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(1+2)-Dimensional Radially Symmetric Wave Maps Revisit*

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Abstract The author gives an alternative and simple proof of the global existence of smooth solutions to the Cauchy problem for wave maps from the (1+2)-dimensional Minkowski space to an arbitrary compact smooth Riemannian manifold without boundary, for arbitrary smooth, radially symmetric data. The author can also treat non-compact manifold under some additional assumptions which generalize the existing ones.

Keywords Cauchy problem, Wave maps, Global smooth solution
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1 Main Result

Let N be a smooth Riemannian k-manifold without boundary. Without loss of generality, we assume that $N \subset \mathbb{R}^n$, isometrically. We consider wave maps $\Phi = (\Phi^1, \cdots, \Phi^n) = \Phi(t, x) : \mathbb{R} \times \mathbb{R}^2 \to N \in \mathbb{R}^n$, satisfying the equation

$$\Box\Phi = \Phi_{tt} - \Delta\Phi = B(\Phi)(\partial_\alpha \Phi, \partial^\alpha \Phi) \perp T_\Phi N, \tag{1.1}$$

where B denotes the second fundamental form of N. Writing $z = (t, x) = (x^\alpha)_{0 \le \alpha \le 2}$, we also let $\partial_\alpha = \frac{\partial}{\partial x^\alpha},\ \alpha = 0, 1, 2$. We raise and lower indices with the Minkowski metric $\eta = (\eta_{\alpha\beta}) = (\eta^{\alpha\beta}) = \mathrm{diag}(-1, 1, 1)$ and tacitly sum over repeated indices.

Due to its mathematical difficulty and important physical background, the topic of wave maps has experienced an incredible advancement in the past several decades. It has at least two physical motivations to study wave maps. One is the nonlinear σ-model which deals with the case N is a sphere and on the other hand, vacuum Einstein equations with $U(1)\times\mathbb{R}$ symmetries reduce to a radially symmetric wave maps from (1+2)-dimensional Minkowski space to the hyperbolic plane. It is Prof. Gu [3] who first gave the regularity result in (1+1)-dimensional case. For an up to date account of the full developments, we refer to the monograph of Geba and Grillakis [2], as well as [4–5, 8].

The main purpose of this paper is to give an alternative proof of the global existence for (1+2)-dimensional wave maps with radial symmetry. We may assume that TN is parallelizable. Let $\overline{e}_1, \cdots, \overline{e}_k$ be a smooth orthonormal frame field such that at any point $p \in N$, the vectors

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$\overline{e}_1(p), \cdots, \overline{e}_k(p)$ form an orthonormal basis for T_pN. We make the following assumptions on the target manifold.

(H1): $\nabla_p \overline{e}_i(p)$ is uniformly bounded,

(H2): $\nabla_p B(p)$ is uniformly bounded.

We note that those assumptions are in particular satisfied for compact manifold.

The main result of this paper is the following.

Theorem 1.1 *Let $N \subset \mathbb{R}^n$ be a smooth Riemannian manifold without boundary. Assume* (H1)–(H2) *satisfied for N. Then for any radially symmetric data*

$$(\Phi_0, \Phi_1) = (\Phi_0(r), \Phi_1(r)) \in C^\infty(\mathbb{R}^2, TN), \quad r = |x|, \tag{1.2}$$

there exists a unique, smooth solution $\Phi = \Phi(t, r)$ to the Cauchy problem (1.1) *and* (1.2)*, defined for all time.*

Our result slightly generalizes the work of Christodoulou-Tahvildar-Zadeh [1] and Struwe [6–7].

As usual, the proof of Theorem 1.1 is divided into two main steps. The first step is to show that small energy implies regularity and the second step is to show that energy can not concentrate. Our proof of the first step is totally different from the work of Christodoulou-Tahvildar-Zadeh [1] and our proof of the second step simplify a little bit of the argument of Struwe [6–7]. In the work of Christodoulou-Tahvildar-Zadeh [1], the first step is achieved by a Hölder estimate using the fundamental solution of the 2-dimensional radially symmetric wave operator which is quite complicated. While we rely on an energetic argument which we call a new div curl lemma and has potential to work for quasilinear problems. This will be pursued in our future work.

2 Intrinsic Setting

Let $\Phi = \Phi(t, r)$ be a smooth radially symmetric wave maps. Then (1.1) and (1.2) can be rewritten as

$$\Phi_{tt} - \Phi_{rr} - \frac{\Phi_r}{r} = 4B(\Phi)(\Phi_u, \Phi_v), \quad t > 0, \tag{2.1}$$

$$\Phi = \Phi_0(r), \quad \Phi_t = \Phi_1(r), \quad t = 0. \tag{2.2}$$

We denote $u = t - r, \ v = t + r$, then

$$\begin{aligned} &t = \frac{1}{2}(u+v), \quad \partial_t = \partial_u + \partial_v, \quad \partial_u = \frac{1}{2}(\partial_t - \partial_r), \\ &r = \frac{1}{2}(v-u), \quad \partial_r = \partial_v - \partial_u, \quad \partial_v = \frac{1}{2}(\partial_t + \partial_r), \\ &\Phi_u = \partial_u \Phi, \quad \Phi_v = \partial_v \Phi, \quad \Phi_{tt} = \partial_t^2 \Phi, \quad \text{etc.} \end{aligned}$$

and we have the energy conservation law

$$E(t) = \frac{1}{2}\int_0^\infty r(|\Phi_t|^2 + |\Phi_r|^2)\mathrm{d}r = E_0 = \frac{1}{2}\int_0^\infty r(|\Phi_0'|^2 + |\Phi_1|^2)\mathrm{d}r. \tag{2.3}$$

Let D be the pull-back covariant derivative in u^*TN, we may write (2.1) as

$$D_t\Phi_t - \frac{1}{r}D_r(r\Phi_r) = 0. \tag{2.4}$$

From $(\overline{e}_i)_{1\le i\le k}$, we obtain a frame $e_i = R_{ij}(\overline{e}_j \circ \Phi), 1 \le i \le k$ for the pull-back bundle, where $R = R(z) = R(R_{ij})$ maybe any smooth orthogonal matrix. Denoting

$$D_\alpha e_i = A^j_{i\alpha} e_j, \quad 0 \le \alpha \le 2 \tag{2.5}$$

with a matrix-valued connection 1-form $A = A_\alpha \mathrm{d}x^\alpha$. We compute the curvature F of D via the commutation relation

$$\begin{aligned} &D_\alpha D_\beta e_i - D_\beta D_\alpha e_i = D_\alpha(A^j_{i\beta}e_j) - D_\beta(A^j_{i\alpha}e_i) \\ &= (\partial_\alpha A^k_{i\beta} - \partial_\beta A^k_{i\alpha} - A^k_{j\alpha}A^j_{i\beta} - A^k_{j\beta}A^j_{i\alpha})e_k \\ &= F^k_{i\alpha\beta}e_k, \end{aligned}$$

or, more precisely,

$$\mathrm{d}A + \frac{1}{2}[A, A] = F. \tag{2.6}$$

In the radially symmetric case, we may choose $R = R(t, r), A = A_0\mathrm{d}t + A_1\mathrm{d}r$. Following Struwe [7], we may impose "exponential gauge" condition $A_1 = 0$, which yields the relation

$$*\mathrm{d}A = -\partial_r A_0 = F_{01}. \tag{2.7}$$

If we normalize $A_0(t, \infty) = 0, \forall t$, from this relation, we obtain

$$A_0 = \int_r^{+\infty} F_{01}\mathrm{d}s. \tag{2.8}$$

By (H1), we get

$$|F_{01}| \le C|\mathrm{d}\Phi|^2. \tag{2.9}$$

Thus we deduce the estimate

$$|A_0| \lesssim \int_r^{+\infty} |\mathrm{d}\Phi|^2\mathrm{d}s \lesssim E_0 r^{-1}. \tag{2.10}$$

Let

$$\Phi_t = q^i_0 e_i, \quad \Phi_r = q^i_1 e_i. \tag{2.11}$$

Using the notation

$$\begin{aligned} D_\alpha\partial_\beta\Phi &= D_\alpha(q^i_\beta e_i) = (\partial_\alpha q^j_\beta + A^j_{i\beta}q^i_\beta)e_j \\ &= (D_\alpha q^i_\beta)^j e_j, \end{aligned} \tag{2.12}$$

we then may write (2.4) in the form

$$D_t q_0 - \frac{1}{r}D_r(rq_1) = \partial_t q_0 + A_0 q_0 - \frac{1}{r}\partial_r(rq_1) = 0. \tag{2.13}$$

Moreover, we have the commutation relation

$$D_r q_0 = D_t q_1 = \partial_t q_1 + A_0 q_1. \tag{2.14}$$

Taking inner product in $\mathbb{R}^n$ of (2.13) with q_1, we get

$$q_1 \cdot \Big(\partial_t q_0 + A_0 q_0 - \partial_r q_1 - \frac{q_1}{r}\Big) = 0, \tag{2.15}$$

which yields

$$\begin{aligned} & - \partial_t (r^{1-\alpha} q_1 q_0) + \frac{1}{2}\partial_r (r^{1-\alpha}(q_0^2 + q_1^2)) + \frac{\alpha}{2} r^{-\alpha}(q_0^2 + q_1^2) \\ & = -r^{1-\alpha}[q_1 \cdot (A_0 q_0) + (A_0 q_1) \cdot q_0]. \end{aligned} \tag{2.16}$$

Noting (2.11), we get

$$-\partial_t (r^{1-\alpha} \Phi_t \Phi_r) + \frac{1}{2}\partial_r (r^{1-\alpha}(|\Phi_t|^2 + |\Phi_r|^2)) + \frac{\alpha}{2} r^{-\alpha}(|\Phi_t|^2 + |\Phi_r|^2) = F_\alpha, \tag{2.17}$$

where

$$F_\alpha = -r^{1-\alpha}[q_1 \cdot (A_0 q_0) + (A_0 q_1) \cdot q_0]. \tag{2.18}$$

By (2.10), we get

$$|F_\alpha| \lesssim E_0 r^{-\alpha} |\Phi_t||\Phi_r|. \tag{2.19}$$

3 New Div Curl Lemma

The purpose of this section is to prove the following Lemma 3.1 and Lemma 3.2. We call Lemma 3.2 a new div curl Lemma.

Lemma 3.1 *Suppose that*

$$\begin{cases} \partial_u F^{11} + \partial_v F^{12} = G^1, & r \ge 0, \ 0 \le t \le T, \\ \partial_u F^{21} - \partial_v F^{22} = G^2, & r \ge 0, \ 0 \le t \le T \end{cases} \tag{3.1}$$

where $F^{11}, F^{12}, F^{21}, F^{22}$ *are all nonnegative. Moreover,*

$$r = 0: \ F^{11} - F^{12} = F^{21} + F^{22} = 0. \tag{3.2}$$

Then there hold

$$\begin{aligned} & \sup_u \int_{|u|}^{2T-u} F^{11}(u,v)\mathrm{d}u + \sup_v \int_{-v}^{\min\{v,2T-v\}} F^{12}(u,v)\mathrm{d}u \\ \lesssim & \int_0^\infty (F^{11} + F^{12})(0,r)\mathrm{d}r + \int_0^T \int_0^\infty |G_1|(t,r)\mathrm{d}r\mathrm{d}t, \end{aligned} \tag{3.3}$$

$$\begin{aligned} & \sup_u \int_{|u|}^{2T-u} F^{21}(u,v)\mathrm{d}u + \sup_v \int_{-v}^{\min\{v,2T-v\}} F^{22}(u,v)\mathrm{d}u \\ \lesssim & \int_0^\infty (F^{21} + F^{22})(T,r)\mathrm{d}r + \int_0^\infty (F^{21} + F^{22})(0,r)\mathrm{d}r \\ & + \int_0^T \int_0^\infty |G_2|(t,r)\mathrm{d}r\mathrm{d}t. \end{aligned} \tag{3.4}$$

Proof We only prove

$$\begin{aligned}&\sup_u \int_{|u|}^{2T-u} F^{11}(u,v)\mathrm{d}u\\ &\lesssim \int_0^\infty (F^{11}+F^{12})(0,r)\mathrm{d}r + \int_0^T \int_0^\infty |G_1|(t,r)\mathrm{d}r\mathrm{d}t.\end{aligned}$$

The other estimates are similar. Noting (3.2), we have

$$\begin{aligned}&\partial_u \int_{|u|}^{2T-u} F^{11}(u,v)\mathrm{d}u\\ &= -F^{11}|_{v=2T-u} - F^{11}|_{v=|u|}\mathrm{sgn}(u) + \int_{|u|}^{2T-u} \partial_u F^{11}\mathrm{d}v\\ &= -F^{11}|_{v=2T-u} - F^{11}|_{v=|u|}\mathrm{sgn}(u) - \int_{|u|}^{2T-u} \partial_v F^{12}\mathrm{d}v + \int_{|u|}^{2T-u} |G_1|\mathrm{d}v\\ &= -F^{11}|_{v=2T-u} - F^{12}|_{v=2T-u} - F^{11}|_{v=|u|}\mathrm{sgn}(u) + F^{12}|_{v=|u|} + \int_{|u|}^{2T-u} |G_1|\mathrm{d}v\\ &\lesssim (F^{11}+F^{12})|_{t=0} + \int_{|u|}^{2T-u} |G_1|\mathrm{d}v.\end{aligned}$$

Integration in u yields the desired estimate.

Lemma 3.2 *Under the assumptions of Lemma* 3.1, *we have*

$$\begin{aligned}&\int_0^T \int_0^\infty (F^{11}F^{22}+F^{12}F^{21})(t,r)\mathrm{d}r\mathrm{d}t\\ &\lesssim \Big(\int_0^\infty (F^{11}+F^{12})(0,r)\mathrm{d}r + \int_0^T \int_0^\infty |G_1(t,r)|\mathrm{d}r\mathrm{d}t\Big)\\ &\quad\cdot \Big(\int_0^\infty (F^{21}+F^{22})(0,r)\mathrm{d}r + \int_0^\infty (F^{21}+F^{22})(T,r)\mathrm{d}r + \int_0^T \int_0^\infty |G_2(t,r)|\mathrm{d}r\mathrm{d}t\Big).\end{aligned}$$

Proof Similarly to the proof of Lemma 3.1, for $\overline{v} \le 2T-u$, we get

$$\partial_u \int_{|u|}^{\overline{v}} F^{11}(u,v)\mathrm{d}v + F^{12}(u,\overline{v}) \lesssim (F^{11}+F^{12})|_{u+\overline{v}=0} + \int_{|u|}^{\overline{v}} G_1\mathrm{d}v.$$

Thus

$$\begin{aligned}&F^{21}(u,v)\partial_u \int_{|u|}^{\overline{v}} F^{11}(u,v)\mathrm{d}v + F^{21}(u,\overline{v})F^{12}(u,\overline{v})\\ &\lesssim F^{21}(u,\overline{v})(F^{11}+F^{12})|_{u+\overline{v}=0} + \Big(\int_{|u|}^{\overline{v}} G_1(u,v)\mathrm{d}v\Big)F^{21}(u,\overline{v}).\end{aligned}$$

By the second equation in (3.1), the first term is

$$
\begin{aligned}
&\partial_u\Big[F^{21}(u,\overline{v})\int_{|u|}^{\overline{v}}F^{11}(u,v)\mathrm{d}v\Big]-\partial_u F^{21}(u,\overline{v})\int_{|u|}^{\overline{v}}F^{11}(u,v)\mathrm{d}v\\
=\,&\partial_u\Big[F^{21}(u,\overline{v})\int_{|u|}^{\overline{v}}F^{11}(u,v)\mathrm{d}v\Big]-\partial_{\overline{v}}F^{22}(u,\overline{v})\int_{|u|}^{\overline{v}}F^{11}(u,v)\mathrm{d}v\\
&-G_2(u,\overline{v})\int_{|u|}^{\overline{v}}F^{11}(u,v)\mathrm{d}v\\
=\,&\partial_u\Big[F^{21}(u,\overline{v})\int_{|u|}^{\overline{v}}F^{11}(u,v)\mathrm{d}v\Big]-\partial_{\overline{v}}\Big[F^{22}(u,\overline{v})\int_{|u|}^{\overline{v}}F^{11}(u,v)\mathrm{d}v\Big]\\
&+F^{22}(u,\overline{v})F^{11}(u,\overline{v})-G_2(u,\overline{v})\int_{|u|}^{\overline{v}}F^{11}(u,v)\mathrm{d}v.
\end{aligned}
$$

Therefore, we get

$$
\begin{aligned}
&F^{22}(u,\overline{v})F^{11}(u,\overline{v})+F^{21}(u,\overline{v})F^{12}(u,\overline{v})\\
\lesssim\,&\partial_{\overline{v}}\Big[F^{22}(u,\overline{v})\int_{|u|}^{\overline{v}}F^{11}(u,v)\mathrm{d}v\Big]-\partial_u\Big[F^{21}(u,\overline{v})\int_{|u|}^{\overline{v}}F^{11}(u,v)\mathrm{d}v\Big]\\
&+G_2(u,\overline{v})\int_{|u|}^{\overline{v}}F^{11}(u,v)\mathrm{d}v+F^{21}(u,\overline{v})(F^{11}+F^{12})|_{u+\overline{v}=0}+\Big(\int_{|u|}^{\overline{v}}G_1(u,v)\mathrm{d}v\Big)F^{21}(u,\overline{v}).
\end{aligned}
$$

Integration in the region $\{0\le u+\overline{v}\le 2T,\ \overline{v}-u\ge 0\}$ and using Lemma 3.1 for the boundary estimate, we get the desired conclusions.

4 Small Energy Implies Regularity

In this section, we assume

$$E_0\le\varepsilon_1 \tag{4.1}$$

is sufficiently small. Under this assumption, we shall prove the solution is smooth. For that purpose, we only need to give the H^2 estimate of the solution, see [4].

Let $\Psi=\Phi_t$. Differentiating the equation (2.1), we get

$$\Psi_{tt}-\Psi_{rr}-\frac{\Psi_r}{r}=4B(\Phi)(\Psi_u,\Phi_v)+4B(\Phi)(\Phi_u,\Psi_v)+4\Psi\cdot B'(\Phi)(\Phi_u,\Phi_v), \tag{4.2}$$

thus

$$
\begin{aligned}
&r\Psi_t\cdot\Big(\Psi_{tt}-\Psi_{rr}-\frac{\Psi_r}{r}\Big)\\
=\,&4r\Psi_t\cdot B(\Phi)(\Psi_u,\Phi_v)+4r\Psi_t\cdot B(\Phi)(\Phi_u,\Psi_v)+4\Psi_t\cdot[\Psi\cdot A'(\Phi)(\Phi_u,\Phi_v)],\\
&\Psi\cdot B(\Phi)=0.
\end{aligned}
$$

Differentiating with respect to t yields

$$\Psi_t\cdot B(\Phi)=-\Psi\cdot[\Psi\cdot A'(\Phi)].$$

So we get

$$\begin{aligned}
&\partial_t\Big[\frac{1}{2}(\Psi_t^2+\Psi_r^2)r\Big]-\partial_r[r\Psi_t\cdot\Psi_r]\\
&=4r\Psi\cdot[\Psi\cdot B'(\Phi)](\Psi_u,\Phi_v)+4r\Psi\cdot[\Psi\cdot B'(\Phi)](\Phi_u,\Psi_v)+4r\Psi_t\cdot[\Psi\cdot B'(\Phi)](\Phi_u,\Phi_v)\\
&=G_1.
\end{aligned}\tag{4.3}$$

In a similar way, let $\widehat{\Psi}=\Phi_r(t,r)$. We get

$$\begin{aligned}
&\partial_t\Big[\frac{1}{2}\Big(\widehat{\Psi}_t^2+\widehat{\Psi}_r^2+\frac{\widehat{\Psi}^2}{r^2}\Big)r\Big]-\partial_r[r\widehat{\Psi}_t\cdot\widehat{\Psi}_r]\\
&=4r\widehat{\Psi}\cdot[\widehat{\Psi}\cdot B'(\Phi)](\widehat{\Psi}_u,\Phi_v)+4r\widehat{\Psi}\cdot[\widehat{\Psi}\cdot B'(\Phi)](\Phi_u,\widehat{\Psi}_v)+4r\widehat{\Psi}_t\cdot[\widehat{\Psi}\cdot B'(\Phi)](\Phi_u,\Phi_v)\\
&=\widetilde{G}_1.
\end{aligned}\tag{4.4}$$

Integrating in t,r, we get

$$\sup_{0\le t\le T}|D^2\Phi|^2_{L^2(\mathbb{R}^2)}(t)\lesssim|\nabla^2u_0|^2_{L^2(\mathbb{R}^2)}+|\nabla u_1|^2_{L^2(\mathbb{R}^2)}+\int_0^T\int_0^\infty(|\widetilde{G}_1|+|G_1|)(t,r)\mathrm{d}r\mathrm{d}t.\tag{4.5}$$

The main purpose of this section is to prove

$$\int_0^T\int_0^\infty(|\widetilde{G}_1|+|G_1|)(t,r)\mathrm{d}r\mathrm{d}t\lesssim\varepsilon_1E_1,\tag{4.6}$$

where

$$E_1=|\nabla^2u_0|^2_{L^2(\mathbb{R}^2)}+|\nabla u_1|^2_{L^2(\mathbb{R}^2)}.\tag{4.7}$$

For that purpose, we use an induction argument and first we assume

$$\int_0^T\int_0^\infty(|\widetilde{G}_1|+|G_1|)(t,r)\mathrm{d}r\mathrm{d}t\lesssim E_1.\tag{4.8}$$

Then we get

$$|D^2\Phi|^2_{L^2(\mathbb{R}^2)}(t)\lesssim E_1.\tag{4.9}$$

To obtain (4.6), we only estimate G_1. The estimate of $\widetilde{G}_1$ can be done in a similar way. By the expression of G_1, we get

$$\begin{aligned}
|G_1|&\lesssim|\Psi|^2(|\Psi_u||\Phi_v|+|\Phi_u||\Psi_v|)r+|\Psi_t|(|\Psi||\Phi_u||\Phi_v|)r\\
&=g_1+g_2.
\end{aligned}\tag{4.10}$$

We first estimate g_1. Let $\frac{1}{10}\le\beta\le\frac{1}{4}$, we have

$$\begin{aligned}
&\int_0^T\int_0^\infty|g_1|\mathrm{d}r\mathrm{d}t\\
&\lesssim\Big[\int_0^T\int_0^\infty(|\Psi_u|^2|\Phi_v|^2+|\Phi_u|^2|\Psi_v|^2)r^{2-\beta}\mathrm{d}r\mathrm{d}t\Big]^{\frac{1}{2}}\Big[\int_0^T\int_0^\infty|\Psi|^4r^\beta\mathrm{d}r\mathrm{d}t\Big]^{\frac{1}{2}}.
\end{aligned}\tag{4.11}$$

By Sobolev-Hardy inequality, we obtain

$$\int_0^\infty |\Psi|^4 r^\beta \mathrm{d}r \lesssim \Big(\int_0^\infty |\Psi|^2 r^{-\beta}\mathrm{d}r\Big)\Big(\int_0^\infty |\Psi|^2 r\mathrm{d}r\Big)^\beta\Big(\int_0^\infty |\Psi_r|^2 r\mathrm{d}r\Big)^{1-\beta}, \tag{4.12}$$

thus

$$\begin{aligned}\int_0^T\int_0^\infty |g_1|\mathrm{d}r\mathrm{d}t &\lesssim \Big[\int_0^T\int_0^\infty (|\Psi_u|^2|\Phi_v|^2+|\Phi_u|^2|\Psi_v|^2)r^{2-\beta}\mathrm{d}r\mathrm{d}t\Big]^{\frac12}\Big[\int_0^T\int_0^\infty |\Psi|^2 r^{-\beta}\mathrm{d}r\mathrm{d}t\Big]^{\frac12}\\ &\lesssim \varepsilon_1^{\frac{\beta}{2}}E_1^{\frac{1-\beta}{2}}.\end{aligned} \tag{4.13}$$

Take $\alpha=\beta$ in (2.17) and integrate for $0\le r\le+\infty,\ 0\le t\le T$. Noting (2.19), we get

$$\begin{aligned}&\frac{1}{20}\int_0^T\int_0^\infty r^{-\beta}(|\Psi|^2+|\widehat{\Psi}|^2)\mathrm{d}r\mathrm{d}t=\Big(\int_0^\infty r^{1-\beta}\Psi\widehat{\Psi}\Big)\Big|_0^T+\int_0^T\int_0^\infty F_\alpha\mathrm{d}r\mathrm{d}t\\ &\le\Big(\int_0^\infty r^{1-\beta}\Psi\widehat{\Psi}\Big)\Big|_0^T+C\varepsilon_1\int_0^T\int_0^\infty r^{-\beta}(|\Psi|^2+|\widehat{\Psi}|^2)\mathrm{d}r\mathrm{d}t.\end{aligned}$$

So when ε_1 is sufficiently small, we get

$$\frac{1}{40}\int_0^T\int_0^\infty r^{-\beta}(|\Psi|^2+|\widehat{\Psi}|^2)\mathrm{d}r\mathrm{d}t\lesssim\Big(\int_0^\infty r^{1-\beta}\Psi\widehat{\Psi}\Big)\Big|_0^T.$$

By Sobolev-Hardy inequality and noting (4.9), we get

$$\Big(\int_0^\infty r^{1-\beta}\Psi\widehat{\Psi}\Big)|_0^T\lesssim\varepsilon_1^{1-\frac{\beta}{2}}E_1^{\frac{\beta}{2}},$$

thus

$$\Big(\int_0^T\int_0^\infty r^{-\beta}(|\Psi|^2+|\widehat{\Psi}|^2)\mathrm{d}r\mathrm{d}t\Big)^{\frac12}\lesssim\varepsilon_1^{\frac12-\frac{\beta}{4}}E_1^{\frac{\beta}{4}}. \tag{4.14}$$

Rewrite (4.3) as

$$\partial_u[r(\partial_v\Psi)^2]+\partial_v[r(\partial_u\Psi)^2]=2G_1. \tag{4.15}$$

Take $\alpha=\beta$ in (2.17) and rewrite it as

$$\partial_u[r^{1-\beta}(\partial_v\Phi)^2]-\partial_v[r^{1-\beta}(\partial_u\Phi)^2]=-2F_\beta+\beta r^{-\beta}(|\Psi|^2+|\widehat{\Psi}|^2)=G_\beta. \tag{4.16}$$

Noting that (4.14) implies

$$\int_0^T\int_0^\infty |G_\beta|(t,r)\mathrm{d}r\mathrm{d}t\lesssim\varepsilon_1^{1-\frac{\beta}{2}}E_1^{\frac{\beta}{2}}. \tag{4.17}$$

We apply Lemma 3.2 to get

$$\int_0^T\int_0^\infty r^{2-\beta}[(\partial_u\Phi)^2(\partial_v\Psi)^2+(\partial_u\Psi)^2(\partial_v\Phi)^2]\mathrm{d}r\mathrm{d}t\lesssim\varepsilon_1^{1-\frac{\beta}{2}}E_1^{1+\frac{\beta}{2}}. \tag{4.18}$$

So we get

$$\int_0^T\int_0^\infty |g_1|\mathrm{d}r\mathrm{d}t\lesssim\varepsilon_1E_1. \tag{4.19}$$

Now we estimate g_2.

$$\int_0^T\int_0^\infty |g_2|\mathrm{d}r\mathrm{d}t \lesssim \int_0^T \Big(\int_0^\infty |\Psi_t|^2 r\mathrm{d}r\Big)^{\frac{1}{2}}\Big(\int_0^\infty |\Psi|^2|\Phi_u|^2|\Phi_v|^2 r\mathrm{d}r\Big)^{\frac{1}{2}}\mathrm{d}t$$
$$\lesssim E_1^{\frac{1}{2}}\int_0^T \Big(\int_0^\infty |\Psi|^2|\Phi_u|^2|\Phi_v|^2 r\mathrm{d}r\Big)^{\frac{1}{2}}\mathrm{d}t. \tag{4.20}$$

We have

$$\int_0^\infty |\Psi|^2|\Phi_u|^2|\Phi_v|^2 r\mathrm{d}r = \int_0^\infty |\Phi_u|^2|\Phi_v|^2\mathrm{d}r\int_0^r |\Psi|^2\lambda\mathrm{d}\lambda$$
$$= -2\int_0^\infty\int_0^r |\Psi|^2\lambda\mathrm{d}\lambda[(\Phi_u\cdot\widehat{\Psi}_u)|\Phi_v|^2 + |\Phi_u|^2(\Phi_v\cdot\widehat{\Psi}_v)]$$
$$\le 2\int_0^\infty \mathrm{d}r\int_0^r |\Psi|^2\lambda^{-\beta}\mathrm{d}\lambda[r^{1+\beta}|\Phi_u||\Phi_v|(|\widehat{\Psi}_u||\Phi_v| + |\widehat{\Psi}_v||\Phi_u|)]$$
$$\le 2\Big(\int_0^\infty |\Psi|^2 r^{-\beta}\mathrm{d}r\Big)\Big(\int_0^\infty (|\widehat{\Psi}_u|^2|\Phi_v|^2 + |\widehat{\Psi}_v|^2|\Phi_u|^2)r^{2-\beta}\mathrm{d}r\Big)^{\frac{1}{2}}$$
$$\Big(\int_0^\infty |\Phi_u|^2|\Phi_v|^2 r^{3\beta}\mathrm{d}r\Big)^{\frac{1}{2}}.$$

Therefore we get

$$\int_0^T\int_0^\infty |g_2|\mathrm{d}r\mathrm{d}t \lesssim E_1^{\frac{1}{2}}\Big(\int_0^T\int_0^\infty |\Psi|^2 r^{-\beta}\mathrm{d}r\mathrm{d}t\Big)^{\frac{1}{2}}$$
$$\Big(\int_0^T\int_0^\infty r^{2-\beta}(|\widehat{\Psi}_u|^2|\Phi_v|^2 + |\widehat{\Psi}_v|^2|\Phi_u|^2)\mathrm{d}r\mathrm{d}t\Big)^{\frac{1}{4}}$$
$$\Big(\int_0^T\int_0^\infty |\Phi_u|^2|\Phi_v|^2 r^{3\beta}\mathrm{d}r\mathrm{d}t\Big)^{\frac{1}{4}}. \tag{4.21}$$

The first term in the right-hand side can be estimated by (4.14), and the second term can be estimated in a way similar to (4.18). Thus, it remains to prove

$$\int_0^T\int_0^\infty |\Phi_u|^2|\Phi_v|^2 r^{3\beta}\mathrm{d}r\mathrm{d}t \lesssim \varepsilon_1^{1+\frac{3\beta}{2}}E_1^{1-\frac{3\beta}{2}}.$$

We have by (4.16) with β replaced by α

$$\partial_u[r^{1-\alpha}(\partial_v\Phi)^2] - \partial_v[r^{1-\alpha}(\partial_u\Phi)^2] = G_\alpha. \tag{4.22}$$

We have

$$\Phi_t\cdot\Big[\Phi_{tt} - \Phi_{rr} - \frac{\Phi_r}{r}\Big] = 0,$$

so we get

$$\partial_u[r^{1-\alpha}(\partial_v\Phi)^2] + \partial_v[r^{1-\alpha}(\partial_u\Phi)^2] = \widehat{G}_\alpha,$$

where

$$\widehat{G}_\alpha = (1-\alpha)r^{-\alpha}[(\partial_v\Phi)^2 - (\partial_u\Phi)^2].$$

Thus, we have

$$\int_0^T \int_0^\infty (|\widehat{G}_\alpha| + |G_\alpha|)\mathrm{d}r\mathrm{d}t \lesssim \varepsilon_1^{1-\frac{\alpha}{2}} E_1^{\frac{\alpha}{2}}.$$

So we apply Lemma 3.2 to get

$$\int_0^T \int_0^\infty r^{2-2\alpha}(\partial_v \Phi)^2(\partial_u \Phi)^2 \mathrm{d}r\mathrm{d}t \lesssim \varepsilon_1^{2-\alpha} E_1^{\alpha}.$$

Take $2 - 2\alpha = 3\beta$, we get

$$\int_0^T \int_0^\infty r^{3\beta}(\partial_v \Phi)^2(\partial_u \Phi)^2 \mathrm{d}r\mathrm{d}t \lesssim \varepsilon_1^{1+\frac{3\beta}{2}} E_1^{1-\frac{3\beta}{2}}.$$

This completes the proof of the regularity with small energy.

5 Energy Do Not Concentrate

Let $\Phi(t, r) : [0, t_0) \times \mathbb{R}^2 \to N \subset \mathbb{R}^n$ be a smooth radially symmetric wave map blowing up at time t_0. Necessarily, blow-up occurs at $r = 0$. Shifting and reversing time and then scaling our space-time coordinates suitably, we may assume that u is a smooth radial solution to (1.1) on $(0, 1] \times \mathbb{R}^2$ blowing up at the origin.

Let $K_S^T = \{z = (t, x) : 0 \le |x| \le t, S \le t \le T\}$ be the forward light cone with lateral boundary $M_S^T = \{(t, x) \in K_S^T, |x| = t\}$. Denoting as

$$e = \frac{1}{2}(|\Phi_t|^2 + |\Phi_r|^2), \quad f = \frac{1}{2}(\Phi_t + \Phi_r)^2$$

the energy and flux density of u, respctively, and denoting as

$$E(\Phi, R) = \int_{B_R(0)} e\mathrm{d}x,$$
$$\mathrm{Flux}(\Phi, S, T) = \int_{M_S^T} F\mathrm{d}\sigma.$$

We have

(1) Energy inequality

$$E(u(t), R) \le E(u(t + \tau), R + |\tau|);$$

(2) Flux decay

$$\mathrm{Flux}(\Phi, S, T) \to 0 \quad \text{as } T \to 0,\ S \to 0;$$

(3) Exterior energy decay

$$E(\Phi(t), t) - E(\Phi(t), \lambda t) \to 0 \quad \text{for any } 0 < \lambda < 1;$$

(4) Decay of time derivatives

$$\frac{1}{T} \int_{K_S^T} |\Phi_t|^2 \mathrm{d}z \to 0 \quad \text{as } T \to 0,\ S \to 0.$$

Take $\alpha = -1$ in (2.17), we get

$$\partial_t(r^2\Phi_t \cdot \Phi_r) - \partial_r(r^2 e) + re = r^2[q_1 \cdot (A_0 q_0) + (A_0 q_1) \cdot q_0] = rF.$$

Integrating in K_S^T, we get

$$\begin{aligned}
&\frac{1}{T}\int_{K_S^T}[\partial_t(r^2\Phi_t \cdot \Phi_r) - \partial_r(r^2 e)]\mathrm{d}z + \frac{1}{T}\int_{K_S^T} e\mathrm{d}z = \frac{1}{T}\int_{K_S^T} F\mathrm{d}z \\
&\lesssim \frac{1}{T}E_0\Big(\int_{K_S^T}|\Phi_t|^2\mathrm{d}z\Big)^{\frac12}\Big(\int_{K_S^T} e\mathrm{d}z\Big)^{\frac12} \\
&\lesssim E_0^{\frac32}\Big(\frac{1}{T}\int_{K_S^T}|\Phi_t|^2\mathrm{d}z\Big)^{\frac12} \to 0,
\end{aligned}$$

and

$$\begin{aligned}
&-\frac{1}{T}\int_{K_S^T}[\partial_t(r^2\Phi_t \cdot \Phi_r) - \partial_r(r^2 e)]\mathrm{d}z \\
&= \frac{1}{T}\Big[-\int_{B(0,T)} r\Phi_t \cdot \Phi_r\mathrm{d}z + \int_{B(0,S)} r\Phi_t \cdot \Phi_r\mathrm{d}z\Big] \\
&\quad + \frac{1}{T}\int_{M_S^T} rF\mathrm{d}\sigma.
\end{aligned}$$

We have

$$\begin{aligned}
&\frac{1}{T}\int_{M_S^T} rF\mathrm{d}\sigma \le \int_{M_S^T} F\mathrm{d}\sigma \to 0, \\
&\Big|\frac{1}{T}\int_{B(0,T)} r\Phi_t \cdot \Phi_r\mathrm{d}z\Big| \le \frac{1}{T}\Big|\int_{B(0,\lambda T)} r\Phi_t \cdot \Phi_r\mathrm{d}x\Big| + \int_{B(0,T)-B(0,\lambda T)} e\mathrm{d}x \\
&\le \lambda\int_{B(0,\lambda T)} e\mathrm{d}x + \int_{B(0,T)-B(0,\lambda T)} e\mathrm{d}x \\
&\le \lambda E_0 + \int_{B(0,T)-B(0,\lambda T)} e\mathrm{d}x \to \lambda E_0,
\end{aligned}$$

$$\frac{1}{T}\Big|\int_{B(0,S)} r\Phi_t \cdot \Phi_r\mathrm{d}x\Big| \le \frac{S}{T}\int_{B(0,S)} e\mathrm{d}x \le \frac{S}{T}E_0 \to 0.$$

Thus, taking limit, we get

$$\limsup_{S=T^2, T\to 0}\frac{1}{T}\int_{K_S^T} e\mathrm{d}z \le \lambda E_0,$$

where $\lambda > 0$ is arbitrary. So we get

$$\limsup_{S=T^2, T\to 0}\frac{1}{T}\int_{K_S^T} e\mathrm{d}z = 0.$$

By the monotonicity of the energy, we prove that

$$E(\Phi(S), |S|) \to 0 \quad \text{as } |S| \to 0.$$

This shows that energy can not concentrate, thus by the conclusion of the last section, we prove Theorem 1.1.

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