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On the Radius of Analyticity of Solutions to 3D Navier-Stokes System with Initial Data in L^p*

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Abstract Given initial data $u_0 \in L^p(\mathbb{R}^3)$ for some p in $\big[3, \frac{18}{5}\big[$, the auhtors first prove that 3D incompressible Navier-Stokes system has a unique solution $u = u_L + v$ with $u_L \stackrel{\text{def}}{=} \mathrm{e}^{t\Delta} u_0$ and $v \in \widetilde{L}^\infty([0,T]; \dot{H}^{\frac{5}{2}-\frac{6}{p}}) \cap \widetilde{L}^1(]0,T[; \dot{H}^{\frac{9}{2}-\frac{6}{p}})$ for some positive time T. Then they derive an explicit lower bound for the radius of space analyticity of v, which in particular extends the corresponding results in [Chemin, J.-Y., Gallagher, I. and Zhang, P., On the radius of analyticity of solutions to semi-linear parabolic system, *Math. Res. Lett.*, **27**, 2020, 1631–1643, Herbst, I. and Skibsted, E., Analyticity estimates for the Navier-Stokes equations, *Adv. in Math.*, **228**, 2011, 1990–2033] with initial data in $\dot{H}^s(\mathbb{R}^3)$ for $s \in [\frac{1}{2}, \frac{3}{2}[$.

Keywords Incompressible Navier-Stokes equations, Radius of analyticity, Littlewood-Paley theory
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1 Introduction

In this paper, we investigate the radius of space analyticity of solutions to the following three-dimensional incompressible Navier-Stokes system with initial data in $L^p(\mathbb{R}^3)$ for some $p \in [3, \frac{18}{5}[$:

$$\begin{cases} \partial_t u + u \cdot \nabla u - \Delta u + \nabla p = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0, \end{cases} \tag{1.1}$$

where $u = (u^1, u^2, u^3)$ denotes the velocity of the fluid, and p the scalar pressure function, which guarantees the divergence free condition of the velocity field. Such a system can be used to describe the evolution of a viscous and incompressible fluid.

The study of analyticity of the solutions to Navier-Stokes system originated from Foias and Temam [6], where they studied the analyticity of periodic solutions of (1.1) in space and time

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with initial data $u_0 \in H^1(\mathbb{T}^3)$. The result was later extended by the authors in [3, 12, 13] to that: There exists a positive time T so that

$$\int_{\mathbb{R}^3} |\xi| \Big(\sup_{t \le T} \mathrm{e}^{\sqrt{t}|\xi|} |\widehat{u}(t,\xi)| \Big)^2 \,\mathrm{d}\xi + \int_0^T \int_{\mathbb{R}^3} |\xi|^3 (\mathrm{e}^{\sqrt{t}|\xi|} |\widehat{u}(t,\xi)|)^2 \,\mathrm{d}\xi \mathrm{d}t < \infty,$$

which in particular implies the strong solution of (1.1) with initial data $u_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ is analytic for positive time t, furthermore, the radius of analyticity $R(u(t))$ is greater than $\sqrt{t}$. With initial data $u_0 \in \dot{H}^s(\mathbb{R}^3)$ for some $s \in]\frac{1}{2}, \frac{3}{2}[$, Herst and Skibsted [10] improved the instantaneous radius of analyticity to that

$$\liminf_{t \to 0} \frac{R(u(t))}{\sqrt{t|\ln t|}} \ge \sqrt{2s-1}. \tag{1.2}$$

In the critical case when $s = \frac{1}{2}$, an explicit lower bound of the radius of analyticity was obtained in [5], which in particular implies that $\lim_{t \to 0} \frac{R(u(t))}{\sqrt{t}} = \infty$.

On the other hand, with initial $u_0 \in L^p(\mathbb{R}^3)$ for some $p \in]3, \infty[$, the local well-posedness of system (1.1) was proved by the authors in [7, 14]. The critical case when $p = 3$ was proved by Kato in [11]. Grujič and Kukavica [9] investigated the space analyticity for such solutions of Navier-Stokes system and obtained similar result as that in [6] for the periodic case.

The goal of this paper is to study the instantaneous smoothing effect, especially the instantaneous radius of space analyticity, of solutions to (1.1) with initial data in $L^p(\mathbb{R}^3)$ for $p \in [3, \frac{18}{5}[$, which was constructed in [7, 11, 14]. We remark that the main idea in [5, 10] depends heavily on the Hilbert structure of the solution space. Therefore the idea in [5, 10] can not be directly applied to the case with initial data in $L^p(\mathbb{R}^3)$.

Before proceeding, we denote $u_L \stackrel{\text{def}}{=} \mathrm{e}^{t\Delta} u_0$. Then we write the integral formulation of (1.1) as follows

$$u = u_L + B(u,u) \quad \text{with } B(u,v) \stackrel{\text{def}}{=} \int_0^t \mathrm{e}^{(t-s)\Delta} \mathbb{P} \nabla \cdot (u \otimes v) \,\mathrm{d}s, \tag{1.3}$$

where $\mathbb{P} \stackrel{\text{def}}{=} \mathrm{Id} + \nabla(-\Delta)^{-1} \mathrm{div}$ designates Leray projection operator. In what follows, we denote $v \stackrel{\text{def}}{=} u - u_L$. Then by virtue of (1.3), we obtain

$$v = B(v,v) + B(v,u_L) + B(u_L,v) + B(u_L,u_L) \tag{1.4}$$

for the bilinear operator $B(\cdot,\cdot)$ defined by (1.3).

In [15], the second author proved that for $u_0 \in L^3(\mathbb{R}^3)$, v defined by (1.4) belongs to $\widetilde{L}^\infty([0,T]; \dot{B}^{\frac{1}{2}}_{\frac{3}{2},\infty}(\mathbb{R}^3)) \cap \widetilde{L}^1(]0,T[;\, \dot{B}^{\frac{3}{2}}_{\frac{3}{2},\infty}(\mathbb{R}^3))$ for any $T < T^*$, which is the lifespan of the solution u constructed in [11]. One may check Definition A.2 for the definition of the function spaces. We are going to prove that $v \in C([0,T]; \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)) \cap L^2(]0,T[; \dot{H}^{\frac{3}{2}}(\mathbb{R}^3))$. Indeed the more precise definition of function space that we are going to work with is defined as follows.

Definition 1.1 *Let $T > 0$ and $p \in [3, \infty[$, $E_p(T)$ is the space of homogeneous tempered distribution u which satisfies*

$$\|u\|_{E_p(T)} \stackrel{\text{def}}{=} \|u\|_{\widetilde{L}^\infty_T(\dot{H}^{\frac{5}{2}-\frac{6}{p}})} + \|u\|_{\widetilde{L}^1_T(\dot{H}^{\frac{9}{2}-\frac{6}{p}})} < \infty,$$

where for $q \in [1, \infty]$ and $s \in \mathbb{R}$,

$$\|u\|_{\widetilde{L}^q_T(\dot{H}^s)} \stackrel{\text{def}}{=} \Big(\sum_{j\in\mathbb{Z}} 2^{2js} \|\Delta_j u\|^2_{L^q_T(L^2)}\Big)^{\frac{1}{2}}. \tag{1.5}$$

One may check the basic facts on Littlewood-Paley theory in the Appendix.

Our first result states as follows, which in particular improve the corresponding result in [15].

Theorem 1.1 *Let $u_0 \in L^p(\mathbb{R}^3)$ with $3 \leq p < \frac{18}{5}$ be a solenoidal vector field. Then for $p = 3$, there exists a positive time T, whereas for $p \in]3, \frac{18}{5}[$, there exists a sufficiently small constant c_0, such that for any T satisfying*

$$T^{\frac{1}{2\gamma}} \|u_0\|_{L^p} \leq c_0 \quad \text{with } \gamma \stackrel{\text{def}}{=} \frac{p}{p-3}, \tag{1.6}$$

system (1.1) has a unique solution $u = u_L + v$ with $v \in E_p(T)$.

Next inspired by [10], we are going to investigate the radius of space analyticity of $v \stackrel{\text{def}}{=} u - u_L$ with $u_0 \in L^p(\mathbb{R}^3)$ for some $p \in \big]3, \frac{18}{5}\big[$. The main result states as follows.

Theorem 1.2 *Let $u_0 \in L^p(\mathbb{R}^3)$ with $3 < p < \frac{18}{5}$ and $A \stackrel{\text{def}}{=} \sqrt{-\Delta}$. Let $u = u_L + v$ be the unique solution of (1.1) determined by Theorem 1.1. Then for any $\varepsilon \leq \varepsilon_1$, there exist constants $t_1 = t_1(\varepsilon, \|u_0\|_{L^p})$ and $c_1 = c_1(\varepsilon, \|u_0\|_{L^p})$ such that*

$$\|e^{\sqrt{\frac{2(1-\varepsilon)}{\gamma}}\sqrt{t\ln|t|}A} v(t)\|_{\dot{H}^{\frac{5}{2}-\frac{6}{p}}} \leq c_1 t^{-\frac{1}{\gamma}} \quad \text{for any } t \in [0, t_0]. \tag{1.7}$$

In particular, we have

$$\liminf_{t\to 0} \frac{\mathrm{rad}(v(t))}{\sqrt{|t\ln t|}} \geq \sqrt{\frac{2}{\gamma}}. \tag{1.8}$$

Remark 1.1 It follows from Sobolev embedding theorem that $\dot{H}^s(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$ for $s = 3\big(\frac{1}{2} - \frac{1}{p}\big)$. We observe that

$$\sqrt{2s-1} = \sqrt{\frac{2}{\gamma}} \qquad \text{with } \gamma \stackrel{\text{def}}{=} \frac{p}{p-3},$$

so that (1.8) is consistent with (1.2) in the case when $p \in]3, \frac{18}{5}[$. Theorem 1.2 in particular improves the regularity of the initial data in [10].

Finally motivated by [5], we consider the case when $p = 3$, which is the critical regularity of the Navier-Stokes system.

Theorem 1.3 *Let* $q\in]2,\infty[$, $u_0\in L^3(\mathbb{R}^3)$ *and* $u_{\eta,L}\stackrel{\text{def}}{=}e^{\eta t\Delta}u_0$. *Let* $u=u_L+v$ *be the unique solution of* (1.1) *determined by Theorem* 1.1. *Then for any* $0<\eta<\varepsilon\le\varepsilon_2$, *there exist constants* $t_2=t_2(\varepsilon,\eta,\|u_0\|_{L^3})$ *and* $c_2=c_2(\varepsilon,\eta,\|u_0\|_{L^3})$ *such that*

$$\|e^{2(-(1-\varepsilon)\ln\|u_{\eta,L}\|_{\widetilde{L}^{2q}_t(\dot{B}^{\frac{1}{q}}_{3,3})})^{\frac12}t^{\frac12}A}v(t)\|_{\dot{H}^{\frac12}}\le c_2\quad\textit{for any } t\in[0,t_2],\tag{1.9}$$

which in particular implies

$$\liminf_{t\to0}\frac{R(v(t))}{t^{\frac12}(-\ln\|u_{\eta,L}\|_{\widetilde{L}^{2q}_t(\dot{B}^{\frac1q}_{3,3})})^{\frac12}}\ge2.\tag{1.10}$$

Remark 1.2 It follows from [1, Theorem 2.40] that $L^3(\mathbb{R}^3)\hookrightarrow\dot{B}^0_{3,3}(\mathbb{R}^3)$, so that

$$\lim_{t\to0}\|u_{\eta,L}\|_{\widetilde{L}^{2q}_t(\dot{B}^{\frac1q}_{3,3})}=0,\tag{1.11}$$

which together with (1.10) ensures that

$$\lim_{t\to0}\frac{R(v(t))}{t^{\frac12}}=\infty.$$

In the rest of this paper, we shall always use the convention that: For $a\lesssim b$, we mean that there is a uniform constant C, which may be different on different lines, such that $a\le Cb$.

Let us complete this section by the sketch of this paper.

In Section 2, we shall present the proof of Theorem 1.1. Section 3 is devoted to the proof of Theorem 1.2. In Section 4, we shall present the proof of Theorem 1.3. Finally in the Appendix, we shall collect some basic facts on Littlewood-Paley theory.

2 The Sobolev Regularity of v

This section is devoted to the proof of Theorem 1.1. The main idea of the proof is inspired by [15]. Indeed due to $p\in[2,\infty[$, we first deduce from [1, Theorem 2.40] that $L^p(\mathbb{R}^3)\hookrightarrow\dot{B}^0_{p,p}(\mathbb{R}^3)$, so that for $u_L\stackrel{\text{def}}{=}e^{t\Delta}u_0$, it follows from Lemma A.2 that

$$\begin{aligned}\|\Delta_ju_L\|_{L^\infty_T(L^p)}+2^{2j}\|\Delta_ju_L\|_{L^1_T(L^p)}&\lesssim\|\Delta_ju_0\|_{L^p}\\&\lesssim c_{j,p}\|u_0\|_{L^p}.\end{aligned}\tag{2.1}$$

Here and all in that follows, we always denote $(c_{j,r})_{j\in\mathbb{Z}}$ to be a generic element of $\ell^r(\mathbb{Z})$ so that $\sum\limits_{j\in\mathbb{Z}}c^r_{j,r}=1$. For simplicity, we shall always denote $(c_{j,2})_{j\in\mathbb{Z}}$ by $(c_j)_{j\in\mathbb{Z}}$ in the rest of this paper.

The proof of Theorem 1.1 will be essentially based on the following three lemmas.

Lemma 2.1 *Let* $u_0\in L^p(\mathbb{R}^3)$ *with* $p\in[2,4[$ *and* $B(\cdot,\cdot)$ *be the bilinear operator determined by* (1.3). *Then there holds*

$$\|B(u_L,u_L)\|_{E_p(T)}\lesssim\|u_0\|^2_{L^p},\tag{2.2}$$

where the norm $\|\cdot\|_{E_p(T)}$ *is defined by Definition* 1.1.

Proof We first get, by applying Bony's decomposition (A.3) to $u_L \otimes u_L$, that

$$u_L \otimes u_L = 2T_{u_L} u_L + R(u_L, u_L).$$

Considering the support properties to the Fourier transforms of the terms above, we have

$$\begin{aligned} \Delta_j(T_{u_L} u_L) &= \sum_{|j-j'|\le 4} \Delta_j(S_{j'-1} u_L \otimes \Delta_{j'} u_L) \\ \Delta_j(R(u_L, u_L)) &= \sum_{j'\ge j-3} \Delta_j(\Delta_{j'} u_L \otimes \widetilde{\Delta}_{j'} u_L). \end{aligned} \tag{2.3}$$

Due to $p \in [2, 4[$, by applying Lemma A.1 and (2.1), we find

$$\begin{aligned} \|\Delta_j(R(u_L, u_L))\|_{L^1_T(L^2)} &\lesssim 2^{(\frac{6}{p}-\frac{3}{2})j} \sum_{j'\ge j-3} \|\Delta_{j'} u_L\|_{L^\infty_T(L^p)} \|\widetilde{\Delta}_{j'} u_L\|_{L^1_T(L^p)} \\ &\lesssim 2^{(\frac{6}{p}-\frac{3}{2})j} \|u_0\|^2_{L^p} \sum_{j'\ge j-3} c_{j',\frac{p}{2}} 2^{-2j'} \\ &\lesssim c_{j,\frac{p}{2}} 2^{(\frac{6}{p}-\frac{7}{2})j} \|u_0\|^2_{L^p}. \end{aligned}$$

While observing that due to $p \in [2, 4[$, one has

$$\|S_{j'-1} u_L\|_{L^\infty_T(L^{\frac{2p}{p-2}})} \lesssim \sum_{\ell\le j-2} 2^{(\frac{6}{p}-\frac{3}{2})\ell} \|\Delta_\ell u_L\|_{L^\infty_T(L^p)} \lesssim c_{j,p} 2^{(\frac{6}{p}-\frac{3}{2})j} \|u_0\|_{L^p},$$

from which and (2.1), we infer

$$\begin{aligned} \|\Delta_j(T_{u_L} u_L)\|_{L^1_T(L^2)} &\lesssim \sum_{|j-j'|\le 4} \|S_{j'-1} u_L\|_{L^\infty_T(L^{\frac{2p}{p-2}})} \|\Delta_{j'} u_L\|_{L^1_T(L^p)} \\ &\lesssim c_{j,\frac{p}{2}} 2^{(\frac{6}{p}-\frac{7}{2})j} \|u_0\|^2_{L^p}. \end{aligned}$$

As a result, it comes out

$$\begin{aligned} \|\Delta_j(u_L \otimes u_L)\|_{L^1_T(L^2)} &\lesssim c_{j,\frac{p}{2}} 2^{(\frac{6}{p}-\frac{7}{2})j} \|u_0\|^2_{L^p} \\ &\lesssim c_j 2^{(\frac{6}{p}-\frac{7}{2})j} \|u_0\|^2_{L^p}, \end{aligned} \tag{2.4}$$

where in the last step, we used again the fact that $p \le 4$.

On the other hand, it follows from (1.3) and Lemma A.2 that

$$\|\Delta_j(B(u_L, u_L))(t)\|_{L^2} \lesssim 2^j \int_0^t \mathrm{e}^{-c(t-s)2^{2j}} \|\Delta_j(u_L \otimes u_L)(s)\|_{L^2}\,\mathrm{d}s,$$

which together with (2.4) ensures that

$$\begin{aligned} \|\Delta_j(B(u_L, u_L))\|_{L^\infty_T(L^2)} + 2^{2j} \|\Delta_j(B(u_L, u_L))\|_{L^1_T(L^2)} &\lesssim 2^j \|\Delta_j(u_L \otimes u_L)\|_{L^1_T(L^2)} \\ &\lesssim c_j 2^{(\frac{6}{p}-\frac{5}{2})j} \|u_0\|^2_{L^p}, \end{aligned}$$

which together with Definition 1.1 leads to (2.2), and we complete the proof of Lemma 2.1.

Remark 2.1 It is easy to observe from the derivation of (2.4) that

$$\|\Delta_j(u_L\otimes u_L)\|_{L^1_T(L^2)}\lesssim c_j2^{-\frac{3}{2}j}\|u_0\|_{L^3}\|u_L\|_{\widetilde{L}^1_T(\dot{B}^2_{3,3})},$$

from which, we deduce that

$$\|B(u_L,u_L)\|_{E_3(T)}\lesssim \|u_0\|_{L^3}\|u_L\|_{\widetilde{L}^1_T(\dot{B}^2_{3,3})}. \tag{2.5}$$

Lemma 2.2 *Let $v,w\in E_p(T)$ with $p\in[3,6[$. Let $\gamma\stackrel{\text{def}}{=}\frac{p}{p-3}$. Then there holds*

$$\|B(v,w)\|_{E_p(T)}\lesssim T^{\frac{1}{\gamma}}\|v\|_{E_p(T)}\|w\|_{E_p(T)}. \tag{2.6}$$

Proof By applying Bony's decomposition (A.3) to vw, we write

$$vw=T_vw+T_wv+R(v,w). \tag{2.7}$$

In view of (2.3), we get, by applying Hölder inequality, that

$$\begin{aligned}\|\Delta_j(T_vw)\|_{L^1_T(L^2)}&\lesssim\sum_{|j'-j|\le 4}\|S_{j'-1}v\|_{L^\infty_T(L^\infty)}\|\Delta_{j'}w\|_{L^1_T(L^2)}\\&\lesssim\sum_{|j'-j|\le 4}c_{j'}2^{(\frac{6}{p}-\frac{9}{2})j}\|S_{j'-1}v\|_{L^\infty_T(L^\infty)}\|w\|_{E_p(T)}.\end{aligned}$$

However due to $p<6$, it follows from Lemma A.1 that

$$\|S_{j'-1}v\|_{L^\infty_T(L^\infty)}\lesssim\sum_{\ell\le j-2}2^{\frac{3}{2}\ell}\|\Delta_\ell v\|_{L^\infty_T(L^2)}\lesssim 2^{(\frac{6}{p}-1)j}\|v\|_{E_p(T)}.$$

As a consequence, we obtain

$$\|\Delta_j(T_vw)\|_{L^1_T(L^2)}\lesssim c_j2^{(\frac{12}{p}-\frac{11}{2})j}\|v\|_{E_p(T)}\|w\|_{E_p(T)}.$$

Similarly, we obtain

$$\|\Delta_j(T_wv)\|_{L^1_T(L^2)}\lesssim c_j2^{(\frac{12}{p}-\frac{11}{2})j}\|v\|_{E_p(T)}\|w\|_{E_p(T)}.$$

Whereas we get, by applying Lemma A.1, that

$$\begin{aligned}\|\Delta_j(R(v,w))\|_{L^1_T(L^2)}&\lesssim 2^{\frac{3}{2}j}\sum_{j'\ge j-3}\|\Delta_{j'}v\|_{L^\infty_T(L^2)}\|\widetilde{\Delta}_{j'}w\|_{L^1_T(L^2)}\\&\lesssim 2^{\frac{3}{2}j}\sum_{j'\ge j-3}c_{j'}2^{(\frac{12}{p}-7)j'}\|v\|_{E_p(T)}\|w\|_{E_p(T)}\\&\lesssim c_j2^{(\frac{12}{p}-\frac{11}{2})j}\|v\|_{E_p(T)}\|w\|_{E_p(T)}.\end{aligned}$$

Therefore we obtain

$$\|\Delta_j(vw)\|_{L^1_T(L^2)}\lesssim c_j2^{(\frac{12}{p}-\frac{11}{2})j}\|v\|_{E_p(T)}\|w\|_{E_p(T)}. \tag{2.8}$$

Thanks to (2.8), we deduce from (1.3) that

$$\begin{aligned}\|\Delta_j(B(v,w))\|_{L^1_T(L^2)} &\lesssim 2^j \Big\| \int_0^t \mathrm{e}^{-c(t-s)2^{2j}} \|\Delta_j(v\otimes w)(s)\|_{L^2}\,\mathrm{d}s\Big\|_{L^1_T}\\ &\lesssim 2^j T^{\frac{1}{\gamma}} \|\mathrm{e}^{-ct2^{2j}}\|_{L^{\frac{p}{3}}} \|\Delta_j(v\otimes w)\|_{L^1_T(L^2)}\\ &\lesssim c_j T^{\frac{1}{\gamma}} 2^{-(\frac{9}{2}-\frac{6}{p})j} \|v\|_{E_p(T)}\|w\|_{E_p(T)}. \end{aligned} \tag{2.9}$$

On the other hand, due to $3 \le p < 6$, it follows from the law of product in Besov spaces (see [1] for instance) that

$$\|vw\|_{\widetilde{L}^\infty_T(\dot{H}^{\frac{7}{2}-\frac{12}{p}})} \lesssim \|v\|_{\widetilde{L}^\infty_T(\dot{H}^{\frac{5}{2}-\frac{6}{p}})} \|w\|_{\widetilde{L}^\infty_T(\dot{H}^{\frac{5}{2}-\frac{6}{p}})},$$

which implies

$$\|\Delta_j(vw)\|_{L^\infty_T(L^2)} \lesssim c_j 2^{(\frac{12}{p}-\frac{7}{2})j} \|v\|_{E_p(T)}\|w\|_{E_p(T)},$$

so that we deduce from (1.3) that

$$\begin{aligned}\|\Delta_j(B(v,w))\|_{L^\infty_T(L^2)} &\lesssim 2^j \Big\| \int_0^t \mathrm{e}^{-c(t-s)2^{2j}} \|\Delta_j(v\otimes w)(s)\|_{L^2}\,\mathrm{d}s\Big\|_{L^\infty_T}\\ &\lesssim 2^j T^{\frac{1}{\gamma}} \|\mathrm{e}^{-ct2^{2j}}\|_{L^{\frac{p}{3}}} \|\Delta_j(v\otimes w)\|_{L^\infty_T(L^2)}\\ &\lesssim c_j T^{\frac{1}{\gamma}} 2^{-(\frac{5}{2}-\frac{6}{p})j} \|v\|_{E_p(T)}\|w\|_{E_p(T)}. \end{aligned}$$

This together with (2.9) ensures (2.6), and we conclude the proof Lemma 2.2.

Lemma 2.3 *Let $u_0 \in L^p(\mathbb{R}^3)$ and $v \in E_p(T)$ with $p \in \big[3, \frac{18}{5}\big[$, then one has*

$$\|B(u_L, v)\|_{E_p(T)} \lesssim T^{\frac{1}{2\gamma}} \|u_0\|_{L^p}\|v\|_{E_p(T)}. \tag{2.10}$$

Proof In view of (A.3), we write

$$vu_L = T_{u_L}v + T_v u_L + R(v, u_L).$$

Yet it follows from Lemma A.1 that

$$\begin{aligned}\|S_{j-1}u_L\|_{L^{2p}_T(L^\infty)} &\lesssim \sum_{\ell\le j-2} 2^{\frac{3}{p}\ell} \|\Delta_\ell u_L\|_{L^{2p}_T(L^p)} \lesssim 2^{\frac{2}{p}j}\|u_0\|_{L^p},\\ \|\Delta_j v\|_{L_T^{\frac{2p}{p+2}}(L^2)} &\le \|\Delta_j v\|^{\frac{1}{2}-\frac{1}{p}}_{L^\infty_T(L^2)} \|\Delta_j v\|^{\frac{1}{2}+\frac{1}{p}}_{L^1_T(L^2)} \lesssim c_j 2^{-(\frac{7}{2}-\frac{4}{p})} \|v\|_{E_p(T)},\end{aligned}$$

so that we deduce

$$\begin{aligned}\|\Delta_j(T_{u_L}v)\|_{L_T^{\frac{2p}{p+3}}(L^2)} &\lesssim \sum_{|j'-j|\le 4} \|S_{j'-1}u_L\|_{L^{2p}_T(L^\infty)} \|\Delta_{j'}v\|_{L_T^{\frac{2p}{p+2}}(L^2)}\\ &\lesssim c_j 2^{-(\frac{7}{2}-\frac{6}{p})j}\|u_0\|_{L^p}\|v\|_{E_p(T)}.\end{aligned}$$

Along the same line, we infer

$$\begin{aligned}\|\Delta_j(R(u_L,v))\|_{L_T^{\frac{2p}{p+3}}(L^2)} &\lesssim 2^{\frac{3}{p}j}\sum_{j'\geq j-3}\|\Delta_{j'}u_L\|_{L_T^{2p}(L^p)}\|\widetilde{\Delta}_{j'}v\|_{L_T^{\frac{2p}{p+2}}(L^2)}\\ &\lesssim 2^{\frac{3}{p}j}\sum_{j'\geq j-3}c_{j'}2^{-(\frac{7}{2}-\frac{3}{p})j'}\|u_0\|_{L^p}\|v\|_{E_p(T)}\\ &\lesssim c_j 2^{-(\frac{7}{2}-\frac{6}{p})j}\|u_0\|_{L^p}\|v\|_{E_p(T)}.\end{aligned}$$

Finally, due to $p<\frac{18}{5}$, we have

$$\begin{aligned}\|S_{j-1}v\|_{L_T^\infty(L^{\frac{2p}{p-2}})} &\lesssim \sum_{\ell\leq j-2}2^{\frac{3}{p}\ell}\|\Delta_\ell v\|_{L_T^\infty(L^2)}\\ &\lesssim \sum_{\ell\leq j-2}c_\ell 2^{\ell(\frac{9}{p}-\frac{5}{2})}\|v\|_{\widetilde{L}_T^\infty(\dot{H}^{\frac{5}{2}-\frac{6}{p}})} \lesssim c_j 2^{(\frac{9}{p}-\frac{5}{2})j}\|v\|_{\widetilde{L}_T^\infty(\dot{H}^{\frac{5}{2}-\frac{6}{p}})},\end{aligned}$$

so that by applying Hölder inequality, we find

$$\begin{aligned}\|\Delta_j(T_v u_L)\|_{L_T^{\frac{2p}{p+3}}(L^2)} &\lesssim \sum_{|j-j'|\leq 4}\|S_{j'-1}v\|_{L_T^\infty(L^{\frac{2p}{p-2}})}\|\Delta_{j'}u_L\|_{L_T^{\frac{2p}{p+3}}(L^p)}\\ &\lesssim \sum_{|j-j'|\leq 4}c_{j'}2^{-(\frac{7}{2}-\frac{6}{p})j'}\|v\|_{E_p(T)}\|u_0\|_{L^p}\\ &\lesssim c_j 2^{-(\frac{7}{2}-\frac{6}{p})j}\|u_0\|_{L^p}\|v\|_{E_p(T)},\end{aligned}$$

where we used the fact that

$$\begin{aligned}\|\Delta_j u_L\|_{L_T^{\frac{2p}{p+3}}(L^p)} &\lesssim \|\Delta_j u_L\|_{L_T^\infty(L^p)}^{\frac{p-3}{2p}}\|\Delta_j u_L\|_{L_T^1(L^p)}^{\frac{p+3}{2p}}\\ &\lesssim 2^{-j(1+\frac{3}{p})}\|u_0\|_{L^p}.\end{aligned}$$

As a consequence, we obtain

$$\|\Delta_j(v\otimes u_L)\|_{L_T^{\frac{2p}{p+3}}(L^2)} \lesssim c_j 2^{-(\frac{7}{2}-\frac{6}{p})j}\|u_0\|_{L^p}\|v\|_{E_p(T)}. \tag{2.11}$$

Then we get, by a similar derivation of (2.9), that

$$\begin{aligned}&\|\Delta_j(B(u_L,v))\|_{L_T^\infty(L^2)} + 2^{2j}\|\Delta_j(B(u_L,v))\|_{L_T^1(L^2)}\\ &\lesssim 2^j\|\Delta_j(u_L\otimes v)\|_{L_T^1(L^2)}\\ &\lesssim 2^j T^{\frac{1}{2\gamma}}\|\Delta_j(u_L\otimes v)\|_{L_T^{\frac{2p}{p+3}}(L^2)}\\ &\lesssim c_j T^{\frac{1}{2\gamma}}2^{-(\frac{5}{2}-\frac{6}{p})j}\|u_0\|_{L^p}\|v\|_{E_p(T)},\end{aligned}$$

from which and Definition 1.1, we conclude the proof of (2.10).

Remark 2.2 It follows from the proof of (2.11) that

$$\|\Delta_j(T_{u_L}v+R(u_L,v))\|_{L_T^1(L^2)} \lesssim c_j 2^{-\frac{3}{2}j}\|u_L\|_{\widetilde{L}_T^6(\dot{B}_{3,3}^{\frac{1}{3}})}\|v\|_{E_p(T)},$$

$$\|\Delta_j(T_v u_L)\|_{L^1_T(L^2)} \lesssim c_j 2^{-\frac{3}{2}j}\|u_L\|_{\widetilde{L}^1_T(\dot{B}^2_{3,3})}\|v\|_{E_p(T)},$$

from which, we infer

$$\|B(u_L,v)\|_{E_p(T)} \lesssim (\|u_L\|_{\widetilde{L}^6_T(\dot{B}^{\frac{1}{3}}_{3,3})} + \|u_L\|_{\widetilde{L}^1_T(\dot{B}^2_{3,3})})\|v\|_{E_p(T)}. \tag{2.12}$$

With the above lemmas, to prove Theorem 1.1, we also need the following lemma from [4].

Lemma 2.4 (see [4, Lemma 2.1]) *Let X be a Banach space. Let L be a continuous linear map from X to X and B be a bilinear map from $X \times X$ to X. We define*

$$\|L\|_{\mathcal{L}(X)} \stackrel{\text{def}}{=} \sup_{\|x\|=1}\|Lx\| \quad \text{and} \quad \|B\|_{\mathcal{B}(X)} \stackrel{\text{def}}{=} \sup_{\|x\|=\|y\|=1}\|B(x,y)\|.$$

If $\|L\|_{\mathcal{L}(X)} < 1$, then for any x_0 in X such that

$$\|x_0\|_X < \frac{(1-\|L\|_{\mathcal{L}(X)})^2}{4\|B\|_{\mathcal{B}(X)}}, \tag{2.13}$$

the equation

$$x = x_0 + Lx + B(x,x)$$

has a unique solution in the ball of center 0 *and radius* $\frac{1-\|L\|_{\mathcal{L}(X)}}{2\|B\|_{\mathcal{B}(X)}}$.

Now we are in a position to complete the proof of Theorem 1.1.

Proof of Theorem 1.1 Let us take $X \stackrel{\text{def}}{=} E_p(T)$, which is defined by Definition 1.1, we denote

$$Lv \stackrel{\text{def}}{=} B(v,u_L) + B(u_L,v) \quad \text{and} \quad x_0 = B(u_L,u_L). \tag{2.14}$$

We first consider the case when $p \in \left]3, \frac{18}{5}\right[$. Then it follows from Lemmas 2.1–2.3 that

$$\|x_0\|_X \le C\|u_0\|^2_{L^p}, \quad \|B\|_{\mathcal{B}(X)} \le CT^{\frac{1}{\gamma}} \quad \text{and} \quad \|L\|_{\mathcal{L}(X)} \le CT^{\frac{1}{2\gamma}}\|u_0\|_{L^p}$$

for γ determined by (1.6). So that under the assumption that $T^{\frac{1}{2\gamma}}\|u_0\|_{L^p} \le c_0$ for some c_0 sufficiently small, there holds $CT^{\frac{1}{2\gamma}}\|u_0\|_{L^p} \le \frac{1}{4}$, which ensures (2.13). Hence we deduce from Lemma 2.4 that (1.4) has a unique solution v in $E_p(T)$. Then by virtue of (1.3), $u = u_L + v$ is the unique solution of (1.1) on $[0,T]$.

While for the case when $p = 3$, we deduce from (2.5), (2.6) and (2.12) that

$$\begin{aligned}&\|x_0\|_X \le C\|u_0\|_{L^3}\|u_L\|_{\widetilde{L}^1_T(\dot{B}^2_{3,3})}, \quad \|B\|_{\mathcal{B}(X)} \le C \quad \text{and}\\ &\|L\|_{\mathcal{L}(X)} \le C(\|u_L\|_{\widetilde{L}^6_T(\dot{B}^{\frac{1}{3}}_{3,3})} + \|u_L\|_{\widetilde{L}^1_T(\dot{B}^2_{3,3})}).\end{aligned}$$

Yet it follows from (1.11) that there exists some positive time T so that

$$C(\|u_L\|_{\widetilde{L}^6_T(\dot{B}^{\frac{1}{3}}_{3,3})} + \|u_L\|_{\widetilde{L}^1_T(\dot{B}^2_{3,3})}) \le \frac{1}{2} \quad \text{and} \quad C^2\|u_0\|_{L^3}\|u_L\|_{\widetilde{L}^1_T(\dot{B}^2_{3,3})} \le \frac{1}{16},$$

which ensures (2.13). Hence we deduce from Lemma 2.4 that (1.4) has a unique solution v in $E_3(T)$. Then by virtue of (1.3), $u = u_L + v$ is the unique solution of (1.1) on $[0,T]$. We finish the proof of Theorem 1.1.

3 The Radius of Analyticity of v for $p \in \left]3, \frac{18}{5}\right[$

In this section, we shall investigate the radius of analyticity of $v \stackrel{\text{def}}{=} u - u_L$ with initial data $u_0 \in L^p(\mathbb{R}^3)$ for some $p \in \left]3, \frac{18}{5}\right[$, namely, we are going to present the proof of Theorem 1.2. The main idea of the proof is inspired by [10]. In order to do so, we first introduce the functional space we are going to work with.

Definition 3.1 *Let $A \stackrel{\text{def}}{=} \sqrt{-\Delta}$, let $\lambda, T > 0$ and $\varepsilon \in]0,1[$. We define the norm of the functional space $\mathfrak{F}_p^\varepsilon(T)$ as*

$$\|u\|_{\mathfrak{F}_p^\varepsilon(T)} \stackrel{\text{def}}{=} \|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}\mathrm{e}^{\lambda\frac{t}{\sqrt{T}}A}u\|_{\widetilde{L}^\infty_T(\dot{H}^{\frac{5}{2}-\frac{6}{p}})} + \|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}\mathrm{e}^{\lambda\frac{t}{\sqrt{T}}A}u\|_{\widetilde{L}^1_T(\dot{H}^{\frac{9}{2}-\frac{6}{p}})}, \tag{3.1}$$

where the norm of $\widetilde{L}^q_T(\dot{H}^s)$ is defined by (1.5).

We start the proof of Theorem 1.2 by recalling the following lemma from [8].

Lemma 3.1 (see [8, Lemma 6.2.7]) *Let $m(\xi)$ be a complex-valued bounded function on $\mathbb{R}^n \setminus \{0\}$ that satisfies for some $A < \infty$,*

$$\Big(\int_{R<|\xi|<2R} |\partial_\xi^\alpha m(\xi)|^2 \,\mathrm{d}\xi\Big)^{\frac{1}{2}} \leq AR^{\frac{n}{2}-|\alpha|}$$

for any multi-index $|\alpha| \leq [\frac{n}{2}] + 1$ and all $R > 0$. Then for all $1 < p < \infty$, any $f \in L^p(\mathbb{R}^n)$, we have

$$\|m(D)f\|_{L^p} \leq C_n \max(p, (p-1)^{-1})(A + \|m\|_{L^\infty})\|f\|_{L^p}.$$

Thanks to the above lemma, we are going to derive the following estimate for the heat semi-group acting on u_0.

Lemma 3.2 *Let $\eta > 0$ and $u_{\eta,L} \stackrel{\text{def}}{=} \mathrm{e}^{\eta t\Delta}u_0$. Then for any $\eta < \varepsilon$, there exists a positive constant $C_{\varepsilon,\eta}$ so that for any $\ell \in \mathbb{Z}$,*

$$\|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}\mathrm{e}^{\lambda\frac{t}{\sqrt{T}}A}\Delta_\ell u_L\|_{L^p} \leq C_{\varepsilon,\eta}\|\Delta_\ell u_{\eta,L}\|_{L^p} \leq C_{\varepsilon,\eta}\mathrm{e}^{-c\eta t2^{2\ell}}\|\Delta_\ell u_0\|_{L^p}. \tag{3.2}$$

Proof Let us denote $m(t,\xi) \stackrel{\text{def}}{=} \mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}\mathrm{e}^{\lambda\frac{t}{\sqrt{T}}|\xi|}\mathrm{e}^{-(1-\eta)t|\xi|^2}$. Observing that

$$-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T} + \lambda\frac{t}{\sqrt{T}}|\xi| - (1-\eta)t|\xi|^2 = -\Big(\frac{\lambda}{2\sqrt{1-\varepsilon}}\sqrt{\frac{t}{T}} - \sqrt{1-\varepsilon}\sqrt{t}|\xi|\Big)^2 + (\eta-\varepsilon)t|\xi|^2.$$

Due to $\eta < \varepsilon$, one has

$$\|m(t,\cdot)\|_{L^\infty} \leq 1. \tag{3.3}$$

While observing that for $\xi \in \mathbb{R}^3 \setminus \{0\}$,

$$\partial_{\xi_i} m(t,\xi) = m(t,\xi)\Big(\lambda\frac{t}{\sqrt{T}}\frac{\xi_i}{|\xi|} - 2(1-\eta)t\xi_i\Big).$$

Notice that $\eta < \varepsilon$, one can always find $\delta > 0$ so that $(1-\varepsilon)(1+\delta)^2 - 1 + \eta < 0$. Then we deduce that

$$\begin{aligned}|\xi||\nabla m(t,\xi)| &\le \lambda\frac{t}{\sqrt{T}}|\xi|\mathrm{e}^{-\delta\lambda\frac{t}{\sqrt{T}}|\xi|}\mathrm{e}^{-(\frac{\lambda}{2\sqrt{1-\varepsilon}}\sqrt{\frac{t}{T}}-\sqrt{1-\varepsilon}(1+\delta)\sqrt{t}|\xi|)^2}\mathrm{e}^{((1-\varepsilon)(1+\delta)^2-1+\eta)t|\xi|^2}\\ &\quad + 2(1-\eta)t|\xi|^2\mathrm{e}^{-(\varepsilon-\eta)t|\xi|^2}\mathrm{e}^{-(\frac{\lambda}{2\sqrt{1-\varepsilon}}\sqrt{\frac{t}{T}}-\sqrt{1-\varepsilon}\sqrt{t}|\xi|)^2}\\ &\le C_{\varepsilon,\eta}.\end{aligned} \tag{3.4}$$

Similarly for $\xi \in \mathbb{R}^n \setminus \{0\}$,

$$\begin{aligned}\partial_{\xi_i}\partial_{\xi_j}m(t,\xi) = m(t,\xi)\Big(&\Big(\lambda\frac{t}{\sqrt{T}}\frac{\xi_i}{|\xi|}-2(1-\eta)t\xi_i\Big)\Big(\lambda\frac{t}{\sqrt{T}}\frac{\xi_j}{|\xi|}-2(1-\eta)t\xi_j\Big)\\ &+\lambda\frac{t}{\sqrt{T}}\partial_{\xi_j}\Big(\frac{\xi_i}{|\xi|}\Big)-2(1-\eta)t\delta_{ij}\Big),\end{aligned}$$

from which, we deduce that

$$\begin{aligned}|\xi|^2|\nabla^2 m(t,\xi)| &\le C\Big(\lambda\frac{t}{\sqrt{T}}|\xi| + \Big(\lambda\frac{t}{\sqrt{T}}|\xi|\Big)^2\Big)\mathrm{e}^{-\delta\lambda\frac{t}{\sqrt{T}}|\xi|}\\ &\quad + C((1-\eta)t|\xi|^2+(1-\eta)^2t^2|\xi|^4)\mathrm{e}^{-(\varepsilon-\eta)t|\xi|^2}\\ &\le C_{\varepsilon,\eta}.\end{aligned} \tag{3.5}$$

By combining (3.3), (3.4) with (3.5) and then applying Lemma 3.1, we conclude the proof of (3.2).

Remark 3.1 In the rest of this section, we always take η in Lemma 3.2 to be $\frac{\varepsilon}{2}$.

Then the key ingredients used to prove Theorem 1.2 will be the following lemmas.

Lemma 3.3 *Let $u_0 \in L^p(\mathbb{R}^3)$ with $3 \le p < 4$. Then there holds*

$$\|B(u_L,u_L)\|_{\mathfrak{F}_p^\varepsilon(T)} \le C_\varepsilon \mathrm{e}^{\frac{\lambda^2}{4(1-\varepsilon)}}\|u_0\|_{L^p}^2. \tag{3.6}$$

Proof Let $u_L^\lambda \stackrel{\text{def}}{=} \mathrm{e}^{\lambda\frac{t}{\sqrt{T}}A}u_L$. Then in view of (2.3), we get, by Hölder inequality and the convex inequality that $|\xi| \le |\xi-\zeta|+|\zeta|$, that

$$\begin{aligned}&\|\mathrm{e}^{-\frac{\lambda^2}{2(1-\varepsilon)}\frac{t}{T}}\mathrm{e}^{\lambda\frac{t}{\sqrt{T}}A}\Delta_j(T_{u_L}u_L)\|_{L^1_T(L^2)}\\ &\lesssim \sum_{|j-j'|\le 4}\|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}S_{j'-1}u_L^\lambda\|_{L^\infty_T(L^{\frac{2p}{p-2}})}\|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}\Delta_{j'}u_L^\lambda\|_{L^1_T(L^p)}.\end{aligned}$$

However due to $p < 4$, we deduce from Lemmas A.1 and 3.2 that

$$\begin{aligned}\|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}S_{j'-1}u_L^\lambda\|_{L^\infty_T(L^{\frac{2p}{p-2}})} &\lesssim \sum_{\ell\le j'-2}2^{(\frac{6}{p}-\frac{3}{2})\ell}\|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}\Delta_\ell u_L^\lambda\|_{L^\infty_T(L^p)}\\ &\le C_\varepsilon\sum_{\ell\le j'-2}2^{(\frac{6}{p}-\frac{3}{2})\ell}\|\Delta_\ell u_{\frac{\varepsilon}{2},L}\|_{L^\infty_T(L^p)}\\ &\le C_\varepsilon c_{j,p}2^{(\frac{6}{p}-\frac{3}{2})j}\|u_0\|_{L^p},\end{aligned} \tag{3.7}$$

where in the last step, we used the fact that $p<4$.

Similarly, we have

$$\|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}\Delta_j u_L^\lambda\|_{L^1_T(L^p)}\leq C_\varepsilon 2^{-2j}c_{j,p}\|u_0\|_{L^p}. \tag{3.8}$$

Hence we obtain

$$\|\mathrm{e}^{-\frac{\lambda^2}{2(1-\varepsilon)}\frac{t}{T}}\mathrm{e}^{\lambda\frac{t}{\sqrt{T}}A}\Delta_j(T_{u_L}u_L)\|_{L^1_T(L^2)}\leq C_\varepsilon 2^{(\frac{6}{p}-\frac{7}{2})j}c_{j,\frac{p}{2}}\|u_0\|^2_{L^p}.$$

Along the same line, we deduce from (2.3) and Lemma A.1 that

$$\begin{aligned}
&\|\mathrm{e}^{-\frac{\lambda^2}{2(1-\varepsilon)}\frac{t}{T}}\mathrm{e}^{\lambda\frac{t}{\sqrt{T}}A}\Delta_j(R(u_L,u_L))\|_{L^1_T(L^2)}\\
&\lesssim 2^{(\frac{6}{p}-\frac{3}{2})j}\sum_{j'\geq j-3}\|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}\Delta_{j'}u_L^\lambda\|_{L^\infty_T(L^p)}\|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}\widetilde{\Delta}_{l'}u_L^\lambda\|_{L^1_T(L^p)}\\
&\leq C_\varepsilon 2^{(\frac{6}{p}-\frac{3}{2})j}\sum_{j'\geq j-3}c_{j,\frac{p}{2}}2^{-2j'}\|u_0\|^2_{L^p}\\
&\leq C_\varepsilon c_{j,\frac{p}{2}}2^{(\frac{6}{p}-\frac{7}{2})j}\|u_0\|^2_{L^p}.
\end{aligned}$$

Hence for $3\leq p<4$, we obtain

$$\|\mathrm{e}^{-\frac{\lambda^2}{2(1-\varepsilon)}\frac{t}{T}}\mathrm{e}^{\lambda\frac{t}{\sqrt{T}}A}\Delta_j(u_L\otimes u_L)\|_{L^1_T(L^2)}\leq C_\varepsilon c_j 2^{(\frac{6}{p}-\frac{7}{2})j}\|u_0\|^2_{L^p}. \tag{3.9}$$

On the other hand, we observe that for any $\varepsilon\in[0,1[$,

$$\|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}\mathrm{e}^{-(1-\varepsilon)tA^2}\mathrm{e}^{\lambda\frac{t}{\sqrt{T}}A}f\|_{L^2}\leq\|f\|_{L^2}, \tag{3.10}$$

from which, we infer

$$\begin{aligned}
&\Big\|\int_0^t\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}\mathrm{e}^{-(t-s)A^2}\mathrm{e}^{\lambda\frac{t}{\sqrt{T}}A}\mathbb{P}\nabla\Delta_j(u_L\otimes u_L)(s)\,\mathrm{d}s\Big\|_{L^2}\\
&\lesssim\int_0^t\|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t-s}{T}}\mathrm{e}^{-(1-\varepsilon)(t-s)A^2}\mathrm{e}^{\lambda\frac{t-s}{\sqrt{T}}A}\mathrm{e}^{-\varepsilon(t-s)A^2}\\
&\quad\times\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{s}{T}}\mathrm{e}^{\lambda\frac{s}{T}A}\mathbb{P}\nabla\Delta_j(u_L\otimes u_L)(s)\|_{L^2}\,\mathrm{d}s\\
&\lesssim\int_0^t\mathrm{e}^{\frac{\lambda^2}{4(1-\varepsilon)}\frac{s}{T}}\mathrm{e}^{-\varepsilon(t-s)2^{2j}}2^j\|\mathrm{e}^{-\frac{\lambda^2}{2(1-\varepsilon)}\frac{s}{T}}\mathrm{e}^{\lambda\frac{s}{\sqrt{T}}A}\Delta_j(u_L\otimes u_L)(s)\|_{L^2}\,\mathrm{d}s.
\end{aligned} \tag{3.11}$$

Thanks to (3.9) and (3.11), we deduce that

$$\begin{aligned}
&\|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}\mathrm{e}^{\lambda\frac{t}{\sqrt{T}}A}\Delta_j(B(u_L,u_L))\|_{L^\infty_T(L^2)}+2^{2j}\|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}\mathrm{e}^{\lambda\frac{t}{\sqrt{T}}A}\Delta_j(B(u_L,u_L))\|_{L^1_T(L^2)}\\
&\lesssim\mathrm{e}^{\frac{\lambda^2}{4(1-\varepsilon)}}2^j\|\mathrm{e}^{-\frac{\lambda^2}{2(1-\varepsilon)}\frac{s}{T}}\mathrm{e}^{\lambda\frac{s}{\sqrt{T}}A}\Delta_j(u_L\otimes u_L)\|_{L^1_T(L^2)}\\
&\leq C_\varepsilon c_j 2^{(\frac{6}{p}-\frac{5}{2})j}\mathrm{e}^{\frac{\lambda^2}{4(1-\varepsilon)}}\|u_0\|^2_{L^p},
\end{aligned}$$

which leads to (3.6). This completes the proof of Lemma 3.3.

Lemma 3.4 *Let $v,w\in\mathfrak{F}^\varepsilon_p(T)$ with $p\in[3,6[$. Then for γ determined by (1.6), one has*

$$\|B(v,w)\|_{\mathfrak{F}^\varepsilon_p(T)}\leq C_\varepsilon\mathrm{e}^{\frac{\lambda^2}{4(1-\varepsilon)}}T^{\frac{1}{\gamma}}\|v\|_{\mathfrak{F}^\varepsilon_p(T)}\|w\|_{\mathfrak{F}^\varepsilon_p(T)}. \tag{3.12}$$

Proof For simplicity, we shall denote $v^\lambda(t,x) \stackrel{\text{def}}{=} \mathrm{e}^{\lambda\frac{t}{\sqrt{T}}A}v(t,x)$. By applying Plancherel equality and convex inequality that $|\xi| \leq |\xi-\zeta|+|\zeta|$, we write

$$\begin{aligned}&\|\mathrm{e}^{-\frac{\lambda^2}{2(1-\varepsilon)}\frac{t}{T}}\mathrm{e}^{\lambda\frac{t}{\sqrt{T}}A}\Delta_j(T_v w)\|_{L^1_T(L^2)}\\ \lesssim &\sum_{|j-j'|\leq 4}\|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}S_{j'-1}v^\lambda\|_{L^\infty_T(L^\infty)}\|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}\Delta_{j'}w^\lambda\|_{L^1_T(L^2)}.\end{aligned}$$

Yet due to $p<6$, we get, by applying Lemma A.1 and Definition 3.1, that,

$$\|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}S_{j'-1}v^\lambda\|_{L^\infty_T(L^\infty)} \lesssim \sum_{\ell\leq j'-2}2^{\frac32\ell}\|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}\Delta_\ell v^\lambda\|_{L^\infty_T(L^2)} \lesssim 2^{(\frac6p-1)j}\|v\|_{\mathfrak{F}^\varepsilon_p(T)},$$

so that

$$\|\mathrm{e}^{-\frac{\lambda^2}{2(1-\varepsilon)}\frac{t}{T}}\mathrm{e}^{\lambda\frac{t}{\sqrt{T}}A}\Delta_j(T_v w)\|_{L^1_T(L^2)} \lesssim c_j 2^{(\frac{12}{p}-\frac{11}{2})j}\|v\|_{\mathfrak{F}^\varepsilon_p(T)}\|w\|_{\mathfrak{F}^\varepsilon_p(T)}.$$

Similarly, we have

$$\|\mathrm{e}^{-\frac{\lambda^2}{2(1-\varepsilon)}\frac{t}{T}}\mathrm{e}^{\lambda\frac{t}{\sqrt{T}}A}\Delta_j(T_w v)\|_{L^1_T(L^2)} \lesssim c_j 2^{(\frac{12}{p}-\frac{11}{2})j}\|v\|_{\mathfrak{F}^\varepsilon_p(T)}\|w\|_{\mathfrak{F}^\varepsilon_p(T)}.$$

Whereas it follows from Lemma A.1 that

$$\begin{aligned}&\|\mathrm{e}^{-\frac{\lambda^2}{2(1-\varepsilon)}\frac{t}{T}}\mathrm{e}^{\lambda\frac{t}{\sqrt{T}}A}\Delta_j(R(v,w))\|_{L^1_T(L^2)}\\ \lesssim\ & 2^{\frac32 j}\sum_{j'\geq j-3}\|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}\Delta_{j'}v^\lambda\|_{L^\infty_T(L^2)}\|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}\widetilde{\Delta}_{j'}w^\lambda\|_{L^1_T(L^2)}\\ \lesssim\ & 2^{\frac32 j}\sum_{j'\geq j-3}c_{j'}2^{(\frac{12}{p}-7)j'}\|v\|_{\mathfrak{F}^\varepsilon_p(T)}\|w\|_{\mathfrak{F}^\varepsilon_p(T)}\\ \lesssim\ & c_j 2^{(\frac{12}{p}-\frac{11}{2})j}\|v\|_{\mathfrak{F}^\varepsilon_p(T)}\|w\|_{\mathfrak{F}^\varepsilon_p(T)}.\end{aligned}$$

As a consequence, we deduce from (2.7) that

$$\|\mathrm{e}^{-\frac{\lambda^2}{2(1-\varepsilon)}\frac{t}{T}}\mathrm{e}^{\lambda\frac{t}{\sqrt{T}}A}\Delta_j(v\otimes w)\|_{L^1_T(L^2)} \lesssim c_j 2^{(\frac{12}{p}-\frac{11}{2})j}\|v\|_{\mathfrak{F}^\varepsilon_p(T)}\|w\|_{\mathfrak{F}^\varepsilon_p(T)}. \tag{3.13}$$

Along the same line, one has

$$\begin{aligned}&\|\mathrm{e}^{-\frac{\lambda^2}{2(1-\varepsilon)}\frac{t}{T}}\mathrm{e}^{\lambda\frac{t}{\sqrt{T}}A}\Delta_j(v\otimes w)\|_{L^\infty_T(L^2)}\\ \lesssim\ & c_j 2^{(\frac{12}{p}-\frac72)j}\|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}\mathrm{e}^{\lambda\frac{t}{\sqrt{T}}A}v\|_{\widetilde{L}^\infty_T(\dot H^{\frac52-\frac6p})}\|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}\mathrm{e}^{\lambda\frac{t}{\sqrt{T}}A}w\|_{\widetilde{L}^\infty_T(\dot H^{\frac52-\frac6p})}.\end{aligned} \tag{3.14}$$

Then we get, by applying (3.10), that

$$\begin{aligned}&\|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}\mathrm{e}^{\lambda\frac{t}{\sqrt{T}}A}\Delta_j(B(v,w))(t)\|_{L^2}\\ \lesssim\ & \int_0^t \|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t-s}{T}}\mathrm{e}^{-(1-\varepsilon)(t-s)A^2}\mathrm{e}^{\lambda\frac{t-s}{\sqrt{T}}A}\mathrm{e}^{-\varepsilon(t-s)A^2}\\ &\times \mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{s}{T}}\mathrm{e}^{\lambda\frac{s}{\sqrt{T}}A}\mathbb{P}\nabla\Delta_j(v\otimes w)(s)\|_{L^2}\,\mathrm{d}s\\ \lesssim\ & \mathrm{e}^{\frac{\lambda^2}{4(1-\varepsilon)}}\int_0^t \mathrm{e}^{-\varepsilon(t-s)2^{2j}}2^j\|\mathrm{e}^{-\frac{\lambda^2}{2(1-\varepsilon)}\frac{s}{T}}\mathrm{e}^{\lambda\frac{s}{\sqrt{T}}A}\Delta_j(v\otimes w)(s)\|_{L^2}\,\mathrm{d}s,\end{aligned} \tag{3.15}$$

which together with (3.13) implies

$$\begin{aligned}&\|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}\mathrm{e}^{\lambda\frac{t}{\sqrt{T}}A}\Delta_j(B(v,w))\|_{L^1_T(L^2)}\\ &\lesssim \mathrm{e}^{\frac{\lambda^2}{4(1-\varepsilon)}}T^{\frac{1}{\gamma}}\|\mathrm{e}^{-\varepsilon t2^{2j}}\|_{L_T^{\frac{p}{3}}}\|\mathrm{e}^{-\frac{\lambda^2}{2(1-\varepsilon)}\frac{s}{T}}\mathrm{e}^{\lambda\frac{s}{\sqrt{T}}A}\Delta_j(v\otimes w)\|_{L^1_T(L^2)}\\ &\le C_\varepsilon c_j 2^{-(\frac{9}{2}-\frac{6}{p})j}T^{\frac{1}{\gamma}}\mathrm{e}^{\frac{\lambda^2}{4(1-\varepsilon)}}\|v\|_{\mathfrak{F}_p^\varepsilon(T)}\|w\|_{\mathfrak{F}_p^\varepsilon(T)}.\end{aligned}$$

As a result, it comes out

$$\|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}\mathrm{e}^{\lambda\frac{t}{\sqrt{T}}A}B(v,w)\|_{\widetilde{L}^1_T(\dot H^{\frac{9}{2}-\frac{6}{p}})}\lesssim C_\varepsilon T^{\frac{1}{\gamma}}\mathrm{e}^{\frac{\lambda^2}{4(1-\varepsilon)}}\|v\|_{\mathfrak{F}_p^\varepsilon(T)}\|w\|_{\mathfrak{F}_p^\varepsilon(T)}. \tag{3.16}$$

Similarly, we deduce from (3.14) that

$$\begin{aligned}&\|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}\mathrm{e}^{\lambda\frac{t}{\sqrt{T}}A}\Delta_j(B(v,w))\|_{L^\infty_T(L^2)}\\ &\lesssim \mathrm{e}^{\frac{\lambda^2}{4(1-\varepsilon)}}\Big\|\int_0^t \mathrm{e}^{-\varepsilon(t-s)2^{2j}}2^j\|\mathrm{e}^{-\frac{\lambda^2}{2(1-\varepsilon)}\frac{s}{T}}\mathrm{e}^{\lambda\frac{s}{\sqrt{T}}A}\Delta_j(vw)(s)\|_{L^2}\,\mathrm{d}s\Big\|_{L^\infty_T}\\ &\lesssim \mathrm{e}^{\frac{\lambda^2}{4(1-\varepsilon)}}2^jT^{\frac{1}{\gamma}}\|\mathrm{e}^{-\varepsilon t2^{2j}}\|_{L_T^{\frac{p}{3}}}\|\mathrm{e}^{-\frac{\lambda^2}{2(1-\varepsilon)}\frac{s}{T}}\mathrm{e}^{\lambda\frac{s}{\sqrt{T}}A}\Delta_j(vw)\|_{L^\infty_T(L^2)}\\ &\le C_\varepsilon c_j 2^{-(\frac{5}{2}-\frac{6}{p})j}T^{\frac{1}{\gamma}}\mathrm{e}^{\frac{\lambda^2}{4(1-\varepsilon)}}\|v\|_{\mathfrak{F}_p^\varepsilon(T)}\|w\|_{\mathfrak{F}_p^\varepsilon(T)},\end{aligned}$$

which implies

$$\|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{\sqrt{T}}}\mathrm{e}^{\lambda\frac{t}{\sqrt{T}}A}B(v,w)\|_{\widetilde{L}^\infty_T(\dot H^{\frac{5}{2}-\frac{6}{p}})}\le C_\varepsilon T^{\frac{1}{\gamma}}\mathrm{e}^{\frac{\lambda^2}{4(1-\varepsilon)}}\|v\|_{\mathfrak{F}_p^\varepsilon(T)}\|w\|_{\mathfrak{F}_p^\varepsilon(T)}.$$

This together with (3.16) ensures (3.12), which completes the proof of Lemma 3.4.

Lemma 3.5 *Let $v\in\mathfrak{F}_p^\varepsilon(T)$ and $u_0\in L^p(\mathbb{R}^3)$ with $3\le p<\frac{18}{5}$. Then there holds*

$$\|B(v,u_L)\|_{\mathfrak{F}_p^\varepsilon(T)}\lesssim C_\varepsilon T^{\frac{1}{2\gamma}}\mathrm{e}^{\frac{\lambda^2}{4(1-\varepsilon)}}\|u_0\|_{L^p}\|v\|_{\mathfrak{F}_p^\varepsilon(T)}. \tag{3.17}$$

Proof By applying Hölder inequality and Plancherel equality, we find

$$\begin{aligned}&\|\mathrm{e}^{-\frac{\lambda^2}{2(1-\varepsilon)}\frac{t}{T}}\mathrm{e}^{\lambda\frac{t}{\sqrt{T}}A}\Delta_j(T_vu_L)\|_{L^1_T(L^2)}\\ &\lesssim \sum_{|j-j'|\le4}\|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}S_{j'-1}v^\lambda\|_{L^\infty_T(L^{\frac{2p}{p-2}})}\|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}\Delta_{j'}u_L^\lambda\|_{L^1_T(L^p)}.\end{aligned}$$

Due to $p<\frac{18}{5}$, we deduce from Lemma A.1 that

$$\begin{aligned}\|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}S_{j'-1}v^\lambda\|_{L^\infty_T(L^{\frac{2p}{p-2}})}&\lesssim\sum_{\ell\le j'-2}2^{\frac{3}{p}\ell}\|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}\Delta_\ell v^\lambda\|_{L^\infty_T(L^2)}\\ &\lesssim\sum_{\ell\le j'-2}c_\ell 2^{(\frac{9}{p}-\frac{5}{2})\ell}\|v\|_{\mathfrak{F}_p^\varepsilon(T)}\lesssim c_j2^{(\frac{9}{p}-\frac{5}{2})j}\|v\|_{\mathfrak{F}_p^\varepsilon(T)},\end{aligned}$$

from which and (3.8), we infer

$$\|\mathrm{e}^{-\frac{\lambda^2}{2(1-\varepsilon)}\frac{t}{T}}\mathrm{e}^{\lambda\frac{t}{\sqrt{T}}A}\Delta_j(T_vu_L)\|_{L^1_T(L^2)}\le C_\varepsilon c_j2^{(\frac{9}{p}-\frac{9}{2})j}\|u_0\|_{L^p}\|v\|_{\mathfrak{F}_p^\varepsilon(T)}. \tag{3.18}$$

Similarly, we have

$$\begin{aligned}
&\|e^{-\frac{\lambda^2}{2(1-\varepsilon)}\frac{t}{T}}e^{\lambda\frac{t}{\sqrt{T}}A}\Delta_j(T_{u_L}v)\|_{L^1_T(L^2)}\\
&\lesssim \sum_{|j-j'|\le 4}\|e^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}S_{j'-1}u_L^\lambda\|_{L^\infty_T(L^\infty)}\|e^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}\Delta_{j'}v^\lambda\|_{L^1_T(L^2)}.
\end{aligned}$$

While we get, by a similar derivation of (3.7), that

$$\begin{aligned}
\|e^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}S_{j'-1}u_L^\lambda\|_{L^\infty_T(L^\infty)} &\lesssim \sum_{\ell\le j'-2}2^{\frac{3}{p}\ell}\|e^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}\Delta_\ell u_L^\lambda\|_{L^\infty_T(L^p)}\\
&\lesssim C_\varepsilon c_{j,p}2^{\frac{3}{p}j}\|u_0\|_{L^p},
\end{aligned}$$

which together with Definition 3.1 ensures that

$$\|e^{-\frac{\lambda^2}{2(1-\varepsilon)}\frac{t}{T}}e^{\lambda\frac{t}{\sqrt{T}}A}\Delta_j(T_{u_L}v)\|_{L^1_T(L^2)}\lesssim C_\varepsilon c_j 2^{(\frac{9}{p}-\frac{9}{2})j}\|u_0\|_{L^p}\|v\|_{\mathfrak{F}^\varepsilon_p(T)}. \tag{3.19}$$

Whereas we get, by applying Lemma A.1, that

$$\begin{aligned}
&\|e^{-\frac{\lambda^2}{2(1-\varepsilon)}\frac{t}{T}}e^{\lambda\frac{t}{\sqrt{T}}A}\Delta_j(R(u_L,v))\|_{L^1_T(L^2)}\\
&\lesssim 2^{\frac{3}{p}j}\sum_{j'\ge j-3}\|e^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}\Delta_{j'}u_L^\lambda\|_{L^\infty_T(L^p)}\|e^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}\widetilde{\Delta}_{j'}v^\lambda\|_{L^1_T(L^2)}\\
&\le C_\varepsilon 2^{\frac{3}{p}j}\sum_{j'\ge j-3}c_{j'}2^{(\frac{6}{p}-\frac{9}{2})j'}\|u_0\|_{L^p}\|v\|_{\mathfrak{F}^\varepsilon_p(T)}\\
&\lesssim C_\varepsilon c_j 2^{(\frac{9}{p}-\frac{9}{2})j}\|u_0\|_{L^p}\|v\|_{\mathfrak{F}^\varepsilon_p(T)}.
\end{aligned}$$

By combining (3.18), (3.19) with the above inequality, we achieve for $3\le p<\frac{18}{5}$,

$$\|e^{-\frac{\lambda^2}{2(1-\varepsilon)}\frac{t}{T}}e^{\lambda\frac{t}{\sqrt{T}}A}\Delta_j(u_L\otimes v)\|_{L^1_T(L^2)}\le C_\varepsilon c_j 2^{(\frac{9}{p}-\frac{9}{2})j}\|u_0\|_{L^p}\|v\|_{\mathfrak{F}^\varepsilon_p(T)}. \tag{3.20}$$

Similarly, for $3\le p<\frac{18}{5}$, we have

$$\begin{aligned}
&\|e^{-\frac{\lambda^2}{2(1-\varepsilon)}\frac{t}{T}}e^{\lambda\frac{t}{\sqrt{T}}A}\Delta_j(u_L\otimes v)\|_{L^\infty_T(L^2)}\\
&\le C_\varepsilon c_j 2^{(\frac{9}{p}-\frac{5}{2})j}\|u_0\|_{L^p}\|e^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}e^{\lambda\frac{t}{\sqrt{T}}A}v\|_{\widetilde{L}^\infty_T(\dot{H}^{\frac{5}{2}-\frac{6}{p}})}.
\end{aligned} \tag{3.21}$$

Then by virtue of (3.10), we infer

$$\begin{aligned}
&\Big\|\int_0^t e^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}e^{-(t-s)A^2}e^{\lambda\frac{t}{\sqrt{T}}A}\mathbb{P}\nabla\Delta_j(v\otimes u_L)\,ds\Big\|_{L^2}\\
&\lesssim e^{\frac{\lambda^2}{4(1-\varepsilon)}}\int_0^t e^{-\varepsilon(t-s)2^{2j}}2^j\|e^{-\frac{\lambda^2}{2(1-\varepsilon)}\frac{s}{T}}e^{\lambda\frac{s}{\sqrt{T}}A}\Delta_j(v\otimes u_L)\|_{L^2}\,ds,
\end{aligned}$$

which together with (3.20) ensures that

$$\begin{aligned}
&\|e^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}e^{\lambda\frac{t}{\sqrt{T}}A}\Delta_j(B(v,u_L))\|_{L^1_T(L^2)}\\
&\lesssim e^{\frac{\lambda^2}{4(1-\varepsilon)}}T^{\frac{1}{2\gamma}}\|e^{-\varepsilon t2^{2j}}\|_{L_T^{\frac{2p}{p+3}}}2^j\|e^{-\frac{\lambda^2}{2(1-\varepsilon)}\frac{s}{T}}e^{\lambda\frac{s}{\sqrt{T}}A}\Delta_j(v\otimes u_L)\|_{L^1_T(L^2)}
\end{aligned}$$

$$\leq C_\varepsilon c_j 2^{(\frac{6}{p}-\frac{9}{2})j} \mathrm{e}^{\frac{\lambda^2}{4(1-\varepsilon)}} T^{\frac{1}{2\gamma}} \|u_0\|_{L^p} \|v\|_{\mathfrak{F}_p^\varepsilon} \tag{3.22}$$

for γ being determined by (1.6). Therefore, we obtain

$$\|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}} \mathrm{e}^{\lambda\frac{t}{\sqrt{T}}A} B(v, u_L)\|_{\widetilde{L}^1_T(\dot{H}^{\frac{9}{2}-\frac{6}{p}})} \leq C_\varepsilon T^{\frac{1}{2\gamma}} \mathrm{e}^{\frac{\lambda^2}{4(1-\varepsilon)}} \|u_0\|_{L^p} \|v\|_{\mathfrak{F}_p^\varepsilon(T)}. \tag{3.23}$$

Along the same line, one has

$$\begin{aligned} &\|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}} \mathrm{e}^{\lambda\frac{t}{\sqrt{T}}A} \Delta_j(B(v, u_L))\|_{L^\infty_T(L^2)} \\ &\lesssim \mathrm{e}^{\frac{\lambda^2}{4(1-\varepsilon)}} T^{\frac{1}{2\gamma}} \|\mathrm{e}^{-\varepsilon t 2^{2j}}\|_{L^{\frac{2p}{p+3}}} 2^j \|\mathrm{e}^{-\frac{\lambda^2}{2(1-\varepsilon)}\frac{s}{T}} \mathrm{e}^{\lambda\frac{s}{\sqrt{T}}A} \Delta_j(v\otimes u_L)\|_{L^\infty_T(L^2)} \\ &\lesssim C_\varepsilon c_j 2^{(\frac{6}{p}-\frac{5}{2})j} T^{\frac{1}{2\gamma}} \mathrm{e}^{\frac{\lambda^2}{4(1-\varepsilon)}} \|u_0\|_{L^p} \|v\|_{\mathfrak{F}_p^\varepsilon}, \end{aligned}$$

which implies

$$\|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}} \mathrm{e}^{\lambda\frac{t}{\sqrt{T}}A} B(v, u_L)\|_{\widetilde{L}^\infty_T(\dot{H}^{\frac{5}{2}-\frac{6}{p}})} \leq C_\varepsilon T^{\frac{1}{2\gamma}} \mathrm{e}^{\frac{\lambda^2}{4(1-\varepsilon)}} \|u_0\|_{L^p} \|v\|_{\mathfrak{F}_p^\varepsilon(T)}.$$

This together with (3.23) ensures (3.17), and we complete the proof of Lemma 3.5.

Now we are in a position to complete the proof of Theorem 1.2.

Proof of Theorem 1.2 Once again we are going to prove Theorem 1.2 via Lemma 2.4. In order to do so, we take $X = \mathfrak{F}_p^\varepsilon(T)$, which is determined by Definition 3.1. Let L and x_0 be determined by (2.14). Then it follows from Lemmas 3.3–3.5 that

$$\begin{aligned} &\|x_0\|_X \leq C_\varepsilon \mathrm{e}^{\frac{\lambda^2}{4(1-\varepsilon)}} \|u_0\|^2_{L^p}, \\ &\|B\|_{\mathcal{B}(X)} \leq C_\varepsilon \mathrm{e}^{\frac{\lambda^2}{4(1-\varepsilon)}} T^{\frac{1}{\gamma}}, \\ &\|L\|_{\mathcal{L}(X)} \leq C_\varepsilon \mathrm{e}^{\frac{\lambda^2}{4(1-\varepsilon)}} T^{\frac{1}{2\gamma}} \|u_0\|_{L^p}. \end{aligned}$$

For $T < 1$ and c sufficiently small, we take T to be so small that

$$C_\varepsilon \mathrm{e}^{\frac{\lambda^2}{4(1-\varepsilon)}} T^{\frac{1}{2\gamma}} \|u\|_{L^p} = c. \tag{3.24}$$

Then we deduce from (3.24) that

$$\lambda \stackrel{\text{def}}{=} \sqrt{\frac{2(1-\varepsilon)}{\gamma}\ln|T| + \ln\frac{c}{C_\varepsilon\|u_0\|_{L^p}}}. \tag{3.25}$$

It is easy to observe that under the condition (3.24), there holds (2.13). Then we deduce from Lemma 2.4 that (1.4) has a unique solution in $v \in \mathfrak{F}_p^\varepsilon(T)$. In particular, for $T \leq t_1(\varepsilon, \|u_0\|_{L^p})$ small enough, it follows from Lemma 2.4 and (3.1), (3.25) that

$$\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}} \|\mathrm{e}^{\sqrt{\frac{2(1-\varepsilon)}{\gamma}}\sqrt{T\ln|T|}A} v(T)\|_{\dot{H}^{\frac{5}{2}-\frac{6}{p}}} \leq \frac{1}{2C_\varepsilon} \mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}} T^{-\frac{1}{\gamma}},$$

which implies (1.7).

Since (1.7) holds for any $\varepsilon > 0$, we deduce (1.8) from (1.7). This finishes the proof of Theorem 1.2.

4 The Critical Case $p = 3$

The goal of this section is to present the proof of Theorem 1.3. The main idea is again to use Lemma 2.4. In order to do so, we first introduce the functional space we are going to work with.

Definition 4.1 *Let $\varepsilon \in]0,1[$, $T > 0$ and $2 < q < \infty$. Let $A \stackrel{\text{def}}{=} \sqrt{-\Delta}$. We define the norm of the functional space $\mathfrak{G}_q^\varepsilon(T)$ as*

$$\|u\|_{\mathfrak{G}_q^\varepsilon(T)} \stackrel{\text{def}}{=} \|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}\mathrm{e}^{\lambda\frac{t}{\sqrt{T}}A}u\|_{\widetilde{L}^\infty_T(\dot H^{\frac12})} + \|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}\mathrm{e}^{\lambda\frac{t}{\sqrt{T}}A}u\|_{\widetilde{L}^q_T(\dot H^{\frac12+\frac2q})}, \tag{4.1}$$

where the norm $\|\cdot\|_{\widetilde{L}^q_T(\dot H^s)}$ is given by (1.5).

In the rest of this section, we always fix some $\varepsilon \in]0,1[$, $0 < \eta < \varepsilon$ and $q \in]2,\infty[$. Then the key ingredients used to prove Theorem 1.3 will be the following lemmas.

Lemma 4.1 *Let $u_0 \in L^3(\mathbb{R}^3)$ and $u_{\eta,L} \stackrel{\text{def}}{=} \mathrm{e}^{\eta t\Delta}u_0$. Then there holds*

$$\|B(u_L,u_L)\|_{\mathfrak{G}_q^\varepsilon(T)} \leq C_{\varepsilon,\eta}\mathrm{e}^{\frac{\lambda^2}{4(1-\varepsilon)}}\|u_{\eta,L}\|^2_{\widetilde{L}^{2q}(\dot B^{\frac1q}_{3,3})}. \tag{4.2}$$

Proof As in the previous section, we still denote $u_L^\lambda \stackrel{\text{def}}{=} \mathrm{e}^{\lambda\frac{t}{\sqrt{T}}A}u_L$. Then in view of (2.3), we get, by Hölder inequality and convex inequality that $|\xi| \leq |\xi-\zeta| + |\zeta|$, that

$$\begin{aligned}&\|\mathrm{e}^{-\frac{\lambda^2}{2(1-\varepsilon)}\frac{t}{T}}\mathrm{e}^{\lambda\frac{t}{\sqrt{T}}A}\Delta_j(T_{u_L}u_L)\|_{L^q_T(L^2)}\\ &\lesssim \sum_{|j-j'|\leq 4}\|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}S_{j'-1}u_L^\lambda\|_{L^{2q}_T(L^6)}\|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}\Delta_{j'}u_L^\lambda\|_{L^{2q}_T(L^3)}.\end{aligned}$$

It follows from Lemmas A.1 and 3.2 that

$$\begin{aligned}\|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}S_{j'-1}u_L^\lambda\|_{L^{2q}_T(L^6)} &\lesssim \sum_{\ell\leq j'-2}2^{\frac{\ell}{2}}\|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}\Delta_\ell u_L^\lambda\|_{L^{2q}_T(L^3)}\\ &\leq C_{\varepsilon,\eta}\sum_{\ell\leq j'-2}c_{\ell,3}2^{\ell(\frac12-\frac1q)}\|u_{\eta,L}\|_{\widetilde{L}^{2q}_T(\dot B^{\frac1q}_{3,3})}\\ &\leq C_{\varepsilon,\eta}c_{j,3}2^{(\frac12-\frac1q)j}\|u_{\eta,L}\|_{\widetilde{L}^{2q}_T(\dot B^{\frac1q}_{3,3})},\end{aligned} \tag{4.3}$$

where in the last step, we used the fact that $q > 2$.

Hence we obtain

$$\|\mathrm{e}^{-\frac{\lambda^2}{2(1-\varepsilon)}\frac{t}{T}}\mathrm{e}^{\lambda\frac{t}{\sqrt{T}}A}\Delta_j(T_{u_L}u_L)\|_{L^q_T(L^2)} \leq C_{\varepsilon,\eta}c_j2^{(\frac12-\frac2q)j}\|u_{\eta,L}\|^2_{\widetilde{L}^{2q}_T(\dot B^{\frac1q}_{3,3})}.$$

Along the same line, we deduce from (2.3) and Lemma A.1 that

$$\begin{aligned}&\|\mathrm{e}^{-\frac{\lambda^2}{2(1-\varepsilon)}\frac{t}{T}}\mathrm{e}^{\lambda\frac{t}{\sqrt{T}}A}\Delta_j(R(u_L,u_L))\|_{L^q_T(L^2)}\\ &\lesssim 2^{\frac{j}{2}}\sum_{j'\geq j-3}\|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}\Delta_{j'}u_L^\lambda\|_{L^{2q}_T(L^3)}\|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}\widetilde{\Delta}_{j'}u_L^\lambda\|_{L^{2q}_T(L^3)}\end{aligned}$$

$$\begin{aligned}&\leq C_{\varepsilon,\eta}2^{\frac{j}{2}}\sum_{j'\geq j-3}c_{j'}2^{-\frac{2}{q}j'}\|u_{\eta,L}\|^2_{\widetilde{L}^{2q}_T(\dot{B}^{\frac{1}{q}}_{3,3})}\\&\leq C_{\varepsilon,\eta}c_j2^{(\frac{1}{2}-\frac{2}{q})j}\|u_{\eta,L}\|^2_{\widetilde{L}^{2q}_T(\dot{B}^{\frac{1}{q}}_{3,3})}.\end{aligned}$$

Hence for $2<q<\infty$, we obtain

$$\|\mathrm{e}^{-\frac{\lambda^2}{2(1-\varepsilon)}\frac{t}{T}}\mathrm{e}^{\lambda\frac{t}{\sqrt{T}}A}\Delta_j(u_L\otimes u_L)\|_{L^q_T(L^2)}\leq C_{\varepsilon,\eta}c_j2^{(\frac{1}{2}-\frac{2}{q})j}\|u_{\eta,L}\|^2_{\widetilde{L}^{2q}_T(\dot{B}^{\frac{1}{q}}_{3,3})}. \tag{4.4}$$

Therefore thanks to (3.11), we infer

$$\begin{aligned}&\|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}\mathrm{e}^{\lambda\frac{t}{\sqrt{T}}A}\Delta_j(B(u_L,u_L))\|_{L^\infty_T(L^2)}\\&\lesssim\mathrm{e}^{\frac{\lambda^2}{4(1-\varepsilon)}}2^j\|\mathrm{e}^{-\varepsilon t2^{2j}}\|_{L_T^{\frac{q}{q-1}}}\|\mathrm{e}^{-\frac{\lambda^2}{2(1-\varepsilon)}\frac{s}{T}}\mathrm{e}^{\lambda\frac{s}{\sqrt{T}}A}\Delta_j(u_L\otimes u_L)\|_{L^q_T(L^2)}\\&\leq C_{\varepsilon,\eta}c_j2^{-\frac{j}{2}}\mathrm{e}^{\frac{\lambda^2}{4(1-\varepsilon)}}\|u_{\eta,L}\|^2_{\widetilde{L}^{2q}_T(\dot{B}^{\frac{1}{q}}_{3,3})}.\end{aligned} \tag{4.5}$$

Along the same line, one has

$$\begin{aligned}&\|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}\mathrm{e}^{\lambda\frac{t}{\sqrt{T}}A}\Delta_j(B(u_L,u_L))\|_{L^q_T(L^2)}\\&\lesssim\mathrm{e}^{\frac{\lambda^2}{4(1-\varepsilon)}}2^j\|\mathrm{e}^{-\varepsilon t2^{2j}}\|_{L^1_T}\|\mathrm{e}^{-\frac{\lambda^2}{2(1-\varepsilon)}\frac{s}{T}}\mathrm{e}^{\lambda\frac{s}{\sqrt{T}}A}\Delta_j(u_L\otimes u_L)\|_{L^q_T(L^2)}\\&\leq C_{\varepsilon,\eta}c_j2^{-(\frac{1}{2}+\frac{2}{q})j}\mathrm{e}^{\frac{\lambda^2}{4(1-\varepsilon)}}\|u_{\eta,L}\|^2_{\widetilde{L}^{2q}_T(\dot{B}^{\frac{1}{q}}_{3,3})},\end{aligned}$$

which together with Definition 4.1 and (4.5) leads to (4.2). This completes the proof of Lemma 4.1.

Lemma 4.2 *Let $v,w\in\mathfrak{G}^\varepsilon_q(T)$. Then one has*

$$\|B(v,w)\|_{\mathfrak{G}^\varepsilon_q(T)}\leq C_\varepsilon\mathrm{e}^{\frac{\lambda^2}{4(1-\varepsilon)}}\|v\|_{\mathfrak{G}^\varepsilon_q(T)}\|w\|_{\mathfrak{G}^\varepsilon_q(T)}. \tag{4.6}$$

Proof Once again we denote $v^\lambda(t,x)\overset{\text{def}}{=}\mathrm{e}^{\lambda\frac{t}{\sqrt{T}}A}v(t,x)$. By applying Plancherel equality and convex inequality that $|\xi|\leq|\xi-\zeta|+|\zeta|$, we write

$$\begin{aligned}&\|\mathrm{e}^{-\frac{\lambda^2}{2(1-\varepsilon)}\frac{t}{T}}\mathrm{e}^{\lambda\frac{t}{\sqrt{T}}A}\Delta_j(T_vw)\|_{L^q_T(L^2)}\\&\lesssim\sum_{|j-j'|\leq4}\|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}S_{j'-1}v^\lambda\|_{L^{2q}_T(L^\infty)}\|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}\Delta_{j'}w^\lambda\|_{L^{2q}_T(L^2)}.\end{aligned}$$

Yet by applying Lemma A.1 and Definition 4.1, one has

$$\|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}S_{j'-1}v^\lambda\|_{L^{2q}_T(L^\infty)}\lesssim\sum_{\ell\leq j'-2}2^{\frac{3}{2}\ell}\|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}\Delta_\ell v^\lambda\|_{L^{2q}_T(L^2)}\lesssim c_j2^{(1-\frac{1}{q})j}\|v\|_{\mathfrak{G}^\varepsilon_q(T)},$$

so that

$$\|\mathrm{e}^{-\frac{\lambda^2}{2(1-\varepsilon)}\frac{t}{T}}\mathrm{e}^{\lambda\frac{t}{\sqrt{T}}A}\Delta_j(T_vw)\|_{L^q_T(L^2)}\lesssim\sum_{|j-j'|\leq4}c_{j'}2^{(\frac{1}{2}-\frac{2}{q})j'}\|v\|_{\mathfrak{G}^\varepsilon_q(T)}\|w\|_{\mathfrak{G}^\varepsilon_q(T)}$$

$$\lesssim c_j 2^{(\frac{1}{2}-\frac{2}{q})j}\|v\|_{\mathfrak{G}^\varepsilon_q(T)}\|w\|_{\mathfrak{G}^\varepsilon_q(T)}.$$

Similarly, we have

$$\|\mathrm{e}^{-\frac{\lambda^2}{2(1-\varepsilon)}\frac{t}{T}}\mathrm{e}^{\lambda\frac{t}{\sqrt{T}}A}\Delta_j(T_w v)\|_{L^q_T(L^2)}\lesssim c_j 2^{(\frac{1}{2}-\frac{2}{q})j}\|v\|_{\mathfrak{G}^\varepsilon_q(T)}\|w\|_{\mathfrak{G}^\varepsilon_q(T)}.$$

Whereas it follows from Lemma A.1 that

$$\begin{aligned}
&\|\mathrm{e}^{-\frac{\lambda^2}{2(1-\varepsilon)}\frac{t}{T}}\mathrm{e}^{\lambda\frac{t}{\sqrt{T}}A}\Delta_j(R(v,w))\|_{L^q_T(L^2)}\\
&\lesssim 2^{\frac{3}{2}j}\sum_{j'\geq j-3}\|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}\Delta_{j'}v^\lambda\|_{L^{2q}_T(L^2)}\|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}\widetilde{\Delta}_{j'}w^\lambda\|_{L^{2q}_T(L^2)}\\
&\lesssim 2^{\frac{3}{2}j}\sum_{j'\geq j-3}c_{j'}2^{-(1+\frac{2}{q})j'}\|v\|_{\mathfrak{G}^\varepsilon_q(T)}\|w\|_{\mathfrak{G}^\varepsilon_q(T)}\\
&\lesssim c_j 2^{(\frac{1}{2}-\frac{2}{q})j}\|v\|_{\mathfrak{G}^\varepsilon_q(T)}\|w\|_{\mathfrak{G}^\varepsilon_q(T)}.
\end{aligned}$$

As a consequence, we deduce from (2.7) that

$$\|\mathrm{e}^{-\frac{\lambda^2}{2(1-\varepsilon)}\frac{t}{T}}\mathrm{e}^{\lambda\frac{t}{\sqrt{T}}A}\Delta_j(v\otimes w)\|_{L^q_T(L^2)}\lesssim c_j 2^{(\frac{1}{2}-\frac{2}{q})j}\|v\|_{\mathfrak{G}^\varepsilon_q(T)}\|w\|_{\mathfrak{G}^\varepsilon_q(T)}. \tag{4.7}$$

Thanks to (3.15) and (4.7), we infer

$$\begin{aligned}
&\|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}\mathrm{e}^{\lambda\frac{t}{\sqrt{T}}A}\Delta_j(B(v,w))\|_{L^q_T(L^2)}\\
&\lesssim \mathrm{e}^{\frac{\lambda^2}{4(1-\varepsilon)}}\Big\|\int_0^t \mathrm{e}^{-\varepsilon(t-s)2^{2j}}2^j\|\mathrm{e}^{-\frac{\lambda^2}{2(1-\varepsilon)}\frac{s}{T}}\mathrm{e}^{\lambda\frac{s}{\sqrt{T}}A}\Delta_j(v\otimes w)(s)\|_{L^2}\,\mathrm{d}s\Big\|_{L^q_T}\\
&\lesssim \mathrm{e}^{\frac{\lambda^2}{4(1-\varepsilon)}}2^j\|\mathrm{e}^{-\varepsilon t2^{2j}}\|_{L^1_T}\|\mathrm{e}^{-\frac{\lambda^2}{2(1-\varepsilon)}\frac{s}{T}}\mathrm{e}^{\lambda\frac{s}{\sqrt{T}}A}\Delta_j(vw)\|_{L^q_T(L^2)}\\
&\leq C_\varepsilon c_j 2^{-(\frac{1}{2}+\frac{2}{q})j}\|v\|_{\mathfrak{G}^\varepsilon_q(T)}\|w\|_{\mathfrak{G}^\varepsilon_q(T)}
\end{aligned}$$

and

$$\begin{aligned}
&\|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}\mathrm{e}^{\lambda\frac{t}{\sqrt{T}}A}\Delta_j(B(v,w))\|_{L^\infty_T(L^2)}\\
&\lesssim \mathrm{e}^{\frac{\lambda^2}{4(1-\varepsilon)}}\Big\|\int_0^t \mathrm{e}^{-\varepsilon(t-s)2^{2j}}2^j\|\mathrm{e}^{-\frac{\lambda^2}{2(1-\varepsilon)}\frac{s}{T}}\mathrm{e}^{\lambda\frac{s}{\sqrt{T}}A}\Delta_j(v\otimes w)(s)\|_{L^2}\,\mathrm{d}s\Big\|_{L^\infty_T}\\
&\lesssim \mathrm{e}^{\frac{\lambda^2}{4(1-\varepsilon)}}2^j\|\mathrm{e}^{-\varepsilon t2^{2j}}\|_{L^{\frac{q}{q-1}}_T}\|\mathrm{e}^{-\frac{\lambda^2}{2(1-\varepsilon)}\frac{s}{T}}\mathrm{e}^{\lambda\frac{s}{\sqrt{T}}A}\Delta_j(vw)\|_{L^q_T(L^2)}\\
&\leq C_\varepsilon c_j 2^{-\frac{j}{2}}\|v\|_{\mathfrak{G}^\varepsilon_q(T)}\|w\|_{\mathfrak{G}^\varepsilon_q(T)},
\end{aligned}$$

which together Definition 4.1 ensures (4.6). This completes the proof of Lemma 4.2.

Lemma 4.3 *Let $v\in\mathfrak{G}^\varepsilon_q(T)$ and $u_0\in L^3(\mathbb{R}^3)$. Then there holds*

$$\|B(v,u_L)\|_{\mathfrak{G}^\varepsilon_q(T)}\leq C_{\varepsilon,\eta}\mathrm{e}^{\frac{\lambda^2}{4(1-\varepsilon)}}\|u_{\eta,L}\|_{\widetilde{L}^{2q}_T(\dot{B}^{\frac{1}{q}}_{3,3})}\|v\|_{\mathfrak{G}^\varepsilon_q(T)}. \tag{4.8}$$

Proof By applying Hölder inequality and Plancherel equality, we find

$$\|\mathrm{e}^{-\frac{\lambda^2}{2(1-\varepsilon)}\frac{t}{T}}\mathrm{e}^{\lambda\frac{t}{\sqrt{T}}A}\Delta_j(T_v u_L)\|_{L^q_T(L^2)}$$

$$\lesssim \sum_{|j-j'|\leq 4} \|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}} S_{j'-1} v^\lambda\|_{L^{2q}_T(L^6)} \|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}} \Delta_{j'} u^\lambda_L\|_{L^{2q}_T(L^3)}.$$

Yet it follows from Lemma A.1 that

$$\begin{aligned}\|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}} S_{j'-1} v^\lambda\|_{L^{2q}_T(L^6)} &\lesssim \sum_{\ell\leq j'-2} 2^\ell \|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}} \Delta_\ell v^\lambda\|_{L^{2q}_T(L^2)}\\ &\lesssim \sum_{\ell\leq j'-2} c_\ell 2^{(\frac{1}{2}-\frac{1}{q})\ell} \|v\|_{\mathfrak{G}^\varepsilon_q(T)} \lesssim c_j 2^{(\frac{1}{2}-\frac{1}{q})j} \|v\|_{\mathfrak{G}^\varepsilon_q(T)},\end{aligned}$$

from which, we infer

$$\begin{aligned}&\|\mathrm{e}^{-\frac{\lambda^2}{2(1-\varepsilon)}\frac{t}{T}} \mathrm{e}^{\lambda\frac{t}{\sqrt{T}}A} \Delta_j(T_v u_L)\|_{L^q_T(L^2)}\\ &\leq C_{\varepsilon,\eta} \sum_{|j-j'|\leq 4} c_{j'} 2^{(\frac{1}{2}-\frac{2}{q})j'} \|v\|_{\mathfrak{G}^\varepsilon_q(T)} \|u_{\eta,L}\|_{\widetilde{L}^{2q}_T(\dot{B}^{\frac{1}{q}}_{3,3})}\\ &\leq C_{\varepsilon,\eta} c_j 2^{(\frac{1}{2}-\frac{2}{q})j} \|u_{\eta,L}\|_{\widetilde{L}^{2q}_T(\dot{B}^{\frac{1}{q}}_{3,3})} \|v\|_{\mathfrak{G}^\varepsilon_q(T)}.\end{aligned} \tag{4.9}$$

Similarly, we have

$$\begin{aligned}&\|\mathrm{e}^{-\frac{\lambda^2}{2(1-\varepsilon)}\frac{t}{T}} \mathrm{e}^{\lambda\frac{t}{\sqrt{T}}A} \Delta_j(T_{u_L} v)\|_{L^q_T(L^2)}\\ &\lesssim \sum_{|j-j'|\leq 4} \|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}} S_{j'-1} u^\lambda_L\|_{L^{2q}_T(L^\infty)} \|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}} \Delta_{j'} v^\lambda\|_{L^{2q}_T(L^2)}.\end{aligned}$$

While we get, by a similar derivation of (3.7), that

$$\begin{aligned}\|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}} S_{j'-1} u^\lambda_L\|_{L^{2q}_T(L^\infty)} &\lesssim \sum_{\ell\leq j'-2} 2^\ell \|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}} \Delta_\ell u^\lambda_L\|_{L^{2q}_T(L^3)}\\ &\leq C_{\varepsilon,\eta} \sum_{\ell\leq j'-2} 2^{(1-\frac{1}{q})\ell} \|u_{\eta,L}\|_{\widetilde{L}^{2q}_T(\dot{B}^{\frac{1}{q}}_{3,3})}\\ &\leq C_{\varepsilon,\eta} 2^{(1-\frac{1}{q})j} \|u_{\eta,L}\|_{\widetilde{L}^{2q}_T(\dot{B}^{\frac{1}{q}}_{3,3})},\end{aligned}$$

which together with Definition 4.1 ensures that

$$\begin{aligned}\|\mathrm{e}^{-\frac{\lambda^2}{2(1-\varepsilon)}\frac{t}{T}} \mathrm{e}^{\lambda\frac{t}{\sqrt{T}}A} \Delta_j(T_{u_L} v)\|_{L^q_T(L^2)} &\leq C_{\varepsilon,\eta} \sum_{|j-j'|\leq 4} c_{j'} 2^{(\frac{1}{2}-\frac{2}{q})j'} \|u_{\eta,L}\|_{\widetilde{L}^{2q}_T(\dot{B}^{\frac{1}{q}}_{3,3})} \|v\|_{\mathfrak{G}^\varepsilon_q(T)}\\ &\leq C_{\varepsilon,\eta} c_j 2^{(\frac{1}{2}-\frac{2}{q})j} \|u_{\eta,L}\|_{\widetilde{L}^{2q}_T(\dot{B}^{\frac{1}{q}}_{3,3})} \|v\|_{\mathfrak{G}^\varepsilon_q(T)}.\end{aligned} \tag{4.10}$$

Whereas we get, by applying Lemma A.1, that

$$\begin{aligned}&\|\mathrm{e}^{-\frac{\lambda^2}{2(1-\varepsilon)}\frac{t}{T}} \mathrm{e}^{\lambda\frac{t}{\sqrt{T}}A} \Delta_j(R(u_L, v))\|_{L^q_T(L^2)}\\ &\lesssim 2^j \sum_{j'\geq j-3} \|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}} \Delta_{j'} u^\lambda_L\|_{L^{2q}_T(L^3)} \|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}} \widetilde{\Delta}_{j'} v^\lambda\|_{L^{2q}_T(L^2)}\\ &\leq C_{\varepsilon,\eta} 2^j \sum_{j'\geq j-3} c_{j'} 2^{-(\frac{1}{2}+\frac{2}{q})j'} \|u_{\eta,L}\|_{\widetilde{L}^{2q}_T(\dot{B}^{\frac{1}{q}}_{3,3})} \|v\|_{\mathfrak{G}^\varepsilon_q(T)}\\ &\leq C_{\varepsilon,\eta} c_j 2^{(\frac{1}{2}-\frac{2}{q})j} \|u_{\eta,L}\|_{\widetilde{L}^{2q}_T(\dot{B}^{\frac{1}{q}}_{3,3})} \|v\|_{\mathfrak{G}^\varepsilon_q(T)}.\end{aligned}$$

By summing up the estimates (4.9), (4.10) with the above inequality, we achieve

$$\|\mathrm{e}^{-\frac{\lambda^2}{2(1-\varepsilon)}\frac{t}{T}}\mathrm{e}^{\lambda\frac{t}{\sqrt{T}}A}\Delta_j(u_L\otimes v)\|_{L^q_T(L^2)}\leq C_{\varepsilon,\eta}c_j2^{(\frac{1}{2}-\frac{2}{q})j}\|u_{\eta,L}\|_{\widetilde{L}^{2q}_T(\dot{B}^{\frac{1}{q}}_{3,3})}\|v\|_{\mathfrak{G}^\varepsilon_q(T)}. \tag{4.11}$$

Then similar to the estimate of (3.22), we infer

$$\begin{aligned}&\|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}\mathrm{e}^{\lambda\frac{t}{\sqrt{T}}A}\Delta_j(B(v,u_L))\|_{L^\infty_T(L^2)}\\ \lesssim\;&\mathrm{e}^{\frac{\lambda^2}{4(1-\varepsilon)}}\|\mathrm{e}^{-\varepsilon t2^{2j}}\|_{L_T^{\frac{q}{q-1}}}2^j\|\mathrm{e}^{-\frac{\lambda^2}{2(1-\varepsilon)}\frac{s}{T}}\mathrm{e}^{\lambda\frac{s}{\sqrt{T}}A}\Delta_j(v\otimes u_L)\|_{L^q_T(L^2)}\\ \lesssim\;&C_{\varepsilon,\eta}c_j2^{-\frac{j}{2}}\mathrm{e}^{\frac{\lambda^2}{4(1-\varepsilon)}}\|u_{\eta,L}\|_{\widetilde{L}^{2q}_T(\dot{B}^{\frac{1}{q}}_{3,3})}\|v\|_{\mathfrak{G}^\varepsilon_q(T)}\end{aligned}$$

and

$$\begin{aligned}&\|\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}\mathrm{e}^{\lambda\frac{t}{\sqrt{T}}A}\Delta_j(B(v,u_L))\|_{L^q_T(L^2)}\\ \lesssim\;&\mathrm{e}^{\frac{\lambda^2}{4(1-\varepsilon)}}\|\mathrm{e}^{-\varepsilon t2^{2j}}\|_{L^{\frac{1}{2}}_T}2^j\|\mathrm{e}^{-\frac{\lambda^2}{2(1-\varepsilon)}\frac{s}{T}}\mathrm{e}^{\lambda\frac{s}{\sqrt{T}}A}\Delta_j(v\otimes u_L)\|_{L^q_T(L^2)}\\ \leq\;&C_{\varepsilon,\eta}c_j2^{-(\frac{1}{2}+\frac{2}{q})j}\mathrm{e}^{\frac{\lambda^2}{4(1-\varepsilon)}}\|u_{\eta,L}\|_{\widetilde{L}^{2q}_T(\dot{B}^{\frac{1}{q}}_{3,3})}\|v\|_{\mathfrak{G}^\varepsilon_q(T)},\end{aligned}$$

which together with Definition 4.1 ensures (4.8), and we complete the proof of Lemma 4.3.

Now we are in a position to complete the proof of Theorem 1.3.

Proof of Theorem 1.3 Once again we are going to prove Theorem 1.3 via Lemma 2.4. In order to do so, we take $X=\mathfrak{G}^\varepsilon_p(T)$, which is determined by Definition 4.1. Let L and x_0 be determined by (2.14). It follows from Lemmas 4.1–4.3 that

$$\begin{aligned}\|x_0\|_X&\leq C_{\varepsilon,\eta}\mathrm{e}^{\frac{\lambda^2}{4(1-\varepsilon)}}\|u_{\eta,L}\|^2_{\widetilde{L}^{2q}_T(\dot{B}^{\frac{1}{q}}_{3,3})},\\ \|B\|_{\mathcal{B}(X)}&\leq C_\varepsilon\mathrm{e}^{\frac{\lambda^2}{4(1-\varepsilon)}},\\ \|L\|_{\mathcal{L}(X)}&\leq C_{\varepsilon,\eta}\mathrm{e}^{\frac{\lambda^2}{4(1-\varepsilon)}}\|u_{\eta,L}\|_{\widetilde{L}^{2q}_T(\dot{B}^{\frac{1}{q}}_{3,3})}.\end{aligned}$$

In view of (1.11), for c sufficiently small, we take T so small that

$$C_{\varepsilon,\eta}\mathrm{e}^{\frac{\lambda^2}{4(1-\varepsilon)}}\|u_{\eta,L}\|_{\widetilde{L}^{2q}_T(\dot{B}^{\frac{1}{q}}_{3,3})}=c, \tag{4.12}$$

which implies that

$$\lambda\stackrel{\text{def}}{=}2\sqrt{(1-\varepsilon)\ln\frac{c}{C_{\varepsilon,\eta}\|u_{\eta,L}\|_{\widetilde{L}^{2q}_T(\dot{B}^{\frac{1}{q}}_{3,3})}}}. \tag{4.13}$$

Observing that under the condition (4.12), for (1.4), there holds (2.13), so that we deduce from Lemma 2.4 that (1.4) has a unique solution v in $\mathfrak{G}^\varepsilon_q(T)$, which in particular satisfies

$$\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}}\|\mathrm{e}^{\lambda\sqrt{T}A}v(T)\|_{\dot{H}^{\frac{1}{2}}}\leq c_{\varepsilon,\eta}\mathrm{e}^{-\frac{\lambda^2}{4(1-\varepsilon)}}.$$

Hence thanks to (1.11) and (4.13), there exists some t_2 so that there holds (1.9), from which, we infer

$$\liminf_{t\to 0} \frac{R(v(t))}{t^{\frac{1}{2}}(-\ln \|u_{\eta,L}\|_{\widetilde{L}^{2q}_T(\dot{B}^{\frac{1}{q}}_{3,3})})^{\frac{1}{2}}} \geq 2\sqrt{1-\varepsilon},$$

which implies (1.10). This finishes the proof of Theorem 1.3.

Appendix A Tool Box on Littlewood-Paley Theory

For the convenience of readers, we shall collect some basic facts on Littlewood-Paley theory in this section. Let us first recall the following dyadic operators from [1]:

$$\Delta_j a \stackrel{\text{def}}{=} \mathcal{F}^{-1}(\varphi(2^{-j}|\xi|)\widehat{a}), \quad S_j a \stackrel{\text{def}}{=} \mathcal{F}^{-1}(\chi(2^{-j}|\xi|)\widehat{a}), \tag{A.1}$$

where $\widehat{a}$ denotes the Fourier transform of a, while $\mathcal{F}^{-1}a$ denotes its inverse Fourier transform, $\chi(\tau)$ and $\varphi(\tau)$ are smooth functions such that

$$\text{Supp } \varphi \subset \Big\{\tau\in\mathbb{R} : \frac{3}{4}\leq|\tau|\leq\frac{8}{3}\Big\} \quad\text{and}\quad \sum_{j\in\mathbb{Z}}\varphi(2^{-j}\tau)=1,\ \forall\tau>0\,,$$
$$\text{Supp } \chi \subset \Big\{\tau\in\mathbb{R} : |\tau|\leq\frac{4}{3}\Big\} \quad\text{and}\quad \chi(\tau)+\sum_{j\geq 0}\varphi(2^{-j}\tau)=1,\ \forall\tau\in\mathbb{R}\,.$$

Definition A.1 (Besov spaces) *Let (p,r) be in $[1,\infty]^2$ and s in $\mathbb{R}$. Let us consider $u\in\mathcal{S}'_h$, which means that u is in $\mathcal{S}'(\mathbb{R}^3)$ and satisfies $\lim_{j\to-\infty}\|S_j u\|_{L^\infty}=0$. We set*

$$\|u\|_{\dot{B}^s_{p,r}} \stackrel{\text{def}}{=} \|(2^{js}\|\Delta_j u\|_{L^p})_{j\in\mathbb{Z}}\|_{\ell^r(\mathbb{Z})}.$$

- *If $s<\frac{3}{p}$ $\big($or $s\leq\frac{3}{p}$ if $r=1\big)$, we define $\dot{B}^s_{p,r}(\mathbb{R}^3)\stackrel{\text{def}}{=}\{u\in\mathcal{S}'_h(\mathbb{R}^3)\mid \|u\|_{\dot{B}^s_{p,r}}<\infty\}$.*
- *If $k\in\mathbb{N}$ and if $\frac{3}{p}+k\leq s<\frac{3}{p}+k+1$ $\big($or $s\leq\frac{3}{p}+k+1$ if $r=1\big)$, then we fine $\dot{B}^s_{p,r}(\mathbb{R}^3)$ as the subset of $u\in\mathcal{S}'_h(\mathbb{R}^3)$ such that $\partial^\beta u$ belongs to $\dot{B}^{s-k}_{p,r}(\mathbb{R}^3)$ whenever $|\beta|=k$.*

In the particular case when $p=r=2$, $\dot{B}^s_{2,2}(\mathbb{R}^3)$ is the classical homogenous Sobolev space $\dot{H}^s(\mathbb{R}^3)$.

In order to obtain a better description of the regularizing effect of the transport-diffusion equation, we need to use Chemin-Lerner type spaces $\widetilde{L}^p_T(\dot{B}^s_{p,r})$).

Definition A.2 *Let $q\in[1,+\infty]$ and $T\in]0,+\infty]$. If $q<\infty$, we define $\widetilde{L}^q(0,T;\dot{B}^s_{p,r})$ as the completion of $C([0,T];\mathcal{S}(\mathbb{R}^3))$ by the norm*

$$\|a\|_{\widetilde{L}^q(0,T;\dot{B}^s_{p,r})} \stackrel{\text{def}}{=} \Big\{\sum_{j\in\mathbb{Z}} 2^{jrs}\Big(\int_0^T \|\Delta_j\, a(t)\|^q_{L^p}\,\mathrm{d}t\Big)^{\frac{r}{q}}\Big\}^{\frac{1}{r}}$$

with the usual change if $q=\infty$. For simplicity, we shall denote $\|a\|_{\widetilde{L}^q_T(\dot{B}^s_{p,r})}\stackrel{\text{def}}{=}\|a\|_{\widetilde{L}^q(0,T;\dot{B}^s_{p,r})}$.

Lemma A.1 *Let $\mathcal{B}$ be a ball of $\mathbb{R}^3$, and $\mathcal{C}$ be a ring of $\mathbb{R}^3$; let $1 \le p_2 \le p_1 \le \infty$ and $1 \le q_2 \le q_1 \le \infty$. Then there holds*:

$$\begin{aligned}&\text{If}\quad \mathrm{Supp}\ \widehat{a} \subset 2^j\mathcal{B} \Rightarrow \|\partial_x^\alpha a\|_{L^{p_1}} \lesssim 2^{j(|\alpha|+3(\frac{1}{p_2}-\frac{1}{p_1}))}\|a\|_{L^{p_2}};\\ &\text{if}\quad \mathrm{Supp}\ \widehat{a} \subset 2^j\mathcal{C} \Rightarrow \|a\|_{L^{p_1}} \lesssim 2^{-jN} \sup_{|\alpha|=N} \|\partial_x^\alpha a\|_{L^{p_1}}.\end{aligned}$$

We also recall the action of the heat semigroup on distribution with the Fourier transform of which is supported in an annulus.

Lemma A.2 (see [1, Lemma 2.4]) *Let $\mathcal{C}$ be an annulus. Then there exist constants c and C, such that for any $p \in [1,\infty]$ and any couple of (t,λ), there holds*

$$\mathrm{Supp}\ \widehat{u} \subseteq \lambda\mathcal{C} \ \Rightarrow\ \|\mathrm{e}^{t\Delta}u\|_{L^p} \le C\mathrm{e}^{-ct\lambda^2}\|u\|_{L^p}. \tag{A.2}$$

To deal with the estimate of product of two distributions, we constantly use the following para-differential decomposition from [2]: For any functions $f, g \in \mathcal{S}'(\mathbb{R}^3)$,

$$fg = T_f g + T_g f + R(f,g), \tag{A.3}$$

where

$$T_f g \stackrel{\text{def}}{=} \sum_{j\in\mathbb{Z}} S_{j-1}f\Delta_j g, \quad R(f,g) \stackrel{\text{def}}{=} \sum_{j\in\mathbb{Z}} \Delta_j f\widetilde{\Delta}_j g \quad \text{with } \widetilde{\Delta}_j g \stackrel{\text{def}}{=} \sum_{j'=j-1}^{j+1} \Delta_{j'} g.$$

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