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A Criterion of Nonparabolicity by the Ricci Curvature*

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Abstract A complete manifold is said to be nonparabolic if it does admit a positive Green's function. To find a sharp geometric criterion for the parabolicity/nonparbolicity is an attractive question inside the function theory on Riemannian manifolds. This paper devotes to proving a criterion for nonparabolicity of a complete manifold weakened by the Ricci curvature. For this purpose, we shall apply the new Laplacian comparison theorem established by the first author to show the existence of a non-constant bounded subharmonic function.

Keywords Nonparabolicity, Subharmonic function, Ricci curvature
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1 Introduction

From the initial work of Yau [30] in 1975, where he proved a Liouville theorem for positive harmonic functions on a complete manifold with nonnegative Ricci curvature, the study of function theory on a complete Riemannian manifold has been one of the central problems in geometric analysis. During the past decades, many significant results have been obtained (refer to [4, 13–14, 16, 31–32]). For example, Greene and Wu gave a systematic study of function theory on a complete manifold with a pole (see, for example, [9]). Anderson [2] and Sullivan [24] independently proved that on a complete simply-connected manifolds with curvature pinched by two negative constants, there are a wealth of nontrivial bounded harmonic functions. Li and Tam in [15–17] related the theory of harmonic functions on a complete non-compact manifold with nonnegative sectional curvature outside a compact to its infinity geometric structure. This indicates that function properties also reflect some geometric structures of a complete manifold. It is not a surprise that Green's functions play an effective role in this aspect.

The existence of a Green's function on a general complete manifold was first proved by Malgrange [20] in 1955 and was re-established by Li and Tam [17] in 1987 in constructive and applicable way. However, on a complete manifold, the existence and nonexistence of a positive Green's function impact different properties of harmonic, subharmonic or superharmonic functions (refer to, for example, [14, 16]). This analytic character divides the class of complete manifolds into two categories. A complete manifold is said to be parabolic if it does not admit a

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positive Green's function. Otherwise it is said to be nonparabolic (see [13, 15, 17]) or hyperbolic (see [8, 10]). It is very interesting that in case of minimal surfaces in $\mathbb{R}^3$, while the catenoid is parabolic, and the double periodic Scherk's surface or the triply periodic Schwarz surface are nonparabolic (see, for example, [8]). Since there are several concepts of hyperbolicity in geometry in different sense, in this paper, we shall use the terminology of nonparabolic to denote a complete manifold with a positive Green's function, like it is used in [14, 17].

To find a criterion for parabolicity/nonparabolicity becomes an attractive topic in this aspect. It is well-known that a complete manifold M is nonparabolic if one of the following equivalent three conditions, which is called "Kelvin-Nevanlinna-Royden criteria", is fulfilled (this is established in [19] by Lysons and Sullivan, also refer to [8]): (a) M has a nonconstant superharmonic function; (b) M admits positive capacity, that is to say, there exists a non-empty precompact set $D \subset M$ such that $\mathrm{Cap}(D, M) > 0$, where $\mathrm{Cap}(D, M) := \inf_u \int_M \|\nabla u\|^2 \mathrm{d}v_M$ with the infimum being taken over all real-valued functions $u \in C_0^\infty(M)$ and $u \geq 1$ in D; (c) the Brownian motion on M is transient, that is, there is a non-empty precompact set $D \subset M$ such that the Brownian motion starting from a point in D leaves D with a positive probability. However, excepting the analytic criteria mentioned above, a geometric characterization for the parabolicity or nonparbolicity is a central question inside the function theory on Riemannian manifolds, as it is mentioned in surveys [10, 13] or in [8]. The geometric characterization can be given as a sufficient or necessary condition involving the volume growth, or bounds on its sectional or Ricci curvature and some other geometric quantities (see, for example, [9–10, 16, 32]).

In 1935, Ahlfors proved in [1] that a rotationally symmetric complete surface M^2 is parabolic if and only if the integral $\int_0^\infty 1/\mathrm{vol}(S_\rho)$ is divergent, where S_ρ stands for the geodesic circle of radius ρ in M^2. Based on this, in 1977 Milnor obtained in [23] a decision criterion for the parabolicity/nonparabolicity of a complete surface by the Gauss curvature. Precisely, let M^2 be a 2-dimensional complete simply-connected surface and $\rho(x)$ denotes the distance function on M from a fixed point o to $x \in M^2$, Milnor proved that if the (Gauss) curvature $K(x) \geq -\frac{1}{\rho^2(x)\log\rho(x)}$ for large $\rho(x)$, then M^2 is parabolic. If $K(x) \leq -\frac{1+\varepsilon}{\rho^2(x)\log\rho(x)}$ outside a geodesic ball $B_o(R)$ for some $R > 0$, where $\varepsilon > 0$ is a constant, then M^2 is nonparabolic. Ahlfors' result has been generalized by several authors ([4, 10, 16, 18, 25–26]) to complete manifolds with dimension bigger than 2, in which a so called volume comparison condition is proposed (see [18, 25]). Meanwhile, Milnor's result is also generalized to the higher dimensional case. It was proved in [9] by Greene and Wu in the geometric way that if (M, o) is a Riemannian manifold with a pole $o \in M$ satisfying the radial sectional curvature $(x) \leq -\frac{1+\varepsilon}{\rho^2(x)\log\rho(x)}$ outside a geodesic ball $B_o(R)$ for some $R > 0$. Then M admits a non-constant bounded subharmonic function and hence is nonparabolic by the Kelvin-Nevanlinna-Royden criteria. A similar criterion for parabolicity by the radial sectional curvature is also obtained in [9]. Ichihara proved in [12] in the probabilistic way that a connected and complete n-dimensional Riemannian manifold is parabolic if its Ricci curvature are bounded from below by the corresponding curvature of a model (for the definition of a model see §2 below) which satisfies the Ahlfors's integral divergence condition, and it is nonparabolic if its sectional curvature are bounded from above by the corresponding curvature of a model which satisfies the Ahlfors's integral convergence condition. We would point out that Palmer and his collaborators did a series of works in characterizing the nonparabolicity (or, in other words, the hyperbolicity) of submanifolds, especially the minimal submanifolds, in a Cartan-Hadamard manifold (see, for example, [8, 21–22]).

Based on the above exhibitions, it gives rise to a natural question: Can the assumption on the (radial) sectional curvature be weakened to an assumption on the Ricci curvature instead? This is not just a technical improvement, a motivation for relaxing the curvature assumption is from the study of the structure of complete Kähler manifolds with the nonpositive holomorphic bisectional curvature. A well understanding of the Ricci curvature for the nonparabolicity is obviously very useful in this study.

The aim of this paper is to generalize the criterion for nonparabolicity of a complete manifold from the (radial) sectional curvature to the Ricci curvature under the circumstance of nonpositive sectional curvature. The key idea is based on the new Laplacian comparison theorem established by the first author in 1994. We shall apply a new version of the new Laplacian comparison theorem to show the existence of a non-constant bounded subharmonic function and obtain a criterion for nonparabolicity of a complete manifold in the level of the Ricci curvature. The main result is as follows.

Theorem 1.1 *Let M be an n-dimensional complete simply-connected manifold with nonpositive curvature and o be a fixed point in M. The distance function between $x \in M \setminus \{o\}$ and o on M is denoted by $\rho = \rho(x)$. If the Ricci curvature of M satisfies*

$$\mathrm{Ric}(x) \le -\frac{1+\varepsilon}{\rho^2 \log \rho} \quad \text{for some } \varepsilon > 0 \text{ outside a geodesic ball } B_o(R) \text{ with } R > 0. \tag{1.1}$$

Then M is nonparabolic.

The paper is organized as follows. In §2, we give some preliminaries and display the new version of the new Laplacian comparison. In §3 we shall prove the existence of a non-constant bounded subharmonic function by the assumption of the Ricci curvature. Some comments and questions are also presented in this section.

2 Preliminaries

It is well-known that various comparison theorems, such as Rauch, Hessian and Laplacian comparison theorems, are widely applicable and play important roles in Riemannian geometry. In 1994, the first author established in [5] the following new Laplacian comparison theorem by using the Jacobian equation, which provides a new tool in exploiting geometric properties of Riemannian manifolds with nonpositive curvature.

Theorem 2.1 (Ding [5]) *Let M and $\widetilde{M}$ be two n-dimensional Riemannian manifolds with nonpositive curvature and*

$$\gamma : [0, b] \to M \quad \text{and} \quad \widetilde{\gamma} : [0, b] \to \widetilde{M}$$

be normal geodesics respectively in M *and* $\widetilde{M}$. *We denote by* ρ (resp. $\widetilde{\rho}$), Δ (resp. $\widetilde{\Delta}$) *and* Ric (*resp.* $\widetilde{\mathrm{Ric}}$) *the distance function from* $x = \gamma(0)$ *in* M (resp. *from* $\widetilde{x} = \widetilde{\gamma}(0)$ *in* $\widetilde{M}$), *the Laplacian operator and the Ricci curvature of* M (resp. $\widetilde{M}$), *respectively. Assume that, for any* $t \in [0, b]$, $\mathrm{Ric}(\dot{\gamma}, \dot{\gamma})(t) \leq \frac{1}{n-1}\widetilde{\mathrm{Ric}}(\dot{\widetilde{\gamma}}, \dot{\widetilde{\gamma}})(t)$. *Then*

$$\Delta\rho(\gamma(t)) \geq \frac{1}{n-1}\widetilde{\Delta}\widetilde{\rho}(\widetilde{\gamma}(t)), \quad \forall t \in (0, b]. \tag{2.1}$$

Corollary 2.1 *Let* M *be an* n*-dimensional Riemannian manifold with nonpositive curvature and* $\widetilde{M}$ *be the space form of nonpositive constant curvature* $-k$ $(k \geq 0)$. *If* $\mathrm{Ric}(M) \leq -k$, *then*

$$\Delta\rho \geq \frac{1}{n-1}\widetilde{\Delta}\widetilde{\rho}. \tag{2.2}$$

A noteworthy point of this new Laplacian comparison theorem is that the inequality in the usual Laplacian comparison theorem are reversed. If we restrict ourselves to convex domains, the non-positivity of the sectional curvature in Theorem 2.1 can be removed. A geometrical proof of this new Laplacian comparison theorem was given by Xin in [29] in 1995. The version in terms of the Bakry-Émery Ricci curvature is displayed in [28]. The new Laplacian comparison theorem has applications to the eigenvalue estimates (see [5]), the rigidity of harmonic maps from bounded symmetric domains (see [29]) and the existence of bounded harmonic functions (see [6]). Some other interesting applications in geometry and probability theory are given in [3, 7, 11, 27] and so on. Theorem 2.1 was actually discovered and proved in 1992 when the first author was a PhD student guided by his respectable advisor, Prof. Gu Chaohao.

Before displaying a new version of Theorem 2.1 which is useful in the sequel, let's state a lemma proved in [9, Lemma 5.15], which is also important in this paper.

Lemma 2.1 *Give* $\varepsilon > 0$, *let* K *be any* C^∞ *function on* $[0, \infty)$ *such that* $K \leq 0$ *and* $K(\rho) \leq -\frac{1+\varepsilon}{\rho^2 \log \rho}$ *for large* ρ. *Then there exists a unique complete Riemannian* (*or Hermitian*) *metric* h *on the disk* $D = \{z \in C \mid |z| < 1\}$ *which is rotationally symmetric and whose Gauss curvature along any geodesic ray from the origin in* D *is* K.

From Lemma 2.1, we see that there exists a unique rotationally symmetric complete Riemannian (or Hermitian) metric h on D such that the Gauss curvature K of h satisfies $K \leq 0$ and $K(\rho) \leq -\frac{1+\varepsilon}{\rho^2 \log \rho}$ for large $\rho = \mathrm{d}_h(O, z)$. This complete surface is denoted by (D, K) throughout the paper. We would point out that the 2-dimensional surface (D, K) can be generalized to higher dimensions. The higher dimensional versions are called models in literature (see, for example, [9]). Let (M, o) be Riemannian manifold with a pole o (i.e., $\exp : T_oM \to M$ is a diffeomorphism). (M, o) is called a model if the metric of M is given by

$$\mathrm{d}s^2 = \mathrm{d}r^2 + f^2(r)\mathrm{d}s^2_{\mathbb{S}^{n-1}(1)},$$

where r denotes the usual radial function on $T_oM \setminus \{0\}$ and the map $(r, \theta) \to r\theta$ is the parametrization $\mathbb{R}^+ \times \{\text{unit sphere}\} \to T_oM \setminus \{0\}$ called the geodesic polar coordinates around o, and f depends only on r with $f(0) = 0$ and $f'(0) = 1$. It is well known that the radial sectional curvature of (M,o) is formulated by $K(r(x)) = -\frac{f''(r(x))}{f(r(x))}$. Especially, when $n=2$ the radial curvature becomes the Gauss curvature and, in this case, (M,o) is denoted by (D,K) in this paper, as indicated above.

The following is a new version of the new Laplacian comparison Theorem 2.1 we needed.

Theorem 2.2 *Let M be an n-dimensional Riemannian manifold with nonpositive curvature, $\widetilde{M}$ be a Riemannian surface and*

$$\gamma : [0,b] \to M \quad \text{and} \quad \widetilde{\gamma} : [0,b] \to \widetilde{M}$$

be normal geodesics respectively in M and $\widetilde{M}$. All the notations have the same meanings in Theorem 2.1. Assume that, (1) there are no conjugate points on γ or $\widetilde{\gamma}$; (2) for any $t \in [0,b]$, $\mathrm{Ric}(\dot{\gamma},\dot{\gamma})(t) \le \widetilde{\mathrm{Ric}}(\dot{\widetilde{\gamma}},\dot{\widetilde{\gamma}})(t)$. Then

$$\Delta\rho(\gamma(t)) \ge \widetilde{\Delta}\widetilde{\rho}(\widetilde{\gamma}(t)), \quad \forall t \in (0,b]. \tag{2.3}$$

Corollary 2.2 *Let M be an n-dimensional Riemannian manifold with nonpositive curvature and o be a fixed point in M. The distance function between $x \in M \setminus \{o\}$ and o on M is denoted by $\rho = \rho(x)$. If the Ricci curvature of M satisfies*

$$\mathrm{Ric}(x) \le -\frac{1+\varepsilon}{\rho^2 \log \rho} \quad \text{outside a geodesic ball } B_o(R) \text{ for some } R > 0.$$

Then

$$\Delta\rho(x) \ge \widetilde{\Delta}\widetilde{\rho}(z)$$

for every $x \in M$ and $z \in \widetilde{M}$ such that $\rho(x) = \widetilde{\rho}(z)$, where $\widetilde{M} = (D,K)$ with K being chosen as $K(\rho(x)) \le 0$ and $\mathrm{Ric}_M(x) \le K(\rho(x)) \le -\frac{1+\varepsilon}{\rho^2 \log \rho}$ for $\rho(x) \ge R$ and $\widetilde{\rho}(z)$ is the distance function in D between z and the origin O.

Corollary 2.3 *Assumptions are the same as in Corollary 2.2 and $f : [0,\infty) \to \mathbf{R}$ is a C^∞ nondecreasing (i.e., $f' \ge 0$) function. Then*

$$\Delta f(\rho(x)) \ge \widetilde{\Delta} f(\widetilde{\rho}(z)) \tag{2.4}$$

for every $x \in M$ and $z \in D$ such that $\rho(x) = \widetilde{\rho}(z)$.

A benefit of this new version of the new Laplacian comparison Theorem 2.1 is the deletion of the factor $\frac{1}{n-1}$ in the comparison inequality. We end this section with presenting a simply proof of Theorem 2.2.

As done in [5], for a parallel orthogonal unit bases $\{e_1(t), \cdots, e_n(t)\}$ along the normal geodesic γ in M (where $e_n(t) = \dot{\gamma}(t)$), the $n-1$ independent normal Jacobian fields $\{U_i(t)\}_{1 \le i \le n-1}$ along γ are presented by

$$\begin{pmatrix} U_1(t) \\ \vdots \\ U_{n-1}(t) \end{pmatrix} = A(t) \begin{pmatrix} e_1(t) \\ \vdots \\ e_{n-1}(t) \end{pmatrix},$$

where $A = A(t) : [0, b] \to \mathrm{gl}(n-1, \mathbf{R})$. Since $\{U_i\}_{1 \le i \le n-1}$satifty the Jacobian equation, we equivalently have that

$$\begin{cases} A_{tt} + AK = 0, \\ A(0) = 0, \quad A_t(0) = I, \end{cases} \tag{2.5}$$

where $K = (K_{ij})_{1 \le i,j \le n-1}$, $K_{ij} = \langle R(\dot{\gamma}, e_i)\dot{\gamma}, e_j \rangle$ in which R stands for the curvature operator. Similarly, we have that a normal Jacobian field $\widetilde{U}(t)$ along $\widetilde{\gamma}$ in $\widetilde{M}$ is presented by $\widetilde{U}(t) = \widetilde{A}(t)\widetilde{e}(t)$, where $\widetilde{e}(t)$ is a parallel unit vector field along $\widetilde{\gamma}$ orthogonal to $\dot{\widetilde{\gamma}}(t)$ and $\widetilde{A}$ satisfies

$$\begin{cases} \widetilde{A}_{tt} + \widetilde{A}\widetilde{K} = 0, \\ \widetilde{A}(0) = 0, \quad \widetilde{A}_t(0) = 1, \end{cases} \tag{2.6}$$

where $\widetilde{K} = \widetilde{K}(t)$ is the Gauss curvature of $\widetilde{M}$ at the point $\widetilde{\gamma}(t)$. One notes that $\widetilde{A}(t)$ is a scalar function of t.

Lemma 2.2 *Assume that A and $\widetilde{A}$ solve respectively* (2.5) *and* (2.6), *and* (1) $A^{-1}(t)$ *and* $\widetilde{A}^{-1}(t)$ *exist on* $(0, b]$; (2) $A^{-1}A_t$ *is semi-definite for every* $t \in (0, b]$; (3) $\mathrm{tr}\, K \le \widetilde{K}$ *for every* $t \in (0, b]$. *Then*

$$\mathrm{tr}(A^{-1}A_t) \ge \widetilde{A}^{-1}\widetilde{A}_t, \quad t \in (0, b]. \tag{2.7}$$

Proof First of all, we choose a small $\varepsilon_0 \ge 0$ such that on $(0, \varepsilon_0)$,

$$\mathrm{tr}(A^{-1}A_t) \ge \mathrm{tr}(\widetilde{A}^{-1}\widetilde{A}_t) \tag{2.8}$$

and

$$(\mathrm{tr}(A^{-1}A_t) - (\widetilde{A}^{-1}\widetilde{A}_t))|_{t=\varepsilon_0} \ge 0. \tag{2.9}$$

In fact, from (2.5) we have that $A(0) = 0$, $A_t(0) = I$, $A_{tt}(0) = 0$, $A_{ttt}(0) = -K(0)$ and hence when $t \to 0^+$, $A(t) \sim tI - t^3\frac{K(0)}{6}$, $A_t(t) \sim I - t^2\frac{K(0)}{2}$, $A^{-1}(t) \sim \frac{I}{t} + t\frac{K(0)}{6}$. Similarly, when $t \to 0^+$, $\widetilde{A}(t) \sim tI - t^3\frac{\widetilde{K}(0)}{6}$, $\widetilde{A}_t(t) \sim I - t^2\frac{\widetilde{K}(0)}{2}$, $\widetilde{A}^{-1}(t) \sim \frac{I}{t} + t\frac{\widetilde{K}(0)}{6}$. Therefore, when $t \to 0^+$,

$$\mathrm{tr}(A^{-1}A_t) - \mathrm{tr}(\widetilde{A}^{-1}\widetilde{A}_t) \sim \begin{cases} \dfrac{n-2}{t}, & \text{when } n \ge 3, \\ 0, & \text{when } n = 2. \end{cases}$$

This confirms our above claim for some small $\varepsilon_0 > 0$ in the case of $n \ge 3$ and $\varepsilon_0 = 0$ of $n = 2$.

What remains to prove is that (2.7) holds for every $t \in (0, a]$. For this purpose, we see that

$$[\mathrm{tr}(A^{-1}A_t) - \widetilde{A}^{-1}\widetilde{A}_t]_t = \widetilde{K} - \mathrm{tr}K + \widetilde{A}^{-1}\widetilde{A}_t\widetilde{A}^{-1}\widetilde{A}_t - \mathrm{tr}(A^{-1}A_tA^{-1}A_t). \tag{2.10}$$

Since $A^{-1}A_t$ is semi-definite, we have that

$$\mathrm{tr}(A^{-1}A_tA^{-1}A_t) \le (\mathrm{tr}(A^{-1}A_t))^2.$$

and it is easy to see that

$$(\widetilde{A}^{-1}\widetilde{A}_t\widetilde{A}^{-1}\widetilde{A}_t) = (\widetilde{A}^{-1}\widetilde{A}_t)^2.$$

Hence from (2.10), we obtain

$$[\mathrm{tr}(A^{-1}A_t) - \widetilde{A}^{-1}\widetilde{A}_t]_t \geq -(\mathrm{tr}(A^{-1}A_t))^2 + (\widetilde{A}^{-1}\widetilde{A}_t)^2. \tag{2.11}$$

Now. setting $p(t) = \mathrm{tr}(A^{-1}A_t)$, $q(t) = \widetilde{A}^{-1}\widetilde{A}_t$ and $h(t) = p(t) - q(t)$, we have the follows from (2.11),

$$h_t(t) + h(t)[p(t) + q(t)] \geq 0. \tag{2.12}$$

Based on this differential inequality, as done in [5], we may arrival at that on $[\varepsilon_0, t]$,

$$h(t)\exp\Big(\int_{\varepsilon_0}^t (p(\tau) + q(\tau))\mathrm{d}\tau\Big) - h(\varepsilon_0) \geq 0,$$

or equivalently,

$$h(t) \geq h(\varepsilon_0)\exp\Big(-\int_{\varepsilon_0}^t (p(\tau) + q(\tau))\mathrm{d}\tau\Big), \quad t \in (\varepsilon_0, b]. \tag{2.13}$$

Since (2.9), we know that $h(\varepsilon_0) \geq 0$. Hence, (2.13) and (2.8)–(2.9) imply that our desired inequality (2.7) holds on $(0, b]$. This completes the proof of Lemma 2.2.

Proof of Theorem 2.2 First of all, one may verify that all the conditions in Lemma 2.2 are satisfied under the conditions of Theorem 2.2 (for example, the non-positivity of the curvature guarantees that $A^{-1}A_t$ is semi-definite). Next, by noting that (refer to [5])

$$\Delta\rho(\gamma(t)) = \mathrm{tr}(A^{-1}A_t)$$

and

$$\widetilde{\Delta}\widetilde{\rho}(\widetilde{\gamma}(t)) = \mathrm{tr}(\widetilde{A}^{-1}\widetilde{A}_t)$$

and applying Lemma 2.2, we may easily obtain the inequality (2.3) in Theorem 2.2. This proves Theorem 2.2.

It is easy to see that Corollary 2.2 is a direct consequence of Theorem 2.2 and Corollary 2.3 is a consequence of Corollary 2.2.

3 The Existence of Bounded Subharmonic Functions

In this section, we shall show our main Theorem 1.1. In fact, Theorem 1.1 is a consequence of the following proposition and the Kelvin-Nevanlinna-Royden criteria.

Proposition 3.1 *Let M be an n-dimensional complete non-compact Riemannian manifold with nonpositive curvature and o be a fixed point in M. The distance function between $x \in M \setminus \{o\}$ and o on M is denoted by $\rho = \rho(x)$. If the Ricci curvature of M satisfies*

$$\mathrm{Ric}(x) \leq -\frac{1+\varepsilon}{\rho^2\log\rho} \quad \textit{outside a geodesic ball } B_o(R) \textit{ for some } R > 0. \tag{3.1}$$

Then there exists a C^∞ bounded non-constant subharmonic function φ on M.

Proof The proof of Proposition 3.1 is constructive. The main idea is, under the assumptions, to construct a bounded function φ on M and then to show such a function is in fact subharmonic by applying the comparison Corollary 2.3.

First of all, because of (3.1), we may choose a nonnegative smooth function $k(\rho)$ on $[0,\infty)$ such that

$$\text{Ricci curvature}(x) \leq -k(\rho(x)) \quad \text{and} \quad -k(\rho) \leq -\frac{1+\varepsilon}{\rho^2 \log \rho} \ \text{ for large } \rho.$$

By Lemma 2.1, the unit disk $D = \{|z| < 1\}$ possesses a unique complete Riemannian metric $h = \eta(r)\mathrm{d}z\mathrm{d}\bar{z}$, where $r = |z|$ and $\eta(r) > 0$ is a suitable function of r, such that its (Gauss) curvature K satisfies $K = k(\widetilde{\rho})$, where $\widetilde{\rho}$ denotes the distance function of h based at the origin $O \in D$. Thus if $\sigma : [0,\infty) \to \mathbf{R}$ is any nondecreasing function, Corollary 2.3 implies that

$$\Delta\sigma(\rho(x)) \geq \widetilde{\Delta}\sigma(\widetilde{\rho}(z)) \tag{3.2}$$

for every $x \in M$ and $z \in D$ such that $\rho(x) = \widetilde{\rho}(z)$.

Next, we choose σ as done in [9] section 6. For the completeness of the proof, we outline the construction of σ as follows. Since $\widetilde{\rho}$ is a rotational symmetric function on D, it is regarded as a C^∞ function: $[0,1) \to [0,\infty)$. Extending $\widetilde{\rho}$ to be a function on $(-1,1)$ in the way that $\widetilde{\rho}(-t) = -\widetilde{\rho}(t)$. As a function on $(-1,1)$, $\widetilde{\rho}$ is still C^∞ since it is the inverse function of the normal geodesic $\gamma : (-1,1) \to D$ that sends 0 to $O \in D$. The desired function $\sigma : \mathbf{R} \to [0,1)$ is now defined by

$$\sigma(\widetilde{\rho}(t)) = t^2 \quad \text{for all } t \in (-1,1),$$

that is to say, σ is the square of the inverse function of $\widetilde{\rho}$ and is a C^∞ even function on $\mathbf{R}$. Now one observes that $\sigma(\widetilde{\rho}(z)) = |z|^2 = r^2$, where $r = |z|$. Meanwhile, one also sees that, with $\rho : M \to [0,\infty)$ as indicated above, $\sigma(\rho) : M \to [0,1)$ is C^∞.

Finally, it is obvious that $\sigma : [0,\infty) \to [0,1)$ is a nondecreasing function (since so is the function $\widetilde{\rho}(t) : [0,1) \to [0,\infty)$). By applying Corollary 2.3 to M and (D,K), we have

$$\Delta\sigma(\rho(x)) \geq \widetilde{\Delta}\sigma(\widetilde{\rho}(z)) = \widetilde{\Delta}(r^2)(z), \tag{3.3}$$

where $z \in D$ and $x \in M$ with $\rho(x) = \widetilde{\rho}(z)$. Because of the fact that

$$\widetilde{\Delta}(r^2) = \frac{4}{\eta(z)} \frac{\partial^2}{\partial z \partial \bar{z}} |z|^2 = \frac{4}{\eta(z)} > 0,$$

we obtain from (3.3) that the bounded function $\varphi = \sigma(\rho)$ is strictly subharmonic. This proves Proposition 3.1.

Combining the Kelvin-Nevanlinna-Royden criteria with Proposition 3.1, we obtain Theorem 1.1 directly. There are lots of complete manifolds that satisfy the conditions in Theorem 1.1, but not the conditions of Greene and Wu's. For example, the product manifold $D \times D$ (here $D = \{z \in C \mid |z| < 1\}$ with the Gauss curvature $K(x) \leq 0$ and $K(x) = -\frac{1+\varepsilon}{r^2(x) \log r(x)}$ outside a geodesic ball for some $\varepsilon > 0$, where $r(x)$ stands for the geodesic distance from O to x in D)

does not satisfies Greene and Wu's condition, but it admits the sectional curvature ≤ 0 and the Ricci curvature $\leq -\frac{1+\varepsilon}{\rho^2(x)\log\rho(x)}$ outside a geodesic ball. Hence $D \times D$ is nonparabolic by our criterion. This indicates that our new criterion is more widely useful than that of Greene and Wu's criterion. Finally, we believe that the new version of the Laplacian comparison Theorem 2.2 deserves to being applied in the study of the structure of Kähler manifolds with nonpositive bisectional curvature.

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