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Revisit of the Faddeev Model in Dimension Two*

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Abstract The Faddeev model is a fundamental model in relativistic quantum field theory used to model elementary particles. The Faddeev model can be regarded as a system of non-linear wave equations with both quasi-linear and semi-linear non-linearities, which is particularly challenging in two space dimensions. A key feature of the system is that there exist undifferentiated wave components in the non-linearities, which somehow causes extra difficulties. Nevertheless, the Cauchy problem in two space dimenions was tackled by Lei-Lin-Zhou (2011) with small, regular, and compactly supported initial data, using Klainerman's vector field method enhanced by a novel angular-radial anisotropic technique. In the present paper, the authors revisit the Faddeev model and remove the compactness assumptions on the initial data by Lei-Lin-Zhou (2011). The proof relies on an improved L^2 norm estimate of the wave components in Theorem 3.1 and a decomposition technique for non-linearities of divergence form.

Keywords Faddeev model in $\mathbb{R}^{1+2}$, Global existence, Null condition
2000 MR Subject Classification 35L05

1 Introduction

Model Problem We are interested in the Faddeev model in two space dimensions, which is a very important model in relativistic quantum field theory modelling elementary particles (see [6–8]). The model equations can be written as follows (see Section 3 for its derivation)

$$\begin{aligned}
&\Box \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} + \frac{n_1 \Box n_1 + n_2 \Box n_2}{1 - n_1^2 - n_2^2} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} + \frac{\partial_\mu (\partial^\mu n_1 \partial^\nu n_2 - \partial^\mu n_2 \partial^\nu n_1)}{1 - n_1^2 - n_2^2} \begin{pmatrix} -\partial_\nu n_2 \\ \partial_\nu n_1 \end{pmatrix} \\
&\quad + \frac{\partial_\nu n_1 \partial^\nu n_1 + \partial_\nu n_2 \partial^\nu n_2}{1 - n_1^2 - n_2^2} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} + \frac{n_1^2 \partial_\nu n_1 \partial^\nu n_1 + n_2^2 \partial_\nu n_2 \partial^\nu n_2 + 2 n_1 n_2 \partial_\nu n_1 \partial^\nu n_2}{(1 - n_1^2 - n_2^2)^2} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \\
&\quad - \frac{(n_1 \partial_\mu n_1 + n_2 \partial_\mu n_2)(\partial^\mu n_1 \partial^\nu n_2 - \partial^\mu n_2 \partial^\nu n_1)}{1 - n_1^2 - n_2^2} \begin{pmatrix} -\partial_\nu n_2 \\ \partial_\nu n_1 \end{pmatrix} = 0. \qquad (1.1)
\end{aligned}$$

In the above, we use $\Box = \partial_\alpha \partial^\alpha$ to denote the wave operator, with $\partial_\alpha = \partial_{x_\alpha}$, $(x^0, x^a) = (t, -x_a)$. We take the metric $\eta = \mathrm{diag}(1, -1, -1)$, and the indices are raised or lowered by the metric η. The Einstein summation convention is adopted for repeated indices. We write $\|\cdot\|_{L^p} = \|\cdot\|_{L^p(\mathbb{R}^2)}$ to denote the usual L^p-norm of a nice function and adopt the abbreviation $\|\cdot\| = \|\cdot\|_{L^2}$ if no

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confusion arises. We use $A \lesssim B$ to indicate $A \le CB$ with C a generic constant, and we use the notation $\langle p \rangle = (1+|p|^2)^{\frac{1}{2}}$. We note that the equations of Faddeev model can be regarded as a quasi-linear system of wave equations.

The initial data of the Faddeev model (1.1), prescribed at $t = t_0 = 0$, are denoted by

$$(n_i, \partial_t n_i)(t_0 = 0) = (n_{i0}, n_{i1}), \quad i = 1, 2. \tag{1.2}$$

The Faddeev model is an important model in quantum field theory with extensive mathematical studies. The investigations on the static Faddeev model or some related problems can be found in the series of works [18–21] by Lin-Yang. On the other hand, the Cauchy problem of the Faddeev model was first tackled by Lei-Lin-Zhou [15] in two space dimensions. Later on, the sharp global regularity for the two dimensional Faddeev model was shown by Geba-Nakanishi-Zhang [10] under some extra assumptions. Recently, the large data global existence for the two (and three) dimensional Faddeev model was studied by Geba-Grillakis [9] and by Zha-Liu-Zhou [27].

We note that the Faddeev model can be regarded as a generalisation of the harmonic maps $\mathbb{R}^{1+n} \to \mathbb{S}^2$. We recall the remarkable pioneering work [11] by Gu on harmonic maps in one space dimension, which is relevant to our study. He succeeded in treating the harmonic maps $\mathbb{R}^{1+1} \to M$, where M is a complete Riemannian manifold of dimension n, including the two dimensional sphere $\mathbb{S}^2$ as a special case, and proved that the solution to the Cauchy problem exists globally.

We recall the seminal works [13–14] by Klainerman, [3] by Christodoulou, [23] by Lindblad-Rodnianski on three dimensional non-linear wave equations, and [1] by Alinhac on two dimensional case. The Cauchy problem of the Faddeev model in three space dimensions and higher can be solved using these classical theories. This problem is particularly tricky in two space dimensions. Nevertheless, Lei-Lin-Zhou proved the global well-posedness of the Cauchy problem of the Faddeev model in two space dimensions with compactly supported initial data. The prime goal of the present paper is to remove the compactness assumptions on the initial data. We would also like to draw one's attention to some recent progress on two dimensional wave equations of [2, 5, 12, 16].

Main Theorem We want to show the existence of global solutions to system (1.1) and to derive the pointwise asymptotic behavior of the solutions. Our main result is stated as follows.

Theorem 1.1 *Consider the Faddeev model* (1.1)*, and let* $N \ge 5$ *be an integer. Then there exists a small* $\varepsilon_0 > 0$*, such that for all initial data satisfying the smallness condition*

$$\sum_{|I| \le N+1} \|\Lambda^I n_{i0}\|_{L^2(\mathbb{R}^2)} + \sum_{|J| \le N} \|\langle |x| \rangle \Lambda^J n_{i1}\|_{L^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)} < \varepsilon \le \varepsilon_0$$

with $\Lambda \in \{\partial_a, r\partial_r, \Omega_{ab}\}$*, the Cauchy problem* (1.1)–(1.2) *admits a global-in-time solution* $n = (n_i)$*, which enjoys the pointwise decay results*

$$|n_i(t,x)| \lesssim \langle t+|x| \rangle^{-\frac{1}{2}} \langle t-|x| \rangle^{-\frac{1}{2}} \log(2+t)^{\frac{1}{2}}, \quad |\partial n_i(t,x)| \lesssim \langle t+|x| \rangle^{-\frac{1}{2}} \langle t-|x| \rangle^{-\frac{1}{2}}. \tag{1.3}$$

In general, the smallness condition on $\|\Lambda^J u_{i1}\|_{L^1(\mathbb{R}^2)}$ is not assumed when treating wave equations, but we will need it in the proof of Proposition 2.6. We note that similar assumptions on the initial data also exist in the remarkable result [12], where the authors removed the compactness assumptions on the celebrated result [1] by Alinhac. In [12], the authors applied a novel weighted L^∞–L^∞ estimate for the wave equations to achieve the goals. As a comparison, we use the energy method to prove Theorem 1.1.

It is known that linear waves in two space dimensions decay at the speed of $\langle t+|x|\rangle^{-\frac{1}{2}}\langle t-|x|\rangle^{-\frac{1}{2}}$, however, there is an extra $\log(2+t)^{\frac{1}{2}}$ growth in (1.3) of Theorem 1.1. We note similar logarithmic growth also occur in the results [24, 26], when using energy method and Klainerman-Sobolev inequality to derive the pointwise decay of the wave solution itself.

To bound the wave solution itself, a natural way is to apply the Hardy inequality. We recall that Hardy inequality

$$\Big\|\frac{w}{r}\Big\|_{L^2(\mathbb{R}^n)} \le C\|\nabla w\|_{L^2(\mathbb{R}^n)}$$

can be used in higher dimensions $n\ge 3$, while the following version of Hardy inequality

$$\Big\|\frac{w}{\langle t-r\rangle}\Big\|_{L^2(\mathbb{R}^2)} \le C\|\nabla w\|_{L^2(\mathbb{R}^2)}$$

requires the function w to be compactly supported. Unfortunately, neither of the above versions of Hardy inequality can be applied to our case. New ideas are demanded in dealing with our problem (1.1).

One key idea is to prove refined estimates on the wave solution itself of [15, Theorem 3.1], which is demonstrated in Proposition 2.6. Since the Faddeev model (1.1) contains also quasi-linear non-linearities, the result in Proposition 2.6 cannot be directly applied in the highest-order case, which is the most difficult part of the analysis. Fortunately, utilising a decomposition can help us conquer this difficulty; see the discussion in the begining of Section 4.2. Importantly, this way can also be used to remove the compactness assumptions in [1].

The rest of the paper is organised as follows. In Section 2, we present some preliminaries on wave equations. Then in Section 3, we illustrate the derivation of the equations of the Faddeev model (1.1). Finally, we demonstrate the proof of Theorem 1.1 in Section 4.

2 Preliminary

We work in the $(1+2)$-dimensional Minkowski spacetime with signature $(+,-,-)$. A point in $\mathbb{R}^{1+2}$ is denoted by $(x^0,x^1,x^2)=(t,-x_1,-x_2)$, and its spacial radius is denoted by $r=\sqrt{x_1^2+x_2^2}$. We use Latin letters $a,b,\cdots\in\{1,2\}$ to represent space indices, while the spacetime indices are denoted by Greek letters $\alpha,\beta,\cdots\in\{0,1,2\}$.

The following vector fields will be used when applying Klainerman's vector field method and Alinhac's ghost weight method

$$\begin{cases}\text{translations:}\quad \partial_\alpha=\partial_{x_\alpha}, \quad \alpha=0,1,2;\\ \text{rotations:}\quad \Omega_{ab}:=x_a\partial_b-x_b\partial_a, \quad a,b=1,2;\\ \text{Lorentz boosts:}\quad L_a:=x_a\partial_t+t\partial_a, \quad a=1,2;\\ \text{scaling vector field:}\quad L_0:=t\partial_t+r\partial_r;\\ \text{good derivatives:}\quad G_a:=\dfrac{x_a}{r}\partial_t+\partial_a, \quad a=1,2.\end{cases}$$

We will use Γ to denote the vector fields in

$$V:=\{\partial_\alpha,\Omega_{ab},L_a,L_0\}. \tag{2.1}$$

For a sufficiently nice function $w = w(t,x)$, we define its energy by

$$E(w,t) := \int_{\mathbb{R}^2} \Big(|\partial_t w|^2 + \sum_a |\partial_a w|^2 \Big) \, \mathrm{d}x. \tag{2.2}$$

The following results on commutators will be frequently used, and one refers to [25, Chapter II] or [17, Section 3] for the proof.

Proposition 2.1 *For any $\Gamma', \Gamma'' \in V$, we have*

$$[\Box, \Gamma'] = C\Box, \quad [\Gamma', \Gamma''] = \sum_{\Gamma \in V} C_\Gamma \Gamma, \tag{2.3}$$

in which $[A,B] = AB - BA$ and C, C_Γ are some constants.

We have the following result, and one refers to [25, Chapter II] for its proof.

Proposition 2.2 *For sufficiently nice function $w(t,x)$, it holds*

$$|\partial w(t,x)| \lesssim \langle t-r \rangle^{-1} |\Gamma w(t,x)|. \tag{2.4}$$

We recall the following estimates on the null forms

$$Q_0(v,w) = \partial_t v \partial_t w + \partial_a v \partial^a w, \quad Q_{\alpha\beta}(v,w) = \partial_\alpha v \partial_\beta w - \partial_\beta v \partial_\alpha w. \tag{2.5}$$

Proposition 2.3 *For sufficiently nice functions v, w, we have the following estimates*

$$\begin{cases} |Q_0(v,w)| + |Q_{\alpha\beta}(v,w)| \lesssim \langle t \rangle^{-1} (|\Gamma v||\partial w| + |\Gamma w||\partial v|), \\ |Q_0(v,w)| + |Q_{\alpha\beta}(v,w)| \lesssim \sum_a (|G_a v||\partial w| + |G_a w||\partial v|), \\ |Q_0(v,w)| + |Q_{\alpha\beta}(v,w)| \lesssim \sum_a |G_a v||\partial w| + \langle t \rangle^{-1} |\partial v||\Gamma w|. \end{cases} \tag{2.6}$$

Proof For the first estimate in (2.6), one finds its proof in [25, Chapter II] for instance.

For the second estimate in (2.6), we use the relation $\partial_a = G_a - (x_a/r)\partial_t$ to derive the desired bounds.

For the third estimate in (2.6), we observe that for $t \geq 1$,

$$\begin{aligned} |G_a w||\partial v| &\leq |G_a w - t^{-1} L_a w||\partial v| + t^{-1} |L_a w||\partial v| \\ &\leq \frac{|x_a(t-r)|}{t\, r} |\partial_t w||\partial v| + t^{-1} |L_a w||\partial v| \lesssim \langle t \rangle^{-1} |\partial v||\Gamma w|, \end{aligned}$$

and for small t the third estimate in (2.6) is obvious.

The null forms enjoy the following commutating property, and one refers to [25, Chapter II] for the proof.

Lemma 2.1 *Let $\Gamma \in V$, then we have*

$$\begin{cases} [\Gamma, Q_{\alpha\beta}(v,w)] = C^{\mu\nu}_{\Gamma,\alpha\beta} Q_{\mu\nu}(v,w), \\ [\Gamma, Q_0(v,w)] = C^{\mu\nu}_{\Gamma,0} Q_{\mu\nu}(v,w) + C_{\Gamma,0} Q_0(v,w), \end{cases}$$

in which $[\Gamma, Q(v,w)] = \Gamma Q(v,w) - Q(\Gamma v, w) - Q(v, \Gamma w)$, and C's are constants.

We state the classical Klainerman-Sobolev inequality in $\mathbb{R}^{1+2}$, which will be frequently used in the analysis to derive the pointwise decay of a function, and whose proof can be found in [25, Chapter II] for instance.

Proposition 2.4 *It holds that*

$$\langle t+r\rangle^{\frac{1}{2}}\langle t-r\rangle^{\frac{1}{2}}|u| \lesssim \sum_{|I|\le 2}\|\Gamma^I u\|_{L^2(\mathbb{R}^2)} \tag{2.7}$$

with $\langle a\rangle = \sqrt{1+|a|^2}$.

The following ghost weight energy estimates by Alinhac [1] will play a vital role in closing the higher-order energy estimates. We first define the generalised energy (ghost weight energy) for a nice function w (with small $\delta>0$ to be fixed in Section 4)

$$E_{\mathrm{gst}}(w,t) = \int_{\mathbb{R}^2}(|\partial_t w|^2 + \sum_a |\partial_a w|^2)\,\mathrm{d}x(t) + \sum_a \int_0^t\int_{\mathbb{R}^2}\frac{|G_a w|^2}{\langle r-\tau\rangle^{1+2\delta}}\,\mathrm{d}x\mathrm{d}\tau. \tag{2.8}$$

More specifically, we will need the following quasi-linear version of ghost weight energy estimates.

Proposition 2.5 *Consider the quasi-linear wave system*

$$\begin{cases}\Box w_1 + Q_1^{\alpha\beta}\partial_\alpha\partial_\beta w_1 + Q_2^{\alpha\beta}\partial_\alpha\partial_\beta w_2 = h_1,\\ \Box w_2 + Q_3^{\alpha\beta}\partial_\alpha\partial_\beta w_2 + Q_4^{\alpha\beta}\partial_\alpha\partial_\beta w_1 = h_2\end{cases} \tag{2.9}$$

with solution (w_1,w_2) decaying sufficiently fast as $|x|\to+\infty$. We assume the smallness condition

$$\sum_{\alpha,\beta,1\le i\le 4}|Q_i^{\alpha\beta}| \ll \frac{1}{100},$$

the symmetry condition

$$Q_i^{\alpha\beta} = Q_i^{\beta\alpha},\quad i=1,2,3,4,$$

and the hyperbolicity condition

$$Q_2^{\alpha\beta} = Q_4^{\alpha\beta}.$$

Then we have

$$E_{\mathrm{gst}}(w_1,t) + E_{\mathrm{gst}}(w_2,t) \lesssim E_{\mathrm{gst}}(w_1,t_0) + E_{\mathrm{gst}}(w_2,t_0) + \int_{t_0}^t\int_{\mathbb{R}^2}\mathcal{R}\,\mathrm{d}x\mathrm{d}\tau, \tag{2.10}$$

in which (with $\rho = \mathrm{e}^{\int_{-\infty}^{r-\tau}\langle s\rangle^{-1-2\delta}\,\mathrm{d}s}$)

$$\begin{aligned}\mathcal{R} = {}& |f_1\,\partial_t w_1| + |f_2\,\partial_t w_2| + |\partial_\alpha Q_1^{\alpha\beta}\partial_\beta w_1\partial_t w_1| + |\partial_t Q_1^{\alpha\beta}\partial_\alpha w_1\partial_\beta w_1| + |\partial_\alpha Q_2^{\alpha\beta}\partial_\beta w_2\partial_t w_1|\\ &+ |\partial_t Q_2^{\alpha\beta}\partial_\alpha w_2\partial_\beta w_1| + |\partial_\alpha Q_3^{\alpha\beta}\partial_\beta w_2\partial_t w_2| + |\partial_t Q_3^{\alpha\beta}\partial_\alpha w_2\partial_\beta w_2| + |\partial_\alpha Q_2^{\alpha\beta}\partial_\beta w_1\partial_t w_2|\\ &+ |Q_1^{\alpha\beta}\partial_\beta w_1\partial_t w_1\partial_\alpha\rho| + |Q_1^{\alpha\beta}\partial_\alpha w_1\partial_\beta w_1\partial_t\rho| + |Q_2^{\alpha\beta}\partial_\beta w_2\partial_t w_1\partial_\alpha\rho| + |Q_2^{\alpha\beta}\partial_\alpha w_1\partial_\beta w_2\partial_t\rho|\\ &+ |Q_3^{\alpha\beta}\partial_\beta w_2\partial_t w_2\partial_\alpha\rho| + |Q_3^{\alpha\beta}\partial_\alpha w_2\partial_\beta w_2\partial_t\rho| + |Q_2^{\alpha\beta}\partial_\beta w_1\partial_t w_2\partial_\alpha\rho|.\end{aligned}$$

Proof The proof follows from the following differential identity

$$\frac{1}{2}\sum_{\alpha,j=1,2}\partial_t((\partial_\alpha w_j)^2\rho) - \partial_a(\partial^a w_j\partial_t w_j\rho) + \sum_{j=1,2}\partial_\alpha(Q_{2j-1}^{\alpha\beta}\partial_\beta w_j\partial_t w_j\rho)$$

$$\begin{aligned}
&-\frac{1}{2}\sum_{j=1,2}\partial_t(Q_{2j-1}^{\alpha\beta}\partial_\alpha w_j\partial_\beta w_j\rho)+\partial_\alpha(Q_2^{\alpha\beta}\partial_\beta w_2\partial_t w_1\rho)+\partial_\alpha(Q_2^{\alpha\beta}\partial_\beta w_1\partial_t w_2\rho)\\
&-\partial_t(Q_2^{\alpha\beta}\partial_\alpha w_1\partial_\beta w_2\rho)+\frac{1}{2}\sum_{a,j=1,2}\frac{|G_a w_j|^2}{\langle r-\tau\rangle^{1+2\delta}}-\sum_{j=1,2}Q_{2j-1}^{\alpha\beta}\partial_\beta w_j\partial_t w_j\partial_\alpha\rho\\
&+\frac{1}{2}\sum_{j=1,2}Q_{2j-1}^{\alpha\beta}\partial_\alpha w_j\partial_\beta w_j\partial_t\rho-Q_2^{\alpha\beta}\partial_\beta w_2\partial_t w_1\partial_\alpha\rho-Q_2^{\alpha\beta}\partial_\beta w_1\partial_t w_2\partial_\alpha\rho\\
&+Q_2^{\alpha\beta}\partial_\alpha w_1\partial_\beta w_2\partial_t\rho-\sum_{j=1,2}\partial_\alpha Q_{2j-1}^{\alpha\beta}\partial_\beta w_j\partial_t w_j\rho-\frac{1}{2}\sum_{j=1,2}\partial_t Q_{2j-1}^{\alpha\beta}\partial_\alpha w_j\partial_\beta w_j\rho\\
&-\partial_\alpha Q_2^{\alpha\beta}\partial_\beta w_2\partial_t w_1\rho-\partial_\alpha Q_2^{\alpha\beta}\partial_\beta w_1\partial_t w_2\rho+\partial_t Q_2^{\alpha\beta}\partial_\alpha w_1\partial_\beta w_2\rho=f_1\partial_t w_1\rho+f_2\partial_t w_2\rho.
\end{aligned}$$

To estimate the L^2 norm of the wave components themselves (with no derivatives), we introduce the following result, which can be regarded as an enhanced version of [15, Theorem 3.1] (see also [4]).

Proposition 2.6 *Suppose w solves the wave equation*

$$\begin{cases}\Box w=f,\\ w(0,\cdot)=w_0,\quad \partial_t w(0,\cdot)=w_1,\end{cases}\tag{2.11}$$

and suppose that

$$\|w_0\|_{L^2(\mathbb{R}^2)}+\|w_1\|_{L^2(\mathbb{R}^2)\cap L^1(\mathbb{R}^2)}<+\infty,\tag{2.12}$$

as well as

$$\int_0^t\|f(\tau,\cdot)\|_{L^2(\mathbb{R}^2)\cap L^1(\mathbb{R}^2)}\,\mathrm{d}\tau\lesssim C_f\langle t\rangle^\beta,\quad \beta\in[0,1).\tag{2.13}$$

Then the following L^2 norm bound is valid

$$\|w\|_{L^2(\mathbb{R}^2)}\lesssim\begin{cases}B(t)+\log^{\frac12}(2+t)C_f, & \beta=0,\\ B(t)+\langle t\rangle^\beta\log^{\frac12}(2+t)C_f, & 0<\beta<1,\end{cases}\tag{2.14}$$

in which $B(t)=\|w_0\|_{L^2(\mathbb{R}^2)}+\log^{\frac12}(2+t)\|w_1\|_{L^2(\mathbb{R}^2)\cap L^1(\mathbb{R}^2)}$.

Proof We revisit its proof in [4, 15].

First, we write (2.11) in the Fourier space to get

$$\begin{cases}\partial_{tt}\widehat{w}(t,\xi)+|\xi|^2\widehat{w}(t,\xi)=\widehat{f}(t,\xi),\\ \widehat{w}(0,\cdot)=\widehat{w}_0,\quad \partial_t\widehat{w}(0,\cdot)=\widehat{w}_1,\end{cases}$$

in which (with $\mathrm{i}=\sqrt{-1}$)

$$\widehat{w}(t,\xi)=\int_{\mathbb{R}^2}w(t,x)\mathrm{e}^{-\mathrm{i}x_a\xi^a}\,\mathrm{d}x.$$

For this second-order ordinary differential equation in t, we can write its solution as

$$\widehat{w}(t,\xi)=\cos(t|\xi|)\widehat{w}_0+\frac{\sin(t|\xi|)}{|\xi|}\widehat{w}_1+\int_0^t\frac{\sin((t-\tau)|\xi|)}{|\xi|}\widehat{f}(\tau)\,\mathrm{d}\tau.$$

Thus we get

$$\begin{aligned}\|w\|_{L^2(\mathbb{R}^2)} &\lesssim \|w_0\|_{L^2(\mathbb{R}^2)} + \Big\|\frac{\sin(t|\xi|)}{|\xi|}\widehat{w}_1\Big\|_{L^2(\mathbb{R}^2)} + \int_0^t \Big\|\frac{\sin((t-\tau)|\xi|)}{|\xi|}\widehat{f}(\tau)\Big\|_{L^2(\mathbb{R}^2)}\,\mathrm{d}\tau \\ &=: \|w_0\|_{L^2(\mathbb{R}^2)} + A_1 + A_2. \end{aligned} \tag{2.15}$$

The terms A_1, A_2 need to be carefully treated.

Next, we try to estimate the term A_1, and we find

$$\begin{aligned} A_1^2 &= \int_{\{\xi:|\xi|\le 1\}} \frac{\sin^2(t|\xi|)}{|\xi|^2}|\widehat{w}_1|^2\,\mathrm{d}\xi + \int_{\{\xi:|\xi|\ge 1\}} \frac{\sin^2(t|\xi|)}{|\xi|^2}|\widehat{w}_1|^2\,\mathrm{d}\xi \\ &\le \|\widehat{w}_1\|^2_{L^\infty(\mathbb{R}^2)} \int_{\{\xi:|\xi|\le 1\}} \frac{\sin^2(t|\xi|)}{|\xi|^2}\,\mathrm{d}\xi + \|\widehat{w}_1\|^2_{L^2(\mathbb{R}^2)}. \end{aligned}$$

In succession, we have

$$\begin{aligned} \int_{\{\xi:|\xi|\le 1\}} \frac{\sin^2(t|\xi|)}{|\xi|^2}\,\mathrm{d}\xi &= \int_{S^1}\mathrm{d}S^1 \int_0^1 \frac{\sin^2(t|\xi|)}{|\xi|}\,\mathrm{d}|\xi| \\ &\lesssim \int_0^t \frac{\sin^2 p}{p}\,\mathrm{d}p \\ &\lesssim \int_0^1 1\,\mathrm{d}p + \int_1^{t+2}\frac{1}{p}\,\mathrm{d}p, \end{aligned}$$

in which we used that $\sin|p| \le |p|$ and $|\sin p| \le 1$. Gathering the above results leads us to

$$A_1^2 \lesssim \log(2+t)\|w_1\|^2_{L^2(\mathbb{R}^2)\cap L^1(\mathbb{R}^2)},$$

which yields

$$A_1 \lesssim \log^{\frac{1}{2}}(2+t)\|w_1\|_{L^2(\mathbb{R}^2)\cap L^1(\mathbb{R}^2)}. \tag{2.16}$$

By the analysis for A_1, we then estimate the term A_2

$$\Big\|\frac{\sin((t-\tau)|\xi|)}{|\xi|}\widehat{f}(\tau)\Big\|_{L^2(\mathbb{R}^2)} \lesssim \log^{\frac{1}{2}}(2+t-\tau)\|f(\tau)\|_{L^2(\mathbb{R}^2)\cap L^1(\mathbb{R}^2)}.$$

Hence we have

$$A_2 \lesssim \log^{\frac{1}{2}}(2+t)\int_0^t \|f(\tau)\|_{L^2(\mathbb{R}^2)\cap L^1(\mathbb{R}^2)}\,\mathrm{d}\tau.$$

By the assumptions on the function f, and gathering the results in the previous steps, we finally complete the proof.

3 The Faddeev Model

The mappings of the Faddeev model from the Minkowski spacetime $(\mathbb{R}^{1+n}, \eta)$ to the unit sphere $\mathbb{S}^2$ are denoted by

$$\mathbf{n} : (\mathbb{R}^{1+n}, \eta) \to \mathbb{S}^2, \tag{3.1}$$

whose Lagrangian density reads as folllows (see [6–8])

$$\mathcal{L}(\mathbf{n}) = \frac{1}{2}\partial_\mu \mathbf{n}\cdot\partial^\mu \mathbf{n} - \frac{1}{4}(\partial_\mu \mathbf{n}\wedge\partial_\nu \mathbf{n})\cdot(\partial^\mu \mathbf{n}\wedge\partial^\nu \mathbf{n}).$$

The corresponding Euler-Lagrange equations turn out to be

$$\mathbf{n} \wedge \partial_\mu \partial^\mu \mathbf{n} + [\partial_\mu(\mathbf{n} \cdot (\partial^\mu \mathbf{n} \wedge \partial^\nu \mathbf{n}))]\partial_\nu \mathbf{n} = 0, \tag{3.2}$$

and one refers to [6–8] and [22] and the references therein.

Denote $\mathbf{n} = (n_1, n_2, n_3)$, and we expand the equations in (3.2) to get

$$\begin{cases} n_2 \Box n_3 - n_3 \Box n_2 = \partial_\mu(\mathbf{n} \cdot (\partial^\mu \mathbf{n} \wedge \partial^\nu \mathbf{n}))\partial_\nu n_1, \\ n_3 \Box n_1 - n_1 \Box n_3 = \partial_\mu(\mathbf{n} \cdot (\partial^\mu \mathbf{n} \wedge \partial^\nu \mathbf{n}))\partial_\nu n_2, \\ n_1 \Box n_2 - n_2 \Box n_1 = \partial_\mu(\mathbf{n} \cdot (\partial^\mu \mathbf{n} \wedge \partial^\nu \mathbf{n}))\partial_\nu n_3. \end{cases} \tag{3.3}$$

These three equations are linearly dependent. Recall that $\mathbf{n}$ lies on the unit sphere $\mathbb{S}^2$, which means

$$n_1^2 + n_2^2 + n_3^2 = 1.$$

We consider the small perturbations around the point $(0, 0, 1) \in \mathbb{S}^2$, and thus n_3 is expected to be close to 1. For this reason, we only consider the first two equations appearing in (3.3), which are

$$\begin{cases} \Box n_1 = \dfrac{n_1}{n_3} \Box n_3 + \dfrac{1}{n_3} \partial_\mu(\mathbf{n} \cdot (\partial^\mu \mathbf{n} \wedge \partial^\nu \mathbf{n}))\partial_\nu n_2, \\ \Box n_2 = \dfrac{n_2}{n_3} \Box n_3 - \dfrac{1}{n_3} \partial_\mu(\mathbf{n} \cdot (\partial^\mu \mathbf{n} \wedge \partial^\nu \mathbf{n}))\partial_\nu n_1. \end{cases} \tag{3.4}$$

Recall the relation

$$n_3 = (1 - n_1^2 - n_2^2)^{\frac{1}{2}},$$

and we replace n_3 appearing in (3.4) by the expressions of n_1, n_2.

First, we note that

$$\partial_\mu(1 - n_1^2 - n_2^2)^{\frac{1}{2}} = -(1 - n_1^2 - n_2^2)^{-\frac{1}{2}}(n_1 \partial_\mu n_1 + n_2 \partial_\mu n_2).$$

In succession, we obtain

$$\begin{aligned} &\Box(1 - n_1^2 - n_2^2)^{\frac{1}{2}} = \partial_\mu \partial^\mu (1 - n_1^2 - n_2^2)^{\frac{1}{2}} \\ &= -(1 - n_1^2 - n_2^2)^{-\frac{3}{2}}(n_1 \partial_\mu n_1 + n_2 \partial_\mu n_2)(n_1 \partial^\mu n_1 + n_2 \partial^\mu n_2) \\ &\quad - (1 - n_1^2 - n_2^2)^{-\frac{1}{2}}(\partial_\mu n_1 \partial^\mu n_1 + \partial_\mu n_2 \partial^\mu n_2 + n_1 \Box n_1 + n_2 \Box n_2). \end{aligned}$$

On the other hand, we find

$$\begin{aligned} &\mathbf{n} \cdot (\partial^\mu \mathbf{n} \wedge \partial^\nu \mathbf{n}) \\ &= n_1(\partial^\mu n_2 \partial^\nu n_3 - \partial^\mu n_3 \partial^\nu n_2) + n_2(\partial^\mu n_3 \partial^\nu n_1 - \partial^\mu n_1 \partial^\nu n_3) + n_3(\partial^\mu n_1 \partial^\nu n_2 - \partial^\mu n_2 \partial^\nu n_1) \\ &= \frac{1}{(1 - n_1^2 - n_2^2)^{\frac{1}{2}}}(\partial^\mu n_1 \partial^\nu n_2 - \partial^\mu n_2 \partial^\nu n_1). \end{aligned}$$

Thus we obtain (1.1), i.e.,

$$\begin{aligned} &\Box \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} + \frac{n_1 \Box n_1 + n_2 \Box n_2}{1 - n_1^2 - n_2^2} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} + \frac{\partial_\mu(\partial^\mu n_1 \partial^\nu n_2 - \partial^\mu n_2 \partial^\nu n_1)}{1 - n_1^2 - n_2^2} \begin{pmatrix} -\partial_\nu n_2 \\ \partial_\nu n_1 \end{pmatrix} \\ &+ \frac{\partial_\nu n_1 \partial^\nu n_1 + \partial_\nu n_2 \partial^\nu n_2}{1 - n_1^2 - n_2^2} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} + \frac{n_1^2 \partial_\nu n_1 \partial^\nu n_1 + n_2^2 \partial_\nu n_2 \partial^\nu n_2 + 2 n_1 n_2 \partial_\nu n_1 \partial^\nu n_2}{(1 - n_1^2 - n_2^2)^2} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \\ &- \frac{(n_1 \partial_\mu n_1 + n_2 \partial_\mu n_2)(\partial^\mu n_1 \partial^\nu n_2 - \partial^\mu n_2 \partial^\nu n_1)}{1 - n_1^2 - n_2^2} \begin{pmatrix} -\partial_\nu n_2 \\ \partial_\nu n_1 \end{pmatrix} = 0. \end{aligned}$$

4 Proof of Theorem 1.1

We rely on the bootstrap method and the vector field method to prove Theorem 1.1. We recall the model equations in (1.1) with Γ^I acted are

$$\begin{cases} \Box\Gamma^I n_1 = \Gamma^I g_1 + \Gamma^I f_1, \\ \Box\Gamma^I n_2 = \Gamma^I g_2 + \Gamma^I f_2, \end{cases} \tag{4.1}$$

in which we used g, f to denote the quasi-linear terms and the semi-linear terms, respectively, with

$$\begin{aligned} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} &= -\frac{n_1\Box n_1 + n_2\Box n_2}{1-n_1^2-n_2^2}\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} - \frac{\partial_\mu(\partial^\mu n_1\partial^\nu n_2 - \partial^\mu n_2\partial^\nu n_1)}{1-n_1^2-n_2^2}\begin{pmatrix} -\partial_\nu n_2 \\ \partial_\nu n_1 \end{pmatrix}, \\ \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} &= -\frac{\partial_\nu n_1\partial^\nu n_1 + \partial_\nu n_2\partial^\nu n_2}{1-n_1^2-n_2^2}\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} - \frac{n_1^2\partial_\nu n_1\partial^\nu n_1 + n_2^2\partial_\nu n_2\partial^\nu n_2 + 2n_1n_2\partial_\nu n_1\partial^\nu n_2}{(1-n_1^2-n_2^2)^2}\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \\ &\quad + \frac{(n_1\partial_\mu n_1 + n_2\partial_\mu n_2)(\partial^\mu n_1\partial^\nu n_2 - \partial^\mu n_2\partial^\nu n_1)}{1-n_1^2-n_2^2}\begin{pmatrix} -\partial_\nu n_2 \\ \partial_\nu n_1 \end{pmatrix}. \end{aligned} \tag{4.2}$$

We note that in the semi-linear part, the first (cubic) term is the worst, so in the sequel we will mainly focus on the estimate of this term.

Based on the local well-posedness theory, we assume that the following estimates hold for all $t \in [t_0 = 0, t_1)$

$$\begin{cases} \|\Gamma^I n_i\| \le C_1\varepsilon\langle t\rangle^\delta, & |I| \le N-1, \\ \|\Gamma^I n_i\| \le C_1\varepsilon\langle t\rangle^{3\delta}, & |I| \le N, \\ E_{\rm gst}(t,\Gamma^I n_i)^{\frac12} \le C_1\varepsilon, & |I| \le N. \end{cases} \tag{4.3}$$

In the above, the parameter $0 < \delta \ll \frac{1}{24}$ is a small number, $C_1 \gg 1$ is a large number to be determined, and $\varepsilon \ll 1$ is sufficiently small such that $C_1\varepsilon \ll \frac14$. We note t_1 is defined by

$$t_1 := \sup\{t > t_0 : (4.3) \text{ holds}\}. \tag{4.4}$$

Our goal is to show the refined estimates under the assumptions in (4.3) that

$$\begin{cases} \|\Gamma^I n_i\| \le \dfrac12 C_1\varepsilon\langle t\rangle^\delta, & |I| \le N-1, \\ \|\Gamma^I n_i\| \le \dfrac12 C_1\varepsilon\langle t\rangle^{3\delta}, & |I| \le N, \\ E_{\rm gst}(t,\Gamma^I n_i)^{\frac12} \le \dfrac12 C_1\varepsilon, & |I| \le N. \end{cases} \tag{4.5}$$

Once this is done, we can derive a contradiction by the definition of t_1 if $t_1 < +\infty$, which means $t_1 = +\infty$. Hence we assert that the solution to (1.1) exists globally.

Combined with the Klainerman-Sobolev inequality in Proposition (2.4), we have the following pointwise estimates.

Proposition 4.1 *Let the estimates in* (4.3) *hold, then the following estimates are valid*

$$\begin{cases} |\Gamma^I n_i| \lesssim C_1\varepsilon\langle t-r\rangle^{-\frac12}\langle t+r\rangle^{-\frac12+3\delta}, & |I| \le N-2, \\ |\Gamma^I n_i| \lesssim C_1\varepsilon\langle t-r\rangle^{-\frac12}\langle t+r\rangle^{-\frac12+\delta}, & |I| \le N-3, \\ |\partial\Gamma^I n_i| \lesssim C_1\varepsilon\langle t-r\rangle^{-\frac12}\langle t+r\rangle^{-\frac12}, & |I| \le N-2. \end{cases} \tag{4.6}$$

We have the following result.

Proposition 4.2 *Under the assumptions in* (4.3)*, we have*

$$\begin{cases}\|\Box\Gamma^I n_i\| \lesssim (C_1\varepsilon)^3\langle t\rangle^{-\frac{3}{4}}, & |I|\le N-1,\\ |\Box\Gamma^I n_i| \lesssim (C_1\varepsilon)^3\langle t\rangle^{-\frac{5}{4}}, & |I|\le N-3.\end{cases} \tag{4.7}$$

Proof We first show the second estimate in (4.7). By recalling (4.1), we apply the triangle inequality to get

$$|\Box\Gamma^I n_1| + |\Box\Gamma^I n_2| \le |\Gamma^I g_1| + |\Gamma^I g_2| + |\Gamma^I f_1| + |\Gamma^I f_2|.$$

Summing over the indices $|I|\le N-3$ yields

$$\sum_{|I|\le N-3}(|\Box\Gamma^I n_1| + |\Box\Gamma^I n_2|) \le \sum_{|I|\le N-3}(|\Gamma^I g_1| + |\Gamma^I g_2| + |\Gamma^I f_1| + |\Gamma^I f_2|).$$

Taking into account of (4.2), we find

$$\begin{aligned}\sum_{|I|\le N-3}(|\Gamma^I g_1| + |\Gamma^I g_2|) \lesssim{}& \sum_{|I|\le N-3}(|\Box\Gamma^I n_1| + |\Box\Gamma^I n_2|)\sum_{|I|\le N-3}(|\Gamma^I n_1| + |\Gamma^I n_2|)^2\\ &+\sum_{|I|\le N-3}|\Gamma^I(\partial_\mu(\partial^\mu n_1\partial^\nu n_2 - \partial^\mu n_2\partial^\nu n_1)\partial_\nu n_2)|\\ &+\sum_{|I|\le N-3}|\Gamma^I(\partial_\mu(\partial^\mu n_1\partial^\nu n_2 - \partial^\mu n_2\partial^\nu n_1)\partial_\nu n_1)|.\end{aligned} \tag{4.8}$$

By the estimates in (4.6), we find

$$\sum_{|I|\le N-3}(|\Gamma^I n_1| + |\Gamma^I n_2|)^2 \lesssim (C_1\varepsilon)^2\langle t\rangle^{-1+6\delta}$$

as well as

$$\begin{aligned}&\sum_{|I|\le N-3}|\Gamma^I(\partial_\mu(\partial^\mu n_1\partial^\nu n_2 - \partial^\mu n_2\partial^\nu n_1)\partial_\nu n_2)|\\ &+\sum_{|I|\le N-3}|\Gamma^I(\partial_\mu(\partial^\mu n_1\partial^\nu n_2 - \partial^\mu n_2\partial^\nu n_1)\partial_\nu n_1)|\\ \lesssim&\sum_{|J|\le N-2}(|\partial\Gamma^J n_1| + |\partial\Gamma^J n_2|)^3 \lesssim (C_1\varepsilon)^3\langle t\rangle^{-\frac{3}{2}}.\end{aligned}$$

Similarly, we get

$$\sum_{|I|\le N-3}(|\Gamma^I f_1| + |\Gamma^I f_2|) \lesssim (C_1\varepsilon)^3\langle t\rangle^{-\frac{3}{2}+3\delta}.$$

Gathering these estimates, we arrive at

$$\begin{aligned}&\sum_{|I|\le N-3}(|\Box\Gamma^I n_1| + |\Box\Gamma^I n_2|)\\ \le{}& C(C_1\varepsilon)^2\langle t\rangle^{-1+6\delta}\sum_{|I|\le N-3}(|\Box\Gamma^I n_1| + |\Box\Gamma^I n_2|) + C(C_1\varepsilon)^3\langle t\rangle^{-\frac{3}{2}+3\delta}.\end{aligned}$$

The smallness of $C_1\varepsilon$ and the smallness of δ lead us to

$$\sum_{|I|\le N-3}(|\Box\Gamma^I n_1|+|\Box\Gamma^I n_2|)\lesssim (C_1\varepsilon)^3\langle t\rangle^{-\frac{5}{4}}.$$

In a similar manner, we can show the first estimate appearing in (4.7). Thus the proof is complete.

4.1 Refined lower-order energy

In this part, we will derive the refined estimates for the lower-order case, which are relatively easier.

We start with one lemma.

Lemma 4.1 *Let the assumptions in* (4.3) *hold, then the following estimate is valid for all* $|I|\le N-1$

$$\|\Gamma^I f_1\|_{L^1\bigcap L^2}+\|\Gamma^I f_2\|_{L^1\bigcap L^2}+\|\Gamma^I g_1\|_{L^1\bigcap L^2}+\|\Gamma^I g_2\|_{L^1\bigcap L^2}\lesssim (C_1\varepsilon)^3\langle t\rangle^{-\frac{5}{4}+4\delta}. \tag{4.9}$$

Proof We note that the bounds of the L^2 part in (4.9) are easier than the L^1 part, so we will only provide details of the estimates for the L^1 part.

First, we find

$$\begin{aligned}
&\sum_{|I|\le N-1,1\le i,j,k\le 2}\|\Gamma^I(n_in_j\Box n_k)\|_{L^1}\\
\lesssim&\sum_{\substack{|I_1|+|I_2|+|I_3|\le N-1\\ 1\le i,j,k\le 2}}\|\Gamma^{I_1}n_i\Gamma^{I_2}n_j\Box\Gamma^{I_3}n_k\|_{L^1}\\
\lesssim&\sum_{\substack{|I_1|+|I_2|\le N-1,|I_3|\le N-3,\\ 1\le i,j,k\le 2}}\|\Gamma^{I_1}n_i\|\|\Gamma^{I_2}n_j\|\|\Box\Gamma^{I_3}n_k\|_{L^\infty}\\
&+\sum_{\substack{|I_2|+|I_3|\le N-1,|I_1|\le N-3,\\ 1\le i,j,k\le 2}}\|\Gamma^{I_1}n_i\|_{L^\infty}\|\Gamma^{I_2}n_j\|\|\Box\Gamma^{I_3}n_k\|\\
\lesssim&(C_1\varepsilon)^3\langle t\rangle^{-\frac{5}{4}+4\delta},
\end{aligned}$$

in which $\frac{N-1}{2}\le N-3$ (i.e., $N\ge 5$) suffices to guarantee the last inequality. Thus we have

$$\begin{aligned}
&\sum_{|I|\le N-1,1\le i,j,k\le 2}\Big\|\Gamma^I\frac{n_in_j\Box n_k}{1-n_1^2-n_2^2}\Big\|_{L^1}\\
\lesssim&\sum_{|I|\le N-1,1\le i,j,k\le 2}\Big\|\frac{\Gamma^I(n_in_j\Box n_k)}{1-n_1^2-n_2^2}\Big\|_{L^1}\\
&+\sum_{\substack{|I_1|\le N-2,1\le|I_2|\le N-1\\ 1\le i,j,k\le 2}}\Big\|\Gamma^{I_1}(n_in_j\Box n_k)\,\Gamma^{I_2}\frac{1}{1-n_1^2-n_2^2}\Big\|_{L^1}\\
\lesssim&(C_1\varepsilon)^3\langle t\rangle^{-\frac{5}{4}+4\delta},
\end{aligned}\tag{4.10}$$

in which we used the observation that $\Gamma^{I_2}\frac{1}{1-n_1^2-n_2^2}$ produces higher-order (better) terms in the last step. Next, we note

$$\sum_{|I|\le N-1,1\le i\le 2}\|\Gamma^I[\partial_\mu(\partial^\mu n_1\partial^\nu n_2-\partial^\mu n_2\partial^\nu n_1)\partial_\nu n_i]\|_{L^1}$$

$$
\begin{aligned}
&\lesssim \sum_{|I_1|+|I_2|\le N-1,1\le i\le 2} \|\Gamma^{I_1}\partial_\mu(\partial^\mu n_1\partial^\nu n_2-\partial^\mu n_2\partial^\nu n_1)\Gamma^{I_2}\partial_\nu n_i\|_{L^1}\\
&\lesssim x\langle t\rangle^{-1} \sum_{\substack{|J_1|+|J_2|\le N,|J_3|\le N-2\\ 1\le i,j,k\le 2}} \|\Gamma^{J_1}\partial n_j\|\|\Gamma^{J_2}n_k\|\|\Gamma^{J_3}\partial n_i\|_{L^\infty}\\
&\quad+\langle t\rangle^{-1} \sum_{\substack{|J_1|+|J_2|\le N-2,|J_3|\le N-1\\ 1\le i,j,k\le 2}} \|\Gamma^{J_1}\partial n_j\|_{L^\infty}\|\Gamma^{J_2}n_k\|\|\Gamma^{J_3}\partial n_i\|\\
&\lesssim (C_1\varepsilon)^3\langle t\rangle^{-\frac32+2\delta}.
\end{aligned}
$$

Since again the terms $\Gamma^J\frac{1}{1-n_1^2-n_2^2}, 1\le |J|\le N-1$ will not cause trouble, we obtain

$$
\begin{aligned}
&\sum_{|I|\le N-1,1\le i\le 2}\Big\|\Gamma^I\frac{\partial_\mu(\partial^\mu n_1\partial^\nu n_2-\partial^\mu n_2\partial^\nu n_1)\partial_\nu n_i}{1-n_1^2-n_2^2}\Big\|_{L^1}\\
&\lesssim \sum_{|I|\le N-1,1\le i\le 2}\Big\|\frac{\Gamma^I[\partial_\mu(\partial^\mu n_1\partial^\nu n_2-\partial^\mu n_2\partial^\nu n_1)\partial_\nu n_i]}{1-n_1^2-n_2^2}\Big\|_{L^1}\\
&\quad+\sum_{\substack{|I_1|\le N-2,1\le|I_2|\le N-1\\ 1\le i\le 2}}\Big\|\Gamma^{I_1}[\partial_\mu(\partial^\mu n_1\partial^\nu n_2-\partial^\mu n_2\partial^\nu n_1)\partial_\nu n_i]\,\Gamma^{I_2}\frac{1}{1-n_1^2-n_2^2}\Big\|_{L^1}\\
&\lesssim (C_1\varepsilon)^3\langle t\rangle^{-\frac32+2\delta}.
\end{aligned}
\tag{4.11}
$$

The estimates (4.10) and (4.11) lead us to

$$
\sum_{|I|\le N-1}(\|\Gamma^I g_1\|_{L^1}+\|\Gamma^I g_2\|_{L^1})\lesssim (C_1\varepsilon)^3\langle t\rangle^{-\frac54+4\delta},
\tag{4.12}
$$

and hence

$$
\sum_{|I|\le N-1}(\|\Gamma^I g_1\|_{L^1\bigcap L^2}+\|\Gamma^I g_2\|_{L^1\bigcap L^2})\lesssim (C_1\varepsilon)^3\langle t\rangle^{-\frac54+4\delta}.
\tag{4.13}
$$

Similarly, we have

$$
\sum_{|I|\le N-1}(\|\Gamma^I f_1\|_{L^1\bigcap L^2}+\|\Gamma^I f_2\|_{L^1\bigcap L^2})\lesssim (C_1\varepsilon)^3\langle t\rangle^{-\frac32+6\delta}.
\tag{4.14}
$$

The proof is done.

Proposition 4.3 *Under the assumptions in* (4.3), *we have*

$$
\begin{cases}
\|\Gamma^I n_i\|\lesssim \varepsilon+(C_1\varepsilon)^3\langle t\rangle^\delta, & |I|\le N-1,\\
E_{\rm gst}(t,\Gamma^I n_i)^{\frac12}\lesssim \varepsilon+(C_1\varepsilon)^3, & |I|\le N-1.
\end{cases}
\tag{4.15}
$$

Proof By the estimates in Lemma 4.1 and Proposition 2.6, we get

$$
\|\Gamma^I n_i\|\lesssim \varepsilon+(C_1\varepsilon)^3\log^{\frac12}(2+t),\quad |I|\le N-1,
\tag{4.16}
$$

which is even stronger than the first one appearing in (4.15).

Similarly, by the estimates in Lemma 4.1 and the ghost weight energy estimates in Proposition 2.5, we easily obtain the second estimate in (4.15).

The proof is complete.

4.2 Refined higher-order energy

This part is devoted to show the refined estimates of the highest-order case.

Before we estimate $\|\Gamma^I n_i\|$ for $|I| = N$, we first introduce the following decomposition. We recall that (the same argument applies to n_2)

$$
\begin{aligned}
&\Box\Gamma^I n_1\\
&=\Gamma^I g_1+\Gamma^I f_1\\
&=\frac{-n_1^2\Box\Gamma^I n_1-n_1n_2\Box\Gamma^I n_2}{1-n_1^2-n_2^2}+\frac{\partial_\mu\Gamma^I(\partial^\mu n_1\partial^\nu n_2-\partial^\mu n_2\partial^\nu n_1)\partial_\nu n_2}{1-n_1^2-n_2^2}\\
&\quad+\Gamma^I g_1+\frac{n_1^2\Box\Gamma^I n_1+n_1n_2\Box\Gamma^I n_2}{1-n_1^2-n_2^2}-\frac{\partial_\mu\Gamma^I(\partial^\mu n_1\partial^\nu n_2-\partial^\mu n_2\partial^\nu n_1)\partial_\nu n_2}{1-n_1^2-n_2^2}+\Gamma^I f_1\\
&=-\partial_\mu\frac{n_1^2\partial^\mu\Gamma^I n_1}{1-n_1^2-n_2^2}+\partial_\mu\frac{n_1^2}{1-n_1^2-n_2^2}\partial^\mu\Gamma^I n_1-\partial_\mu\frac{n_1n_2\partial^\mu\Gamma^I n_2}{1-n_1^2-n_2^2}+\partial_\mu\frac{n_1n_2}{1-n_1^2-n_2^2}\partial^\mu\Gamma^I n_2\\
&\quad+\partial_\mu\frac{\Gamma^I(\partial^\mu n_1\partial^\nu n_2-\partial^\mu n_2\partial^\nu n_1)\partial_\nu n_2}{1-n_1^2-n_2^2}-\Gamma^I(\partial^\mu n_1\partial^\nu n_2-\partial^\mu n_2\partial^\nu n_1)\partial_\mu\frac{\partial_\nu n_2}{1-n_1^2-n_2^2}\\
&\quad+\Gamma^I g_1+\frac{n_1^2\Box\Gamma^I n_1+n_1n_2\Box\Gamma^I n_2}{1-n_1^2-n_2^2}-\frac{\partial_\mu\Gamma^I(\partial^\mu n_1\partial^\nu n_2-\partial^\mu n_2\partial^\nu n_1)\partial_\nu n_2}{1-n_1^2-n_2^2}+\Gamma^I f_1.
\end{aligned}
$$

We reorganise the terms to get

$$
\begin{aligned}
&\Box\Gamma^I n_1\\
&=\partial_\mu\Big(\frac{-n_1^2\partial^\mu\Gamma^I n_1-n_1n_2\partial^\mu\Gamma^I n_2+\Gamma^I(\partial^\mu n_1\partial^\nu n_2-\partial^\mu n_2\partial^\nu n_1)\partial_\nu n_2}{1-n_1^2-n_2^2}\Big)\\
&\quad+\partial_\mu\frac{n_1^2}{1-n_1^2-n_2^2}\partial^\mu\Gamma^I n_1+\partial_\mu\frac{n_1n_2}{1-n_1^2-n_2^2}\partial^\mu\Gamma^I n_2\\
&\quad-\Gamma^I(\partial^\mu n_1\partial^\nu n_2-\partial^\mu n_2\partial^\nu n_1)\partial_\mu\frac{\partial_\nu n_2}{1-n_1^2-n_2^2}\\
&\quad+\Gamma^I g_1+\frac{n_1^2\Box\Gamma^I n_1+n_1n_2\Box\Gamma^I n_2}{1-n_1^2-n_2^2}-\frac{\partial_\mu\Gamma^I(\partial^\mu n_1\partial^\nu n_2-\partial^\mu n_2\partial^\nu n_1)\partial_\nu n_2}{1-n_1^2-n_2^2}+\Gamma^I f_1. \qquad (4.17)
\end{aligned}
$$

We next introduce the new variables

$$(m^0, m^1, m^2, m^3),$$

which are solutions to the equations

$$
\begin{aligned}
&\Box m^\mu=\frac{-n_1^2\partial^\mu\Gamma^I n_1-n_1n_2\partial^\mu\Gamma^I n_2+\Gamma^I(\partial^\mu n_1\partial^\nu n_2-\partial^\mu n_2\partial^\nu n_1)\partial_\nu n_2}{1-n_1^2-n_2^2},\\
&(m^\mu,\partial_t m^\mu)(t_0)=(0,0),
\end{aligned}
\qquad (4.18)
$$

and

$$
\begin{aligned}
\Box m^3 &= \partial_\mu \frac{n_1^2}{1-n_1^2-n_2^2}\partial^\mu \Gamma^I n_1 + \partial_\mu \frac{n_1 n_2}{1-n_1^2-n_2^2}\partial^\mu \Gamma^I n_2 \\
&\quad - \Gamma^I(\partial^\mu n_1 \partial^\nu n_2 - \partial^\mu n_2 \partial^\nu n_1)\,\partial_\mu \frac{\partial_\nu n_2}{1-n_1^2-n_2^2} \\
&\quad + \Gamma^I g_1 + \frac{n_1^2 \Box \Gamma^I n_1 + n_1 n_2 \Box \Gamma^I n_2}{1-n_1^2-n_2^2} \\
&\quad - \frac{\partial_\mu \Gamma^I(\partial^\mu n_1 \partial^\nu n_2 - \partial^\mu n_2 \partial^\nu n_1)\partial_\nu n_2}{1-n_1^2-n_2^2} + \Gamma^I f_1, \\
(m^3, \partial_t m^3)(t_0) &= (m_0^3, m_1^3),
\end{aligned}
\tag{4.19}
$$

in which

$$
\begin{aligned}
&(m_0^3, m_1^3) \\
&= \Big(\Gamma^I n_1, \partial_t \Gamma^I n_1 - \frac{-n_1^2 \partial^0 \Gamma^I n_1 - n_1 n_2 \partial^0 \Gamma^I n_2 + \Gamma^I(\partial^0 n_1 \partial^\nu n_2 - \partial^0 n_2 \partial^\nu n_1)\partial_\nu n_2}{1-n_1^2-n_2^2}\Big)(t_0).
\end{aligned}
$$

We note that the relation between n_1 and (m^μ, m^3) reads as follows

$$
\Gamma^I n_1 = \partial_\mu m^\mu + m^3. \tag{4.20}
$$

Thus, to estimate the unknown $\Gamma^I n_1$, it suffices to estimate the new variables (m^μ, m^3). We comment that this strategy can also be applied to remove the compactness assumptions on the model problem studied in [1].

Proposition 4.4 *Under the assumptions in* (4.3), *we have*

$$
\|\Gamma^I n_i\| \lesssim \varepsilon + (C_1\varepsilon)^3 \langle t\rangle^{3\delta}, \quad |I| = N. \tag{4.21}
$$

Proof We only provide the proof for $\|\Gamma^I n_1\|$ with $|I| = N$.

Step 1 Bounds for $\|\partial m^\mu\|$. Recall the equations in (4.18), and the energy estimates for waves imply

$$
\begin{aligned}
&E(m^\mu, t)^{\frac12} \\
&\lesssim E(m^\mu, t_0)^{\frac12} + \int_{t_0}^t \Big\| \frac{-n_1^2 \partial^\mu \Gamma^I n_1 - n_1 n_2 \partial^\mu \Gamma^I n_2 + \Gamma^I(\partial^\mu n_1 \partial^\nu n_2 - \partial^\mu n_2 \partial^\nu n_1)\partial_\nu n_2}{1-n_1^2-n_2^2} \Big\| \,\mathrm{d}\tau \\
&\lesssim \int_{t_0}^t \sum_i \|\partial \Gamma^I n_i\| \sum_{i, |J| \le N-2} \Big\| \frac{n_i^2 + |\partial \Gamma^J n_i|^2}{1-n_1^2-n_2^2} \Big\|_{L^\infty} \mathrm{d}\tau \\
&\lesssim (C_1\varepsilon)^3 \int_{t_0}^t \langle \tau\rangle^{-1+2\delta}\,\mathrm{d}\tau \lesssim (C_1\varepsilon)^3 \langle t\rangle^{2\delta}.
\end{aligned}
$$

Step 2 Bounds for $\|m^3\|$.

We rely on Proposition 2.6 to achieve this, so we only need to bound the right-hand side of equation (4.19), i.e.,

$$
\int_{t_0}^t \Big\| \partial_\mu \frac{n_1^2}{1-n_1^2-n_2^2}\partial^\mu \Gamma^I n_1 + \partial_\mu \frac{n_1 n_2}{1-n_1^2-n_2^2}\partial^\mu \Gamma^I n_2
$$

$$- \Gamma^I(\partial^\mu n_1 \partial^\nu n_2 - \partial^\mu n_2 \partial^\nu n_1)\, \partial_\mu \frac{\partial_\nu n_2}{1-n_1^2-n_2^2} + \Gamma^I g_1 + \frac{n_1^2 \Box \Gamma^I n_1 + n_1 n_2 \Box \Gamma^I n_2}{1-n_1^2-n_2^2}$$
$$- \frac{\partial_\mu \Gamma^I(\partial^\mu n_1 \partial^\nu n_2 - \partial^\mu n_2 \partial^\nu n_1)\partial_\nu n_2}{1-n_1^2-n_2^2} + \Gamma^I f_1 \Big\|_{L^1 \bigcap L^2} \mathrm{d}\tau.$$

We first estimate the cubic term $\Gamma^I(n_i \partial_\mu n_j \partial^\mu n_k)$ in $\Gamma^I f_1$, which is the worst term in $\Gamma^I f_1$. We find

$$\begin{aligned}
&\sum_{i,j,k} \|\Gamma^I(n_i \partial_\mu n_j \partial^\mu n_k)\|_{L^1} \\
&\lesssim \sum_{i,j,k,|I_1|+|I_2|+|I_3|\le N} \|\Gamma^{I_1} n_i \partial_\mu \Gamma^{I_2} n_j \partial^\mu \Gamma^{I_3} n_k\|_{L^1} \\
&\lesssim \sum_{\substack{a,i,j,k,|I_1|\le N-2 \\ |I_2|\le N, |I_3|\le N-2}} \| |\Gamma^{I_1} n_i|\, |G_a \Gamma^{I_2} n_j|\, |\partial \Gamma^{I_3} n_k| \|_{L^1} \\
&\quad + \langle\tau\rangle^{-1} \sum_{\substack{i,j,k,|I_1|\le N-2 \\ |I_2|\le N, |I_3|\le N-2}} \| |\Gamma^{I_1} n_i|\, |\partial \Gamma^{I_2} n_j|\, |\Gamma \Gamma^{I_3} n_k| \|_{L^1} \\
&\quad + \langle\tau\rangle^{-1} \sum_{\substack{i,j,k,|I_1|\le N \\ |I_2|\le N-2, |I_3|\le N-3}} \| |\Gamma^{I_1} n_i|\, |\Gamma \Gamma^{I_2} n_j|\, |\partial \Gamma^{I_3} n_k| \|_{L^1} \\
&\lesssim \sum_{\substack{a,i,j,k,|I_1|\le N-2 \\ |I_2|\le N, |I_3|\le N-2}} \|\Gamma^{I_1} n_i\| \Big\| \frac{G_a \Gamma^{I_2} n_j}{\langle\tau - r\rangle^{\frac12+\delta}} \Big\| \|\langle\tau - r\rangle^{\frac12+\delta} \partial \Gamma^{I_3} n_k\|_{L^\infty} \\
&\quad + \langle\tau\rangle^{-1} \sum_{\substack{i,j,k,|I_1|\le N-2 \\ |I_2|\le N, |I_3|\le N-2}} \|\Gamma^{I_1} n_i\|_{L^\infty} \|\partial \Gamma^{I_2} n_j\| \|\Gamma \Gamma^{I_3} n_k\| \\
&\quad + \langle\tau\rangle^{-1} \sum_{\substack{i,j,k,|I_1|\le N \\ |I_2|\le N-2, |I_3|\le N-2}} \|\Gamma^{I_1} n_i\| \|\Gamma \Gamma^{I_2} n_j\| \|\partial \Gamma^{I_3} n_k\|_{L^\infty} \\
&\lesssim (C_1\varepsilon)^3 \langle\tau\rangle^{-\frac32+4\delta} + (C_1\varepsilon)^2 \langle\tau\rangle^{-\frac12+2\delta} \sum_{a,j,|I_2|\le N} \Big\| \frac{G_a \Gamma^{I_2} n_j}{\langle\tau - r\rangle^{\frac12+\delta}} \Big\|
\end{aligned}$$

In the same way, we obtain

$$\begin{aligned}
&\sum_{i,j,k} \|\Gamma^I(n_i \partial_\mu n_j \partial^\mu n_k)\|_{L^1 \bigcap L^2} \\
&\lesssim (C_1\varepsilon)^3 \langle\tau\rangle^{-\frac32+4\delta} + (C_1\varepsilon)^2 \langle\tau\rangle^{-\frac12+2\delta} \sum_{a,j,|I_2|\le N} \Big\| \frac{G_a \Gamma^{I_2} n_j}{\langle\tau - r\rangle^{\frac12+\delta}} \Big\|
\end{aligned}$$

Thus we proceed to get

$$\begin{aligned}
&\int_{t_0}^t \sum_{i,j,k} \|\Gamma^I(n_i \partial_\mu n_j \partial^\mu n_k)\|_{L^1 \bigcap L^2} \mathrm{d}\tau \\
&\lesssim (C_1\varepsilon)^3 + (C_1\varepsilon)^2 \Big(\int_{t_0}^t \langle\tau\rangle^{-1+4\delta} \mathrm{d}\tau \Big)^{\frac12} \sum_{a,j,|I_2|\le N} \Big(\int_{t_0}^t \Big\| \frac{G_a \Gamma^{I_2} n_j}{\langle\tau - r\rangle^{\frac12+\delta}} \Big\|^2 \mathrm{d}\tau \Big)^{\frac12} \\
&\lesssim (C_1\varepsilon)^3 \langle t\rangle^{2\delta}. \qquad (4.22)
\end{aligned}$$

Very similarly, we can show

$$\begin{aligned}&\int_{t_0}^t \Big\|\partial_\mu \frac{n_1^2}{1-n_1^2-n_2^2}\partial^\mu \Gamma^I n_1 + \partial_\mu \frac{n_1 n_2}{1-n_1^2-n_2^2}\partial^\mu \Gamma^I n_2\\ &\quad - \Gamma^I(\partial^\mu n_1 \partial^\nu n_2 - \partial^\mu n_2 \partial^\nu n_1)\partial_\mu \frac{\partial_\nu n_2}{1-n_1^2-n_2^2} + \Gamma^I f_1\Big\|_{L^1\bigcap L^2} \mathrm{d}\tau\\ &\lesssim (C_1\varepsilon)^3 \langle t\rangle^{2\delta}. \end{aligned} \tag{4.23}$$

In this step, we are only left with estimating

$$\int_{t_0}^t \Big\|\Gamma^I g_1 + \frac{n_1^2 \Box \Gamma^I n_1 - n_1 n_2 \Box \Gamma^I n_2}{1-n_1^2-n_2^2} - \frac{\partial_\mu \Gamma^I(\partial^\mu n_1 \partial^\nu n_2 - \partial^\mu n_2 \partial^\nu n_1)\partial_\nu n_2}{1-n_1^2-n_2^2}\Big\|_{L^1\bigcap L^2} \mathrm{d}\tau.$$

We observe that

$$\begin{aligned}&\Big\|\Gamma^I g_1 + \frac{n_1^2 \Box \Gamma^I n_1 - n_1 n_2 \Box \Gamma^I n_2}{1-n_1^2-n_2^2} - \frac{\partial_\mu \Gamma^I(\partial^\mu n_1 \partial^\nu n_2 - \partial^\mu n_2 \partial^\nu n_1)\partial_\nu n_2}{1-n_1^2-n_2^2}\Big\|_{L^1\bigcap L^2}\\ &\lesssim \sum_{|I_1|+|I_2|\le N, |I_1|\le N-1} \Big\|\Box \Gamma^{I_1} n_1\, \Gamma^{I_2} \frac{n_1^2}{1-n_1^2-n_2^2}\Big\|_{L^1\bigcap L^2}\\ &\quad + \sum_{|I_1|+|I_2|\le N, |I_1|\le N-1} \Big\|\Box \Gamma^{I_1} n_2\, \Gamma^{I_2} \frac{n_1 n_2}{1-n_1^2-n_2^2}\Big\|_{L^1\bigcap L^2}\\ &\quad + \sum_{|I_1|+|I_2|\le N, |I_1|\le N-1} \Big\|\partial_\mu \Gamma^{I_1}(\partial^\mu n_1 \partial^\nu n_2 - \partial^\mu n_2 \partial^\nu n_1)\, \Gamma^{I_2} \frac{\partial_\nu n_2}{1-n_1^2-n_2^2}\Big\|_{L^1\bigcap L^2}. \end{aligned}$$

By recalling the results in Proposition 4.2, we easily get

$$\begin{aligned}&\sum_{|I_1|+|I_2|\le N, |I_1|\le N-1} \Big\|\Box \Gamma^{I_1} n_1\, \Gamma^{I_2} \frac{n_1^2}{1-n_1^2-n_2^2}\Big\|_{L^1\bigcap L^2}\\ &\quad + \sum_{|I_1|+|I_2|\le N, |I_1|\le N-1} \Big\|\Box \Gamma^{I_1} n_2\, \Gamma^{I_2} \frac{n_1 n_2}{1-n_1^2-n_2^2}\Big\|_{L^1\bigcap L^2}\\ &\lesssim (C_1\varepsilon)^3 \langle \tau\rangle^{-\frac{5}{4}+6\delta}. \end{aligned}$$

We note that

$$\begin{aligned}&\sum_{|I_1|+|I_2|\le N, |I_1|\le N-1} \Big\|\partial_\mu \Gamma^{I_1}(\partial^\mu n_1 \partial^\nu n_2 - \partial^\mu n_2 \partial^\nu n_1)\, \Gamma^{I_2} \frac{\partial_\nu n_2}{1-n_1^2-n_2^2}\Big\|_{L^1\bigcap L^2}\\ &\lesssim \sum_{\substack{|J|+|I_2|\le N+1\\ |J|\le N, |I_2|\le N, \mu}} \Big\|\Gamma^J(\partial^\mu n_1 \partial^\nu n_2 - \partial^\mu n_2 \partial^\nu n_1)\, \Gamma^{I_2} \frac{\partial_\nu n_2}{1-n_1^2-n_2^2}\Big\|_{L^1\bigcap L^2}. \end{aligned}$$

The way we show (4.22) leads us to

$$\sum_{\substack{|J|+|I_2|\le N+1\\ |J|\le N, |I_2|\le N, \mu}} \int_{t_0}^t \Big\|\Gamma^J(\partial^\mu n_1 \partial^\nu n_2 - \partial^\mu n_2 \partial^\nu n_1)\, \Gamma^{I_2} \frac{\partial_\nu n_2}{1-n_1^2-n_2^2}\Big\|_{L^1\bigcap L^2} \mathrm{d}\tau \lesssim (C_1\varepsilon)^3 \langle t\rangle^{2\delta}.$$

Thus we obtain

$$\int_{t_0}^t \Big\|\Gamma^I g_1 + \frac{n_1^2 \Box \Gamma^I n_1 - n_1 n_2 \Box \Gamma^I n_2}{1-n_1^2-n_2^2} - \frac{\partial_\mu \Gamma^I(\partial^\mu n_1 \partial^\nu n_2 - \partial^\mu n_2 \partial^\nu n_1)\partial_\nu n_2}{1-n_1^2-n_2^2}\Big\|_{L^1\bigcap L^2} \mathrm{d}\tau$$

$$\lesssim (C_1\varepsilon)^3\langle t\rangle^{2\delta}. \tag{4.24}$$

The combination of (4.23)–(4.24) and Proposition 2.6 yields

$$\|m^3\| \lesssim \varepsilon + (C_1\varepsilon)^3\langle t\rangle^{3\delta}. \tag{4.25}$$

Step 3 Bounds for $\|\Gamma^I n_i\|$.

By the estimates in the first two steps, we arrive at

$$\|\Gamma^I n_1\| \lesssim \|\partial_\mu m^\mu\| + \|m^3\| \lesssim \varepsilon + (C_1\varepsilon)^3\langle t\rangle^{3\delta}, \quad |I| = N. \tag{4.26}$$

The same also holds for n_2, thus the proof is complete.

Recall the expressions of g_1, g_2 in (4.2), and we rewrite them as

$$\begin{cases} g_1 = P_1^{\alpha\beta}\partial_\alpha\partial_\beta n_1 + P_2^{\alpha\beta}\partial_\alpha\partial_\beta n_2, \\ g_2 = P_3^{\alpha\beta}\partial_\alpha\partial_\beta n_2 + P_4^{\alpha\beta}\partial_\alpha\partial_\beta n_1, \end{cases} \tag{4.27}$$

in which

$$\begin{cases} P_1^{\alpha\beta} = \dfrac{n_1^2\eta^{\alpha\beta} - \partial_\nu n_2\partial^\nu n_2\eta^{\alpha\beta} + \partial^\alpha n_2\partial^\beta n_2}{1-n_1^2-n_2^2}, \\ P_2^{\alpha\beta} = \dfrac{n_1n_2\eta^{\alpha\beta} + \partial_\nu n_1\partial^\nu n_2\eta^{\alpha\beta}}{1-n_1^2-n_2^2} - \dfrac{\partial^\alpha n_1\partial^\beta n_2 + \partial^\alpha n_2\partial^\beta n_1}{2(1-n_1^2-n_2^2)}, \\ P_3^{\alpha\beta} = \dfrac{n_2^2\eta^{\alpha\beta} + \partial^\alpha n_1\partial^\beta n_1 - \partial_\nu n_1\partial^\nu n_1\eta^{\alpha\beta}}{1-n_1^2-n_2^2}, \\ P_2^{\alpha\beta} = \dfrac{n_1n_2\eta^{\alpha\beta} + \partial_\nu n_1\partial^\nu n_2\eta^{\alpha\beta}}{1-n_1^2-n_2^2} - \dfrac{\partial^\alpha n_1\partial^\beta n_2 + \partial^\alpha n_2\partial^\beta n_1}{2(1-n_1^2-n_2^2)}. \end{cases} \tag{4.28}$$

Thus the model equations (1.1) can be written as

$$\begin{cases} \Box n_1 + P_1^{\alpha\beta}\partial_\alpha\partial_\beta n_1 + P_2^{\alpha\beta}\partial_\alpha\partial_\beta n_2 + f_1 = 0, \\ \Box n_2 + P_3^{\alpha\beta}\partial_\alpha\partial_\beta n_2 + P_4^{\alpha\beta}\partial_\alpha\partial_\beta n_1 + f_2 = 0. \end{cases} \tag{4.29}$$

Acting Γ^I with $|I| = N$ to the equations, we further get

$$\begin{cases} \Box\Gamma^I n_1 + P_1^{\alpha\beta}\partial_\alpha\partial_\beta\Gamma^I n_1 + P_2^{\alpha\beta}\partial_\alpha\partial_\beta\Gamma^I n_2 = -\Gamma^I f_1 + Q_1, \\ \Box\Gamma^I n_2 + P_3^{\alpha\beta}\partial_\alpha\partial_\beta\Gamma^I n_2 + P_4^{\alpha\beta}\partial_\alpha\partial_\beta\Gamma^I n_1 = -\Gamma^I f_2 + Q_2, \end{cases} \tag{4.30}$$

in which

$$\begin{cases} Q_1 = P_1^{\alpha\beta}\partial_\alpha\partial_\beta\Gamma^I n_1 + P_2^{\alpha\beta}\partial_\alpha\partial_\beta\Gamma^I n_2 - \Gamma^I g_1, \\ Q_2 = P_3^{\alpha\beta}\partial_\alpha\partial_\beta\Gamma^I n_2 + P_4^{\alpha\beta}\partial_\alpha\partial_\beta\Gamma^I n_1 - \Gamma^I g_2. \end{cases} \tag{4.31}$$

We note that

$$P_i^{\alpha\beta} = P_i^{\beta\alpha}, \quad i = 1,2,3,4$$

and

$$P_2^{\alpha\beta} = P_4^{\alpha\beta},$$

which guarantee the hyperbolicity of the quasi-linear system.

We first show the estimates for the source terms in (4.30).

Lemma 4.2 *For $|I| = N$ we have*

$$\int_{t_0}^{t} (\| - \Gamma^I f_1 + Q_1\| + \| - \Gamma^I f_2 + Q_2\|) \, \mathrm{d}\tau \lesssim (C_1\varepsilon)^3. \tag{4.32}$$

Proof We will only provide the detailed estimates for the term $-\Gamma^I f_1 + Q_1$ as the term $-\Gamma^I f_2 + Q_2$ can be bounded in the same way.

We recall that the estimates of (4.22) and (4.23) can be applied to show

$$\int_{t_0}^{t} \| - \Gamma^I f_1\| \, \mathrm{d}\tau \lesssim (C_1\varepsilon)^3,$$

so we will only need to consider

$$\int_{t_0}^{t} \|Q_1\| \, \mathrm{d}\tau.$$

We observe that

$$\begin{aligned}
\Gamma^I g_1 &= \sum_{I_1+I_2=I} \Gamma^{I_1} \frac{-n_1^2}{1-n_1^2-n_2^2} \Gamma^{I_2} \Box n_1 + \sum_{I_1+I_2=I} \Gamma^{I_1} \frac{-n_1 n_2}{1-n_1^2-n_2^2} \Gamma^{I_2} \Box n_2 \\
&\quad + \sum_{I_1+I_2=I} \Gamma^{I_1} \frac{\partial_\nu n_2}{1-n_1^2-n_2^2} \Gamma^{I_2} \partial_\mu (\partial^\mu n_1 \partial^\nu n_2 - \partial^\mu n_2 \partial^\nu n_1) \\
&= \frac{-n_1^2}{1-n_1^2-n_2^2} \Gamma^I \Box n_1 + \frac{-n_1 n_2}{1-n_1^2-n_2^2} \Gamma^I \Box n_2 \\
&\quad + \frac{\partial_\nu n_2}{1-n_1^2-n_2^2} \Gamma^I \partial_\mu (\partial^\mu n_1 \partial^\nu n_2 - \partial^\mu n_2 \partial^\nu n_1) \\
&\quad + \sum_{\substack{I_1+I_2=I \\ |I_2|<|I|}} \Gamma^{I_1} \frac{-n_1^2}{1-n_1^2-n_2^2} \Gamma^{I_2} \Box n_1 + \sum_{\substack{I_1+I_2=I \\ |I_2|<|I|}} \Gamma^{I_1} \frac{-n_1 n_2}{1-n_1^2-n_2^2} \Gamma^{I_2} \Box n_2 \\
&\quad + \sum_{\substack{I_1+I_2=I \\ |I_2|<|I|}} \Gamma^{I_1} \frac{\partial_\nu n_2}{1-n_1^2-n_2^2} \Gamma^{I_2} \partial_\mu (\partial^\mu n_1 \partial^\nu n_2 - \partial^\mu n_2 \partial^\nu n_1).
\end{aligned}$$

For the third term in the right-hand side of the above equation, the commutator estimates yield

$$\begin{aligned}
&\frac{\partial_\nu n_2}{1-n_1^2-n_2^2} \Gamma^I \partial_\mu (\partial^\mu n_1 \partial^\nu n_2 - \partial^\mu n_2 \partial^\nu n_1) \\
&= \frac{\partial_\nu n_2}{1-n_1^2-n_2^2} \partial_\mu \Gamma^I (\partial^\mu n_1 \partial^\nu n_2 - \partial^\mu n_2 \partial^\nu n_1) \\
&\quad + \sum_{|J| \le N-1} C^\alpha_{J,\mu} \frac{\partial_\nu n_2}{1-n_1^2-n_2^2} \partial_\alpha \Gamma^J (\partial^\mu n_1 \partial^\nu n_2 - \partial^\mu n_2 \partial^\nu n_1) \\
&= \frac{\partial_\nu n_2}{1-n_1^2-n_2^2} \partial_\mu (\partial^\mu \Gamma^I n_1 \partial^\nu n_2 - \partial^\mu n_2 \partial^\nu \Gamma^I n_1) \\
&\quad + \frac{\partial_\nu n_2}{1-n_1^2-n_2^2} \partial_\mu (\partial^\mu n_1 \partial^\nu \Gamma^I n_2 - \partial^\mu \Gamma^I n_2 \partial^\nu n_1) \\
&\quad + \sum_{\substack{|I_1|+|I_2| \le |I| \\ |I_1|<N, |I_2|<N}} C_{I_1,I_2} \frac{\partial_\nu n_2}{1-n_1^2-n_2^2} \partial_\mu (\partial^\mu \Gamma^{I_1} n_1 \partial^\nu \Gamma^{I_2} n_2 - \partial^\mu \Gamma^{I_2} n_2 \partial^\nu \Gamma^{I_1} n_1)
\end{aligned}$$

$$+\sum_{|J|\le N-1} C^{\alpha}_{J,\mu}\frac{\partial_\nu n_2}{1-n_1^2-n_2^2}\partial_\alpha\Gamma^J(\partial^\mu n_1\partial^\nu n_2-\partial^\mu n_2\partial^\nu n_1),$$

in which C's are constants. Gathering the above two identities and recalling (4.31) give us

$$\begin{aligned}
Q_1 &= P_1^{\alpha\beta}\partial_\alpha\partial_\beta\Gamma^I n_1+P_2^{\alpha\beta}\partial_\alpha\partial_\beta\Gamma^I n_2-\Gamma^I g_1\\
&=\sum_{\substack{I_1+I_2=I\\|I_2|<|I|}}\Gamma^{I_1}\frac{-n_1^2}{1-n_1^2-n_2^2}\Gamma^{I_2}\Box n_1+\sum_{\substack{I_1+I_2=I\\|I_2|<|I|}}\Gamma^{I_1}\frac{-n_1n_2}{1-n_1^2-n_2^2}\Gamma^{I_2}\Box n_2\\
&\quad+\sum_{\substack{I_1+I_2=I\\|I_2|<|I|}}\Gamma^{I_1}\frac{\partial_\nu n_2}{1-n_1^2-n_2^2}\Gamma^{I_2}\partial_\mu(\partial^\mu n_1\partial^\nu n_2-\partial^\mu n_2\partial^\nu n_1)\\
&\quad+\sum_{\substack{|I_1|+|I_2|\le|I|\\|I_1|<N,|I_2|<N}} C_{I_1,I_2}\frac{\partial_\nu n_2}{1-n_1^2-n_2^2}\partial_\mu(\partial^\mu\Gamma^{I_1}n_1\partial^\nu\Gamma^{I_2}n_2-\partial^\mu\Gamma^{I_2}n_2\partial^\nu\Gamma^{I_1}n_1)\\
&\quad+\sum_{|J|\le N-1} C^{\alpha}_{J,\mu}\frac{\partial_\nu n_2}{1-n_1^2-n_2^2}\partial_\alpha\Gamma^J(\partial^\mu n_1\partial^\nu n_2-\partial^\mu n_2\partial^\nu n_1).
\end{aligned}$$

We find all of the terms to be estimated are null terms, so the analysis in Lemma 4.1 can be used to deduce

$$\int_{t_0}^t \|Q_1\|\,\mathrm{d}\tau\lesssim (C_1\varepsilon)^3.$$

The proof is complete.

Proposition 4.5 *Under the assumptions in* (4.3), *we have*

$$E_{\rm gst}(t,\Gamma^I n_i)^{\frac12}\lesssim \varepsilon+(C_1\varepsilon)^2,\quad |I|=N. \tag{4.33}$$

Proof According to the ghost weight energy estimates in Proposition 2.5, we only need to show

$$\int_{t_0}^t\int_{\mathbb{R}^2}\mathcal{R}\,\mathrm{d}x\mathrm{d}\tau\lesssim (C_1\varepsilon)^4,$$

in which (with $w_1=\Gamma^I n_1, w_2=\Gamma^I n_2$)

$$\begin{aligned}
\mathcal{R} &= |(-\Gamma^I f_1+Q_1)\,\partial_t w_1|+|(-\Gamma^I f_2+Q_2)\,\partial_t w_2|+|\partial_\alpha P_1^{\alpha\beta}\partial_\beta w_1\partial_t w_1|+|\partial_t P_1^{\alpha\beta}\partial_\alpha w_1\partial_\beta w_1|\\
&\quad+|\partial_\alpha P_2^{\alpha\beta}\partial_\beta w_2\partial_t w_1|+|\partial_t P_2^{\alpha\beta}\partial_\alpha w_2\partial_\beta w_1|+|\partial_\alpha P_3^{\alpha\beta}\partial_\beta w_2\partial_t w_2|+|\partial_t P_3^{\alpha\beta}\partial_\alpha w_2\partial_\beta w_2|\\
&\quad+|\partial_\alpha P_2^{\alpha\beta}\partial_\beta w_1\partial_t w_2|+|P_1^{\alpha\beta}\partial_\alpha w_1\partial_t w_1\partial_\beta\rho|+|P_1^{\alpha\beta}\partial_\alpha w_1\partial_\beta w_1\partial_t\rho|+|P_2^{\alpha\beta}\partial_\beta w_2\partial_t w_1\partial_\alpha\rho|\\
&\quad+|P_2^{\alpha\beta}\partial_\alpha w_1\partial_\beta w_2\partial_t\rho|+|P_3^{\alpha\beta}\partial_\alpha w_2\partial_t w_2\partial_\beta\rho|+|P_3^{\alpha\beta}\partial_\alpha w_2\partial_\beta w_2\partial_t\rho|+|P_2^{\alpha\beta}\partial_\beta w_1\partial_t w_2\partial_\alpha\rho|\\
&=:\mathcal{R}_{11}+\mathcal{R}_{12}+\mathcal{R}_{13}+\cdots+\mathcal{R}_{44}.
\end{aligned}$$

We divide these terms into three classes

- Class I: $\mathcal{R}_{11},\mathcal{R}_{12}$.
- Class II: $\mathcal{R}_{13},\mathcal{R}_{14},\mathcal{R}_{21},\mathcal{R}_{22},\mathcal{R}_{23},\mathcal{R}_{24},\mathcal{R}_{31},\mathcal{R}_{33},\mathcal{R}_{41},\mathcal{R}_{43}$.
- Class III: $\mathcal{R}_{32},\mathcal{R}_{34},\mathcal{R}_{42},\mathcal{R}_{44}$.

In each class, we will only illustrate the details of the estimates for one representative term, and others can be estimates analogously.

By the estimates in Lemma 4.2, we get

$$\int_{t_0}^t \int_{\mathbb{R}^2} \mathcal{R}_{11}\,\mathrm{d}x\mathrm{d}\tau \lesssim \int_{t_0}^t \| -\Gamma^I f_1 + Q_1\|\, \|\partial_t \Gamma^I n_1\|\,\mathrm{d}\tau \lesssim (C_1\varepsilon)^4,$$

and similarly

$$\int_{t_0}^t \int_{\mathbb{R}^2} \mathcal{R}_{12}\,\mathrm{d}x\mathrm{d}\tau \lesssim (C_1\varepsilon)^4.$$

Next, we treat the term $\mathcal{R}_{13}$. We first ignore the denominator in $P_1^{\alpha\beta}$, and find

$$\begin{aligned}&\partial_\alpha(n_1^2\eta^{\alpha\beta} - \partial_\nu n_2\partial^\nu n_2\eta^{\alpha\beta} + \partial^\alpha n_2\partial^\beta n_2)\partial_\beta w_1\partial_t w_1\\ &= 2n_1\partial_\alpha n_1\partial^\alpha w_1\partial_t w_1 - 2\partial_\nu\partial_\alpha n_2\partial^\nu n_2\partial^\alpha w_1\partial_t w_1\\ &\quad + \partial_\alpha\partial^\alpha n_2\partial^\beta n_2\partial_\beta w_1\partial_t w_1 + \partial^\alpha n_2\partial_\alpha\partial^\beta n_2\partial_\beta w_1\partial_t w_1,\end{aligned}$$

in which we note each term in the right-hand side is null. By Proposition 2.3, we have

$$\begin{aligned}&\int_{\mathbb{R}^2} |2n_1\partial_\alpha n_1\partial^\alpha w_1\partial_t w_1|\,\mathrm{d}x\\ &\lesssim \sum_a \int_{\mathbb{R}^2} |n_1||\partial n_1||G_a w_1||\partial_t w_1|\,\mathrm{d}x + \int_{\mathbb{R}^2} \langle\tau\rangle^{-1}|n_1||\Gamma n_1||\partial w_1||\partial_t w_1|\,\mathrm{d}x\\ &\lesssim \sum_a \|\langle\tau - r\rangle|n_1||\partial n_1|\|_{L^\infty}\Big\|\frac{G_a w_1}{\langle\tau-r\rangle}\Big\|\|\partial_t w_1\| + \langle\tau\rangle^{-1}\||n_1||\Gamma n_1|\|_{L^\infty}\|\partial w_1\|\|\partial_t w_1\|\\ &\lesssim (C_1\varepsilon)^3\sum_a \langle\tau\rangle^{-1+6\delta}\Big\|\frac{G_a w_1}{\langle\tau-r\rangle}\Big\| + (C_1\varepsilon)^4\langle\tau\rangle^{-2+6\delta},\end{aligned}$$

and we proceed to get

$$\begin{aligned}&\int_{t_0}^t\int_{\mathbb{R}^2} |2n_1\partial_\alpha n_1\partial^\alpha w_1\partial_t w_1|\,\mathrm{d}x\mathrm{d}\tau\\ &\lesssim (C_1\varepsilon)^3\sum_a\Big(\int_{t_0}^t \langle\tau\rangle^{-2+12\delta}\,\mathrm{d}\tau\Big)^{\frac12}\Big(\int_{t_0}^t\Big\|\frac{G_a w_1}{\langle\tau-r\rangle}\Big\|^2\mathrm{d}\tau\Big)^{\frac12} + (C_1\varepsilon)^4\int_{t_0}^t\langle\tau\rangle^{-2+6\delta}\,\mathrm{d}\tau\\ &\lesssim (C_1\varepsilon)^4.\end{aligned}$$

In the same manner, we can treat other terms in $\mathcal{R}_{13}$ and show

$$\int_{t_0}^t \int_{\mathbb{R}^2} \mathcal{R}_{13}\,\mathrm{d}x\mathrm{d}\tau \lesssim (C_1\varepsilon)^4.$$

Thus, similarly we get the same bound for other terms in this class.

Now we estimate the term $\mathcal{R}_{32}$. By the smallness of n_1, n_2, we have

$$\begin{aligned}\mathcal{R}_{32} &\lesssim |n_1^2\partial_\alpha w_1\partial^\alpha\rho\partial_t w_1| + |\partial_\nu n_2\partial^\nu n_2\partial_\alpha w_1\partial^\alpha\rho\partial_t w_1| + |\partial^\alpha n_2\partial_\alpha w_1\partial^\beta n_2\partial_\beta\rho\partial_t w_1|\\ &\lesssim |n_1^2||\partial_\alpha w_1\partial^\alpha\rho\partial_t w_1| + |\partial_\nu n_2\partial^\nu n_2||\partial w_1||\partial_t w_1| + |\partial n_2||\partial^\alpha n_2\partial_\alpha w_1||\partial_t w_1|,\end{aligned}$$

in which we used the relation $|\partial\rho| \lesssim 1$ (recall $\rho = \mathrm{e}^{\int_{-\infty}^{r-\tau}\langle s\rangle^{-1-2\delta}\,\mathrm{d}s}$). Since the last two terms can be bounded in the same way as we did for the term $\mathcal{R}_{13}$, so we will only estimate the first term in the right-hand side of the above inequality. We observe that

$$\partial_t\rho = \mathrm{e}^{\int_{-\infty}^{r-\tau}\langle s\rangle^{-1-2\delta}\,\mathrm{d}s}\frac{-1}{\langle r-\tau\rangle^{1+2\delta}},\quad \partial_a\rho = \mathrm{e}^{\int_{-\infty}^{r-\tau}\langle s\rangle^{-1-2\delta}\,\mathrm{d}s}\frac{1}{\langle r-\tau\rangle^{1+2\delta}}\frac{x_a}{r},$$

which lead to

$$G_a\rho = \frac{x_a}{r}\partial_t\rho + \partial_a\rho = 0.$$

By the estimates for null forms in Proposition 2.3, we have

$$\begin{aligned}&\int_{\mathbb{R}^2} |n_1^2||\partial_\alpha w_1 \partial^\alpha \rho \partial_t w_1|\,\mathrm{d}x\\ &\lesssim \sum_a \|\langle r-\tau\rangle n_1^2\|_{L^\infty}\Big\|\frac{G_a w_1}{\langle r-\tau\rangle}\Big\|\|\partial_t w_1\|\\ &\lesssim (C_1\varepsilon)^3\sum_a \langle\tau\rangle^{-1+6\delta}\Big\|\frac{G_a w_1}{\langle r-\tau\rangle}\Big\|.\end{aligned}$$

By the smallness of δ, we get

$$\begin{aligned}&\int_{t_0}^t\int_{\mathbb{R}^2} |n_1^2||\partial_\alpha w_1 \partial^\alpha \rho \partial_t w_1|\,\mathrm{d}x\mathrm{d}\tau\\ &\lesssim (C_1\varepsilon)^3\sum_a\Big(\int_{t_0}^t \langle\tau\rangle^{-2+12\delta}\,\mathrm{d}\tau\Big)^{\frac12}\Big(\int_{t_0}^t\Big\|\frac{G_a w_1}{\langle r-\tau\rangle}\Big\|^2\mathrm{d}\tau\Big)^{\frac12} \lesssim (C_1\varepsilon)^4.\end{aligned}$$

Thus, we get the same bound for other terms in this class.

The proof is done.

We are now ready to provide the proof for our main result.

Proof of Theorem 1.1 By the refined estimates in Propositions 4.3–4.5, we can choose $C_1 \gg 1$ very large, and $\varepsilon \ll 1$ sufficiently small, such that the estimates in (4.5) hold. This means the solution to the Faddeev model (1.1) exists globally.

The pointwise decay in (1.3) can be seen from (4.15)–(4.16) and the Klainerman-Sobolev inequality in Proposition 2.4.

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