y_l}\frac{\partial g^i}{\partial x_j}.$$

For $\sigma = n+1,\cdots,m$, denote

$$\widetilde{b}_{\sigma 1} = b_{\sigma 1}, \quad \widetilde{b}_{\sigma 2} = b_{\sigma 2} - a_{kj}\frac{\partial^2 g^l}{\partial x_k\partial x_h}\frac{\partial x_h}{\partial y_l}\frac{\partial g^\sigma}{\partial x_j}.$$

By (1.5) it is easy to see

$$|\widetilde{b}_{\sigma 1}| + |\partial_{y_\sigma}\widetilde{b}_{\sigma 1}| \le C_*$$

and

$$\widetilde{b}_{\sigma 1}\ge 0 \quad \text{on } g^\sigma = y_\sigma = 0.$$

By means of (1.3) and (1.4), it follows that

$$\begin{aligned}|\widetilde{b}_{\sigma 2}| &\le C_*\sqrt{y_\sigma} + \Big|a_{kj}\frac{\partial^2 g^l}{\partial x_k\partial x_h}\frac{\partial x_h}{\partial y_l}\frac{\partial g^\sigma}{\partial x_j}\Big|\\ &= C_*\sqrt{y_\sigma} + |a_{kj}p_k\partial_j g^\sigma| \quad \Big(\text{where } p_k = \frac{\partial^2 g^l}{\partial x_k\partial x_h}\frac{\partial x_h}{\partial y_l}\Big)\\ &\le C_*\sqrt{y_\sigma} + \sqrt{a_{kj}\partial_k g^\sigma\partial_j g^\sigma}\sqrt{a_{kj}p_kp_j} \le \widetilde{C}C_*\sqrt{y_\sigma}\end{aligned}$$

for some constant $\widetilde{C}$ depending only on $|g^\sigma|_{C^2}$ and $\frac{1}{\det(\partial_\tau g^\sigma)}$. In the sequel we also denote $\widetilde{C}C_*$ by C_*. It is easy to see that (1.3)–(1.4) tell us

$$C_*\Big(\sum_{i=1}^n\xi_i^2 + \sum_{i=n+1}^{n+m}y_i\xi_i^2\Big) \ge \widetilde{a}_{ij}\xi_i\xi_j \ge \frac{1}{C_*}\Big(\sum_{i=1}^n\xi_i^2 + \sum_{i=n+1}^{n+m}y_i\xi_i^2\Big), \quad \forall\xi\in\mathbb{R}^{n+m}.$$

Under the above coordinate transformation, we only need to consider the following model equation

$$\partial_i(a_{ij}\partial_j u) + b_i\partial_i u = f \quad \text{in } Q_{n,m}(1), \tag{1.7}$$

where $Q_{n,m}(r)$ is defined as

$$Q_{n,m}(r) = \{x \in \mathbb{R}^{n+m} \mid -r < x_i < r,\ i = 1, \cdots, n,\ 0 < x_i < r^2,\ i = n+1, \cdots, n+m\}.$$

In the definition of $Q_{n,m}(r)$, $n = 0$ is allowed and $m \geq 1$. Moreover, a_{ij}, b_i satisfy the following structure conditions

(1)

$$C_*^{-1}\Big(\sum_{i=1}^{n}\xi_i^2 + \sum_{i=n+1}^{n+m} x_i\xi_i^2\Big) \leq a_{ij}\xi_i\xi_j \leq C_*\Big(\sum_{i=1}^{n}\xi_i^2 + \sum_{i=n+1}^{n+m} x_i\xi_i^2\Big), \quad \forall \xi \in \mathbb{R}^{n+m}. \tag{1.8}$$

(2) For $i = n+1, \cdots, n+m$, let $b_i = b_{i1} + b_{i2}$. Then we need

$$b_{i1} \geq 0 \quad \text{on } x_i = 0, \quad |b_{i2}| \leq C_*\sqrt{x_i}, \tag{1.9}$$

also

$$\sum_{i=1}^{n+m}|b_i| + \sum_{i=n+1}^{n+m}|\partial_i b_{i1}| \leq C_*. \tag{1.10}$$

The condition (1.8) implies

$$\begin{aligned}
&\frac{1}{C_*}x_i \leq a_{ii} \leq C_* x_i, \quad n+1 \leq i \leq n+m,\\
&|a_{ij}| \leq C_*\sqrt{x_j}, \quad 1 \leq i \leq n, \quad n+1 \leq j \leq n+m,\\
&|a_{ij}| \leq C_*\sqrt{x_i x_j}, \quad n+1 \leq i \leq n+m, \quad n+1 \leq j \leq n+m.
\end{aligned}$$

The study of the Hölder regularity for uniformly elliptic equation dates back to the late 1930s due to its relation to the Hilbert's 19$^{\text{th}}$ problem. The Hölder regularity was first proven by Morrey [9] in two dimensions in the late 1930s, and by De Giorgi [2] and Nash [10] in higher dimensions in the late 1950s. Now the Hölder regularity for uniformly elliptic equation is well known.

For $m = 1$, (1.7) is modelled by the following equation

$$\Delta_x u + y u_{yy} + a u_y = f \quad \text{in } \mathbb{R}^{n+1}_+. \tag{1.11}$$

Usually, people call (1.11) Keldysh type degenerate elliptic equation. There is an interesting phenomenon for Keldysh type degenerate elliptic equation. The coefficient a determines the formulation of the well-posed problem. Exactly speaking, in $C(\overline{\mathbb{R}^{n+1}_+})$, the well-posed problem of (1.11) is distinguished into the following two cases.

(1) $a < 1$. We need prescribe boundary condition on $y = 0$.

(2) $a \geq 1$. We can't prescribe boundary condition on $y = 0$.

Such a phenomenon was discovered by Keldyš in [8]. Later, Fichera [3–4] and Oleĭnik [11–12] established general theory on second order linear elliptic equations with nonnegative characteristic form. See also the book [13] and the references therein.

In the present paper, we are interested in the Hölder regularity for the special degenerate elliptic equation as in (1.7)–(1.10). For $m = 1$, the non-divergence form of (1.7) is investigated in [1] which studies the free boundary problem associated to Gauss curvature flow with flat sides. Under a slightly weaker condition

$$\frac{b_{n+1}(x',0)}{\partial_{n+1}a_{(n+1)(n+1)}(x',0)} \geq -1+\nu \tag{1.12}$$

for some positive constant ν, the boundary Hölder a priori estimates are established in [1] for solution $u \in C^2(\overline{Q_{n,1}(1)})$. Also, for $n = 1, m = 1$, the boundary Hölder a priori estimates are established in [6] which studies isometric embedding of Alexandrov-Nirenberg surfaces. For $n \geq 2, m = 1$, the boundary Hölder estimates are established in [7] for a class of degenerate semi-linear elliptic equations in general bounded domains. There are also many other related studies for $m = 1$ which can't be exhausted in the present paper.

For $m \geq 2$, to authors' knowledge, the only known Hölder estimates for this type of degenerate elliptic equation in polytope type domain are done by [15]. In [15], the non-divergence form of (1.7) is studied for $n = 0$, $m \geq 2$. Under a similar condition as (1.12)

$$\frac{b_i(x)}{\partial_i a_{ii}(x)} \geq -1+\nu \quad \text{on } x_i = 0 \tag{1.13}$$

for some positive constant ν, the boundary Hölder a priori estimates are established for $u \in C^2(\overline{Q_{0,m}(1)})$ by some probability method.

Our main result is as following.

Theorem 1.2 *Let* (1.8)–(1.10) *be assumed and* $f \in L^q(Q_{n,m}(1))$ *for some* $q > \frac{n+2m}{2}$. *Suppose* $u \in C^2(Q_{n,m}(1)) \cap L^\infty(Q_{n,m}(1))$ *satisfies* (1.7). *Then there exists some* $\alpha \in (0,1)$ *such that*

$$\begin{aligned} &|u(x)-u(\widetilde{x})| \\ &\leq C\Big(\sup_{Q_{n,m}(1)}|u| + \|f\|_{L^q(Q_{n,m}(1))}\Big)|x-\widetilde{x}|^\alpha, \quad \forall x,\widetilde{x} \in Q_{n,m}\Big(\frac{1}{2}\Big), \end{aligned} \tag{1.14}$$

where α *and* C *are positive constants depending only on* q *and* C_* *in* (1.8) *and* (1.10).

Remark 1.1 Although, we don't impose boundary condition for (1.7). Theorem 1.2 tells us that we can Hölder continuously extend the solution u from interior up to the boundary.

The domains we studied are much complicated than that in [1] and for Theorem 1.1 and Theorem 1.2, no a priori regularity of solutions on the boundary is needed except the solutions in $L^\infty(\Omega)$ or in $L^\infty(Q_{n,m}(1))$. It should be emphasized that in [1, 15] under the assumption (1.12) or (1.13), the solutions to corresponding problems might be not C^2-smooth up to the boundary. See the following two examples.

Example 1.1 (see [14]) Consider the following degenerate elliptic equation

$$u_{xx} + yu_{yy} + bu_y = 0 \quad \text{in } \mathbb{R}^2_+. \tag{1.15}$$

Let $u(x, y) = F\big(\frac{x}{2\sqrt{y}}\big) = F(z)$. Then a direct computation yields that $F(z)$ solves the following ODE

$$(1 + z^2)F'' + (3 - 2b)zF' = 0 \quad \text{in } \mathbb{R}. \tag{1.16}$$

A direct integration of (1.16) yields that

$$F(z) = \int_0^z \mathrm{e}^{-(3-2b)\int_0^s \frac{t}{1+t^2}\mathrm{d}t}\mathrm{d}s = \int_0^z (1 + s^2)^{-\frac{3-2b}{2}}\mathrm{d}s$$

solves (1.16) with $F(0) = 0$, $F'(0) = 1$. Hence for $b < 1$, $F(z)$ is a non-constant bounded smooth function. This implies $u(x, y)$ is bounded in $\mathbb{R}^2_+$ and is dis-continuous at $(0, 0)$.

Example 1.2 Consider the following degenerate elliptic equation

$$xu_{xx} + yu_{yy} + au_x + bu_y = 0 \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^+. \tag{1.17}$$

Let $u(x, y) = F\big(\ln\big(\frac{x}{y}\big)\big) = F(z)$. Then a direct computation yields that $F(z)$ solves the following ODE

$$F'' + \Big(\frac{a-1}{1+\mathrm{e}^z} + \frac{(1-b)\mathrm{e}^z}{1+\mathrm{e}^z}\Big)F' = 0 \quad \text{in } \mathbb{R}. \tag{1.18}$$

A direct integration of (1.18) yields that

$$F(z) = \int_0^z \frac{\mathrm{e}^{(1-a)s}}{(1+\mathrm{e}^s)^{2-a-b}}\mathrm{d}s \tag{1.19}$$

solves (1.18). In the present case, we also know $F(z)$ is smooth and uniformly bounded in $(-\infty, +\infty)$ for $a, b < 1$. This implies $u(x, y)$ is bounded in $\mathbb{R}^+ \times \mathbb{R}^+$ and is dis-continuous at $(0, 0)$.

Remark 1.2 Examples 1.1–1.2 satisfy all the assumptions of Theorem 2.1 in [1] and Corollary 1.2 in [15] except that $u \in C^2(\overline{Q_{n,m}(1)})$ apriorily. It is very interesting if one can construct counterexamples for $a \geq 1, b < 1$ in Example 1.2.

2 The Proof for Theorem 1.2

In this section, we give a boundary Hölder estimates for a class of degenerate elliptic equation by De Giorgi iteration. Some ideas come from [6] and [7]. In [6], the proof is based on some regularity of the solutions up to the boundary. In [7], additional regularity of the coefficients a_{ij}, b_i is assumed. In the present case, we want to generalize these results in [6–7] to polytope type domains.

In order to prove Theorem 1.2, we introduce a weighted Sobolve space $\widetilde{W}^{1,2}(Q_{n,m}(1))$ which is the completion of $C^1(\overline{Q_{n,m}(1)})$ under the norm

$$\Big(\int_{Q_{n,m}(1)} \sum_{i=1}^n u_i^2 + \sum_{i=n+1}^{n+m} x_i u_i^2 + u^2 \mathrm{d}x\Big)^{\frac{1}{2}}.$$

We first give the following weighted Sobolev embedding inequality and Poincaré inequality which are generalizations of the corresponding inequalities in [6–7].

Lemma 2.1 (1) *For all* $u \in C^1(\overline{Q_{n,m}(1)})$ *with* $u = 0$ *on* $\partial Q_{n,m}(1) \cap (\mathbb{R}^n \times (\mathbb{R}^+)^m)$, *there is a universal constant* C *independent of* $Q_{n,m}(1)$ *such that*

$$\Big(\int_{Q_{n,m}(1)} |u|^{\frac{2(n+2m)}{n+2m-2}} \mathrm{d}x\Big)^{\frac{n+2m-2}{n+2m}} \le C \int_{Q_{n,m}(1)} \Big(\sum_{i=1}^{n} u_i^2 + \sum_{i=n+1}^{n+m} x_i u_i^2\Big) \mathrm{d}x. \tag{2.1}$$

(2) *For any* $\varepsilon > 0$, *there exists a constant* C_ε *such that*

$$\int_{Q_{n,m}(1)} u^2 \mathrm{d}x \le C_\varepsilon \int_{Q_{n,m}(1)} \Big(\sum_{i=1}^{n} u_i^2 + \sum_{i=n+1}^{n+m} x_i u_i^2\Big) \mathrm{d}x \tag{2.2}$$

for all $u \in C^1(\overline{Q_{n,m}(1)})$ *subject to* $|\{x \in Q_{n,m}(1) \mid u(x) = 0\}| \ge \varepsilon$.

Proof Let $G = Q_{n,m}(1)$. Define a transform $T : G \to T(G)$ by

$$T(x) = y, \quad y_i = x_i, \quad 1 \le i \le n, \quad y_i = 2\sqrt{x_i}, \quad n+1 \le i \le n+m.$$

Lift $T(G)$ in $\mathbb{R}^{n+2m}$ by defining

$$\widetilde{T(G)} = \{(y, z) \in \mathbb{R}^{n+m} \times \mathbb{R}^m : \ y \in T(G), \ 0 < z_i < y_{n+i}, \ i = 1, \cdots m\}.$$

Then

$$\int_G |u|^p \mathrm{d}x = \Big(\frac{1}{2}\Big)^m \int_{T(G)} |u \circ T^{-1}|^p \Big(\prod_{i=n+1}^{n+m} y_i\Big) \mathrm{d}y = \Big(\frac{1}{2}\Big)^m \|u \circ T^{-1}\|^p_{L^p(\widetilde{T(G)})} \tag{2.3}$$

and

$$\begin{aligned} \int_G \Big(\sum_{i=1}^{n} u_{x_i}^2 + \sum_{i=n+1}^{n+m} x_i u_{x_i}^2\Big) \mathrm{d}x &= \Big(\frac{1}{2}\Big)^m \sum_{i=1}^{n+m} \int_{T(G)} u_{y_i}^2 \Big(\prod_{i=n+1}^{n+m} y_i\Big) \mathrm{d}y \\ &= \Big(\frac{1}{2}\Big)^m \|\widetilde{\nabla}(u \circ T^{-1})\|^2_{L^2(\widetilde{T(G)})}, \end{aligned} \tag{2.4}$$

where $\widetilde{\nabla} = (\nabla_y, \nabla_z)$ is the gradient in $\mathbb{R}^{n+2m}$.

Now let us consider the first part of the present lemma. Let $u \in C^1(\overline{G})$ with $u = 0$ on $\partial G \cap_{i=n+1}^{n+m} \{x_i > 0\}$. Set

$$\widetilde{u}(x) = \begin{cases} u(x) & \text{for } x \in G, \\ 0 & \text{for } x \in \mathbb{R}^n \times (\mathbb{R}^+)^m \setminus G. \end{cases}$$

Then, define $v(y) = \widetilde{u}(x)$ and

$$w(y, z) = v(y_1, \cdots, y_n, \max\{y_{n+1}, z_1\}, \cdots, \max\{y_{n+m}, z_m\}) \quad \text{in } \mathbb{R}^n \times (\mathbb{R}^+)^{2m}.$$

Then, we can extend w to $\mathbb{R}^{n+2m}$ by even extensions first with respect to the plane $y_i = 0$, $i = n+1, \cdots, n+m$ and then to the plane $z_i = 0$, $i = 1, \cdots, m$. By the Sobolev embedding theorem, we have $w \in H^1(\mathbb{R}^{n+2m}) \subset L^{\frac{2(n+2m)}{n+2m-2}}(\mathbb{R}^{n+2m})$ and

$$\Big(\int_{\mathbb{R}^{n+2m}} w^{\frac{2(n+2m)}{n+2m-2}} \mathrm{d}y\mathrm{d}z\Big)^{\frac{n+2m-2}{n+2m}} \le C \int_{\mathbb{R}^{n+2m}} |\widetilde{\nabla} w|^2 \mathrm{d}y\mathrm{d}z.$$

Therefore by (2.3) and (2.4), we obtain

$$\Big(\int_G u^{\frac{2(n+2m)}{n+2m-2}}\mathrm{d}x\Big)^{\frac{n+2m-2}{n+2m}} \le \Big(C_1\int_{\mathbb{R}^{n+2m}} w^{\frac{2(n+2m)}{n+2m-2}}\mathrm{d}y\mathrm{d}z\Big)^{\frac{n+2m-2}{n+2m}} \le C\int_{\mathbb{R}^{n+2m}}|\widetilde{\nabla}w|^2\mathrm{d}y\mathrm{d}z$$
$$\le C'\int_{\widetilde{T(G)}}|\widetilde{\nabla}w|^2\mathrm{d}y\mathrm{d}z = 2^m C'\int_G\Big(\sum_{i=1}^n u_{x_i}^2+\sum_{i=n+1}^{n+m}x_iu_{x_i}^2\Big)\mathrm{d}x,$$

where C_1, C and C' are universal positive constants, independent of u.

Next, we consider the second part of the present lemma. Suppose that $u\in C^1(\overline{G})$ with $|\{x\in G:\ u(x)=0\}|\ge\varepsilon>0$. Then, it is easy to see that

$$|\{y:\ v(y)=0\}|\ge C\varepsilon$$

and

$$|\{(y,z)\in\widetilde{T(G)}:\ w(y,z)=0\}|\ge C\varepsilon$$

for another universal constant C. By the well-known Poincaré inequality, we get

$$\int_{\widetilde{T(G)}} w^2\mathrm{d}y\mathrm{d}z\le C_\varepsilon\int_{\widetilde{T(G)}}|\widetilde{\nabla}w|^2\mathrm{d}y\mathrm{d}z,$$

where C_ε is a positive constant depending only on ε. Then

$$\int_G u^2\mathrm{d}x=\Big(\frac{1}{2}\Big)^m\int_{\widetilde{T(G)}} w^2\mathrm{d}y\mathrm{d}z\le\Big(\frac{1}{2}\Big)^m C_\varepsilon\int_{\widetilde{T(G)}}|\widetilde{\nabla}w|^2\mathrm{d}y\mathrm{d}z$$
$$\le C'_\varepsilon\int_G\Big(\sum_{i=1}^n u_{x_i}^2+\sum_{i=n+1}^{n+m}x_iu_{x_i}^2\Big)\mathrm{d}x.$$

This completes the proof of the present lemma.

Now we begin to discuss the boundary Hölder regularity of $u\in L^\infty(Q_{n,m}(1))\cap C^2(Q_{n,m}(1))$ which satisfies (1.7). As a first step, we show that such a solution $u\in\widetilde{W}^{1,2}(Q_{n,m}(r))$, $\forall r\in(0,1)$, has no trace on the boundary $x_i=0$, $i=n+1,\cdots,n+m$ in general. So we need to bypass this obstacle in the present circumstance.

Lemma 2.2 *Let the assumptions of Theorem* 1.2 *be satisfied. Then,* $u\in\widetilde{W}^{1,2}(Q_{n,m}(r))$, $\forall r\in(0,1)$ *and satisfies*

$$\int\varphi^2\Big(\sum_{i=1}^n u_{x_i}^2+\sum_{i=n+1}^{n+m}x_iu_{x_i}^2\Big)\mathrm{d}x$$
$$\le C\|u\|_\infty^2\Big(\int\Big(\varphi^2+\sum_{i=1}^n\varphi_{x_i}^2+\sum_{i=n+1}^{n+m}x_i\varphi_i^2+\sum_{i=n+1}^{n+m}\varphi|\varphi_i|\Big)\mathrm{d}x+1\Big)+C\int\varphi^2f^2\mathrm{d}x,$$

where C is a positive constant depending only on C_* in (1.8) *and* (1.10) *and* $0\le\varphi\in C_0^\infty(\overline{\mathbb{R}^n\times(\mathbb{R}^+)^m})$ *subject to*

$$\varphi\equiv1\quad in\ Q_{n,m}(r),\qquad\varphi\equiv0\quad in\ Q_{n,m}^c\Big(\frac{1+r}{2}\Big).\tag{2.5}$$

Proof Consider a smooth cut-off function $0 \le \chi_\varepsilon \le 1$ such

$$\chi_\varepsilon(t) = \begin{cases} 0, & 0 \le t \le \varepsilon, \\ 1, & t \ge 2\varepsilon \end{cases}$$

and

$$|D^k \chi_\varepsilon| \le \frac{C_k}{\varepsilon^k}, \quad k = 1, 2, \cdots .$$

Let

$$\eta_\varepsilon(x) = \prod_{i=n+1}^{n+m} \chi_\varepsilon(x_i).$$

We multiply (1.7) by $-\varphi^2\eta_\varepsilon^2 u$ and integrate by parts. Then

$$\begin{aligned}\int_{Q_{n,m}(1)} \varphi^2\eta_\varepsilon^2 a_{ij}u_iu_j &= -2\int_{Q_{n,m}(1)} \eta_\varepsilon^2\varphi u a_{ij}\varphi_i u_j - 2\int_{Q_{n,m}(1)} \varphi^2\eta_\varepsilon u a_{ij}u_j\partial_i\eta_\varepsilon \\ &\quad + \int_{Q_{n,m}(1)} \varphi^2\eta_\varepsilon^2 u b_i u_i - \int_{Q_{n,m}(1)} \varphi^2\eta_\varepsilon^2 u f. \end{aligned} \tag{2.6}$$

Next, the Cauchy inequality implies, for $\delta > 0$ to be determined,

$$\begin{aligned}&2\eta_\varepsilon^2\varphi u a_{ij}\varphi_i u_j \le \delta\eta_\varepsilon^2\varphi^2 a_{ij}u_iu_j + \frac{1}{\delta}a_{ij}\varphi_i\varphi_j\eta_\varepsilon^2u^2, \\ &2\eta_\varepsilon\varphi^2 u a_{ij}\partial_i\eta_\varepsilon u_j \le \delta\eta_\varepsilon^2\varphi^2 a_{ij}u_iu_j + \frac{1}{\delta}\varphi^2u^2a_{ij}\partial_i\eta_\varepsilon\partial_j\eta_\varepsilon.\end{aligned}$$

Also one has

$$\begin{aligned}&\int_{Q_{n,m}(1)} \varphi^2u^2a_{ij}\partial_i\eta_\varepsilon\partial_j\eta_\varepsilon \mathrm{d}x \le C\|u\|_\infty^2\|\varphi\|_\infty^2 \sum_{i=n+1}^{n+m}\int_{Q_{n,m}(1)} x_i(\partial_i\eta_\varepsilon)^2\mathrm{d}x \\ &\le C\|u\|_\infty^2\|\varphi\|_\infty^2 \sum_{i=n+1}^{n+m}\int_{Q_{n,m}(1)\cap\{\varepsilon\le x_i\le 2\varepsilon\}} \frac{1}{\varepsilon}\mathrm{d}x \le C'\|u\|_\infty^2.\end{aligned}$$

Therefore, by (1.8), one has

$$\begin{aligned}&\frac{1}{C_*}(1-\widetilde{C}\delta)\int \eta_\varepsilon^2\varphi^2\Big(\sum_{i=1}^n u_{x_i}^2 + \sum_{i=n+1}^{n+m} x_iu_{x_i}^2\Big) \\ &\le \frac{C_*}{\delta}\int\Big(\sum_{i=1}^n \varphi_{x_i}^2 + \sum_{i=n+1}^{n+m} x_i\varphi_{x_i}^2\Big)\eta_\varepsilon^2u^2 + \int \varphi^2\eta_\varepsilon^2 u b_i u_i - \int \varphi^2\eta_\varepsilon^2 u f + C\|u\|_\infty^2.\end{aligned}$$

Next, for the b_i-term, $i = 1, \cdots, n$, we have, by $|b_i| \le C_*$ in (1.10),

$$\Big|\int \eta_\varepsilon^2\varphi^2 u b_i u_i\Big| \le \frac{\delta}{C_*}\int \eta_\varepsilon^2\varphi^2u_i^2 + \frac{C_*^3}{\delta}\int \eta_\varepsilon^2\varphi^2u^2.$$

For the b_i-term, $i = n+1, \cdots, n+m$, one has

$$\int_{Q_{n,m}(1)} \eta_\varepsilon^2\varphi^2 b_i u u_i = \frac{1}{2}\int_{Q_{n,m}(1)} \eta_\varepsilon^2\varphi^2 b_{i1}\partial_i(u^2) + \int_{Q_{n,m}(1)} \eta_\varepsilon^2\varphi^2 b_{i2}uu_i$$

$$= -\frac{1}{2}\int_{Q_{n,m}(1)} \partial_i(\eta_\varepsilon^2\varphi^2 b_{i1})u^2 + \int_{Q_{n,m}(1)} \eta_\varepsilon^2\varphi^2 b_{i2}uu_i.$$

Therefore, by (1.10), for $i = n+1, \cdots, n+m$,

$$\begin{aligned}&\int_{Q_{n,m}(1)} \eta_\varepsilon^2\varphi^2 b_i uu_i \\ &\le \Big(C_* + \frac{C_*^3}{\delta}\Big)\int(\varphi^2 + \varphi|\varphi_i|)\eta_\varepsilon^2 u^2 + C\|u\|_\infty^2 + \frac{\delta}{C_*}\int_{Q_{n,m}(1)} \eta_\varepsilon^2\varphi^2 x_i u_i^2.\end{aligned}$$

By a simple substitution and taking δ small enough, then let $\varepsilon \to 0$, we obtain the desired result in the present lemma.

Lemma 2.3 *Let $u \in \widetilde{W}^{1,2}(Q_{n,m}(1))$. Denote*

$$A(\delta) = \sum_{i=1}^{m}\int_{Q_{n,m}(1)\cap\{x_{n+i}=\delta\}} \Big(\sum_{k=1}^{n} u_{x_k}^2 + \sum_{j=1}^{m} x_{n+j}u_{x_{n+j}}^2\Big).$$

Then

$$\varliminf_{\delta\to 0^+} \delta A(\delta) = 0.$$

Proof If it is not true, it follows that

$$A(\delta) \ge \frac{c_0}{\delta}$$

for some positive constant $c_0 > 0$. This implies

$$+\infty = \int_0^1 \frac{c_0}{\delta}\mathrm{d}\delta \le \int_0^1 A(\delta)\mathrm{d}\delta \le C\|u\|^2_{\widetilde{W}^{1,2}(Q_{n,m}(1))}$$

which yields a contradiction.

Lemma 2.4 *Let the assumptions of Theorem 1.2 be satisfied. Then, for any φ as mentioned in Lemma 2.2, there holds*

$$\begin{aligned}&\int \varphi^2\Big(\sum_{i=1}^{n} u_{x_i}^2 + \sum_{i=n+1}^{n+m} x_i u_{x_i}^2\Big)\mathrm{d}x \\ &\le C\int\Big(\varphi^2 + \sum_{i=1}^{n}\varphi_{x_i}^2 + \sum_{i=n+1}^{n+m} x_i\varphi_i^2 + \sum_{i=n+1}^{n+m}\varphi|\varphi_i|\Big)u^2 + \int\varphi^2 f^2, \qquad (2.7)\end{aligned}$$

where C is a positive constant depending only on C_* in (1.8) *and* (1.10).

Proof By Lemma 2.2, one knows $u \in \widetilde{W}^{1,2}(Q_{n,m}(r))$, $r \in (0,1)$. We multiply (1.7) by $-\varphi^2 u$ and integrate by parts. Let $G \subset \mathbb{R}^n \times (\mathbb{R}^+)^m$ be a bounded domain such that $\varphi = 0$ in $\mathbb{R}^n \times (\mathbb{R}^+)^m \setminus G$. Set

$$G_\delta = G \cap_{i=n+1}^{n+m} \{x_i > \delta\}.$$

Then

$$\int_{G_\delta} \varphi^2 a_{ij}u_iu_j = \int_{\partial G_\delta} \varphi^2 u a_{ij}u_j\nu_i - 2\int_{G_\delta} \varphi u a_{ij}\varphi_i u_j + \int_{G_\delta} \varphi^2 u b_i u_i - \int_{G_\delta} \varphi^2 uf.$$

For the boundary integral, we first note $\varphi = 0$ on $\partial G_\delta \bigcap_{i=n+1}^{n+m} \{x_i > \delta\}$. Next, on $\partial G_\delta \cap \{x_{n+m} = \delta\}$, $\nu_1 = \cdots = \nu_{n+m-1} = 0$. One has

$$\Big| \int_{\partial G_\delta \cap \{x_{n+m}=\delta\}} \varphi^2 u a_{(n+m)j} u_j \nu_{n+m} \Big| \le C_\varphi \|u\|_\infty \sqrt{\delta} \int_{\partial G_\delta \cap \{x_{n+m}=\delta\}} \sqrt{a_{jj}} |u_j| \mathrm{d}S$$
$$\le C_\varphi \|u\|_\infty \Big(\int_{\partial G_\delta \cap \{x_{n+m}=\delta\}} \delta a_{jj} |u_j|^2 \mathrm{d}S \Big)^{\frac{1}{2}}.$$

Notice that

$$a_{jj} \le C, \quad j = 1, \cdots, n, \quad a_{jj} \le C x_j, \quad j = n+1, \cdots, n+m.$$

By Lemmas 2.2–2.3, after taking a subsequence, we have

$$\Big| \int_{\partial G_\delta} \varphi^2 u a_{ij} u_j \nu_i \Big| = o(1), \quad \text{as } \delta \to 0.$$

By Cauchy inequality, for $\varepsilon > 0$ to be determined, one has

$$\frac{1}{C_*}(1-\varepsilon) \int_{G_\delta} \varphi^2 \Big(\sum_{i=1}^{n} u_{x_i}^2 + \sum_{i=n+1}^{n+m} x_i u_{x_i}^2 \Big)$$
$$\le \frac{C_*}{\varepsilon} \int_{G_\delta} \Big(\sum_{i=1}^{n} \varphi_{x_i}^2 + \sum_{i=n+1}^{n+m} x_i \varphi_{x_i}^2 \Big) u^2 + \int_{G_\delta} \varphi^2 u b_i u_i - \int_{G_\delta} \varphi^2 u f + o(1)$$

and

$$\sum_{i \le n} \Big| \int_{G_\delta} \varphi^2 u b_i u_i \Big| \le \frac{\varepsilon}{C_*} \sum_{i \le n} \int_{G_\delta} \varphi^2 |u_i|^2 + \frac{2C_*^3}{\varepsilon} \int_{G_\delta} \varphi^2 u^2.$$

On the other hand, for $i = n+1, \cdots, n+m$, one has

$$\int_{G_\delta} \varphi^2 b_i u u_i = \frac{1}{2} \int_{G_\delta} \varphi^2 b_{i1} \partial_i (u^2) + \int_{G_\delta} \varphi^2 b_{i2} u \partial_i u$$
$$= -\frac{1}{2} \int_{G_\delta} \partial_i (\varphi^2 b_{i1}) u^2 + \int_{\partial G_\delta} \varphi^2 b_{i1} u^2 \nu_i + \int_{G_\delta} \varphi^2 b_{i2} u \partial_i u.$$

For fixed $i = n+1, \cdots, n+m$, on $x_i = \delta$, $\nu_i = -1$, by (1.10), one has

$$b_{i1}(x_1, \cdots, x_{i-1}, \delta, x_{i+1}, \cdots, x_{n+m}) - b_{i1}(x_1, \cdots, x_{i-1}, 0, x_{i+1}, \cdots, x_{n+m}) \ge -C\delta. \tag{2.8}$$

Hence

$$\int_{\partial G_\delta} \varphi^2 b_{i1} u^2 \nu_i \le C \int_{\partial G_\delta \cap \{x_i = \delta\}} \delta u^2 = O(\delta). \tag{2.9}$$

Therefore, by (1.10),

$$\int_{G_\delta} \varphi^2 b_i u u_i \le O(\delta) - \frac{1}{2} \int_{G_\delta} (\varphi^2 \partial_i b_{i1} + 2\varphi \varphi_i b_{i1}) u^2 + C_* \int_{G_\delta} \varphi \sqrt{x_i} u |u_i|$$
$$\le O(\delta) + \Big(C_* + \frac{C_*^3}{\varepsilon} \Big) \int_{G_\delta} (\varphi^2 + \varphi |\varphi_n|) u^2 + \frac{\varepsilon}{C_*} \int_{G_\delta} \varphi^2 x_i u_i^2.$$

By a simple substitution and taking $\varepsilon = \frac{1}{4}$, then let $\delta \to 0$, we obtain

$$\begin{aligned}&\int \varphi^2\Big(\sum_{i=1}^{n} u_{x_i}^2 + \sum_{i=n+1}^{n+m} x_i u_{x_i}^2\Big)\\ &\le C\int \Big(\varphi^2 + \sum_{i=1}^{n}\varphi_{x_i}^2 + \sum_{i=n+1}^{n+m} x_i\varphi_i^2 + \sum_{i=n+1}^{n+m}\varphi|\varphi_i|\Big)u^2 + C_2\int \varphi^2|uf|.\end{aligned}$$

Another application of the Cauchy inequality implies the desired result.

Lemma 2.5 *Let (1.8), (1.10) be assumed and $f \in L^q(Q_{n,m}(R))$, for some $R \in (0,1]$ and $q > \frac{n+2m}{2}$. Suppose $u \in C^2(Q_{n,m}(R)) \cap L^\infty(Q_{n,m}(R))$ satisfies*

$$\partial_i(a_{ij}u_j) + b_i u_i \ge f \quad \text{in } Q_{n,m}(R). \tag{2.10}$$

Then, for any $\theta \in (0,1)$,

$$\sup_{Q_{n,m}(\theta R)} u^+ \le C\Big\{\Big(\frac{1}{|Q_{n,m}(R)|}\int_{Q_{n,m}(R)} u^2\Big)^{\frac{1}{2}} + R^2\Big(\frac{1}{|Q_{n,m}(R)|}\int_{Q_{n,m}(R)} |f|^q\Big)^{\frac{1}{q}}\Big\}, \tag{2.11}$$

where C is a positive constant depending only on q, θ and C_*.

Proof For simplicity, we assume $R = 1$. Let φ be a smooth cut-off function as mentioned in Lemma 2.2, and set $\overline{u} = (u-k)^+$ for some $k \ge 0$. Multiply the differential inequality (2.10) by $-\varphi^2\overline{u}$ and integrate in $Q_{n,m}(1)$. Proceeding as in the proof of Lemma 2.4, we have

$$\begin{aligned}&\int \varphi^2\Big(\sum_{i=1}^{n} \overline{u}_{x_i}^2 + \sum_{i=n+1}^{n+m} x_i \overline{u}_{x_i}^2\Big)\\ &\le C\int \Big(\varphi^2 + \sum_{i=1}^{n}\varphi_{x_i}^2 + \sum_{i=n+1}^{n+m} x_i\varphi_i^2 + \sum_{i=n+1}^{n+m}\varphi|\varphi_i|\Big)\overline{u}^2 + \int \varphi^2\overline{u}f\end{aligned}$$

and then

$$\begin{aligned}&\int \Big(\sum_{i=1}^{n} (\varphi\overline{u})_{x_i}^2 + \sum_{i=n+1}^{n+m} x_i (\varphi\overline{u})_{x_i}^2\Big)\\ &\le C\int \Big(\varphi^2 + \sum_{i=1}^{n}\varphi_{x_i}^2 + \sum_{i=n+1}^{n+m} x_i\varphi_i^2 + \sum_{i=n+1}^{n+m}\varphi|\varphi_i|\Big)\overline{u}^2 + \int \varphi^2\overline{u}f.\end{aligned}$$

Lemma 2.1(1) implies

$$\begin{aligned}&\Big(\int \varphi^{\frac{2(n+2m)}{n+2m-2}}\overline{u}^{\frac{2(n+2m)}{n+2m-2}}\Big)^{\frac{n+2m-2}{n+2m}}\\ &\le C\int \Big(\varphi^2 + \sum_{i=1}^{n}\varphi_{x_i}^2 + \sum_{i=n+1}^{n+m} x_i\varphi_i^2 + \sum_{i=n+1}^{n+m}\varphi|\varphi_i|\Big)\overline{u}^2 + \int \varphi^2\overline{u}f.\end{aligned}$$

By the Hölder inequality, we have

$$\int \varphi^2\overline{u}f \le \Big(\int (\varphi\overline{u})^{\frac{2(n+2m)}{n+2m-2}}\Big)^{\frac{n+2m-2}{2(n+2m)}}\Big(\int (\varphi f)^q\Big)^{\frac{1}{q}}|\{\varphi\overline{u} \ne 0\}|^{1-\frac{n+2m-2}{2(n+2m)}-\frac{1}{q}}$$

$$\leq \frac{1}{2}\Big(\int (\varphi\overline{u})^{\frac{2(n+2m)}{n+2m-2}}\Big)^{\frac{n+2m-2}{n+2m}} + \frac{1}{2}\|f\|^2_{L^q}|\{\varphi\overline{u}\neq 0\}|^{\frac{n+2m+2}{n+2m}-\frac{2}{q}},$$

and hence

$$\Big(\int (\varphi\overline{u})^{\frac{2(n+2m)}{n+2m-2}}\Big)^{\frac{n+2m-2}{n+2m}}$$
$$\leq C\int \Big(\varphi^2+\sum_{i=1}^{n}\varphi^2_{x_i}+\sum_{i=n+1}^{n+m}x_i\varphi_i^2+\sum_{i=n+1}^{n+m}\varphi|\varphi_i|\Big)\overline{u}^2 + \|f\|^2_{L^q}|\{\varphi\overline{u}\neq 0\}|^{\frac{n+2m+2}{n+2m}-\frac{2}{q}}.$$

By the Hölder inequality again, we have

$$\int (\varphi\overline{u})^2 \leq \Big(\int (\varphi\overline{u})^{\frac{2(n+2m)}{n+2m-2}}\Big)^{\frac{n+2m-2}{n+2m}}|\{\varphi\overline{u}\neq 0\}|^{\frac{2}{n+2m}},$$

and hence

$$\int \varphi^2\overline{u}^2 \leq \|f\|^2_{L^q}|\{\varphi\overline{u}\neq 0\}|^{\frac{n+2m+4}{n+2m}-\frac{2}{q}}$$
$$+ C|\{\varphi\overline{u}\neq 0\}|^{\frac{2}{n+2m}}\int \Big(\varphi^2+\sum_{i=1}^{n}\varphi^2_{x_i}+\sum_{i=n+1}^{n+m}x_i\varphi_i^2+\sum_{i=n+1}^{n+m}\varphi|\varphi_i|\Big)\overline{u}^2.$$

In the following, we take

$$\varepsilon = \min\Big\{\frac{2}{n+2m},\frac{4}{n+2m}-\frac{2}{q}\Big\} > 0.$$

Then

$$\int \varphi^2\overline{u}^2 \leq \|f\|^2_{L^q}|\{\varphi\overline{u}\neq 0\}|^{1+\varepsilon}$$
$$+ C|\{\varphi\overline{u}\neq 0\}|^{\varepsilon}\int \Big(\varphi^2+\sum_{i=1}^{n}\varphi^2_{x_i}+\sum_{i=n+1}^{n+m}x_i\varphi_i^2+\sum_{i=n+1}^{n+m}\varphi|\varphi_i|\Big)\overline{u}^2.$$

Set, for any $r\in(0,1]$ and $k\geq 0$,

$$A(k,r) = \{x\in Q_{n,m}(r):\, u(x)\geq k\}.$$

For any $0<r<R<1$, we take a cut-off function φ such that $\varphi=1$ in $Q_{n,m}(r)$ and $\varphi=0$ in $Q_{n,m}(1)\setminus Q_{n,m}(R)$. Then

$$\varphi^2+\sum_{i=1}^{n}\varphi^2_{x_i}+\sum_{i=n+1}^{n+m}x_i\varphi_i^2+\sum_{i=n+1}^{n+m}\varphi|\varphi_i| \leq \frac{C}{(R-r)^2},$$

and hence

$$\int_{A(k,r)}(u-k)^2 \leq C\Big\{\frac{1}{(R-r)^2}\int_{A(k,R)}(u-k)^2|A(k,R)|^{\varepsilon} + \|f\|^2_{L^q}|A(k,R)|^{1+\varepsilon}\Big\}.$$

For any $h>k\geq 0$, we have

$$\int_{A(h,R)}(u-h)^2 \leq \int_{A(k,R)}(u-k)^2$$

and

$$|A(h,R)| = |G_R \cap \{u-k > h-k\}| \le \frac{1}{(h-k)^2}\int_{A(k,R)} (u-k)^2.$$

Hence,

$$\begin{aligned}\int_{A(h,r)} (u-h)^2 &\le C\Big\{\frac{1}{(R-r)^2}\int_{A(h,R)} (u-h)^2 + \|f\|_{L^q}^2 |A(h,R)|\Big\}|A(h,R)|^{\varepsilon}\\ &\le C\Big\{\frac{1}{(R-r)^2} + \frac{1}{(h-k)^2}\|f\|_{L^q}^2\Big\}\frac{1}{(h-k)^{2\varepsilon}}\Big(\int_{A(k,R)} (u-k)^2\Big)^{1+\varepsilon}.\end{aligned}$$

In summary, we obtain, for any $0 < r < R < 1$ and $0 \le k < h$,

$$\|(u-h)^+\|_{L^2(G_r)} \le C\Big\{\frac{1}{R-r} + \frac{1}{h-k}\|f\|_{L^q(Q_{n,m}(1))}\Big\}\frac{1}{(h-k)^{\varepsilon}}\|(u-k)^+\|_{L^2(G_R)}^{1+\varepsilon}.$$

For any $\theta \in (0,1)$, a standard iteration yields

$$\sup_{Q_{n,m}(\theta)} u^+ \le C\{\|u^+\|_{L^2(Q_{n,m}(1))} + \|f\|_{L^q(Q_{n,m}(1))}\}.$$

This is the desired result.

Next, we shall give a lower bound for positive supersolutions.

Lemma 2.6 *Let* (1.8), (1.10) *be assumed and* $f \in L^q(Q_{n,m}(1))$ *for some* $q > \frac{n+2m}{2}$. *Suppose* $u \in C^2(Q_{n,m}(1)) \cap L^\infty(Q_{n,m}(1))$ *is positive and satisfies*

$$\partial_i(a_{ij}u_j) + b_i u_i \le f \quad \text{in } Q_{n,m}(1).$$

Then, for any $\varepsilon \in (0,1)$, *there exist constants* $\delta > 0$ *and* $C > 1$, *depending only on* q, ε *and* C_* *in* (1.8) *and* (1.10), *such that, if*

$$\Big|\Big\{x \in Q_{n,m}(1) : u(x) \ge \frac{1}{2}\Big\}\Big| \ge \varepsilon |Q_{n,m}(1)|,$$

and

$$\|f\|_{L^q(Q_{n,m}(1))} \le \delta,$$

then

$$\inf_{Q_{n,m}(\frac{1}{2})} u \ge \frac{1}{C}. \tag{2.12}$$

Proof Let φ be a nonnegative smooth cut-off function with support in $Q_{n,m}(1) \bigcup\limits_{i=n+1}^{n+m} \{x_i = 0\}$. Let $G_\gamma = Q_{n,m}(1) \bigcap\limits_{i=n+1}^{n+m} \{x_i > \gamma\}$. Then

$$\int_{G_\gamma} a_{ij}u_i\varphi_j - \int_{G_\gamma} b_i u_i \varphi - \int_{\partial G_\gamma} a_{ij}\varphi u_i \nu_j \ge -\int_{G_\gamma} f\varphi. \tag{2.13}$$

By Lemma 2.3, one knows

$$\int_{\partial G_\gamma} a_{ij}\varphi u_j \nu_i = -\sum_{i=n+1}^{n+m} \int_{\partial G_\gamma \cap \{x_i = \gamma\}} a_{ij}\varphi u_j \to 0, \quad \text{as } \gamma \to 0.$$

If f is not identically zero, we take $\delta = \|f\|_{L^q(B_1)}$. Otherwise, we take an arbitrary $\delta > 0$. Replacing φ by $\varphi/(u+\delta)$ in (2.13), we have

$$-\int_{G_\gamma} a_{ij}\frac{u_i u_j}{(u+\delta)^2}\varphi + \int_{G_\gamma} a_{ij}\frac{u_i}{u+\delta}\varphi_j - \int_{G_\gamma} b_i\frac{u_i}{u+\delta}\varphi \geq -\int_{G_\gamma}\frac{f}{u+\delta}\varphi + o_\gamma(1).$$

Here we use $o_\gamma(1)$ denote $o(1)$ as $\gamma \to 0$. Then setting

$$v = \log\frac{1}{u+\delta},$$

we get

$$-\int_{G_\gamma} a_{ij}v_i v_j\varphi - \int_{G_\gamma} a_{ij}v_i\varphi_j + \int_{G_\gamma} b_i v_i\varphi \geq -\int_{G_\gamma}\frac{f}{u+\delta}\varphi + o_\gamma(1).$$

In particular, v satisfies

$$\int_{G_\gamma} a_{ij}v_i\varphi_j - \int_{G_\gamma} b_i v_i\varphi \leq \int_{G_\gamma}\frac{f}{u+\delta}\varphi + o_\gamma(1).$$

The choice of δ implies $\|f/\delta\|_{L^q(Q_{n,m}(1))} \leq 1$. Then, letting $\gamma \to 0$, for any $\theta \in \left(\frac{1}{2}, 1\right)$, Lemma 2.5 implies

$$\sup_{Q_{n,m}(\frac{1}{2})}(v^+)^2 \leq C\Big\{\int_{Q_{n,m}(\theta)}(v^+)^2 + 1\Big\}, \tag{2.14}$$

where C is a positive constant depending only on q, θ and C_* in (1.8) and (1.10).

Now, replace φ in (2.13) by

$$\Big(\frac{1}{u+\delta} - 1\Big)^+\varphi^2.$$

Then, we have

$$\begin{aligned}\int_{G_\gamma} a_{ij}\partial_i v^+\partial_j v^+\varphi^2 \leq {}& -2\int_{G_\gamma}\varphi(1-u-\delta)^+ a_{ij}\partial_i v^+\varphi_j\\ &+\int_{G_\gamma}\varphi^2(1-u-\delta)^+ b_i\partial_i v^+\\ &+\int_{G_\gamma}\varphi^2\frac{f}{u+\delta}(1-u-\delta)^+ + o_\gamma(1).\end{aligned}$$

We now consider the b_i-term for $i = n+1, \cdots, n+m$ and write

$$\begin{aligned}\int_{G_\gamma}\varphi^2(1-u-\delta)^+ b_i\partial_i v^+ &= \int_{G_\gamma}\varphi^2(1-u-\delta)^+ b_{i1}\partial_i v^+ + \int_{G_\gamma}\varphi^2(1-u-\delta)^+ b_{i2}\partial_i v^+\\ &\leq \int_{G_\gamma}\varphi^2(1-u-\delta)^+ b_{i1}\partial_i v^+ + \varepsilon\int_{G_\gamma}\varphi^2 x_i(\partial_i v^+)^2 + \frac{C}{\varepsilon}\int_{G_\gamma}\varphi^2.\end{aligned}$$

Also

$$\begin{aligned}\int_{G_\gamma}\varphi^2(1-u-\delta)^+ b_{i1}\partial_i v^+ &= \int_{G_\gamma}\varphi^2 b_{i1}\partial_i\Big[\Big(\log\frac{1}{u+\delta}\Big)^+ - (1-u-\delta)^+\Big]\\ &= -\int_{G_\gamma}\partial_i(\varphi^2 b_{i1})\Big[\Big(\log\frac{1}{u+\delta}\Big)^+ - (1-u-\delta)^+\Big]\end{aligned}$$

$$+\int_{\partial G_\gamma}\varphi^2 b_{i1}\Big[\Big(\log\frac{1}{u+\delta}\Big)^+-(1-u-\delta)^+\Big]\nu_i.$$

Note that $\varphi=0$ or $\nu_i=0$ on $\partial G_\gamma\setminus\{x_i=0\}$ and for $u+\delta<1$,

$$\Big(\log\frac{1}{u+\delta}\Big)^+>(1-u-\delta)^+. \tag{2.15}$$

By (1.9) and (1.10), similar to (2.8) and (2.9), one has

$$\int_{\partial G_\gamma}\varphi^2 b_{i1}\Big[\Big(\log\frac{1}{u+\delta}\Big)^+-(1-u-\delta)^+\Big]\nu_i\le O(\gamma).$$

This implies

$$\begin{aligned}&\int_{G_\gamma}\varphi^2(1-u-\delta)^+b_i\partial_i v^+\\ &\le\int_{G_\gamma}|\partial_i(\varphi^2 b_{i1})|v^++\varepsilon\int_{G_\gamma}\varphi^2x_i(\partial_i v^+)^2+\frac{C}{\varepsilon}\int_{G_\gamma}\varphi^2+O(\gamma),\end{aligned}$$

and hence

$$\begin{aligned}\int_{G_\gamma}a_{ij}\partial_i v^+\partial_j v^+\varphi^2\le&-2\int_{G_\gamma}\varphi(1-u-\delta)^+a_{ij}\partial_i v^+\varphi_j+\sum_{i=1}^n\int_{G_\gamma}\varphi^2(1-u-\delta)^+b_i\partial_i v^+\\ &+\int_{G_\gamma}|\partial_i(\varphi^2b_{i1})|v^++\varepsilon\int_{G_\gamma}\varphi^2x_i(\partial_i v^+)^2\\ &+\frac{C}{\varepsilon}\int_{G_\gamma}\varphi^2+O(\gamma)+\int_{G_\gamma}\varphi^2\frac{f}{u+\delta}(1-u-\delta)^+.\end{aligned}$$

By proceeding as in the proof of Lemma 2.4 and $\gamma\to 0$, we have

$$\begin{aligned}&\int\varphi^2\Big(\sum_{i=1}^n(v_{x_i}^+)^2+\sum_{i=n+1}^{n+m}x_i(v_{x_i}^+)^2\Big)\mathrm{d}x\\ &\le C\Big\{\int\Big(\varphi^2+\sum_{i=1}^n\varphi_{x_i}^2+\sum_{i=n+1}^{n+m}x_i\varphi_i^2\Big)+\sum_{i=n+1}^{n+m}\int(\varphi+|\varphi_i|)\varphi v^++\int\varphi^2\frac{f}{\delta}\Big\}.\end{aligned}$$

The choice of δ implies $\|f/\delta\|_{L^q(Q_{n,m}(1))}\le 1$. Hence, for any $\theta_1<\theta_2<1$, we take $\varphi=1$ in $Q_{n,m}(\theta_1)$ and $\varphi=0$ in $Q_{n,m}(1)\setminus Q_{n,m}(\theta_2)$. Then, for any $\tau\in(0,1)$ to be determined, we have

$$\int_{Q_{n,m}(\theta_1)}\Big(\sum_{i=1}^n(v_{x_i}^+)^2+\sum_{i=n+1}^{n+m}x_i(v_{x_i}^+)^2\Big)\le\frac{C_\tau}{(\theta_2-\theta_1)^2}+\tau\int_{Q_{n,m}(\theta_2)}(v^+)^2. \tag{2.16}$$

Note

$$\begin{aligned}&|\{x\in Q_{n,m}(\theta_1):\,v^+=0\}|\\ &\ge|\{x\in Q_{n,m}(1):\,u+\delta\ge1\}|-|Q_{n,m}(1)|+|Q_{n,m}(\theta_1)|\\ &\ge|Q_{n,m}(\theta_1)|-(1-\varepsilon)|Q_{n,m}(1)|=\Big(1-\frac{1-\varepsilon}{\theta_1^{n+2m}}\Big)|Q_{n,m}(\theta_1)|\ge\frac12\varepsilon|Q_{n,m}(\theta_1)|,\end{aligned}$$

by taking θ_1 such that

$$\theta_0 \equiv \max\Big\{\frac{1}{2}, \Big(\frac{1-\varepsilon}{1-\frac{\varepsilon}{2}}\Big)^{\frac{1}{n+2m}}\Big\} < \theta_1 < 1.$$

Then Lemma 2.1(2) implies

$$\int_{Q_{n,m}(\theta_1)} (v^+)^2 \le C \int_{Q_{n,m}(\theta_1)} \Big(\sum_{i=1}^{n} (v^+_{x_i})^2 + \sum_{i=n+1}^{n+m} x_i (v^+_{x_i})^2\Big) \quad \text{for all } \theta_1 \ge \theta_0. \tag{2.17}$$

It must be emphasized that C in (2.17) depends on ε through θ_0, and is independent of θ_1. By combining (2.16) and (2.17), we have

$$\int_{Q_{n,m}(\theta_1)} (v^+)^2 \le \frac{C_\tau}{(\theta_2-\theta_1)^2} + C\tau \int_{Q_{n,m}(\theta_2)} (v^+)^2.$$

Now choose τ such that $C\tau = \frac{1}{2}$. We obtain, for any $\theta_0 < \theta_1 < \theta_2 < 1$,

$$\int_{Q_{n,m}(\theta_1)} (v^+)^2 \le \frac{C_\tau}{(\theta_2-\theta_1)^2} + \frac{1}{2} \int_{Q_{n,m}(\theta_2)} (v^+)^2.$$

A standard iteration yields, for any $\theta_0 < \theta < 1$,

$$\int_{Q_{n,m}(\theta)} (v^+)^2 \le \frac{C}{(1-\theta)^2}. \tag{2.18}$$

By combining (2.14) and (2.18) and fixing $\theta \in (\theta_0, 1)$, we obtain

$$\sup_{Q_{n,m}(\frac{1}{2})} (v^+)^2 \le C,$$

and hence

$$\inf_{Q_{n,m}(\frac{1}{2})} u + \delta \ge \mathrm{e}^{-C}.$$

We note that the constant C above is independent of δ. If $f \equiv 0$, we simply let $\delta \to 0$. Otherwise, taking $\delta = \frac{\mathrm{e}^{-C}}{2}$, we have the desired estimate.

Now, we are ready to prove the Hölder regularity up to the boundary.

Proof of Theorem 1.2 Set, for any $r \le 1$,

$$M(r) = \sup_{Q_{n,m}(r)} u, \quad m(r) = \inf_{Q_{n,m}(r)} u,$$

and

$$\omega(r) = M(r) - m(r).$$

We now claim, for any $r \le 1$,

$$\omega\Big(\frac{r}{2}\Big) \le \sigma\omega(r) + Cr^{1-\frac{n+2m}{2q}} \|f\|_{L^q(Q_{n,m}(r))}, \tag{2.19}$$

where $\sigma \in (0,1)$ and $C > 1$ are constants depending only on q and C_* in (1.8) and (1.10). If (2.19) is true, then by a simple iteration, we have, for any $r \le \frac{1}{2}$,

$$\omega(r) \le Cr^\alpha \{\omega(1) + \|f\|_{L^q(Q_{n,m}(1))}\},$$

where $\alpha \in (0,1)$ and $C > 1$ are constants depending only on q and C_* in (1.8) and (1.10).

We now prove (2.19) for $r = 1$. The general case follows from a simple scaling. Let $\varepsilon = \frac{1}{2}$ and δ be determined as in Lemma 2.6. If

$$\delta\omega(1) \le \|f\|_{L^q(Q_{n,m}(1))},$$

then

$$\omega\Big(\frac{1}{2}\Big) \le \omega(1) \le \frac{1}{\delta}\|f\|_{L^q(Q_{n,m}(1))}. \tag{2.20}$$

Next, we assume

$$\|f\|_{L^q(Q_{n,m}(1))} \le \delta\omega(1).$$

We note that $\frac{u}{\omega(1)}$ satisfies

$$\partial_i\Big(a_{ij}\partial_j\Big(\frac{u}{\omega(1)}\Big)\Big) + b_i\partial_i\Big(\frac{u}{\omega(1)}\Big) = \frac{f}{\omega(1)} \quad \text{in } Q_{n,m}(1).$$

Hence

$$\Big\|\frac{f}{\omega(1)}\Big\|_{L^q(Q_{n,m}(1))} \le \delta$$

by the previous assumption. We consider the following two cases:

$$\Big|\Big\{x \in Q_{n,m}(1) : \frac{u - m(1)}{M(1) - m(1)} \ge \frac{1}{2}\Big\}\Big| \ge \frac{1}{2}|Q_{n,m}(1)| \tag{2.21}$$

and

$$\Big|\Big\{x \in Q_{n,m}(1) : \frac{M(1) - u}{M(1) - m(1)} \ge \frac{1}{2}\Big\}\Big| \ge \frac{1}{2}|Q_{n,m}(1)|. \tag{2.22}$$

If (2.21) holds, we apply Lemma 2.6 to $\frac{u-m(1)}{M(1)-m(1)}$ and get

$$m\Big(\frac{1}{2}\Big) - m(1) \ge \frac{1}{C}(M(1) - m(1)).$$

If (2.22) holds, we apply Lemma 2.6 to $\frac{M(1)-u}{M(1)-m(1)}$ and get

$$M(1) - M\Big(\frac{1}{2}\Big) \ge \frac{1}{C}(M(1) - m(1)).$$

Since $m\big(\frac{1}{2}\big) \ge m(1)$ and $M\big(\frac{1}{2}\big) \le M(1)$, we have in both cases

$$M\Big(\frac{1}{2}\Big) - m\Big(\frac{1}{2}\Big) \le \Big(1 - \frac{1}{C}\Big)(M(1) - m(1)),$$

and hence

$$\omega\Big(\frac{1}{2}\Big) \le \sigma\omega(1) \tag{2.23}$$

for some constant $\sigma \in (0,1)$. We have (2.19) by combining (2.20) and (2.23).

First we consider $m = 1$. Consider two points $\overline{x}, \widetilde{x} \in Q_{n,m}\big(\frac{1}{2}\big)$ and $|\overline{x}-\widetilde{x}|+\overline{x}_{n+1}+\widetilde{x}_{n+1} << 1$.

(1) $r^2 = |\overline{x}-\widetilde{x}|^{\frac{1}{4}} \ge \max(\overline{x}_{n+1}, \widetilde{x}_{n+1})$. Then

$$\overline{x}, \widetilde{x} \in G_{r,(\frac{\overline{x}'+\widetilde{x}'}{2},0)} = \Big\{x \in Q_{n,m}(1) \mid \Big|x' - \frac{\overline{x}'+\widetilde{x}'}{2}\Big| < r, 0 < x_{n+1} < r^2\Big\}.$$

By (2.19), one knows

$$|u(\overline{x})-u(\widetilde{x})| \le Cr^{\alpha}(\omega(1)+\|f\|_{L^q(Q_{n,m}(1))}) \le C(\omega(1)+\|f\|_{L^q(Q_{n,m}(1))})|\widetilde{x}-\overline{x}|^{\frac{\alpha}{8}}.$$

(2) $|\overline{x}-\widetilde{x}|^{\frac{1}{4}} \le \max(\overline{x}_{n+1}, \widetilde{x}_{n+1}) = \overline{x}_{n+1} \le \varepsilon_0$ for some ε_0 small enough to be determined later. Then one knows

$$\begin{aligned}\widetilde{x}_{n+1} &\ge \overline{x}_{n+1} - |\widetilde{x}_{n+1}-\overline{x}_{n+1}| \\ &\ge \overline{x}_{n+1} - |\overline{x}-\widetilde{x}| \\ &\ge (1-\varepsilon_0^3)x_{n+1} \ge \Big(\frac{1}{2}\Big)\overline{x}_{n+1}\end{aligned}$$

for ε_0 small. Let $\lambda = \overline{x}_{n+1}$ and $v(y) = u\big(\lambda^{\frac{1}{2}}y' + \frac{\overline{x}'+\widetilde{x}'}{2}, \lambda y_{n+1}\big)$. Then $v(y)$ solves

$$\partial_i(\widetilde{a}_{ij}(y)\partial_j v) + \widetilde{b}_i(y)\partial_i v = \widetilde{f}(y), \tag{2.24}$$

where

$$\begin{aligned}&\widetilde{a}_{ij}(y) = a_{ij}(x), \quad \widetilde{a}_{i(n+1)}(y) = \frac{1}{\lambda^{\frac{1}{2}}}a_{i(n+1)}(x), \quad \widetilde{a}_{(n+1)(n+1)}(y) = \frac{1}{\lambda}a_{(n+1)(n+1)}(x),\\ &\widetilde{b}_i(y) = \lambda^{\frac{1}{2}}b_i(x), \quad \widetilde{b}_{n+1}(y) = b_{n+1}(x), \quad \widetilde{f}(y) = \lambda f(x), \quad 1 \le i,j \le n.\end{aligned}$$

Then by (1.8)–(1.10), one knows (2.24) is a uniformly elliptic equation with bounded measurable coefficients in $\Sigma = \big\{y \mid |y'| \le 4, \frac{1}{2} \le y_{n+1} \le 8\big\}$. Then by [5, Theorem 8.24], there exists $\beta(C^*, n) \in (0,1)$ such that

$$\|v\|_{C^{\beta}(\Sigma')} \le C(\|v\|_{L^2(\Sigma)} + \|\widetilde{f}\|_{L^q(\Sigma)}) \tag{2.25}$$

for some constant $C = C(C^*, n, q)$ and $\Sigma' = \{y \mid |y'| \le 2, 1 \le y_n \le 4\}$. The assumption $|\overline{x}-\widetilde{x}|^{\frac{1}{4}} \le \overline{x}_{n+1} = \lambda$ implies

$$|\overline{x}-\widetilde{x}|^{\frac{3}{2}} \ge \frac{|\overline{x}'-\widetilde{x}'|^2}{\lambda}, \quad |\overline{x}-\widetilde{x}|^{\frac{3}{2}} \ge \frac{|\overline{x}_{n+1}-\widetilde{x}_{n+1}|^2}{\lambda^2}.$$

Hence,

$$\begin{aligned}\frac{|u(\overline{x})-u(\widetilde{x})|}{|\overline{x}-\widetilde{x}|^{\frac{3}{4}\beta}} &\le C_0\frac{|u(\overline{x})-u(\widetilde{x})|}{(\lambda^{-1}|\overline{x}'-\widetilde{x}'|^2 + \lambda^{-2}|\widetilde{x}_{n+1}-\overline{x}_{n+1}|^2)^{\frac{\beta}{2}}} \\ &= C_0\frac{|v(\overline{y})-v(\widetilde{y})|}{|\overline{y}-\widetilde{y}|^{\beta}} \le C(\|v\|_{L^2(\Sigma)} + \|\widetilde{f}\|_{L^q(\Sigma)}) \\ &\le C_1(\|u\|_{L^\infty(Q_{n,m}(1))} + \lambda^{1-\frac{n+2}{2q}}\|f\|_{L^q(Q_{n,m}(1))}).\end{aligned} \tag{2.26}$$

The above arguments complete the proof of Theorem 1.2 for $m = 1$.

Now we prove the present theorem by induction. Similarly, consider two points $\overline{x}, \widetilde{x} \in Q_{n,m}\big(\frac{1}{2}\big)$. Also we use $x = (x', x'')$, where $x' = (x_1, \cdots, x_n)$ and $x'' = (x_{n+1}, \cdots, x_{n+m})$.

(1) $r^2 = |\overline{x} - \widetilde{x}|^{\frac{1}{4}} \geq \max(|\overline{x}''|, |\widetilde{x}''|)$. Then

$$\overline{x}, \widetilde{x} \in G_{r,(\frac{\overline{x}'+\widetilde{x}'}{2},0)} = \Big\{x \in Q_{n,m}(1) \mid \Big|x' - \frac{\overline{x}'+\widetilde{x}'}{2}\Big| < r, 0 < x_{n+i} < r^2,\ i = 1, \cdots, m\Big\}.$$

By (2.19), one knows

$$|u(\overline{x}) - u(\widetilde{x})| \leq Cr^{\alpha}(\omega(1) + \|f\|_{L^q(Q_{n,m}(1))}) \leq C(\omega(1) + \|f\|_{L^q(Q_{n,m}(1))})|\widetilde{x} - \overline{x}|^{\frac{\alpha}{8}}.$$

(2) $|\overline{x} - \widetilde{x}|^{\frac{1}{4}} \leq \max(|\overline{x}''|, |\widetilde{x}''|) \leq \varepsilon_0$ for some positive ε_0 small enough. A similar argument as in $m = 1$ implies

$$\frac{1}{2}|\widetilde{x}''| \leq |\overline{x}''| \leq 2|\widetilde{x}''|.$$

Let $\lambda = |\overline{x}''|$ and $v(y) = u(\lambda^{\frac{1}{2}}y' + \frac{\overline{x}'+\widetilde{x}'}{2}, \lambda y'')$. Then $v(y)$ solves

$$\partial_i(\widetilde{a}_{ij}(y)\partial_j v) + \widetilde{b}_i(y)\partial_i v = \widetilde{f}(y), \tag{2.27}$$

where

$$\begin{aligned}
&\widetilde{a}_{ij}(y) = a_{ij}(x), \quad \widetilde{a}_{iJ}(y) = \frac{1}{\lambda^{\frac{1}{2}}}a_{iJ}(x), \quad 1 \leq i, j \leq n,\ n+1 \leq J \leq n+m,\\
&\widetilde{a}_{IJ}(y) = \frac{1}{\lambda}a_{IJ}(x), \quad n+1 \leq I, J \leq n+m, \quad \widetilde{b}_i(y) = \lambda^{\frac{1}{2}}b_i(x), \quad 1 \leq i \leq n,\\
&\widetilde{b}_I(y) = b_I(x), \quad n+1 \leq I \leq n+m, \quad \widetilde{f}(y) = \lambda f(x).
\end{aligned}$$

Without loss of generality, we may also assume $\overline{x}_{n+1} \geq \frac{1}{\sqrt{m}}|\overline{x}''|$. Then by (1.8)–(1.10), one knows that (2.27) is an elliptic equation with bounded measurable coefficients such that n and m are replaced by $n+1$ and $m-1$ respectively in (1.8)–(1.10) in

$$\widehat{\Sigma} = \Big\{y \mid |y'| < 4,\ \frac{1}{8m} < y_{n+1} < 8,\ 0 < y_{n+i} < 8,\ i = 2, \cdots, m\Big\}.$$

Then by our induction assumption, there exists $\beta(C^*, n, m) \in (0, 1)$ such that

$$\|v\|_{C^{\beta}(\widehat{\Sigma}')} \leq C(\|v\|_{L^2(\Sigma)} + \|\widetilde{f}\|_{L^q(\Sigma)}) \tag{2.28}$$

for some constant $C = C(C^*, n, m, q)$ and

$$\widehat{\Sigma} = \Big\{y \mid |y'| < 2,\ \frac{1}{4m} < y_{n+1} < 4,\ 0 < y_{n+i} < 4,\ i = 2, \cdots, m\Big\}.$$

By the assumption in this case, one knows

$$\begin{aligned}
\frac{|u(\overline{x}) - u(\widetilde{x})|}{|\overline{x} - \widetilde{x}|^{\frac{3}{4}\beta}} &\leq C_0\frac{|u(\overline{x}) - u(\widetilde{x})|}{(\lambda^{-1}|\overline{x}' - \widetilde{x}'|^2 + \lambda^{-2}|\widetilde{x}'' - \overline{x}''|^2)^{\frac{\beta}{2}}}\\
&= C_0\frac{|v(\overline{y}) - v(\widetilde{y})|}{|\overline{y} - \widetilde{y}|^{\beta}} \leq C(\|v\|_{L^2(\widehat{\Sigma})} + \|\widetilde{f}\|_{L^q(\widehat{\Sigma})})\\
&\leq C_1(\|u\|_{L^\infty(Q_{n,m}(1))} + \lambda^{1-\frac{n+2m}{2q}}\|f\|_{L^q(Q_{n,m}(1))}).
\end{aligned} \tag{2.29}$$

This ends the proof of the present theorem.

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