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Stability of Multiplier Ideal Sheaves*

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Abstract In the present article, the authors find and establish stability of multiplier ideal sheaves, which is more general than strong openness.

Keywords Plurisubharmonic function, Multiplier ideal sheaf, Strong openness and stability, Coherent analytic sheaf, L^2 estimate
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1 Introduction

Multiplier ideal sheaves have been playing an important role in several complex variables and complex geometry, since its introduction in 1990's (e.g., see [4, 17, 23–24]). The basic properties of a multiplier ideal sheaf include its coherence, integral closedness and Nadel's vanishing theorem.

Later on, Demailly (see [4–5]) proposed the strong openness conjecture which asserts that a multiplier ideal sheaf satisfies the strong openness property. The conjecture was also stated by Siu, Y.-T. in [22] and many others. In 2013, the first author and the third author solved the conjecture (see [13]).

In this paper, we study a more general openness property called stability for a multiplier ideal sheaf. The paper was posted on arXiv (see [8]).

Let $D \subset \mathbb{C}^n$ be a bounded pseudoconvex domain, $o \in D$ be the origin of $\mathbb{C}^n$ and $\varphi \in \mathrm{Psh}(D)$ be a plurisubharmonic function on D. The multiplier ideal sheaf $\mathscr{I}(\varphi)$ consists of germs of holomorphic functions f such that $|f|^2 \mathrm{e}^{-\varphi}$ is locally integrable, which is a coherent sheaf of ideals (see [5]).

Demailly's strong openness conjecture (SOC for short) (see [4]) If $(f, o) \in \mathscr{I}(\varphi)_o$, then there exists $\delta > 0$ such that $(f, o) \in \mathscr{I}((1+\delta)\varphi)_o$.

Here δ seems to be dependent on the germ (f, o), however, by the coherence of the multiplier ideal sheaf, δ is actually independent of the germ (f, o).

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Note that $\mathscr{I}(\varphi)_o$ is finitely generated by $(f_j)_{j=1,\cdots,k_0}$. Let $\mathscr{I}(\varphi)_o=(f_1,\cdots,f_{k_0})$. The truth of SOC implies that there exists $\delta_j>0$ such that $(f_j,o)\in\mathscr{I}((1+\delta_j)\varphi)_o$ for any $1\le j\le k_0$. Then, SOC is equivalent to the equality $\mathscr{I}(\varphi)_o=\mathscr{I}((1+\delta_0)\varphi)_o$, where $\delta_0=\min\limits_{1\le j\le k_0}\{\delta_j\}$ which is independent of the germ (f,o).

As $\mathscr{I}((1+\delta_0)\varphi)_o\subset\mathscr{I}_+(\varphi)_o:=\bigcup\limits_{\delta>0}\mathscr{I}((1+\delta)\varphi)_o\subset\mathscr{I}(\varphi)_o$, SOC is also equivalent to the equality $\mathscr{I}(\varphi)_o=\mathscr{I}_+(\varphi)_o$.

Another reformulation of the strong openness conjecture is that $\{p\in\mathbb{R}:|f|^2\mathrm{e}^{-p\varphi}$ is locally integrable$\}$ is open. When $\mathscr{I}(\varphi)_o$ is trivial, the so-called openness conjecture was solved by Berndtsson in [2]. Such assertions originate from the fundamental fact in calculus: the set $\big\{p\in\mathbb{R}:\frac{1}{|x|^{pc}}=\mathrm{e}^{-p\varphi}$ is locally integrable at the origin$\big\}$ is open, which is actually $=\big\{p<\frac{1}{c}\big\}$, where $\varphi=c\log|x|, c>0$. This explains why openness and strong openness are named so.

In [13], Guan and Zhou proved the above SOC. Moreover, they also established an effectiveness lower bound for δ in the conjecture in [14].

In the present article, we obtain the following stability of multiplier ideal sheaves.

Theorem 1.1 *Let $(\varphi_j)_{j\in\mathbb{N}^+}$ be a sequence of negative plurisubharmonic functions on D, which is convergent to $\varphi\in\mathrm{Psh}(D)$ in Lebesgue measure, and $\mathscr{I}(\varphi_j)_o\subset\mathscr{I}(\varphi)_o$. Let $(F_j)_{j\in\mathbb{N}^+}$ be a sequence of holomorphic functions on D with $(F_j,o)\in\mathscr{I}(\varphi)_o$, which is compactly convergent to a holomorphic function F. Then, $|F_j|^2\mathrm{e}^{-\varphi_j}$ converges to $|F|^2\mathrm{e}^{-\varphi}$ in the L^1 norm near o. In particular, there exists $\varepsilon_0>0$ such that $\mathscr{I}(\varphi_j)_o=\mathscr{I}((1+\varepsilon_0)\varphi_j)_o=\mathscr{I}(\varphi)_o$ for any large enough j.*

The last conclusion in the above theorem can be obtained by [14, Proposition 1.8] and by the finite generation of $\mathscr{I}(\varphi)_o$.

The following proposition can be deduced from [16, Theorem 4.1.8]. Here we give another proof by using our main theorem.

Proposition 1.1 *Let $(\varphi_j)_{j\in\mathbb{N}^+}$ be a sequence of negative plurisubharmonic functions on D. If φ_j is convergent to $\varphi\in\mathrm{Psh}(D)$ in Lebesgue measure, then φ_j converges to φ in the L^p_{loc} $(0<p<\infty)$ norm.*

Proof It suffices to prove when $p\in\mathbb{N}^+$. By scaling, we can assume the Lelong number $\nu(\varphi,o)<1$. Thus, $\mathscr{I}(\varphi_j)_o\subset\mathscr{I}(\varphi)_o=\mathcal{O}_o$. Then, the desired result follows from Theorem 1.1 and the inequality

$$\frac{1}{p!}\int_D|\varphi_j-\varphi|^p\mathrm{d}\lambda_n\le\int_D|\mathrm{e}^{-\varphi_j}-\mathrm{e}^{-\varphi}|\mathrm{d}\lambda_n,$$

which follows from the inequality $\frac{1}{p!}(a-b)^p\le(\mathrm{e}^{a-b}-1)\mathrm{e}^b$ for any $a\ge b\ge0$.

As an application of Theorem 1.1, we can conclude the following semi-continuity of complex singularity exponents.

Corollary 1.1 (see [6, Main Theorem 0.2]) *Let X be a complex manifold, $K\subset X$ be a compact subset and φ be a plurisubharmonic function on X. If $c<c_K(\varphi)$ (complex singularity exponent of φ on K) and (φ_j) is a sequence of plurisubharmonic functions on X which is con-*

vergent to φ *in* L^1_{loc} *norm, then* $\mathrm{e}^{-2c\varphi_j}$ *converges to* $\mathrm{e}^{-2c\varphi}$ *in* L^1 *norm over some neighborhood* U *of* K.

Indeed, by subtracting a constant, we can assume that φ is negative on K. As $\int_K \varphi_j \mathrm{d}\lambda_n \le \int_K |\varphi - \varphi_j| \mathrm{d}\lambda_n + \int_K \varphi \mathrm{d}\lambda_n$, we obtain that φ_j is also negative on K. Then, Corollary 1.1 is a special case of Theorem 1.1 when $\mathscr{I}(\varphi)_o = \mathcal{O}_o$.

With the additional conditions $\varphi_j \le \varphi$ and $F_j = F$, Theorem 1.1 reduces to the main result in [18].

If $\varphi = \log |g|$ with $|g|^2 := |g_1|^2 + \cdots + |g_J|^2$ for holomorphic functions $g_1, \cdots, g_J$ on a concentric polydisk $\Delta^n \times \Delta$, then it follows that the following holds.

Corollary 1.2 (see [19, Main Theorem]) *Assume that* $\int_{\Delta^n} |g(z,0)|^{-\delta} < \infty$. *Then there exists a smaller concentric polydisk* $\Delta'^n \times \Delta'$ *so that the function* $c \mapsto \int_{\Delta'^n} |g(z,c)|^{-\delta}$ *is finite and continuous for* $c \in \Delta' \subset\subset \Delta$.

2 Lemmas Used in the Proof of Main Results

Let $L^2_{\mathcal{O}}(D)$ be the Hilbert space of homomorphic functions on D with finite L^2 norm, i.e.,

$$L^2_{\mathcal{O}}(D) := \Big\{ f \in \mathcal{O}(D) \Big|\ \|f\|^2_D = \int_D |f|^2 \mathrm{d}\lambda_n < \infty \Big\},$$

whose inner product is defined to be $(f, g) = \int_D f \cdot \overline{g} \mathrm{d}\lambda_n,\ \forall f, g \in L^2_{\mathcal{O}}(D)$.

We are now in a position to prove the following lemma.

Lemma 2.1 *Let* $I \subset \mathcal{O}_o$ *be an ideal and* $(e_k)_{k \in \mathbb{N}^+}$ *be an orthonormal basis of*

$$\mathcal{H}_I := \{ f \in L^2_{\mathcal{O}}(D) \mid (f, o) \in I \},$$

a closed subspace of $L^2_{\mathcal{O}}(D)$. *Then, there exist a neighborhood* $U_0 \subset\subset D$ *of* o, *an integer* $k_0 > 0$ *and some constant* $C_0 > 1$ *such that*

$$\sum_{k=1}^{\infty} |e_k|^2 \le C_0 \cdot \sum_{k=1}^{k_0} |e_k|^2 \quad \text{on } U_0.$$

Proof It follows from the strong Noetherian property of coherent analytic sheaves that the sequence of ideal sheaves generated by the holomorphic functions

$$(e_k(z)\overline{e_k(\overline{w})})_{k \le N}, \quad N = 1, 2, \cdots$$

on $D \times D^*$ is locally stationary, where $D^* := \{ \overline{w} \mid w \in D \}$.

Let $B \subset\subset D$ be a ball centered at o. Then there exists $k_0 > 0$ such that for any $N \ge k_0$, we have $(e_k(z)\overline{e_k(\overline{w})})_{k \le N} = (e_k(z)\overline{e_k(\overline{w})})_{k \le k_0}$ on $B \times B$.

Complete (e_k) to an orthonormal basis $(\widetilde{e}_\alpha)$ of $L^2_{\mathcal{O}}(D)$. Then, $\sum\limits_{k=1}^{\infty} |e_k(z)|^2$ is a subsum of $\sum\limits_{\alpha=1}^{\infty} |\widetilde{e}_\alpha(z)|^2$, which is known to converge uniformly on compact subsets by the theory of

Bergman kernel. Thus, it follows from the inequality

$$\Big|\sum_{k=p}^{q} e_k(z)\overline{e_k(\overline{w})}\Big| \le \Big(\sum_{k=p}^{q}|e_k(z)|^2\sum_{k=p}^{q}|e_k(\overline{w})|^2\Big)^{\frac{1}{2}}$$

that $\sum\limits_{k=1}^{\infty} e_k(z)\overline{e_k(\overline{w})}$ is uniformly convergent on every compact subset of $D\times D^*$.

By the closedness of the sections of coherent analytic sheaves under the topology of compact convergence (see [7]), $\sum\limits_{k=1}^{\infty} e_k(z)\overline{e_k(\overline{w})}$ is a section of the coherent ideal sheaf generated by $(e_k(z)\overline{e_k(\overline{w})})_{k\le k_0}$ over $B\times B$. Then, there exist a smaller ball $B_0\subset\subset B$ centered at o and functions $a_k(z,w)\in\mathcal{O}(\overline{B_0\times B_0})$, $1\le k\le k_0$, such that on $\overline{B_0\times B_0}$,

$$\sum_{k=1}^{\infty} e_k(z)\overline{e_k(\overline{w})} = \sum_{k=1}^{k_0} a_k(z,w)e_k(z)\overline{e_k(\overline{w})}.$$

Finally, by restricting to the conjugate diagonal $w=\overline{z}$, we get

$$\sum_{k=1}^{\infty}|e_k|^2 \le C_0\cdot\sum_{k=1}^{k_0}|e_k|^2 \quad \text{on } B_0.$$

In order to prove Theorem 1.1, we also need the following lemma (see the Appendix 4 for a proof), whose various forms already appear in [10–12, 14].

Lemma 2.2 *Let $B\in(0,+\infty)$ be arbitrarily given and t_0 be a positive number. Let $D\subset\mathbb{C}^n$ be a bounded pseudoconvex domain containing the origin o. Let ψ be a negative plurisubharmonic function on D such that $\psi(o)=-\infty$ and φ be a plurisubharmonic function on D. Then, for any bounded holomorphic function F on $\{\psi<-t_0\}$ satisfying*

$$\int_D \frac{1}{B}\mathbb{1}_{\{-t_0-B<\psi<-t_0\}}|F|^2\mathrm{e}^{-\varphi}\mathrm{d}\lambda_n \le C_1 < +\infty, \tag{2.1}$$

there exists a holomorphic function $\widetilde{F}$ on D such that

$$\int_D |\widetilde{F}-b(\psi)F|^2\mathrm{e}^{-\varphi+v(\psi)}\mathrm{d}\lambda_n \le (1-\mathrm{e}^{-(t_0+B)})C_1, \tag{2.2}$$

which implies

$$(\widetilde{F}-F,o)\in\mathscr{I}(\varphi)_o,$$

where $C_1>0$ is a constant, $b(t)=1-\int_{-\infty}^{t}\frac{1}{B}\mathbb{1}_{\{-t_0-B<s<-t_0\}}\mathrm{d}s$ and $v(t)=\int_0^t(1-b(s))\mathrm{d}s$.

It is clear that $\mathbb{1}_{\{-t_0<t<+\infty\}}\le 1-b(t)\le\mathbb{1}_{\{-t_0-B<t<+\infty\}}$ and $\max\{t,-t_0-B\}\le v(t)\le\max\{t,-t_0\}$.

The following lemma is well known in real analysis (see the proof of [15, Theorem 13.44]).

Lemma 2.3 *Let $(f_j)_{j\in\mathbb{N}^+}$ be a sequence of functions in $L^p_{\mathrm{loc}}(D)$ $(p>1)$, which is convergent to f in Lebesgue measure. If there exists some constant $M>0$ such that*

$$\Big(\int_D |f_j|^p\mathrm{d}\lambda_n\Big)^{\frac{1}{p}} < M,$$

then

$$\int_D |f_j - f| \mathrm{d}\lambda_n \to 0, \quad j \to \infty.$$

3 Proof of Main Result

Let $I \subset \mathcal{O}_o$ be an ideal and φ be a negative plurisubharmonic function on some bounded psedoconvex domain $D \subset \mathbb{C}^n$ with $\varphi(o) = -\infty$. Choose e_k, k_0, U_0, C_0 as in Lemma 2.1, C_1 as in Lemma 2.2 and put

$$C = C_\varepsilon(\varphi) := \Big[(k_0(1-\mathrm{e}^{-\varepsilon(t_0+1)})C_0C_1)^{-\frac{1}{2}} - 1 - \Big(\frac{\sum\limits_{k=1}^{k_0}\int_D \mathbb{1}_{\{\varphi<-t_0\}}|e_k|^2\mathrm{d}\lambda_n}{k_0(1-\mathrm{e}^{-\varepsilon(t_0+1)})C_1}\Big)^{\frac{1}{2}}\Big]^{-1},$$

where $\varepsilon \in (0,1]$ and t_0 are two positive numbers. When choosing t_0 large enough and ε small enough, C could be positive.

At first, we obtain the following estimation of the weighted L^2 norm near the singularities of plurisubharmonic weight related to SOC.

Proposition 3.1 *Assume that $\mathscr{I}(\varphi)_o \subset I \subset \mathcal{O}_o$. If $C > 0$, then*

$$\int_{U_0\cap\{\varphi<-(t_0+1)\}} \Big(\sum_{k=1}^{\infty}|e_k|^2\Big)\mathrm{e}^{-\varphi}\mathrm{d}\lambda_n \le C^2.$$

Proof Thanks to the strong openness, replacing B, t_0, φ, ψ by $\varepsilon, \varepsilon t_0, (1+\varepsilon)\varphi, \varepsilon\varphi$ respectively for small enough $\varepsilon > 0$ (shrinking D if necessary) in Lemma 2.2, it follows that, for any $1 \le k \le k_0$, there exists a holomorphic function $F_k \in \mathcal{O}(D)$ such that

$$\int_D |F_k - b(\varepsilon\varphi)e_k|^2\mathrm{e}^{-\varphi}\mathrm{d}\lambda_n \le (1-\mathrm{e}^{-\varepsilon(t_0+1)})C_1. \tag{3.1}$$

By Minkowski's inequality, we obtain

$$\begin{aligned}&\Big(\sum_{k=1}^{k_0}\int_D |F_k|^2\mathrm{d}\lambda_n\Big)^{\frac{1}{2}}\\ &\le \Big(\sum_{k=1}^{k_0}\int_D |F_k - b(\varepsilon\varphi)e_k|^2\mathrm{e}^{-\varphi}\mathrm{d}\lambda_n\Big)^{\frac{1}{2}} + \Big(\sum_{k=1}^{k_0}\int_D |b(\varepsilon\varphi)e_k|^2\mathrm{d}\lambda_n\Big)^{\frac{1}{2}}.\end{aligned} \tag{3.2}$$

It follows from (3.1) and $0 \le b(\varepsilon\varphi) \le \mathbb{1}_{\{\varphi<-t_0\}}$ that

$$\begin{aligned}&\Big(\sum_{k=1}^{k_0}\int_D |F_k|^2\mathrm{d}\lambda_n\Big)^{\frac{1}{2}}\\ &\le (k_0(1-\mathrm{e}^{-\varepsilon(t_0+1)})C_1)^{\frac{1}{2}} + \Big(\sum_{k=1}^{k_0}\int_D \mathbb{1}_{\{\varphi<-t_0\}}|e_k|^2\mathrm{d}\lambda_n\Big)^{\frac{1}{2}}.\end{aligned} \tag{3.3}$$

By Lemma 2.2, we know that $(F_k - e_k, o) \in \mathscr{I}((1+\varepsilon)\varphi)_o \subset \mathscr{I}(\varphi)_o \subset I$ and $(F_k, o) \in I$.

Hence, we have

$$F_k = \sum_{j=1}^{\infty} a_k^j e_j, \quad a_k^j \in \mathbb{C},\ 1 \le k \le k_0$$

and

$$\int_D |F_k|^2 \mathrm{d}\lambda_n = \sum_{j=1}^{\infty} |a_k^j|^2, \quad 1 \le k \le k_0.$$

Thus, we deduce from Lemma 2.1 that, on U_0,

$$\begin{aligned}
&\Big(\sum_{k=1}^{k_0} |F_k - e_k|^2\Big)^{\frac{1}{2}} \ge \Big(\sum_{k=1}^{k_0} |e_k|^2\Big)^{\frac{1}{2}} - \Big(\sum_{k=1}^{k_0} |F_k|^2\Big)^{\frac{1}{2}} \\
&\ge \Big(\frac{1}{C_0}\Big)^{\frac{1}{2}} \Big(\sum_{k=1}^{\infty} |e_k|^2\Big)^{\frac{1}{2}} - \Big(\sum_{k=1}^{k_0}\sum_{j=1}^{\infty} |a_k^j|^2\Big)^{\frac{1}{2}} \Big(\sum_{k=1}^{\infty} |e_k|^2\Big)^{\frac{1}{2}} \\
&= \Big(\Big(\frac{1}{C_0}\Big)^{\frac{1}{2}} - \Big(\sum_{k=1}^{k_0} \int_D |F_k|^2 \mathrm{d}\lambda_n\Big)^{\frac{1}{2}}\Big) \Big(\sum_{k=1}^{\infty} |e_k|^2\Big)^{\frac{1}{2}} \\
&\ge \Big(\Big(\frac{1}{C_0}\Big)^{\frac{1}{2}} - (k_0(1 - \mathrm{e}^{-\varepsilon(t_0+1)})C_1)^{\frac{1}{2}} \\
&\quad - \Big(\sum_{k=1}^{k_0} \int_D \mathbb{1}_{\{\varphi < -t_0\}} |e_k|^2 \mathrm{d}\lambda_n\Big)^{\frac{1}{2}}\Big) \Big(\sum_{k=1}^{\infty} |e_k|^2\Big)^{\frac{1}{2}}.
\end{aligned} \tag{3.4}$$

Denote by

$$A := \Big(\frac{1}{C_0}\Big)^{\frac{1}{2}} - (k_0(1 - \mathrm{e}^{-\varepsilon(t_0+1)})C_1)^{\frac{1}{2}} - \Big(\sum_{k=1}^{k_0} \int_D \mathbb{1}_{\{\varphi < -t_0\}} |e_k|^2 \mathrm{d}\lambda_n\Big)^{\frac{1}{2}}.$$

Since $C_\varepsilon(\varphi) > 0$ and

$$A \cdot C_\varepsilon(\varphi) = (k_0(1 - \mathrm{e}^{-\varepsilon(t_0+1)})C_1)^{\frac{1}{2}} > 0,$$

it follows that $A > 0$.

Then from (3.4) we obtain

$$\begin{aligned}
&A^2 \cdot \Big(\int_{\{\varphi < -(t_0+1)\} \cap U_0} \Big(\sum_{k=1}^{\infty} |e_k|^2\Big) \mathrm{e}^{-\varphi} \mathrm{d}\lambda_n\Big) \\
&\le \int_{\{\varphi < -(t_0+1)\} \cap U_0} \Big(\sum_{k=1}^{k_0} |F_k - e_k|^2\Big) \mathrm{e}^{-\varphi} \mathrm{d}\lambda_n \\
&= \sum_{k=1}^{k_0} \int_{\{\varphi < -(t_0+1)\} \cap U_0} |F_k - e_k|^2 \mathrm{e}^{-\varphi} \mathrm{d}\lambda_n.
\end{aligned} \tag{3.5}$$

Note that

$$\sum_{k=1}^{k_0} |F_k - b(\varepsilon\varphi) e_k|^2 \big|_{\{\varphi < -(t_0+1)\} \cap U_0} = \sum_{k=1}^{k_0} |F_k - e_k|^2.$$

It follows from Lemma 2.2 that

$$\begin{aligned}
&\sum_{k=1}^{k_0} \int_{\{\varphi < -(t_0+1)\} \cap U_0} |F_k - e_k|^2 \mathrm{e}^{-\varphi} \mathrm{d}\lambda_n \\
&\le \sum_{k=1}^{k_0} \int_D |F_k - b(\varepsilon\varphi) e_k|^2 \mathrm{e}^{-\varphi} \mathrm{d}\lambda_n
\end{aligned}$$

$$\leq k_0(1-\mathrm{e}^{-\varepsilon(t_0+1)})C_1. \tag{3.6}$$

Combining inequalities (3.5) and (3.6), we have

$$\begin{aligned}
&\int_{\{\varphi<-(t_0+1)\}\cap U_0}\Big(\sum_{k=1}^{\infty}|e_k|^2\Big)\mathrm{e}^{-\varphi}\mathrm{d}\lambda_n \leq \frac{k_0(1-\mathrm{e}^{-\varepsilon(t_0+1)})C_1}{A^2}\\
&=\Big[(k_0(1-\mathrm{e}^{-\varepsilon(t_0+1)})C_0C_1)^{-\frac{1}{2}}-1-\Big(\frac{\sum\limits_{k=1}^{k_0}\int_D \mathbb{1}_{\{\varphi<-t_0\}}|e_k|^2\mathrm{d}\lambda_n}{k_0(1-\mathrm{e}^{-\varepsilon(t_0+1)})C_1}\Big)^{\frac{1}{2}}\Big]^{-2}\\
&=C_\varepsilon^2(\varphi).
\end{aligned} \tag{3.7}$$

For our proof of Theorem 1.1, it is necessary to prove the following result.

If $I=\mathscr{I}(\varphi)_o$ for some negative plurisubharmonic function φ on D, then any orthonormal basis of the L^2 space $\mathcal{H}^2(D,\varphi)$, which consists of holomorphic functions f on D such that $|f|^2\mathrm{e}^{-\varphi}$ is integrable on D, is contained in $\mathcal{H}_I$.

Since $\mathscr{I}(\varphi)$ is generated by any orthonormal basis of $\mathcal{H}^2(D,\varphi)$ (see [5, Proposition 5.7]), we may assume that $e_k (1\leq k\leq k_0)$ are the generators of $I=\mathscr{I}(\varphi)_o$ and bounded on D in Lemma 2.1 (shrinking D and B_0 if necessary).

Lemma 3.1 *Let e_k $(1\leq k\leq k_0)$ be generators of $I=\mathscr{I}(\varphi)_o$ with bounded $\sum\limits_{k=1}^{k_0}|e_k|$ on D, which is in the unit ball $B(o;1)$ and*

$$\sum_{k=1}^{k_0}\int_D |e_k|^2\mathrm{e}^{-(1+\varepsilon_0)\varphi}\mathrm{d}\lambda_n<\infty.$$

Then, for any $M>0$, there exists $t_0\gg 0$ such that for any negative plurisubharmonic function ψ on D with $\mathscr{I}(\psi)_o\subset\mathscr{I}(\varphi)_o$ and

$$\sum_{k=1}^{k_0}\int_D \mathbb{1}_{\{\widetilde{\psi}<-t_0\}}|e_k|^2\mathrm{d}\lambda_n \leq 2\sum_{k=1}^{k_0}\int_D \mathbb{1}_{\{\widetilde{\varphi}<-t_0\}}|e_k|^2\mathrm{d}\lambda_n, \tag{3.8}$$

we have

$$\int_{U_0\cap\{|z|<\mathrm{e}^{-\frac{(1+\varepsilon_0)(1+\frac{\varepsilon_0}{2})(t_0+1)}{\frac{\varepsilon_0}{2}}}\}}\Big(\sum_{k=1}^{\infty}|e_k|^2\Big)\mathrm{e}^{-\widetilde{\psi}}\mathrm{d}\lambda_n<M,$$

where

$$\widetilde{\varphi}=\varphi+\frac{\frac{\varepsilon_0}{2}}{(1+\varepsilon_0)(1+\frac{\varepsilon_0}{2})}\frac{\log|z|}{2},\quad \widetilde{\psi}=\psi+\frac{\frac{\varepsilon_0}{2}}{(1+\varepsilon_0)(1+\frac{\varepsilon_0}{2})}\frac{\log|z|}{2}.$$

Proof By Hölder inequality, we have

$$\begin{aligned}
&\int_U |F|^2\mathrm{e}^{-(1+\frac{\varepsilon_0}{2})\widetilde{\varphi}}\mathrm{d}\lambda_n\\
&\leq\Big(\int_U |F|^2\mathrm{e}^{-(1+\varepsilon_0)\varphi}\mathrm{d}\lambda_n\Big)^{\frac{1+\frac{\varepsilon_0}{2}}{1+\varepsilon_0}}\Big(\int_U |F|^2\mathrm{e}^{-\frac{\log|z|}{2}}\mathrm{d}\lambda_n\Big)^{\frac{\frac{\varepsilon_0}{2}}{1+\varepsilon_0}},
\end{aligned}$$

which implies $\mathscr{I}((1+\varepsilon_0)\varphi)_o \subset \mathscr{I}\big(\big(1+\frac{\varepsilon_0}{2}\big)\widetilde{\varphi}\big)_o \subset \mathscr{I}(\widetilde{\varphi})_o \subset \mathscr{I}(\varphi)_o$, i.e.,

$$\mathscr{I}\Big(\Big(1+\frac{\varepsilon_0}{2}\Big)\widetilde{\varphi}\Big)_o = \mathscr{I}(\widetilde{\varphi})_o = \mathscr{I}(\varphi)_o.$$

As

$$\sum_{k=1}^{k_0}\int_D |e_k|^2 \mathrm{e}^{-(1+\varepsilon_0)\varphi}\mathrm{d}\lambda_n < \infty,$$

there exists $t_0 \gg 0$ such that $0 < C_{\frac{\varepsilon_0}{2}}(\widetilde{\varphi}) < \frac{\sqrt{M}}{2}$, and $0 < C_{\frac{\varepsilon_0}{2}}(\widetilde{\psi}) \le 2\cdot C_{\frac{\varepsilon_0}{2}}(\widetilde{\varphi})$ by (3.8).

Since $\widetilde{\psi} \le \frac{\frac{\varepsilon_0}{2}}{(1+\varepsilon_0)(1+\frac{\varepsilon_0}{2})}\log|z|$ on D, we have

$$\{|z| < \mathrm{e}^{-\frac{(1+\varepsilon_0)(1+\frac{\varepsilon_0}{2})(t_0+1)}{\frac{\varepsilon_0}{2}}}\} \subset \{\widetilde{\psi} < -(t_0+1)\}.$$

Then, by Proposition 3.1, we obtain that

$$\begin{aligned}&\int_{U_0\cap\{|z|<\mathrm{e}^{-\frac{(1+\varepsilon_0)(1+\frac{\varepsilon_0}{2})(t_0+1)}{\frac{\varepsilon_0}{2}}}\}}\Big(\sum_{k=1}^{\infty}|e_k|^2\Big)\mathrm{e}^{-\widetilde{\psi}}\mathrm{d}\lambda_n\\ \le&\int_{U_0\cap\{\widetilde{\psi}<-(t_0+1)\}}\Big(\sum_{k=1}^{\infty}|e_k|^2\Big)\mathrm{e}^{-\widetilde{\psi}}\mathrm{d}\lambda_n \le C^2_{\frac{\varepsilon_0}{2}}(\widetilde{\psi}) < M.\end{aligned}$$

Proof of Theorem 1.1 As every sequence which is convergent in Lebesgue measure has a subsequence which is convergent almost everywhere, it is sufficient to prove the result for the case that φ_j is convergent to φ almost everywhere.

By the truth of SOC, there exists $\varepsilon_0 > 0$ such that $\mathscr{I}(\varphi) = \mathscr{I}((1+\varepsilon_0)\varphi)$ on a neighborhood D of o. Without loss of generality, we may assume that the unit ball $B(o;1) \supset D$.

Since F_j is compactly convergent to a holomorphic function F, by shrinking D, we can assume that $\int_D |F_j|^2\mathrm{d}\lambda_n$ is bounded with respect to j.

Let $e_k,\ 1\le k\le k_0$, be as in Lemma 3.1. Then, we infer from $(F_j,o)\in\mathscr{I}(\varphi)_o$ and Lemma 2.1 that there exist complex numbers a_j^k such that $F_j = \sum\limits_{k=1}^{\infty} a_j^k e_k$, and $\sum\limits_{k=1}^{\infty}|a_j^k|^2 = \int_D |F_j|^2\mathrm{d}\lambda_n$ is bounded with respect to j.

Since φ_j is convergent to φ almost everywhere, it follows from the dominated convergence theorem that

$$\sum_{k=1}^{k_0}\int_D \mathbb{1}_{\{\widetilde{\varphi}_j<-t_0\}}|e_k|^2\mathrm{d}\lambda_n \le 2\sum_{k=1}^{k_0}\int_D \mathbb{1}_{\{\widetilde{\varphi}<-t_0\}}|e_k|^2\mathrm{d}\lambda_n,$$

where $\widetilde{\varphi_j} = \varphi_j + \frac{\frac{\varepsilon_0}{2}}{(1+\varepsilon_0)(1+\frac{\varepsilon_0}{2})}\frac{\log|z|}{2}$.

By Lemma 3.1, there exists a neighborhood $V_0 \subset\subset D$ of o and $M > 0$ such that

$$\int_{V_0}\sum_{k=1}^{\infty}|e_k|^2\mathrm{e}^{-\varphi_j}\mathrm{d}\lambda_n < M.$$

Let $\varepsilon\in(0,\varepsilon_0)$. By replacing φ with $\big(1+\frac{\varepsilon}{2}\big)\varphi$ and φ_j with $\big(1+\frac{\varepsilon}{2}\big)\varphi_j$, we have

$$\int_{\widetilde{V}_0}\sum_{k=1}^{\infty}|e_k|^2\mathrm{e}^{-(1+\frac{\varepsilon}{2})\varphi_j}\mathrm{d}\lambda_n < \widetilde{M}$$

for some neighborhood $\widetilde{V}_0 \supset o$ and some constant $\widetilde{M}$ which are independent of φ_j.

As $\sum\limits_{k=1}^{\infty} |a_j^k|^2$ is bounded with respect to j, by Schwarz inequality, it follows that

$$\int_{\widetilde{V}_0} |F_j|^2 \mathrm{e}^{-(1+\frac{\varepsilon}{2})\varphi_j} \mathrm{d}\lambda_n \le \int_{\widetilde{V}_0} \Big(\sum_{k=1}^{\infty} |a_j^k|^2\Big) \cdot \Big(\sum_{k=1}^{\infty} |e_k|^2\Big) \mathrm{e}^{-(1+\frac{\varepsilon}{2})\varphi_j} \mathrm{d}\lambda_n$$

is bounded with respect to j.

Then, by Lemma 2.3, we obtain that $F_j \mathrm{e}^{-\varphi_j}$ converges to $F\mathrm{e}^{-\varphi}$ in the L^1_{loc} norm on $\widetilde{V}_0$, as j goes to infinity.

By replacing φ_j with $(1+\varepsilon_0)\varphi_j$, we obtain the second assertion from the first one.

4 Appendix

For the sake of completeness, this section is devoted to the proof of Lemma 2.2.

4.1 L^2 estimates for some $\overline{\partial}$ equations

For the sake of convenience, we will recall some known facts on L^2 estimates for some $\overline{\partial}$ equations. Here, $\overline{\partial}^*$ means the Hilbert adjoint operator of $\overline{\partial}$.

Lemma 4.1 (see [20], see also [1]) *Let $\Omega \subset\subset \mathbb{C}^n$ be a domain with C^∞ boundary $b\Omega$, $\Phi \in C^\infty(\overline{\Omega})$, Let ρ be a C^∞ defining function for Ω such that $|\mathrm{d}\rho| = 1$ on $b\Omega$. Let η be a smooth function on $\overline{\Omega}$. For any $(0,1)$-form $\alpha = \sum\limits_{j=1}^{n} \alpha_{\overline{j}} \mathrm{d}\overline{z}^j \in \mathrm{Dom}_\Omega(\overline{\partial}^*) \cap C^\infty_{(0,1)}(\overline{\Omega})$,*

$$\begin{aligned}
&\int_\Omega \eta |\overline{\partial}^*_\Phi \alpha|^2 \mathrm{e}^{-\Phi} \mathrm{d}\lambda_n + \int_\Omega \eta |\overline{\partial}\alpha|^2 \mathrm{e}^{-\Phi} \mathrm{d}\lambda_n \\
={}& \sum_{i,j=1}^{n} \int_\Omega \eta |\overline{\partial}_j \alpha_{\overline{j}}|^2 \mathrm{d}\lambda_n + \sum_{i,j=1}^{n} \int_{b\Omega} \eta (\partial_i \overline{\partial}_j \rho) \alpha_{\overline{i}} \overline{\alpha_{\overline{j}}} \mathrm{e}^{-\Phi} \mathrm{d}S \\
&+ \sum_{i,j=1}^{n} \int_\Omega \eta (\partial_i \overline{\partial}_j \Phi) \alpha_{\overline{i}} \overline{\alpha_{\overline{j}}} \mathrm{e}^{-\Phi} \mathrm{d}\lambda_n + \sum_{i,j=1}^{n} \int_\Omega -(\partial_i \overline{\partial}_j \eta) \alpha_{\overline{i}} \overline{\alpha_{\overline{j}}} \mathrm{e}^{-\Phi} \mathrm{d}\lambda_n \\
&+ 2\mathrm{Re}(\overline{\partial}^*_\Phi \alpha, \alpha \llcorner (\overline{\partial}\eta)^\sharp)_{\Omega,\Phi},
\end{aligned} \tag{4.1}$$

where $\mathrm{d}\lambda_n$ is the Lebesgue measure on $\mathbb{C}^n$, and $\alpha \llcorner (\overline{\partial}\eta)^\sharp = \sum\limits_j \alpha_{\overline{j}} \partial_j \eta$.

The symbols and notations can be referred to [26] (also [20–21, 25]).

Lemma 4.2 (see [1], see also [26]) *Let $\Omega \subset\subset \mathbb{C}^n$ be a strictly pseudoconvex domain with C^∞ boundary $b\Omega$ and $\Phi \in C^\infty(\overline{\Omega})$. Let λ be a $\overline{\partial}$ closed smooth form of bidgree $(n,1)$ on $\overline{\Omega}$. Assume the inequality*

$$|(\lambda, \alpha)_{\Omega,\Phi}|^2 \le C \int_\Omega |\overline{\partial}^*_\Phi \alpha|^2 \frac{\mathrm{e}^{-\Phi}}{\mu} \mathrm{d}\lambda_n < \infty,$$

where $\frac{1}{\mu}$ is an integrable positive function on Ω and C is a constant, holds for all $(n,1)$-form $\alpha \in \mathrm{Dom}_\Omega(\overline{\partial}^*) \cap \mathrm{Ker}(\overline{\partial}) \cap C^\infty_{(n,1)}(\overline{\Omega})$. Then there is a solution u to the equation $\overline{\partial}u = \lambda$ such that

$$\int_\Omega |u|^2 \mu \mathrm{e}^{-\Phi} \mathrm{d}\lambda_n \le C.$$

4.2 Proof of Lemma 2.2

For the sake of completeness, let us recall some steps in our proof in [11] (see also [10, 12]) with some slight modifications for a proof of Lemma 2.2.

It suffices to consider the case that D is strongly pseudoconvex domain, φ, ψ are plurisubharmonic functions on an open set U containing $\overline{D}$ and F is a bounded holomorphic function on $U \cap \{\psi < -t_0\}$. Then it follows from inequality (2.1) that

$$\int_{\{\psi<-t_0\}\cap D} |F|^2 \le \int_{\{\psi<-t_0-\frac{B}{2}\}\cap D} |F|^2 + \int_{\{-t_0-B<\psi<-t_0\}\cap D} |F|^2 < +\infty. \tag{4.2}$$

Then it follows from method of convolution (see e.g. [3]) that there exist smooth plurisubharmonic functions ψ_m and φ_m on an open set $U \subset \overline{D}$ decreasing convergent to ψ and φ respectively, such that

$$\sup_m \sup_D \psi_m < 0 \quad \text{ and } \quad \sup_m \sup_D \varphi_m < +\infty.$$

Let $\varepsilon \in (0, \frac{1}{8}B)$ and $\{v_\varepsilon\}_{\varepsilon\in(0,\frac{1}{8}B)}$ be a family of smooth increasing convex functions on $\mathbb{R}$, which are continuous functions on $\mathbb{R} \cup \{-\infty\}$ such that

(1) $v_\varepsilon(t) = t$ for $t \ge -t_0 - \varepsilon$, $v_\varepsilon(t) = \text{constant}$ for $t < -t_0 - B + \varepsilon$;

(2) $v''_\varepsilon(t)$ are pointwise convergent to $\frac{1}{B}\mathbb{1}_{\{-t_0-B,-t_0\}}$, when $\varepsilon \to 0$ and

$$0 \le v''_\varepsilon(t) \le \frac{2}{B}\mathbb{1}_{\{-t_0-B+\varepsilon,-t_0-\varepsilon\}}$$

for any $t \in \mathbb{R}$;

(3) $v'_\varepsilon(t)$ are pointwise convergent to $1-b(t)$, which is a continuous function on $\mathbb{R}\cup\{-\infty\}$), when $\varepsilon \to 0$ and $0 \le v'_\varepsilon(t) \le 1$ for any $t \in \mathbb{R}$.

One can construct the family $\{v_\varepsilon\}_{\varepsilon\in(0,\frac{1}{8}B)}$ by setting

$$\begin{aligned} v_\varepsilon(t) :&= \int_{-\infty}^{t}\Big(\int_{-\infty}^{t_1}\Big(\frac{1}{B-4\varepsilon}\mathbb{1}_{\{-t_0-B+2\varepsilon,-t_0-2\varepsilon\}} * \rho_{\frac{1}{4}\varepsilon}\Big)(s)\mathrm{d}s\Big)\mathrm{d}t_1 \\ &\quad - \int_{-\infty}^{0}\Big(\int_{-\infty}^{t_1}\Big(\frac{1}{B-4\varepsilon}\mathbb{1}_{\{-t_0-B+2\varepsilon,-t_0-2\varepsilon\}} * \rho_{\frac{1}{4}\varepsilon}\Big)(s)\mathrm{d}s\Big)\mathrm{d}t_1, \end{aligned} \tag{4.3}$$

where $\rho_{\frac{1}{4}\varepsilon}$ is the kernel of convolution satisfying $\mathrm{Supp}(\rho_{\frac{1}{4}\varepsilon}) \subset \big(-\frac{1}{4}\varepsilon, \frac{1}{4}\varepsilon\big)$. Then it follows that

$$v''_\varepsilon(t) = \frac{1}{B-4\varepsilon}\mathbb{1}_{\{-t_0-B+2\varepsilon,-t_0-2\varepsilon\}} * \rho_{\frac{1}{4}\varepsilon}(t)$$

and

$$v'_\varepsilon(t) = \int_{-\infty}^{t}\Big(\frac{1}{B-4\varepsilon}\mathbb{1}_{\{-t_0-B+2\varepsilon,-t_0-2\varepsilon\}} * \rho_{\frac{1}{4}\varepsilon}\Big)(s)\mathrm{d}s.$$

It suffices to consider the case that

$$\int_D \frac{1}{B}\mathbb{1}_{\{-t_0-B<\psi<-t_0\}}|F|^2 \mathrm{e}^{-\psi-\varphi}\mathrm{d}\lambda_n < +\infty. \tag{4.4}$$

Let $\eta = s(-v_\varepsilon(\psi_m))$ and $\phi = u(-v_\varepsilon(\psi_m))$, where $0 \le s \in C^\infty((0,+\infty))$ and $u \in C^\infty((0,+\infty))$ with $\lim\limits_{t\to+\infty} u(t) = 0$ satisfy $u''s - s'' > 0$ and $s' - u's = 1$. It follows from $\sup\limits_m \sum\limits_D \psi_m < 0$

that $\phi = u(-v_\varepsilon(\psi_m))$ are uniformly bounded on D with respect to m and ε, and $u(-v_\varepsilon(\psi))$ are uniformly bounded on D with respect to ε.

Now, put $\Phi = \phi + \varphi_{m'}$ and let $\alpha = \sum\limits_{j=1}^{n} \alpha_j \mathrm{d}\overline{z}^j \in \mathrm{Dom}_D(\overline{\partial}^*) \cap \mathrm{Ker}(\overline{\partial}) \cap C^\infty_{(0,1)}(\overline{D})$. By Cauchy-Schwarz inequality, it follows that

$$2\mathrm{Re}(\overline{\partial}^*_\Phi \alpha, \alpha \llcorner (\overline{\partial}\eta)^\sharp)_{D,\Phi} \geq -\int_D g^{-1} |\overline{\partial}^*_\Phi \alpha|^2 \mathrm{e}^{-\Phi} \mathrm{d}\lambda_n + \sum_{j,k=1}^{n} \int_D (-g(\partial_j \eta)\overline{\partial}_k \eta) \alpha_{\overline{j}} \overline{\alpha_{\overline{k}}} \mathrm{e}^{-\Phi} \mathrm{d}\lambda_n. \tag{4.5}$$

Using Lemma 4.1 and inequality (4.5), since $s \geq 0$ and ψ_m is a plurisubharmonic function on $\overline{D}_v$, we get

$$\begin{aligned}\int_D (\eta + g^{-1}) |\overline{\partial}^*_\Phi \alpha|^2 \mathrm{e}^{-\Phi} \mathrm{d}\lambda_n &\geq \sum_{j,k=1}^{n} \int_D (-\partial_j \overline{\partial}_k \eta + \eta \partial_j \overline{\partial}_k \Phi - g(\partial_j \eta)\overline{\partial}_k \eta) \alpha_{\overline{j}} \overline{\alpha_{\overline{k}}} \mathrm{e}^{-\Phi} \mathrm{d}\lambda_n \\ &\geq \sum_{j,k=1}^{n} \int_D (-\partial_j \overline{\partial}_k \eta + \eta \partial_j \overline{\partial}_k \phi - g(\partial_j \eta)\overline{\partial}_k \eta) \alpha_{\overline{j}} \overline{\alpha_{\overline{k}}} \mathrm{e}^{-\Phi} \mathrm{d}\lambda_n, \end{aligned} \tag{4.6}$$

where g is a positive continuous function on D. Next, we need some calculations to determine g.

Since

$$\partial_j \overline{\partial}_k \eta = -s'(-v_\varepsilon(\psi_m)) \partial_j \overline{\partial}_k (v_\varepsilon(\psi_m)) + s''(-v_\varepsilon(\psi_m)) \partial_j v_\varepsilon(\psi_m) \overline{\partial}_k v_\varepsilon(\psi_m) \tag{4.7}$$

and

$$\partial_j \overline{\partial}_k \phi = -u'(-v_\varepsilon(\psi_m)) \partial_j \overline{\partial}_k v_\varepsilon(\psi_m) + u''(-v_\varepsilon(\psi_m)) \partial_j v_\varepsilon(\psi_m) \overline{\partial}_k v_\varepsilon(\psi_m) \tag{4.8}$$

for any $1 \leq j, k \leq n$, we have

$$\begin{aligned} &\sum_{1\leq j,k\leq n} (-\partial_j \overline{\partial}_k \eta + \eta \partial_j \overline{\partial}_k \phi - g(\partial_j \eta)\overline{\partial}_k \eta) \alpha_{\overline{j}} \overline{\alpha_{\overline{k}}} \\ &= (s' - su') \sum_{1\leq j,k\leq n} \partial_j \overline{\partial}_k v_\varepsilon(\psi_m) \alpha_{\overline{j}} \overline{\alpha_{\overline{k}}} \\ &\quad + ((u''s - s'') - gs'^2) \sum_{1\leq j,k\leq n} \partial_j (-v_\varepsilon(\psi_m)) \overline{\partial}_k (-v_\varepsilon(\psi_m)) \alpha_{\overline{j}} \overline{\alpha_{\overline{k}}} \\ &= (s' - su') \sum_{1\leq j,k\leq n} (v'_\varepsilon(\psi_m) \partial_j \overline{\partial}_k \psi_m + v''_\varepsilon(\psi_m) \partial_j(\psi_m) \overline{\partial}_k(\psi_m)) \alpha_{\overline{j}} \overline{\alpha_{\overline{k}}} \\ &\quad + ((u''s - s'') - gs'^2) \sum_{1\leq j,k\leq n} \partial_j (-v_\varepsilon(\psi_m)) \overline{\partial}_k (-v_\varepsilon(\psi_m)) \alpha_{\overline{j}} \overline{\alpha_{\overline{k}}}. \end{aligned} \tag{4.9}$$

We omit composite item $-v_\varepsilon(\psi_m)$ after $s' - su'$ and $(u''s - s'') - gs'^2$ in the above equalities.

Let $g = \frac{u''s - s''}{s'^2}(-v_\varepsilon(\psi_m))$. It follows that $\eta + g^{-1} = \big(s + \frac{s'^2}{u''s - s''}\big)(-v_\varepsilon(\psi_m))$. By inequalities (4.6), we infer from $v'_\varepsilon \geq 0$ and $s' - su' = 1$ that

$$\int_D (\eta + g^{-1}) |\overline{\partial}^*_\Phi \alpha|^2 \mathrm{e}^{-\Phi} \mathrm{d}\lambda_n \geq \int_D (v''_\varepsilon \circ \psi_m) |\alpha \llcorner (\overline{\partial}\psi_m)^\sharp|^2 \mathrm{e}^{-\Phi} \mathrm{d}\lambda_n. \tag{4.10}$$

As F is holomorphic on $\{\psi < -t_0\} \supset\supset \mathrm{Supp}(v'_\varepsilon(\psi_m))$, then $\lambda := \overline{\partial}[(1 - v'_\varepsilon(\psi_m))F]$ is well-defined and smooth on D. Combining the definition of contraction with Cauchy-Schwarz inequality and inequality (4.10), it follows that

$$\begin{aligned}|(\lambda, \alpha)_{D,\Phi}|^2 &= |(v''_\varepsilon(\psi_m)\overline{\partial}\psi_m F, \alpha)_{D,\Phi}|^2 \\ &= |(v''_\varepsilon(\psi_m)F, \alpha \llcorner (\overline{\partial}\psi_m)^\sharp)_{D,\Phi}|^2 \\ &\le \int_D v''_\varepsilon(\psi_m)|F|^2 \mathrm{e}^{-\Phi} \mathrm{d}\lambda_n \int_D v''_\varepsilon(\psi_m)|\alpha \llcorner (\overline{\partial}\psi_m)^\sharp|^2 \mathrm{e}^{-\Phi} \mathrm{d}\lambda_n \\ &\le \Big(\int_D v''_\varepsilon(\psi_m)|F|^2 \mathrm{e}^{-\Phi} \mathrm{d}\lambda_n\Big)\Big(\int_D (\eta + g^{-1})|\overline{\partial}^*_\Phi \alpha|^2 \mathrm{e}^{-\Phi} \mathrm{d}\lambda_n\Big). \end{aligned} \tag{4.11}$$

Let $\mu := (\eta + g^{-1})^{-1}$. Using Lemma 4.2, we have locally L^1 functions $u_{m,m',\varepsilon}$ on D such that $\overline{\partial} u_{m,m',\varepsilon} = \lambda$ and

$$\int_D |u_{m,m',\varepsilon}|^2 (\eta + g^{-1})^{-1} \mathrm{e}^{-\Phi} \mathrm{d}\lambda_n \le \int_D (v''_\varepsilon(\psi_m))|F|^2 \mathrm{e}^{-\Phi} \mathrm{d}\lambda_n. \tag{4.12}$$

Assume that we can choose η and ϕ such that $\mathrm{e}^{v_\varepsilon \circ \psi_m} \mathrm{e}^{\phi} = (\eta + g^{-1})^{-1}$. Then inequality (4.12) becomes

$$\int_D |u_{m,m',\varepsilon}|^2 \mathrm{e}^{v_\varepsilon(\psi_m) - \varphi_{m'}} \mathrm{d}\lambda_n \le \int_D v''_\varepsilon(\psi_m)|F|^2 \mathrm{e}^{-\phi - \varphi_{m'}} \mathrm{d}\lambda_n. \tag{4.13}$$

Let $F_{m,m',\varepsilon} := -u_{m,m',\varepsilon} + (1 - v'_\varepsilon(\psi_m))F$. Then inequality (4.13) becomes

$$\begin{aligned}&\int_D |F_{m,m',\varepsilon} - (1 - v'_\varepsilon(\psi_m))F|^2 \mathrm{e}^{v_\varepsilon(\psi_m) - \varphi_{m'}} \mathrm{d}\lambda_n \\ &\le \int_D (v''_\varepsilon(\psi_m))|F|^2 \mathrm{e}^{-\phi - \varphi_{m'}} \mathrm{d}\lambda_n. \end{aligned} \tag{4.14}$$

Considering a compactly convergent subsequence of $F_{m,m',\varepsilon}$ (also denoted by $F_{m,m',\varepsilon}$), and taking limits

$$\lim_{m' \to +\infty} \lim_{\varepsilon \to 0+0} \lim_{m \to +\infty} F_{m,m',\varepsilon}$$

(denoted by $\widetilde{F}$), one can obtain

$$\int_D |\widetilde{F} - b(\psi)F|^2 \mathrm{e}^{v(\psi) - \varphi} \mathrm{d}\lambda_n \le \Big(\sup_D \mathrm{e}^{-u(-v(\psi))}\Big) C_1. \tag{4.15}$$

4.3 ODE system

It suffices to find η and ϕ such that $(\eta + g^{-1}) = \mathrm{e}^{-\psi_m} \mathrm{e}^{-\phi}$ on D. As $\eta = s(-v_\varepsilon(\psi_m))$ and $\phi = u(-v_\varepsilon(\psi_m))$, we have $(\eta + g^{-1})\mathrm{e}^{v_\varepsilon(\psi_m)} \mathrm{e}^{\phi} = (s + \frac{s'^2}{u''s - s''})\mathrm{e}^{-t} \mathrm{e}^{u} \circ (-v_\varepsilon(\psi_m))$.

Summarizing the above discussion about s and u, we are naturally led to a system of ODEs (see [9–12]):

$$\begin{aligned}&\Big(s + \frac{s'^2}{u''s - s''}\Big)\mathrm{e}^{u-t} = 1, \\ &s' - su' = 1, \end{aligned} \tag{4.16}$$

where $t \in [0, +\infty)$ and $\mathbf{C} = 1$.

It is not hard to solve the ODE system (4.16) and get $u = -\log(1 - \mathrm{e}^{-t})$ and $s = \frac{t}{1-\mathrm{e}^{-t}} - 1$. It follows that $s \in C^\infty((0, +\infty))$ satisfies $s \geq 0$, $\lim\limits_{t\to+\infty} u(t) = 0$ and $u \in C^\infty((0, +\infty))$ satisfies $u''s - s'' > 0$.

As $u = -\log(1 - \mathrm{e}^{-t})$ is decreasing with respect to t, then it follows from $0 \geq v(t) \geq \max\{t, -t_0 - B_0\} \geq -t_0 - B_0$ for any $t \leq 0$ that

$$\sup_D \mathrm{e}^{-u(-v(\psi))} \leq \sup_{t\in(0,t_0+B]} \mathrm{e}^{-u(t)} = \sup_{t\in(0,t_0+B]} (1 - \mathrm{e}^{-t}) = 1 - \mathrm{e}^{-(t_0+B)}. \tag{4.17}$$

Thus, we conclude the proof of Lemma 2.2.

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