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Hayward Quasilocal Energy of Tori*

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Abstract In this paper, the authors show that one cannot dream of the positivity of the Hayward energy in the general situation. They consider a scenario of a spherically symmetric constant density star matched to the Schwarzschild solution, representing momentarily static initial data. It is proved that any topological tori within the star, distorted or not, have strictly positive Hayward energy. Surprisingly we find analytic examples of 'thin' tori with negative Hayward energy in the outer neighborhood of the Schwarzschild horizon. These tori are swept out by rotating the standard round circles in the static coordinates but they are distorted in the isotropic coordinates. Numerical results also indicate that there exist horizontally dragged tori with strictly negative Hayward energy in the region between the boundary of the star and the Schwarzschild horizon.

Keywords Quasilocal energy, Positivity, Toroidal topology
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1 Introduction

Finding a suitable notion of quasilocal energy-momentum for finite spacetime domains at quasilocal level is one of the most challenging problems in classical general relativity (see [16]). Even though there was high expectation in the 1980's, this problem has proven to be surprisingly difficult and we have no ultimately satisfied expression yet (see [20]). However, there are various 'lists of criteria of reasonableness' in the literature among which expect that the quasilocal energy should be nonnegative under certain energy conditions (see [4]). This expectation is inspired by the successful proof of the positivity of the total gravitational energy (see [1, 17–18, 22].

The existing candidates for the quasilocal energy are mixture of advantages and difficulties (see [20]). In 1993, Brown and York introduced a notion of quasilocal energy following the Hamiltonian-Jacobi method (see [3]). The energy expression can be viewed as the total mean curvatures comparison in the physical space and the reference space. When the surface in question is a topological sphere with positive Gauss curvature and positive mean curvature, Shi and Tam proved that the Brown-York energy is nonnegative (see [19]). One important feature of the Brown-York energy is that it requires a flat reference via isometric embedding. The issue of isometric embedding is a very beautiful and challenging problem in mathematics. However, it

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is difficult to compute the precise value of the energy of a generic surface for working relativists. Hawking had a different definition whose advantages are simplicity and calculability (see [6]).

Let $(\widetilde{M}, \widetilde{g})$ be a spacetime. Assume that (Σ, σ) is a spacelike closed 2-surface with the induced 2-metric σ. Consider the ingoing $(-)$ and outgoing $(+)$ null geodesic congruences from Σ. Let $\theta_\pm$ be the null expansions. Then the Hawking energy (see [6]) of the 2-surface Σ is defined as

$$E_{\text{Hawking}}(\Sigma) = \frac{1}{8\pi}\sqrt{\frac{|\Sigma|}{16\pi}}\int_\Sigma (\mathrm{Scal}_\sigma + \theta_+\theta_-)\mathrm{d}\sigma.$$

Here Scal_σ is the scalar curvature and $|\Sigma|$ is the area of Σ with respect to the 2-metric σ.

For a 2-surface Σ embedded in a spacelike hypersurface (M, g), there is an energy expression called the Geroch energy (see [5])

$$E_{\text{Geroch}}(\Sigma) = \frac{1}{8\pi}\sqrt{\frac{|\Sigma|}{16\pi}}\int_\Sigma \Big(\mathrm{Scal}_\sigma - \frac{1}{2}H^2\Big)\mathrm{d}\sigma,$$

where H is the mean curvature of (Σ, σ) in the hypersurface (M, g). The Geroch energy is very useful. Notably, it is monotonically increasing along the inverse mean curvature flow and this plays a key role in the proof of the Riemannian Penrose inequality (see [10]).

Unfortunately, the value of either the Hawking energy or the Geroch energy is too small in some sense. Even in the flat $\mathbb{R}^3$, $E_{\text{Geroch}}(\Sigma)$ is strictly negative unless Σ is round. And it is also shown in [2, Section 2] that the Hawking energy is never positive for arbitrary spheres in a totally umbilical spacelike hypersurface in a conformally flat Einstein space.

To overcome these drawbacks, Hayward proposed a measure of energy by adding additional terms based on the double-null foliation. His energy becomes zero for any generic 2-surface in the flat Minkowski spacetime $\mathbb{R}^{3,1}$. Let $\theta^\pm$ and $(\sigma)^\pm_{ij}$ be the expansions and the shear tensors of the outgoing and incoming null geodesic congruences, respectively. Then the Hayward energy reads

$$E_{\text{Hayward}}(\Sigma) = \frac{1}{8\pi}\sqrt{\frac{|\Sigma|}{16\pi}}\int_\Sigma \Big(\mathrm{Scal}_\sigma + \theta^+\theta^- - \frac{1}{2}(\sigma)^+_{AB}(\sigma)^{AB}_-\Big)\mathrm{d}\sigma.$$

The above energy expression (see [20, Eqn (6.5), Page 61]) is different from the original one Hayward suggested in [8] which contains an additional anoholonomicity term ω^k, being the projection onto Σ of the commutators of the null normal vectors to Σ. This anoholonomicity indeed is a boost-gauge-dependent quantity (see [20, Page 61]). Throughout this paper, we use the energy expression as Eqn (6.5) in [20, Page 61] and still call it the Hayward energy. There are other notions of the quasilocal energy in the literature (e.g. the Kijowski energy (see [13]) and the Wang-Yau energy (see [21])). For a review and a more detailed discussion of the various energy expressions, see e.g. [20].

Without much fanfare, the Hayward energy has some nice properties. In particular, it behaves quite well for the gravitational collapse of the Oppenheimer-Snyder dust cloud with uniform density (see [9]). It has been shown that the Hayward energy is conserved and remains positive during the collapse for any 2-surface. It should be emphasized that even though the spacetime is assumed to be spherically symmetric, there are no restrictions on the topology and the symmetry of the surface for the results in [9].

Most known results of the quasilocal energy in the literature are concerned with 2-surfaces with spherical topology. There seems to be no ‘a priori’ restriction on the topology of the surface. There do exist trapped surfaces or minimal surfaces with toroidal topology (see [11–12, 14]). The ideal construction of quasilocal energy should work for any closed orientable 2-surfaces (see [20]).

It seems to bring much more attentions to the investigation of the Hayward energy. Our ambition in this paper is quite modest. We cannot dream of positivity of the Hayward energy in the general situation. We consider a scenario of a spherically symmetric constant density star matched to the Schwarzschild solution but the test surfaces are chosen to be certain types of tori—the simplest 2-surfaces with nontrivial topology. This 'constant density star model' represents momentarily static initial data and it was used to study the relationship between the minimal energy density and the size of the star in [15]. Precisely, we show that any topological tori entirely within the star, distorted or not, have strictly positive Hayward energy. The energy expression for tori outside the star is also given and we prove that standard 'thin tori' in the isotropic coordinates must have positive Hayward energy. Surprisingly we find analytic examples of 'thin' tori with negative Hayward energy in the outer neighborhood of the Schwarzschild horizon. These tori are swept out by rotating the standard round circles in the static coordinates but they are distorted in the isotropic coordinates. Numerical results also indicate that there exist horizontally dragged tori with strictly negative Hayward energy in the region between the boundary of the star and the Schwarzschild horizon.

2 Tori Within the Star

We consider a spherically symmetric initial data (M, g) for a constant density star matched to the Schwarzschild solution, which presents a time slice in a static spacetime $(\widetilde{M}, \widetilde{g})$. In the isotropic coordinates $\{x^i\}$, the 3-metric g reads (see [15])

$$\begin{aligned} g &= \Phi^4(R)((\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2) \\ &= \Phi^4(R)(\mathrm{d}R^2 + R^2\mathrm{d}\Theta^2 + R^2\sin^2\Theta\mathrm{d}\varphi^2), \end{aligned} \tag{2.1}$$

where

$$\Phi(R) = \begin{cases} \dfrac{(1+\beta R_0^2)^{\frac{3}{2}}}{\sqrt{1+\beta R^2}} & \text{for } R \le R_0, \\ 1 + \beta \dfrac{R_0^3}{R} & \text{for } R > R_0. \end{cases} \tag{2.2}$$

Here $R = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$ is the spherical radial coordinate and R_0 is the Euclidean radius of the star.

The nonzero components of the Riemann curvature tensors of g are

$$\begin{aligned} R_{R\Theta R\Theta} &= -2R\Phi^2(-R(\Phi')^2 + \Phi(\Phi' + R\Phi'')), \\ R_{R\varphi R\varphi} &= -2R\Phi^2\sin^2\Theta(-R(\Phi')^2 + \Phi(\Phi' + R\Phi'')), \\ R_{\Theta\varphi\Theta\varphi} &= -4R^3\sin^2\Theta\ \Phi^2\Phi'(\Phi + R\Phi'), \end{aligned} \tag{2.3}$$

where the prime $'$ denotes the differentiation with respect to R.

In this section, we prove that any generic topological tori lying entirely within the star must have positive Hayward energy. Without loss of generality, from now on we assume that $R_0 = 1$ for convenience. In this case, the conformal factor $\Phi = \frac{(1+\beta)^{\frac{3}{2}}}{\sqrt{1+\beta R^2}}$ for $R \le 1$ and $\Phi = 1 + \frac{\beta}{R}$ for $R > 1$. With this choice, the asymptotic mass of the spacetime is $m = 2\beta$.

Let Σ be a generic topological torus lying within the star. Denote by $\{e_2, e_3\}$ the orthonormal frame tangential to Σ. Then the contracted version of the Gauss equation for (Σ, σ) in (M, g) reads

$$\mathrm{Scal}_\sigma - H^2 + |A|_\sigma^2 = \sum_{i,j=2,3} R(e_i, e_j, e_i, e_j). \tag{2.4}$$

Here H and A are the mean curvature and the second fundamental form of (Σ, σ) in (M, g) respectively, and $R(e_i, e_j, e_i, e_j)$ is the curvature of the 3-metric g acting on the frame $\{e_i\}$.

Note that (M, g) is momentarily static, i.e. the extrinsic curvature in $(\widetilde{M}, \widetilde{g})$ vanishing. By (2.4), the Hayward quasilocal energy can be rewritten as

$$E_{\mathrm{Hayward}}(\Sigma) = \frac{1}{8\pi}\sqrt{\frac{|\Sigma|}{16\pi}}\int_\Sigma 2R(e_2, e_3, e_2, e_3)\mathrm{d}\sigma.$$

From (2.1)–(2.3), it is easy to verify that

$$R_{ijkl} = \frac{4\beta}{(1+\beta)^6}(g_{ik}g_{jl} - g_{jk}g_{il}) = \frac{\mathrm{Scal}_g}{6}(g_{ik}g_{jl} - g_{jk}g_{il}).$$

This indicates that the star has positive constant sectional curvature. The integrand $R(e_2, e_3, e_2, e_3)$ turns out to be the sectional curvature of the 3-metric g with respect to the tangent plane of a generic topological torus. It immediately follows that $E_{\mathrm{Hayward}}(\Sigma) > 0$.

3 Tori Outside the Star

Assume that we are given a family of coordinate tori, denoted by Σ, of major radius a and minor radius b, with the x^3-axis as the symmetry axis. They are swept out by rotating the standard circles $\{(x^1, x^3) \mid (x^1 - a)^2 + (x^3)^2 = b^2\}$ along the x^3-axis, as illustrated in Figure 1.

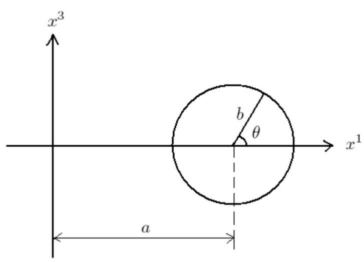

Figure 1 Standard circle in the $x^1 - x^3$-plane.

Indeed, these tori are parameterized as

$$\begin{aligned}
x^1 &= (a + b\cos\theta)\cos\varphi,\\
x^2 &= (a + b\cos\theta)\sin\varphi, \quad \theta \in [0, 2\pi),\ \varphi \in [0, 2\pi),\\
x^3 &= b\sin\theta.
\end{aligned} \tag{3.1}$$

The induced 2-metric σ reads

$$\sigma = \Phi^4(R)\big(b^2\mathrm{d}\theta^2 + (a + b\cos\theta)^2\mathrm{d}\varphi^2\big)$$

$$= \sigma_{22}\mathrm{d}\theta^2 + \sigma_{33}\mathrm{d}\varphi^2,$$

where $\sigma_{22} = \Phi^4 b^2$ and $\sigma_{33} = \Phi^4(a + b\cos\theta)^2$. The area form of Σ is

$$\mathrm{d}\sigma = \sqrt{\det\sigma}\mathrm{d}\theta \wedge \mathrm{d}\varphi = \Phi^4 b(a + b\cos\theta)\mathrm{d}\theta \wedge \mathrm{d}\varphi.$$

Denote by $\{e_2, e_3\}$ the orthonormal frame tangential to the surface Σ. By the chain rule, one has

$$\begin{aligned} e_2 &= \frac{1}{\sqrt{\sigma_{22}}}\frac{\partial}{\partial\theta} = \frac{1}{\Phi^2 b}\Big(- \frac{ab\sin\theta}{\sqrt{a^2 + b^2 + 2ab\cos\theta}}\frac{\partial}{\partial R} - \frac{b(b + a\cos\theta)}{a^2 + b^2 + 2ab\cos\theta}\frac{\partial}{\partial\Theta}\Big) \\ &= (e_2)^R\frac{\partial}{\partial R} + (e_2)^\Theta\frac{\partial}{\partial\Theta}, \\ e_3 &= \frac{1}{\sqrt{\sigma_{33}}}\frac{\partial}{\partial\varphi} = \frac{1}{\Phi^2(a + b\cos\theta)}\frac{\partial}{\partial\varphi} \\ &= (e_3)^\varphi\frac{\partial}{\partial\varphi}. \end{aligned}$$

Here

$$\begin{aligned} (e_2)^R &= -\frac{1}{\Phi^2}\frac{a\sin\theta}{\sqrt{a^2 + b^2 + 2ab\cos\theta}}, \\ (e_2)^\Theta &= -\frac{1}{\Phi^2}\frac{b + a\cos\theta}{a^2 + b^2 + 2ab\cos\theta}, \\ (e_3)^\varphi &= \frac{1}{\Phi^2(a + b\cos\theta)}. \end{aligned}$$

In this section, we calculate the Hayward energy for the coordinate tori (3.1) outside the star. In this case, the conformal factor $\Phi = 1 + \frac{\beta}{R}$ and

$$R(e_2, e_3, e_2, e_3) = \frac{R\beta(a^2 + 4b^2 + 8ab\cos\theta + 3a^2\cos 2\theta)}{(R + \beta)^6},$$

where $R = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$.

The Hayward energy for Σ is

$$\begin{aligned} E_{\text{Hayward}}(\Sigma) &= \frac{1}{8\pi}\sqrt{\frac{|\Sigma|}{16\pi}}\int_\Sigma 2R(e_2, e_3, e_2, e_3)\mathrm{d}\sigma \\ &= \frac{1}{4}\sqrt{\frac{|\Sigma|}{16\pi}}\int_0^{2\pi}\frac{2b\beta(a + b\cos\theta)(a^2 + 4b^2 + 3a^2\cos 2\theta + 8ab\cos\theta)}{(a^2 + 2ab\cos\theta + b^2)^{\frac{3}{2}}(\sqrt{a^2 + 2ab\cos\theta + b^2} + \beta)^2}\mathrm{d}\theta. \end{aligned} \tag{3.2}$$

For small b, one has

$$\begin{aligned} E_{\text{Hayward}}(\Sigma) &= \frac{1}{4}\sqrt{\frac{|\Sigma|}{16\pi}}\int_0^{2\pi}\Big[\frac{2\beta(1 + 3\cos 2\theta)}{(a + \beta)^2}b + \mathcal{O}(b^2)\Big]\mathrm{d}\theta \\ &= \frac{1}{4}\sqrt{\frac{|\Sigma|}{16\pi}}\Big[\frac{4\pi\beta b}{(a + \beta)^2} + \mathcal{O}(b^2)\Big]. \end{aligned}$$

This shows that standard 'thin tori' in the isotropic coordinates outside the star have positive Hayward energy.

For general a and b, it is not easy to figure out the triangle integral. We calculate the Hayward energy (3.2) numerically and the results indicate that the Hayward energy is positive for the coordinate tori (3.1). For instance, we show the plot of $E_{\text{Hayward}}(\Sigma)$ with respect to a ($4 \le a \le 10$) and b ($0.01 \le b \le 1$) for $\beta = 3 + 2\sqrt{2}$ in Figure 2. The number $3 + 2\sqrt{2}$ is the minimal value of β to construct marginally trapped tori in the star (see [12, Section III]).

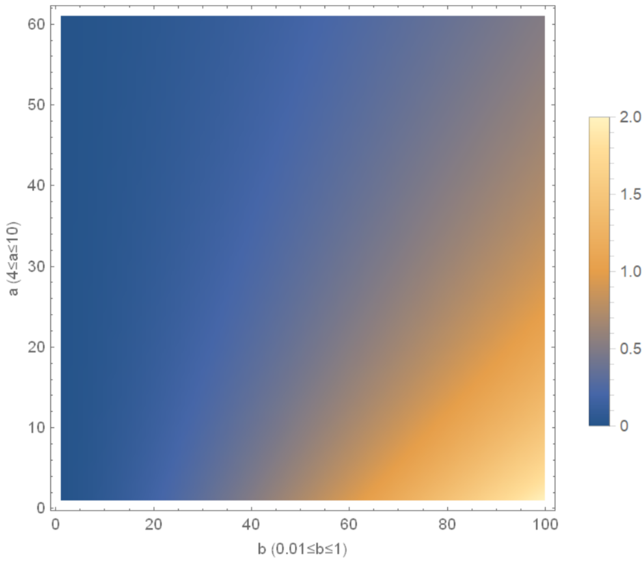

Figure 2 Plot of $E_{\text{Hayward}}(\Sigma)$ of the coordinate tori with respect to a and b for $\beta = 3 + 2\sqrt{2}$.

4 Distorted Thin Tori with Negative Energy

In the previous sections, we have checked many cases that tori have positive Hayward energy. Do we really have confidence to conjecture that the positivity property for the Hayward quasilocal energy holds for tori in our scenario? Surprisingly, we find examples of distorted tori in the outer neighborhood of the Schwarzschild horizon with strictly negative Hayward energy. This could even be done in an analytic approach in the static coordinates.

The spatial Schwarzschild metric g^m can be written in the static spherical coordinates $\{\widehat{R}, \Theta, \varphi\}$ as

$$g^m = \frac{1}{1 - \frac{2m}{\widehat{R}}} \mathrm{d}\widehat{R}^2 + \widehat{R}^2 \mathrm{d}\Theta^2 + \widehat{R}^2 \sin^2 \Theta \mathrm{d}\varphi^2.$$

Denote by

$$\check{e}_1 = \sqrt{1 - \frac{2m}{\widehat{R}}} \frac{\partial}{\partial \widehat{R}}, \quad \check{e}_2 = \frac{1}{\widehat{R}} \frac{\partial}{\partial \Theta}, \quad \check{e}_3 = \frac{1}{\widehat{R} \sin \Theta} \frac{\partial}{\partial \varphi} \tag{4.1}$$

the orthonormal frame. In this frame, the nonzero components of the Riemann curvature tensors are

$$\begin{aligned} &R(\check{e}_1, \check{e}_2, \check{e}_1, \check{e}_2) = R(\check{e}_1, \check{e}_3, \check{e}_1, \check{e}_3) = -\frac{m}{\widehat{R}^3}, \\ &R(\check{e}_2, \check{e}_3, \check{e}_2, \check{e}_3) = \frac{2m}{\widehat{R}^3}. \end{aligned}$$

We consider a family of coordinate tori $\widehat{\Sigma}$ in the static coordinates which are parameterized as

$$\begin{aligned} y^1 &= (a + b \cos \theta) \cos \varphi, \\ y^2 &= (a + b \cos \theta) \sin \varphi, \quad \theta \in [0, 2\pi), \ \varphi \in [0, 2\pi), \\ y^3 &= b \sin \theta. \end{aligned} \tag{4.2}$$

Here $\widehat{R} = \sqrt{(y^1)^2 + (y^2)^2 + (y^3)^2}$ and the tori are swept out by rotating the standard circles $\{(y^1, y^3) \mid (y^1 - a)^2 + (y^3)^2 = b^2\}$ along the y^3-axis. It can be obtained from (4.2) that the induced 2-metric $\widehat{\sigma}$ on $\widehat{\Sigma}$ is

$$\widehat{\sigma} = \widehat{\sigma}_{22}\mathrm{d}\theta^2 + \widehat{\sigma}_{33}\mathrm{d}\varphi^2,$$

where

$$\begin{aligned}
\widehat{\sigma}_{22} &= \frac{b^2\big((a^2+b^2+2ab\cos\theta)^{\frac{3}{2}} - 2m(a\cos\theta+b)^2\big)}{(a^2+b^2+2ab\cos\theta)^{\frac{3}{2}} - 2m(a^2+b^2+2ab\cos\theta)},\\
\widehat{\sigma}_{33} &= (a+b\cos\theta)^2.
\end{aligned}$$

The area form of $\widehat{\Sigma}$ with respect to the induced 2-metric $\widehat{\sigma}$ is

$$\begin{aligned}
\mathrm{d}\widehat{\sigma} &= \sqrt{\det\widehat{\sigma}}\mathrm{d}\theta\wedge\mathrm{d}\varphi\\
&= b(a+b\cos\theta)\sqrt{\frac{(a^2+b^2+2ab\cos\theta)^{\frac{3}{2}} - 2m(a\cos\theta+b)^2}{(a^2+b^2+2ab\cos\theta)^{\frac{3}{2}} - 2m(a^2+b^2+2ab\cos\theta)}}\mathrm{d}\theta\wedge\mathrm{d}\varphi.
\end{aligned}$$

Again, denote by $\{\widehat{e}_2, \widehat{e}_3\}$ the orthonormal frame tangential to the surface $\widehat{\Sigma}$. By the chain rule, one has

$$\frac{\partial}{\partial\theta} = -\frac{ab\sin\theta}{\sqrt{a^2+b^2+2ab\cos\theta}}\frac{\partial}{\partial\widehat{R}} - \frac{b(b+a\cos\theta)}{a^2+b^2+2ab\cos\theta}\frac{\partial}{\partial\Theta}.$$

Then

$$\begin{aligned}
\widehat{e}_2 &= \frac{1}{\sqrt{\widehat{\sigma}_{22}}}\Big(-\frac{ab\sin\theta}{\sqrt{a^2+b^2+2ab\cos\theta}}\frac{\partial}{\partial\widehat{R}} - \frac{b(b+a\cos\theta)}{a^2+b^2+2ab\cos\theta}\frac{\partial}{\partial\Theta}\Big),\\
\widehat{e}_3 &= \frac{1}{\sqrt{\widehat{\sigma}_{33}}}\frac{\partial}{\partial\varphi}.
\end{aligned}$$

In terms of the orthonormal frame (4.1),

$$\begin{aligned}
\widehat{e}_2 &= (\widehat{e}_2)^1\check{e}_1 + (\widehat{e}_2)^2\check{e}_2,\\
\widehat{e}_3 &= \check{e}_3,
\end{aligned}$$

where

$$\begin{aligned}
(\widehat{e}_2)^1 &= -\frac{a\sin\theta(a^2+b^2+2ab\cos\theta)^{\frac{1}{4}}}{((a^2+b^2+2ab\cos\theta)^{\frac{3}{2}} - 2m(b+a\cos\theta)^2)^{\frac{1}{2}}},\\
(\widehat{e}_2)^2 &= -\frac{(b+a\cos\theta)\sqrt{(a^2+b^2+2ab\cos\theta)^{\frac{1}{2}} - 2m}}{((a^2+b^2+2ab\cos\theta)^{\frac{3}{2}} - 2m(b+a\cos\theta)^2)^{\frac{1}{2}}}.
\end{aligned}$$

Then

$$\begin{aligned}
&R(\widehat{e}_2, \widehat{e}_3, \widehat{e}_2, \widehat{e}_3)\\
&= R((\widehat{e}_2)^1\check{e}_1 + (\widehat{e}_2)^2\check{e}_2, \check{e}_3, (\widehat{e}_2)^1\check{e}_1 + (\widehat{e}_2)^2\check{e}_2, \check{e}_3)\\
&= \frac{m}{r^3}(2((\widehat{e}_2)^2)^2 - ((\widehat{e}_2)^1)^2)\\
&= \frac{m(\sqrt{a^2+2ab\cos\theta+b^2} - 2m)\Big(4(a\cos\theta+b)^2 - \frac{2a^2\sin^2\theta}{1-\frac{2m}{\sqrt{a^2+2ab\cos\theta+b^2}}}\Big)}{2(a^2+2ab\cos\theta+b^2)^{\frac{3}{2}}((a^2+2ab\cos\theta+b^2)^{\frac{3}{2}} - 2m(a\cos\theta+b)^2)}
\end{aligned}$$

$$= \frac{m(4(a\cos\theta+b)^2(\sqrt{a^2+2ab\cos\theta+b^2}-2m)-2a^2\sin^2\theta\sqrt{a^2+2ab\cos\theta+b^2})}{2(a^2+2ab\cos\theta+b^2)^{\frac{3}{2}}((a^2+2ab\cos\theta+b^2)^{\frac{3}{2}}-2m(a\cos\theta+b)^2)}. \tag{4.3}$$

Finally, the Hayward energy of $\widehat{\Sigma}$ can be rewritten as

$$\begin{aligned} E_{\text{Hayward}}(\widehat{\Sigma}) &= \frac{1}{8\pi}\sqrt{\frac{|\widehat{\Sigma}|}{16\pi}}\int_{\widehat{\Sigma}} 2R(\widehat{e}_2,\widehat{e}_3,\widehat{e}_2,\widehat{e}_3)\mathrm{d}\widehat{\sigma} \\ &= \frac{1}{4}\sqrt{\frac{|\widehat{\Sigma}|}{16\pi}}\int_0^{2\pi} \frac{bm(a+b\cos\theta)\Big(4(a\cos\theta+b)^2-\frac{2a^2\sin^2\theta}{1-\frac{2m}{\sqrt{a^2+2ab\cos\theta+b^2}}}\Big)}{(a^2+2ab\cos\theta+b^2)^2\sqrt{\frac{(a^2+2ab\cos\theta+b^2)^{\frac{3}{2}}-2m(a\cos\theta+b)^2}{\sqrt{a^2+2ab\cos\theta+b^2}-2m}}}\mathrm{d}\theta. \end{aligned} \tag{4.4}$$

Let us expand the integrand in the energy expression (4.4) for small b,

$$\begin{aligned} &\frac{bm(a+b\cos\theta)\Big(4(a\cos\theta+b)^2-\frac{2a^2\sin^2\theta}{1-\frac{2m}{\sqrt{a^2+2ab\cos\theta+b^2}}}\Big)}{(a^2+2ab\cos\theta+b^2)^2\sqrt{\frac{(a^2+2ab\cos\theta+b^2)^{\frac{3}{2}}-2m(a\cos\theta+b)^2}{\sqrt{a^2+2ab\cos\theta+b^2}-2m}}} \\ &= \Bigg(\frac{m(4a^2\cos^2\theta-\frac{2a^2\sin^2\theta}{1-\frac{2m}{a}})}{a^3\sqrt{\frac{a^3-2a^2m\cos^2\theta}{a-2m}}}\Bigg)b+\mathcal{O}(b^2). \end{aligned}$$

Setting $a=2.1m$, numerical integral shows that

$$\int_0^{2\pi}\Bigg(\frac{m\Big(4a^2\cos^2\theta-\frac{2a^2\sin^2\theta}{1-\frac{2m}{a}}\Big)}{a^3\sqrt{\frac{a^3-2a^2m\cos^2\theta}{a-2m}}}\Bigg)\mathrm{d}\theta = -\frac{6.46}{m} < 0.$$

This indicates that in the outer neighborhood of the Schwarzschild horizon ($\widehat{R}=2m=4\beta$), standard 'thin tori' in the static coordinates have strictly negative Hayward energy. We can indeed to prove the existence of many of such 'thin tori' with strictly negative energy in the following analytic way. Let

$$a=(2+2\lambda)m \quad \text{and} \quad b=\lambda m. \tag{4.5}$$

The curvature term (4.3) yields that

$$\lim_{\lambda\to 0^+} R(\widehat{e}_2,\widehat{e}_3,\widehat{e}_2,\widehat{e}_3) = -\frac{1}{8m^2}.$$

If one takes λ sufficiently small, the integrand in the energy expression (4.4) is strictly negative and therefore the Hayward energy of such a torus is negative. According to (4.5), the smallness of the parameter λ means both the torus is 'very thin' and its location is in the outer neighborhood of the Schwarzschild horizon.

The relation of the change of coordinates between the isotropic coordinate and the static coordinate is

$$\widehat{R} = R\Big(1+\frac{m}{2R}\Big)^2$$

and further the relation between the Cartesian coordinates $\{x^i\}$ and $\{y^i\}$ is

$$x^1 = \frac{y^1}{2}\Big(1+\Big(1-\frac{2m}{\widehat{R}}\Big)^{\frac{1}{2}}-\frac{m}{\widehat{R}}\Big),$$

$$x^2 = \frac{y^2}{2}\left(1 + \left(1 - \frac{2m}{\widehat{R}}\right)^{\frac{1}{2}} - \frac{m}{\widehat{R}}\right),$$
$$x^3 = \frac{y^3}{2}\left(1 + \left(1 - \frac{2m}{\widehat{R}}\right)^{\frac{1}{2}} - \frac{m}{\widehat{R}}\right).$$

If one transforms the standard circle

$$\{(y^1, y^3) \mid (y^1 - a)^2 + (y^3)^2 = b^2\} \tag{4.6}$$

to the $\{x^i\}$ coordinates, the circle is horizontally dragged, as illustrated in Figure 3.

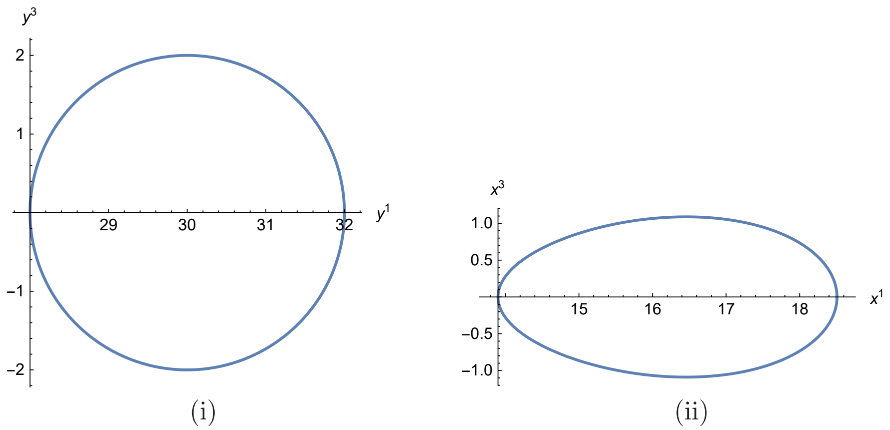

Figure 3 Graphs of the circle (4.6) in different coordinates. For $a = 30$, $b = 2$ and $m = 2\beta = 6 + 4\sqrt{2}$, (i) is the graph of the circle (4.6) in the $y^1 - y^3$-plane; (ii) is the graph of the circle (4.6) in the $x^1 - x^3$-plane.

This motivates us to conjecture that the horizontal dragging effect may lead to negative contribution to the Hayward energy. The ε-horizontally dragged torus $\overline{\Sigma}$ in the isotropic coordinates $\{x^i\}$ is swept out by rotating an ellipse along the x^3-axis, as illustrated in Figure 4.

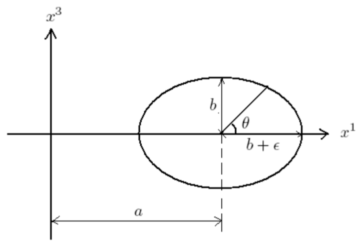

Figure 4 Horizontally dragged circle in the $x^1 - x^3$-plane.

More precisely, the horizontally dragged torus can be parameterized as

$$\begin{aligned} x^1 &= (a + (b + \varepsilon)\cos\theta)\cos\varphi, \\ x^2 &= (a + (b + \varepsilon)\cos\theta)\sin\varphi, \quad \theta \in [0, 2\pi),\ \varphi \in [0, 2\pi), \\ x^3 &= b\sin\theta. \end{aligned} \tag{4.7}$$

It can be obtained from (4.7) that the induced 2-metric $\overline{\sigma}$ on $\overline{\Sigma}$ is

$$\overline{\sigma} = \overline{\sigma}_{22}\mathrm{d}\theta^2 + \overline{\sigma}_{33}\mathrm{d}\varphi^2,$$

where

$$\begin{aligned}\overline{\sigma}_{22} = &\frac{(2b^2 - \varepsilon(2b+\varepsilon)\cos 2\theta + 2b\varepsilon + \varepsilon^2)}{8(2a^2 + 4a\cos\theta(b+\varepsilon) + 2b^2 + \varepsilon(2b+\varepsilon)\cos 2\theta + 2b\varepsilon + \varepsilon^2)^2}\\ &\cdot\ (\sqrt{2}\sqrt{2a^2 + 4a\cos\theta(b+\varepsilon) + 2b^2 + \varepsilon(2b+\varepsilon)\cos 2\theta + 2b\varepsilon + \varepsilon^2} + 2\beta)^4,\\ \overline{\sigma}_{33} = &\frac{(a + (b+\varepsilon)\cos\theta)^2}{4(2a^2 + 4a(b+\varepsilon)\cos\theta + 2b^2 + \varepsilon(2b+\varepsilon)\cos 2\theta + 2b\varepsilon + \varepsilon^2)^2}\\ &\cdot (\sqrt{2}\sqrt{2a^2 + 4a(b+\varepsilon)\cos\theta + 2b^2 + \varepsilon(2b+\varepsilon)\cos 2\theta + 2b\varepsilon + \varepsilon^2} + 2\beta)^4.\end{aligned}$$

Denote by $\{\overline{e}_2, \overline{e}_3\}$ the orthonormal frame tangential to the surface $\overline{\Sigma}$. By the chain rule, one has

$$\begin{aligned}\frac{\partial}{\partial\theta} = &-\frac{(a(b+\varepsilon) + \varepsilon(2b+\varepsilon)\cos\theta)\sin\theta}{\sqrt{a^2 + b^2 + 2a(b+\varepsilon)\cos\theta + \varepsilon(2b+\varepsilon)\cos^2\theta}}\frac{\partial}{\partial R}\\ &-\frac{2b(b+\varepsilon+a\cos\theta)}{2a^2 + 2b^2 + 2b\varepsilon + \varepsilon^2 + 4a(b+\varepsilon)\cos\theta + \varepsilon(2b+\varepsilon)\cos 2\theta}\frac{\partial}{\partial\Theta}.\end{aligned}$$

Then we have

$$\begin{aligned}\overline{e}_2 = &\frac{1}{\sqrt{\overline{\sigma}_{22}}}\Big[-\frac{(a(b+\varepsilon) + \varepsilon(2b+\varepsilon)\cos\theta)\sin\theta}{\sqrt{a^2 + b^2 + 2a(b+\varepsilon)\cos\theta + \varepsilon(2b+\varepsilon)\cos^2\theta}}\frac{\partial}{\partial R}\\ &-\frac{2b(b+\varepsilon+a\cos\theta)}{2a^2 + 2b^2 + 2b\varepsilon + \varepsilon^2 + 4a(b+\varepsilon)\cos\theta + \varepsilon(2b+\varepsilon)\cos 2\theta}\frac{\partial}{\partial\Theta}\Big]\\ = &(\overline{e}_2)^R\frac{\partial}{\partial R} + (\overline{e}_2)^\Theta\frac{\partial}{\partial\Theta},\\ \overline{e}_3 = &\frac{1}{\sqrt{\overline{\sigma}_{33}}}\frac{\partial}{\partial\varphi} = (\overline{e}_3)^\varphi\frac{\partial}{\partial\varphi}\end{aligned}$$

and

$$\begin{aligned}&R(\overline{e}_2, \overline{e}_3, \overline{e}_2, \overline{e}_3)\\ = &((\overline{e}_2)^R)^2((\overline{e}_3)^\varphi)^2 R_{R\varphi R\varphi} + ((\overline{e}_2)^\Theta)^2((\overline{e}_3)^\varphi)^2 R_{\Theta\varphi\Theta\varphi}\\ = &\frac{16\sqrt{2}\beta\sqrt{2a^2 + 4a\cos\theta(b+\varepsilon) + 2b^2 + \varepsilon(2b+\varepsilon)\cos 2\theta + 2b\varepsilon + \varepsilon^2}}{(2b^2 - \varepsilon(2b+\varepsilon)\cos 2\theta + 2b\varepsilon + \varepsilon^2)}\\ &\cdot (\sqrt{2}\sqrt{2a^2 + 4a\cos\theta(b+\varepsilon) + 2b^2 + \varepsilon(2b+\varepsilon)\cos 2\theta + 2b\varepsilon + \varepsilon^2} + 2\beta)^{-6}\\ &\cdot (4a^2\cos 2\theta(3b^2 + 2b\varepsilon + \varepsilon^2) + 4a^2b^2 - 8a^2b\varepsilon - 4a^2\varepsilon^2 + 4a\varepsilon^3\cos 3\theta + 8ab^2\varepsilon\cos 3\theta\\ &+ 4a\cos\theta(8b^3 + 6b^2\varepsilon - 3b\varepsilon^2 - \varepsilon^3) + 12ab\varepsilon^2\cos(3\theta) + \varepsilon^4\cos 4\theta\\ &+ 16b^4 + 32b^3\varepsilon + 4b^2\varepsilon^2\cos 4\theta + 12b^2\varepsilon^2 + 4b\varepsilon^3\cos 4\theta - 4b\varepsilon^3 - \varepsilon^4).\end{aligned}$$

Setting $\beta = 3+2\sqrt{2}$, $a = 3.5$, $b = 0.5$ and $\varepsilon = 1$, numerical result shows that $E_{\mathrm{Hayward}}(\overline{\Sigma}) = -3.17526$.

To see the effect of the horizontal dragging parameter ε to the Hayward energy, we plot the value of $E_{\mathrm{Hayward}}(\overline{\Sigma})$ for different ε with fixed a, b and β in Figure 5.

We also plot $E_{\mathrm{Hayward}}(\overline{\Sigma})$ for different a and b with fixed $\varepsilon = 1$ in Figure 6 from which we numerically found many distorted tori with negative Hayward energy in the region between the boundary of the star and the Schwarzschild horizon. They are horizontally dragged.

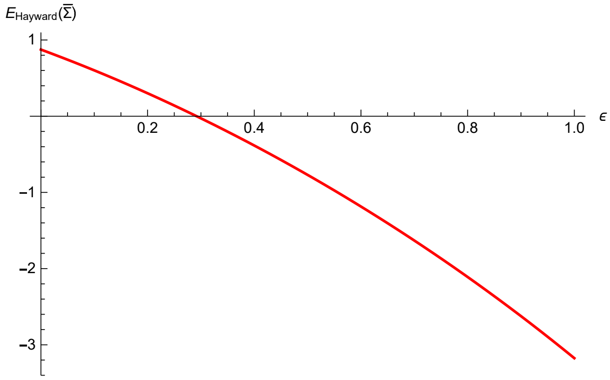

Figure 5 Graph of $E_{\text{Hayward}}(\overline{\Sigma})$ with respect to ε for $a = 3.5, b = 0.5$ and $\beta = 3 + 2\sqrt{2}$.

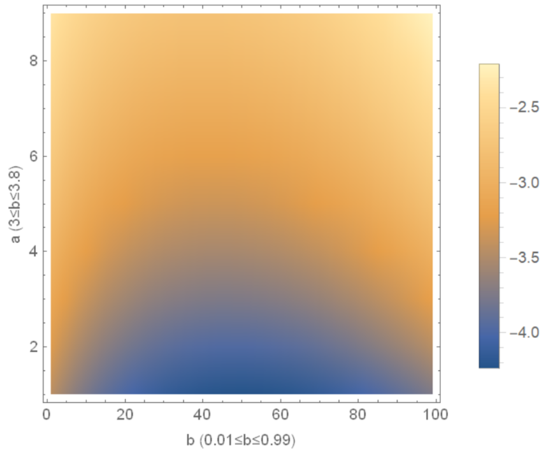

Figure 6 Plot of $E_{\text{Hayward}}(\overline{\Sigma})$ with respect to a and b for $\varepsilon = 1$ and $\beta = 3 + 2\sqrt{2}$.

5 Summary

We consider a scenario of a spherically symmetric constant density star matched to the Schwarzschild solution, representing momentarily static initial data. The positivity of the Hayward quasilocal energy of tori is investigated. For generic tori entirely within the star, distorted or not, they must have positive Hayward energy.

Standard 'thin tori' in the isotropic coordinates outside the star also have positive Hayward energy. It is nature to conjecture that the Hayward quasilocal energy of tori is always positive in our scenario. However, in the static coordinates, examples of tori with negative Hayward energy have been found in both analytic and numerical ways. They are located in the outer neighborhood of the Schwarzschild horizon. These tori are swept out by the standard circles in the static coordinates but they look horizontally dragged in the isotropic coordinates. The horizontal dragging effect to the negative contribution to the Hayward energy is revealed for the finding of distorted tori with negative Hayward energy in the region between the boundary of the star and the Schwarzschild horizon. The results are not so disastrous. When the major radius of the torus is sufficiently large and the torus goes to spatial infinity, the Hayward energy becomes positive. In fact, for large a, one has $E_{\text{Hayward}}(\Sigma) = \frac{\beta}{2}(\frac{b\pi}{a})^{\frac{3}{2}} + \mathcal{O}(a^{-\frac{5}{2}})$. The physical significance of these examples of tori in the momentarily static data with negative energy is that the local dominant energy density does not necessarily guarantee the positivity of the quasilocal energy.

The metric is discontinuous at the boundary of the star where the curvatures have a jump. There are marginally trapped tori numerically constructed which are partially inside the star and partially outside the star (see [12, Section IV]). They are swept out by a distorted circle and both the induced 2-metric and the unit normal are nonsmooth across the boundary of the star. There should be an influence of the gravitational action at the nonsmooth boundary and certain subtle constraints should be imposed appropriately [7]. It is valuable to seek a framework to defining a notion of quasilocal energy for surfaces with corners. But currently we are not able to provide any advice about how to move on.

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