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Darboux Transformation and Exact Solutions for Two Dimensional $A^{(2)}_{2n-1}$ Toda Equations*

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Abstract The Darboux transformation for the two dimensional $A^{(2)}_{2n-1}$ Toda equations is constructed so that it preserves all the symmetries of the corresponding Lax pair. The expression of exact solutions of the equation is obtained by using Darboux transformation.

Keywords Two dimensional affine Toda equation, Exact solutions, Darboux transformation
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1 Introduction

For any finite dimensional simple Lie algebra or any affine Kac-Moody algebra, there is a two dimensional Toda equation (see [14]). All of them are integrable systems. These equations have been studied by various methods such as inverse scattering, Hirota method, Darboux transformation etc. (see [1–3, 6, 11–12, 15–16]). They have also important applications in Toda field theory (see [4, 7]) and in both Riemannian and affine geometry (see [8–9, 13, 20–22]).

For any affine Kac-Moody algebra g, the two dimensional Toda equation is of form (see [13])

$$w_{j,xt} = \exp\Big(\sum_{i=1}^{n} c_{ji} w_i\Big) - v_j \exp\Big(\sum_{i=1}^{n} c_{0i} w_i\Big), \quad j = 1, \cdots, n, \tag{1.1}$$

where $C = (c_{ij})_{0\le i,j\le n}$ is the generalized Cartan matrix of the Kac-Moody algebra g and $v = (v_0, v_1, \cdots, v_n)^T$ satisfies $Cv = 0$.

When $g = A^{(2)}_{2n-1}$ ($n \ge 3$), the generalized Cartan matrix is

$$C = (c_{ij})_{0\le i,j\le n} = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & & & \\ 0 & 2 & -1 & 0 & 0 & & & \\ -1 & -1 & 2 & -1 & 0 & & & \\ 0 & 0 & -1 & 2 & -1 & & & \\ & & & \ddots & \ddots & \ddots & & \\ & & & & -1 & 2 & -1 & 0 \\ & & & & 0 & -1 & 2 & -1 \\ & & & & 0 & 0 & -2 & 2 \end{pmatrix} \tag{1.2}$$

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and the corresponding two dimensional affine Toda equation is

$$\begin{aligned} w_{1,xt} &= \mathrm{e}^{2w_1-w_2} - \mathrm{e}^{-w_2}, \\ w_{j,xt} &= \mathrm{e}^{-w_{j-1}+2w_j-w_{j+1}} - 2\mathrm{e}^{-w_2}, \quad j=2,\cdots,n-1, \\ w_{n,xt} &= \mathrm{e}^{-2w_{n-1}+2w_n} - 2\mathrm{e}^{-w_2}. \end{aligned} \tag{1.3}$$

By setting $w_j = -(u_1+u_2+\cdots+u_j)$ $(j=1,\cdots,n)$, it becomes

$$\begin{aligned} u_{1,xt} &= \mathrm{e}^{u_2+u_1} - \mathrm{e}^{u_2-u_1}, \quad u_{2,xt} = \mathrm{e}^{u_2+u_1} + \mathrm{e}^{u_2-u_1} - \mathrm{e}^{u_3-u_2}, \\ u_{j,xt} &= \mathrm{e}^{u_j-u_{j-1}} - \mathrm{e}^{u_{j+1}-u_j}, \quad j=3,\cdots,n-1, \\ u_{n,xt} &= \mathrm{e}^{u_n-u_{n-1}} - \mathrm{e}^{-2u_n}. \end{aligned} \tag{1.4}$$

It was known that all the equations in (1.1) are integrable (see [13]). For $g = A_n^{(1)}$, its Lax pair has a unitary symmetry and a cyclic symmetry of order n. There is a Darboux transformation of degree one which keeps its symmetries and has many geometric applications (see [9, 11–12]).

There are also some work for the two dimensional Toda equations with other Kac-Moody algebras (see [2–3, 10, 17]), although the symmetries are more complicated. Each of these equations has a unitary symmetry, a reality symmetry and a cyclic symmetry.

In the above systems, the number of independent functions is greatly less than the entries in the coefficient matrices of the Lax pair. This leads to the difficulty in getting explicit solutions. For the two dimensional Toda equations with $g = A_n^{(1)}$, $A_{2n}^{(2)}$ and $A_{2n-1}^{(2)}$, the Darboux transformation for complex solutions was presented in [18–19]. For the two dimensional (real) affine Toda equations with $g = A_{2n}^{(2)}$, $C_n^{(1)}$ and $D_{l+1}^{(2)}$, the binary Darboux transformation (in integral form) was given by [17] and the Darboux transformation (in differential form) was given by [24–25].

In this paper, we present the construction of Darboux transformation $V \to \widetilde{V}$ for the two dimensional $A_{2n-1}^{(2)}$ Toda equation and obtain its explicit real solutions. Here the diagonal matrix V contains the potentials. The degree of Darboux transformation must be high enough to keep all the symmetries of the Lax pair. For the two dimensional Toda equations with $g = A_{2n}^{(2)}$, $C_n^{(1)}$ and $D_{n+1}^{(2)}$, the order of the cyclic symmetry is the same as the order of the matrices. The symmetries guarantee that $\widetilde{V}$ is still diagonal. However, for the two dimensional $A_{2n-1}^{(2)}$ Toda equation, the order of the cyclic symmetry is strictly less than the order of the matrices. This leads to the fact that the form of the coefficients of the Lax pair cannot be fully determined by the above mentioned symmetries. In fact, the symmetries only guarantee that $\widetilde{V}$ is block-diagonal. Therefore, an extra constant matrix should be multiplied before the standard Darboux matrix so that $\widetilde{V}$ is diagonal and is in the same form as V.

In Section 2, the symmetries for the Lax pair of Toda equations are discussed. In Section 3, the general construction of Darboux transformation is reviewed. In Section 4, the Darboux transformation for the two dimensional $A_{2n-1}^{(2)}$ Toda equation is constructed. Explicit expression for the new solutions is given.

2 Two Dimensional $A_{2n-1}^{(2)}$ Toda Equation and Its Lax Pair

Each two dimensional affine Toda equation has the Lax pair

$$\Phi_x = (\lambda J + P)\Phi, \quad \Phi_t = \frac{1}{\lambda} Q\Phi, \tag{2.1}$$

where

$$P = V_x V^{-1}, \quad Q = VJ^TV^{-1}, \tag{2.2}$$

J is a constant $L\times L$ diagonal matrix, $V(x,t)$ is an $L\times L$ matrix-function containing potentials.

The integrability condition of (2.1) is

$$P_t + [J,Q] = 0, \tag{2.3}$$

or equivalently,

$$(V_xV^{-1})_t + [J, VJ^TV^{-1}] = 0. \tag{2.4}$$

With specific symmetries on J and V, (2.4) contains all two dimensional affine Toda equations.

For the two dimensional $A^{(2)}_{2n-1}$ Toda equation, $L = 2n$. Moreover, each $2n\times 2n$ matrix M is written as an $(2n-1)\times(2n-1)$ block matrix $(M_{jk})_{1\le j,k\le 2n-1}$ so that M_{11} is a 2×2 matrix. The matrices corresponding to the two dimensional $A^{(2)}_{2n-1}$ Toda equation are $J = (J_{jk})_{1\le j,k\le 2n-1}$ with non-zero entries $J_{12} = \begin{pmatrix}1\\1\end{pmatrix}$, $J_{2n-1,1} = \begin{pmatrix}1 & 1\end{pmatrix}$ and $J_{j,j+1} = 1$ for $j = 2,\cdots,2n-2$; $V = (V_{jk})_{1\le j,k\le 2n-1}$ with non-zero entries $V_{11} = \begin{pmatrix}\mathrm{e}^{u_1} & \\ & \mathrm{e}^{-u_1}\end{pmatrix}$, $V_{jj} = \mathrm{e}^{u_j}$ and $V_{2n+1-j,2n+1-j} = \mathrm{e}^{-u_j}$ for $j = 2,\cdots,n$. Written explicitly, they are

$$J = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1\\ 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \tag{2.5}$$

$$V = \operatorname{diag}(\mathrm{e}^{u_1}, \mathrm{e}^{-u_1}, \mathrm{e}^{u_2}, \mathrm{e}^{u_3}, \cdots, \mathrm{e}^{u_n}, \mathrm{e}^{-u_n}, \cdots, \mathrm{e}^{-u_2}). \tag{2.6}$$

For any $(2n-1)\times(2n-1)$ block matrix A or any $(2n-1)\times 1$ block vector v divided as above, and for any integers j and k, define $A_{jk} = A_{j'k'}$ and $v_j = v_{k'}$ when $j\equiv j' \bmod (2n-1)$, $k\equiv k' \bmod (2n-1)$. Especially, δ_{jk} equals 1 if $j\equiv k \bmod (2n-1)$ and equals 0 otherwise.

Let $\omega = \exp\left(\frac{2\pi\mathrm{i}}{2n-1}\right)$, $\Omega = \operatorname{diag}\left(\begin{pmatrix}1 & \\ & 1\end{pmatrix}, \omega^{-1}, \cdots, \omega^{-2n}\right)$ where "diag" refers to block-diagonal matrix. Clearly, $\Omega^{2n-1} = I$.

Let $K = (K_{jk})_{1\le j,k\le 2n-1}$ with non-zero entries $K_{11} = \begin{pmatrix} & 1\\ 1 & \end{pmatrix}$ and $K_{j,2n+1-j} = 1$ for $j = 2,\cdots,2n-1$. Then

$$K^2 = I, \quad \Omega K\Omega = K \tag{2.7}$$

and

$$\Omega J\Omega^{-1} = \omega J, \quad KJK = J^T, \quad \Omega V\Omega^{-1} = V, \quad V^TKV = K. \tag{2.8}$$

By direct computation, we have the following lemma.

Lemma 2.1 *If $V(x,t)$ satisfies*

$$\overline{V}=V,\quad \Omega V\Omega^{-1}=V,\quad V^TKV=K, \tag{2.9}$$

then $P=V_xV^{-1}$ and $Q=VJ^TV^{-1}$ satisfy

$$\begin{aligned}&\overline{P}=P,\quad \Omega P\Omega^{-1}=P,\quad KPK=-P^T,\\&\overline{Q}=Q,\quad \Omega Q\Omega^{-1}=\omega^{-1}Q,\quad KQK=Q^T.\end{aligned} \tag{2.10}$$

The symmetries of the Lax pair lead to the symmetries of its solutions. This fact is shown in the following lemma and will be used in constructing Darboux transformations.

Lemma 2.2 *Suppose V satisfies* (2.9), *Φ is a solution of the Lax pair* (2.1) *with $\lambda=\lambda_0$, then*

(i) *$\Omega\Phi$ is a solution of* (2.1) *with $\lambda=\omega\lambda_0$.*

(ii) *$\overline{\Phi}$ is a solution of* (2.1) *with $\lambda=\overline{\lambda}_0$.*

(iii) *$\Psi=K\Phi$ is a solution of the adjoint Lax pair*

$$\Psi_x=-(\lambda J^T+P^T)\Psi,\quad \Psi_t=-\frac{1}{\lambda}Q^T\Psi \tag{2.11}$$

with $\lambda=-\lambda_0$.

(iv) *For any solution F of* (2.1) *and any solution Ψ of* (2.11), *$(\Psi^TF)_x=0$, $(\Psi^TF)_t=0$.*

3 General Construction of Darboux Matrix

A matrix

$$G(x,t,\lambda)=\sum_{j=0}^{r}G_j(x,t)\lambda^{r-j}, \tag{3.1}$$

where $G_0,\cdots,G_r$ are $L\times L$ matrices, is called a Darboux matrix of degree r for the Lax pair (2.1) if there exists a diagonal matrix $\widetilde{V}(x,t)$ satisfying

$$\overline{\widetilde{V}}=\widetilde{V},\quad \Omega\widetilde{V}\Omega^{-1}=\widetilde{V},\quad \widetilde{V}^TK\widetilde{V}=K, \tag{3.2}$$

such that for any solution Φ of (2.1), $\widetilde{\Phi}=G\Phi$ satisfies

$$\widetilde{\Phi}_x=(\lambda J+\widetilde{P})\widetilde{\Phi},\quad \widetilde{\Phi}_t=\frac{1}{\lambda}\widetilde{Q}\widetilde{\Phi} \tag{3.3}$$

with

$$\widetilde{P}=\widetilde{V}_x\widetilde{V}^{-1},\quad \widetilde{Q}=\widetilde{V}J^T\widetilde{V}^{-1}. \tag{3.4}$$

If a Darboux matrix is constructed, new solutions of the two dimensional affine Toda equations can be obtained from a known one.

When the symmetries in (3.2) are not considered, the Darboux transformation can be constructed by generalizing the method in [23] (see also [5, 8]) as follows.

Let $\lambda_1, \cdots, \lambda_r, \widehat{\lambda}_1, \cdots, \widehat{\lambda}_r$ be distinct complex constants. Let H_j be a column solution of the Lax pair (2.1) with $\lambda = \lambda_j$, $\widehat{H}_j$ be a column solution of the adjoint Lax pair (2.11) with $\lambda = \widehat{\lambda}_j$. Let E be an $L \times L$ real constant invertible matrix satisfying

$$EJ = JE, \quad E\Omega = \Omega E, \quad E^T KE = K. \tag{3.5}$$

Let

$$\Gamma_{jk} = \frac{\widehat{H}_j^T H_k}{\lambda_k - \widehat{\lambda}_j}, \quad j, k = 1, \cdots, r, \quad \Gamma = (\Gamma_{jk})_{1 \le j,k \le r}, \quad \check{\Gamma} = \Gamma^{-1}. \tag{3.6}$$

Then

$$G(\lambda) = \prod_{l=1}^{r} (\lambda - \widehat{\lambda}_l) E \Big(I - \sum_{j,k=1}^{r} \frac{H_j \check{\Gamma}_{jk} \widehat{H}_k^T}{\lambda - \widehat{\lambda}_k} \Big) \tag{3.7}$$

is a Darboux matrix without considering symmetries. In fact we have $G(\lambda_j)H_j = 0$, $G(\widehat{\lambda}_j)\Psi_j = 0$ for any solution Ψ_j of (2.1) with $\lambda = \widehat{\lambda}_j$ satisfying $\widehat{H}_j^T \Psi_j = 0$. ((iv) of Lemma 2.2 guarantees that $\widehat{H}_j^T \Psi_j = 0$ holds identically if it holds at one point.) It can be checked directly that

$$G(\lambda)^{-1} = \prod_{l=1}^{r} (\lambda - \widehat{\lambda}_l)^{-1} \Big(I + \sum_{j,k=1}^{r} \frac{H_j \check{\Gamma}_{jk} \widehat{H}_k^T}{\lambda - \lambda_j} \Big) E^{-1}. \tag{3.8}$$

Since $G(\lambda)$ is a polynomial of λ of degree r, we can write

$$G(\lambda) = \lambda^r E + \lambda^{r-1} G_1 + \cdots + \lambda G_{r-1} + G_r. \tag{3.9}$$

From (3.7),

$$\begin{aligned} G_1 &= -\Big(\sum_{l=1}^{r} \widehat{\lambda}_l \Big) E - \sum_{j,k=1}^{r} E H_j \check{\Gamma}_{jk} \widehat{H}_k^T, \\ G_r &= (-1)^r \Big(\prod_{l=1}^{r} \widehat{\lambda}_l \Big) E \Big(I + \sum_{j,k=1}^{r} \frac{H_j \check{\Gamma}_{jk} \widehat{H}_k^T}{\widehat{\lambda}_k} \Big). \end{aligned} \tag{3.10}$$

Lemma 3.1 *Suppose $G(\lambda)$ is given by* (3.7), *then* (3.3) *holds where*

$$\widetilde{P} = EPE^{-1} - [J, G_1]E^{-1}, \quad \widetilde{Q} = G_r Q G_r^{-1}. \tag{3.11}$$

Moreover, if $\widetilde{V}$ satisfies

$$\widetilde{V} = G_r V, \tag{3.12}$$

then $\widetilde{P} = \widetilde{V}_x \widetilde{V}^{-1}$ and $\widetilde{Q} = \widetilde{V} J^T \widetilde{V}^{-1}$ satisfy (3.11).

Proof (3.6) leads to

$$\Gamma_{jk,x} = \widehat{H}_j^T J H_k, \quad \Gamma_{jk,t} = -\frac{1}{\widehat{\lambda}_j \lambda_k} \widehat{H}_j^T Q H_k. \tag{3.13}$$

By the definition (3.11), it can be verified by direct computation that

$$
\begin{aligned}
&(\lambda J+\widetilde{P})G-G(\lambda J+P)-G_x=0,\\
&\lambda^{-1}\widetilde{Q}G-\lambda^{-1}GQ-G_t=0
\end{aligned}
\tag{3.14}
$$

hold for all λ, which are equivalent to (3.3).

Clearly, $\widetilde{V}J^T\widetilde{V}^{-1}=G_rQG_r^{-1}$ is true. It is only necessary to prove that $\widetilde{V}_x\widetilde{V}^{-1}=EPE^{-1}-[J,G_1]E^{-1}$ holds. Taking $\lambda=0$ in

$$
G_x+G(\lambda J+P)=(\lambda J+EPE^{-1}-[J,G_1]E^{-1})G,
\tag{3.15}
$$

which is the first equation of (3.14), we get

$$
G_{r,x}+G_rP=(EPE^{-1}-[J,G_1]E^{-1})G_r.
\tag{3.16}
$$

On the other hand,

$$
G_{r,x}+G_rP=\widetilde{V}_x\widetilde{V}^{-1}G_r
\tag{3.17}
$$

holds by differentiating (3.12). Hence $\widetilde{V}_x\widetilde{V}^{-1}=EPE^{-1}-[J,G_1]E^{-1}$.

4 Darboux Matrix for Two Dimensional $A^{(2)}_{2n-1}$ Toda Equation

Instead of the Darboux transformation with all complex spectral parameters, here we use one real spectral parameter, which reduces the degree of Darboux transformation from $4n-2$ to $2n-1$.

Let $r=2n-1$. Let $\mu\in\mathbf{R}\backslash\{0\}$, H be a real column solution of the Lax pair (2.1) with $\lambda=\mu$. Let $\lambda_j=\omega^{j-1}\mu$, $\widehat{\lambda}_j=-\overline{\lambda}_j$ $(j=1,\cdots,2n-1)$, then Lemma 2.2 implies that $H_j=\Omega^{j-1}H$ is a solution of the Lax pair (2.1) with $\lambda=\lambda_j$, and $\widehat{H}_j=K\overline{H}_j$ is a solution of the adjoint Lax pair (2.11) with $\lambda=-\overline{\lambda}_j$.

By (3.6)–(3.7),

$$
\Gamma_{jk}=\frac{H_j^*KH_k}{\lambda_k+\overline{\lambda}_j},\quad j,k=1,\cdots,2n-1,
\tag{4.1}
$$

$$
G(\lambda)=\prod_{l=1}^{2n-1}(\lambda+\overline{\lambda}_l)E\Big(I-\sum_{j,k=1}^{2n-1}\frac{H_j\check{\Gamma}_{jk}H_k^*K}{\lambda+\overline{\lambda}_k}\Big).
\tag{4.2}
$$

Lemma 4.1 *$G(\lambda)$ satisfies*

$$
\overline{G(\overline{\lambda})}=G(\lambda),
\tag{4.3}
$$

$$
\Omega G(\lambda)\Omega^{-1}=G(\omega\lambda),
\tag{4.4}
$$

$$
G(-\overline{\lambda})^*KG(\lambda)=(\mu^{4n-2}-\lambda^{4n-2})K.
\tag{4.5}
$$

Therefore, $\widetilde{V}=-\mu^{-2n+1}G(0)V$ *satisfies*

$$
\overline{\widetilde{V}}=\widetilde{V},\quad \Omega\widetilde{V}\Omega^{-1}=\widetilde{V},\quad \widetilde{V}^TK\widetilde{V}=K.
\tag{4.6}
$$

Proof From $\overline{\lambda}_k = \lambda_{2-k}$, $\overline{H}_k = H_{2-k}$, we have $\overline{\Gamma}_{jk} = \Gamma_{2-j,2-k}$. Let $C_1 = (\delta_{j,2-k})_{1\le j,k\le 2n-1}$, then $\overline{\Gamma} = C_1\Gamma C_1^{-1}$, which leads to $\overline{\check{\Gamma}} = C_1\check{\Gamma}C_1^{-1}$, i.e., $\overline{\check{\Gamma}}_{jk} = \check{\Gamma}_{2-j,2-k}$. Then (4.3) is derived by direct computation.

Likewise, $\omega\lambda_k = \lambda_{k+1}$ and $\Omega H_k = H_{k+1}$ lead to $\Gamma_{j+1,k-1} = \omega\Gamma_{jk}$, which is equivalent to $C_2\Gamma C_2 = \omega\Gamma$ where $C_2 = (\delta_{j,k-1})_{1\le j,k\le 2n-1}$. Hence $C_2\check{\Gamma}C_2 = \omega\check{\Gamma}$, i.e., $\check{\Gamma}_{j+1,k-1} = \omega\check{\Gamma}_{jk}$. This leads to (4.4) by using (3.5).

From $\Gamma^* = \Gamma$ we have $\check{\Gamma}^* = \check{\Gamma}$, which gives (4.5).

Finally, (4.3)–(4.5) lead to

$$\overline{G(0)} = G(0), \quad \Omega G(0)\Omega^{-1} = G(0), \quad G(0)^*KG(0) = \mu^{4n-2}K. \tag{4.7}$$

Hence $\widetilde{V}$ satisfies the symmetries (4.6) as V does.

Remark 4.1 The symmetries (4.6) guarantee that $\widetilde{V}$ is of form

$$\widetilde{V} = \operatorname{diag}(\widetilde{V}_{11}, \widetilde{V}_{22}, \cdots, \widetilde{V}_{2n-1,2n-1}) \tag{4.8}$$

such that $\widetilde{V}_{jj} = \mathrm{e}^{\widetilde{u}_j}$, $\widetilde{V}_{2n+1-j,2n+1-j} = \mathrm{e}^{-\widetilde{u}_j}$ $(j = 2, \cdots, n)$. However, it only implies that $\widetilde{V}_{11}$ is a 2×2 matrix. In order to get a solution of the Toda equation (1.4), we need $\widetilde{V}_{11} = \begin{pmatrix} \mathrm{e}^{\widetilde{u}_1} & \\ & \mathrm{e}^{-\widetilde{u}_1} \end{pmatrix}$ for some function $\widetilde{u}_1$ as in (2.6). The following theorem will show that this will be realized if E is chosen appropriately.

Theorem 4.1 *Suppose $(u_1, \cdots, u_n)$ is a solution of the two dimensional $A^{(2)}_{2n-1}$ Toda equation (1.4). Let $E = \operatorname{diag}(E_1, 1, 1, \cdots, 1)$ with $E_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Let μ be a positive constant, and H be a real column solution of the Lax pair (2.1) with $\lambda = \mu$. Write $H = \begin{pmatrix} h_1 \\ \vdots \\ h_{2n-1} \end{pmatrix}$, $h_1 = \begin{pmatrix} h_{11} \\ h_{12} \end{pmatrix}$, where $h_{11}, h_{12}, h_2, h_3, \cdots, h_{2n-1}$ are real scalar functions. Then $G(\lambda)$ given by (3.7) is a Darboux transformation for the Lax pair (2.1) so that $\widetilde{V} = G_rV$ is of form (4.8). Moreover, let*

$$\alpha_k = \sum_{a=1}^{k}(-1)^{k+a}\Xi_a - \sum_{a=k+1}^{2n-1}(-1)^{k+a}\Xi_a, \quad k = 1, 2, \cdots, 2n-1, \tag{4.9}$$

where $\Xi_1 = 2h_{11}h_{12}$, $\Xi_k = h_kh_{2-k}$ $(k = 2, \cdots, 2n-1)$. Then $(\widetilde{u}_1, \cdots, \widetilde{u}_n)$ where

$$\begin{aligned} \widetilde{u}_1 &= u_1 + \ln\frac{h_{12}}{h_{11}}, \\ \widetilde{u}_k &= u_k + \ln\frac{\alpha_{2n-k}}{\alpha_{2n+1-k}}, \quad k = 2, \cdots, n \end{aligned} \tag{4.10}$$

is a new solution of (1.4) when all α_k's $(k = 1, \cdots, n)$ are positive.

Proof First, E satisfies (3.5). Let $\xi_m = ((\xi_m)_1, \cdots, (\xi_m)_{2n-1})^T$ $(m = 1, \cdots, 2n-1)$ with

$$(\xi_m)_k = (2n-1)^{-\frac{1}{2}}\omega^{-(k-1)(m-1)}, \tag{4.11}$$

then $(\xi_1, \cdots, \xi_{2n-1})$ is an orthonormal basis of $\mathbf{C}^{2n-1}$ since

$$\sum_{j=1}^{2n-1} \omega^{jk} = \begin{cases} 0, & \text{if } k \not\equiv 0 \bmod(2n-1), \\ 2n-1, & \text{if } k \equiv 0 \bmod(2n-1). \end{cases} \tag{4.12}$$

From (4.1),

$$\Gamma_{jk} = \frac{H^T \Omega^{-j+1} K \Omega^{k-1} H}{\omega^{-j+1}\mu + \omega^{k-1}\mu} = \omega^{j-1} \frac{H^T K \Omega^{j+k-2} H}{(1+\omega^{j+k-2})\mu}. \tag{4.13}$$

Write $H = (h_1, \cdots, h_{2n-1})^T$ where $h_2, \cdots, h_{2n-1}$ are scalars and h_1 is a 2×1 column, then

$$\Gamma \xi_m = \frac{2n-1}{2\mu} \alpha_m \xi_{1-m}, \tag{4.14}$$

where

$$\alpha_m = \frac{2\mu}{2n-1} \sum_{j=1}^{2n-1} \frac{H^T K \Omega^j H}{(1+\omega^j)\mu} \omega^{-(m-1)j} = \frac{2\mu}{2n-1} \sum_{j,a=1}^{2n-1} \frac{\omega^{-j(m+a-2)}}{(1+\omega^j)\mu} h_{2-a}^T K_{2-a,a} h_a. \tag{4.15}$$

Hence

$$\Gamma^{-1} \xi_m = \frac{2\mu}{2n-1} \alpha_{1-m}^{-1} \xi_{1-m}, \tag{4.16}$$

i.e.,

$$\check{\Gamma}_{jk} \omega^{-(k-1)(m-1)} = \frac{2\mu}{2n-1} \alpha_{1-m}^{-1} \omega^{m(j-1)}. \tag{4.17}$$

Moreover, for $m = 1, 2, \cdots, 2n-1$,

$$\begin{aligned} \alpha_{1-m} &= \frac{2\mu}{2n-1} \sum_{j,a=1}^{2n-1} \frac{\omega^{-j(a-m-1)}}{(1+\omega^j)\mu} h_{2-a}^T K_{2-a,a} h_a \\ &\xlongequal{j\to -j,\, a\to 2-a} \frac{2\mu}{2n-1} \sum_{j,a=1}^{2n-1} \frac{\omega^{-j(m+a-2)}}{(1+\omega^j)\mu} h_a^T K_{a,2-a} h_{2-a} = \alpha_m. \end{aligned} \tag{4.18}$$

Here the fact $h_a^T K_{a,2-a} h_{2-a} = h_{2-a}^T K_{2-a,a} h_a$ is used since K is symmetric.

With $\Xi_a = h_a^T K_{a,2-a} h_{2-a}$, (4.15) leads to

$$\begin{aligned} \alpha_k &= \frac{2\mu}{2n-1} \lim_{\varepsilon \to 1-0} \sum_{j,a=1}^{2n-1} \frac{\omega^{-j(k-a)}}{(1+\varepsilon\omega^j)\mu} h_a^T K_{a,2-a} h_{2-a} \\ &= \frac{2}{2n-1} \lim_{\varepsilon \to 1-0} \sum_{j,a=1}^{2n-1} \sum_{p=0}^{\infty} \Xi_a (-\varepsilon)^p \omega^{(p-k+a)j}. \end{aligned} \tag{4.19}$$

For a given p, owing to (4.12), the summation on j in the last expression is nonzero only when $p \equiv a - k \bmod(2n-1)$. Hence, let $p = (2n-1)s + k - a$ and use (4.12),

$$\alpha_k = 2 \lim_{\varepsilon \to 1-0} \sum_{a=1}^{2n-1} \sum_{(2n-1)s \geq a-k} \Xi_a (-\varepsilon)^{(2n-1)s+k-a}$$

$$= 2\lim_{\varepsilon\to 1-0}\Big(\sum_{a=1}^{k}\Xi_a\sum_{s=0}^{\infty}(-\varepsilon)^{(2n-1)s+k-a} + \sum_{a=k+1}^{2n-1}\Xi_a\sum_{s=1}^{\infty}(-\varepsilon)^{(2n-1)s+k-a}\Big)$$
$$= 2\lim_{\varepsilon\to 1-0}\Big(\sum_{a=1}^{k}\Xi_a\frac{(-\varepsilon)^{k-a}}{1+\varepsilon^{2n-1}} + \sum_{a=k+1}^{2n-1}\Xi_a\frac{(-\varepsilon)^{k-a+2n-1}}{1+\varepsilon^{2n-1}}\Big)$$
$$= \sum_{a=1}^{k}(-1)^{k+a}\Xi_a - \sum_{a=k+1}^{2n-1}(-1)^{k+a}\Xi_a,\quad k=1,\cdots,2n-1. \tag{4.20}$$

Especially, when $k=1$, we have

$$\sum_{a=2}^{2n-1}(-1)^{1+a}\Xi_a = \sum_{a=2}^{2n-1}(-1)^{1+a}h_ah_{2-a} \xlongequal{a\to 2n+3-a} -\sum_{a=2}^{2n-1}(-1)^{1+a}h_{2-a}h_a, \tag{4.21}$$

which means $\sum\limits_{a=2}^{2n-1}(-1)^{1+a}\Xi_a=0$. Hence

$$\alpha_1 = \Xi_1 = 2h_{11}h_{12}. \tag{4.22}$$

By using (4.17) and the fact $\prod\limits_{l=1}^{2n-1}\omega^{l-1}=\omega^{(n-1)(2n-1)}=1$, (3.7) gives

$$\mu^{-2n+1}(E^{-1}G_{2n-1})_{ab} = \delta_{ab} - \sum_{j,k=1}^{2n-1}\frac{\omega^{-(a-1)(j-1)}h_a\check{\Gamma}_{jk}h^T_{2-b}K_{2-b,b}\omega^{-(b-1)(k-1)}}{\omega^{-k+1}\mu}$$
$$= \delta_{ab} - \frac{2}{2n-1}\sum_{j=1}^{2n-1}\omega^{-(a-1)(j-1)}\alpha^{-1}_{2-b}\omega^{-(1-b)(j-1)}h_ah^T_{2-b}K_{2-b,b}$$
$$= \delta_{ab} - \frac{2}{2n-1}\sum_{j=1}^{2n-1}\alpha^{-1}_{2-b}\omega^{(j-1)(-a+b)}h_ah^T_{2-b}K_{2-b,b}$$
$$= \delta_{ab} - 2\alpha^{-1}_{2-a}h_ah^T_{2-a}K_{2-b,b}\delta_{ab}. \tag{4.23}$$

This implies that $E^{-1}G_{2n-1}$ is a block-diagonal matrix:

$$\mu^{-2n+1}E^{-1}G_{2n-1} = \operatorname{diag}(g_1,g_2,\cdots,g_{2n-1}), \tag{4.24}$$

where

$$g_a = 1 - 2\alpha^{-1}_{2-a}h_ah^T_{2-a}K_{2-a,a}. \tag{4.25}$$

Note that $g_2,\cdots,g_{2n-1}$ are scalars and g_1 is a 2×2 matrix. We should prove that g_1 is diagonal so that V can be diagonal.

For $k=2,\cdots,2n-1$, (4.20) gives

$$\alpha_{2-k}+\alpha_{1-k} = \sum_{a=1}^{2n+1-k}(-1)^{2n+1-k+a}\Xi_a - \sum_{a=2n+2-k}^{2n-1}(-1)^{2n+1-k+a}\Xi_a$$
$$+\sum_{a=1}^{2n-k}(-1)^{2n-k+a}\Xi_a - \sum_{a=1}^{2n-1}(-1)^{2n-k+a}\Xi_a = 2\Xi_{2n+1-k} = 2\Xi_k. \tag{4.26}$$

When $k=1$,

$$\alpha_{2-k}+\alpha_{1-k}=\alpha_1+\alpha_{2n-1}=\Xi_1-\sum_{a=2}^{2n-1}(-1)^{1+a}\Xi_a+\sum_{a=1}^{2n-1}(-1)^{2n-1+a}\Xi_a=2\Xi_1. \tag{4.27}$$

Hence (4.26) also holds for $k=1$.

For $k=2,\cdots,2n-1$, $h_k h_{2-k}^T K_{2-k,k}=\Xi_k$, hence $g_k=1-2\Xi_k\alpha_{2-k}^{-1}=-\alpha_{1-k}\alpha_{2-k}^{-1}$ by using (4.26). Moreover, (4.22) leads to

$$g_1=I-2h_1h_1^TK_{11}\alpha_1^{-1}=\begin{pmatrix}0 & -\dfrac{h_{11}}{h_{12}}\\ -\dfrac{h_{12}}{h_{11}} & 0\end{pmatrix}. \tag{4.28}$$

Hence $-\mu^{2n-1}G_{2n-1}$ is a diagonal matrix,

$$-\mu^{-2n+1}G_{2n-1}=\operatorname{diag}\Big(\frac{h_{12}}{h_{11}},\frac{h_{11}}{h_{12}},\frac{\alpha_{2n-2}}{\alpha_{2n-1}},\frac{\alpha_{2n-3}}{\alpha_{2n-2}},\cdots,\frac{\alpha_1}{\alpha_2}\Big). \tag{4.29}$$

Note that $(g_1)_{11}(g_1)_{22}=\frac{h_{12}}{h_{11}}\frac{h_{11}}{h_{12}}=1$, and $g_kg_{2n+1-k}=\frac{\alpha_{1-k}\alpha_{k-1}}{\alpha_{2-k}\alpha_k}=1$ $(k=2,\cdots,n)$ due to (4.18). Hence $\widetilde{V}$ is of form (2.6) where u_j's are replaced by $\widetilde{u}_j$'s when all α_j's are positive. Therefore, the transformation (4.10) gives a solution of the two dimensional $A_{2n-1}^{(2)}$ Toda equation. The theorem is proved.

Remark 4.2 The solutions obtained here are local ones. Locally, the condition that all α_k $(k=1,\cdots,n)$ are positive can be guaranteed by suitable choice of the solution of the Lax pair.

Example 4.1 When $n=3$, $(u_1,u_2,u_3)=\big(0,-\frac{3}{5}\ln 2,-\frac{1}{5}\ln 2\big)$ is a solution of the equation (1.4). For this seed solution, the Lax pair (2.1) with $\lambda=\mu\in\mathbf{R}$ becomes

$$H_x=\mu JH,\quad H_t=\mu^{-1}VJ^TV^{-1}H, \tag{4.30}$$

where

$$J=\begin{pmatrix}0&0&1&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\1&1&0&0&0&0\end{pmatrix},\quad V=\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&\gamma^{-3}&0&0&0\\0&0&0&\gamma^{-1}&0&0\\0&0&0&0&\gamma&0\\0&0&0&0&0&\gamma^3\end{pmatrix} \tag{4.31}$$

and $\gamma=\sqrt[5]{2}$. The general solution of (4.30) is $H=(h_{11},h_{12},h_2,h_3,h_4,h_5)^T$ with

$$\begin{aligned}h_{11}&=\mathrm{Re}\Big(\sum_{j=0}^{4}c_j\mathrm{e}^{\omega^j\gamma\mu x+\omega^{-j}\gamma\mu^{-1}t}+c_5\Big),\\ h_{12}&=\mathrm{Re}\Big(\sum_{j=0}^{4}c_j\mathrm{e}^{\omega^j\gamma\mu x+\omega^{-j}\gamma\mu^{-1}t}-c_5\Big),\\ h_k&=\mathrm{Re}\Big(\gamma^{k-1}\sum_{j=0}^{4}c_j\omega^{j(k-1)}\mathrm{e}^{\omega^j\gamma\mu x+\omega^{-j}\gamma\mu^{-1}t}\Big),\quad k=2,3,4,5,\end{aligned} \tag{4.32}$$

where $\omega = \exp\left(\frac{2\pi i}{5}\right)$ and $c_0, c_1, c_2, c_3, c_4, c_5$ are complex constants. Then

$$\begin{aligned} &\Xi_1 = 2h_{11}h_{12}, \quad \Xi_2 = \Xi_5 = h_2h_5, \quad \Xi_3 = \Xi_4 = h_3h_4, \\ &\alpha_1 = \alpha_5 = \Xi_1, \quad \alpha_2 = \alpha_4 = -\Xi_1 + 2\Xi_2, \quad \alpha_3 = \Xi_1 - 2\Xi_2 + 2\Xi_3. \end{aligned} \tag{4.33}$$

Hence Theorem 4.1 implies that

$$\begin{aligned} &\widetilde{u}_1 = \ln\frac{h_{12}}{h_{11}}, \quad \widetilde{u}_2 = -\frac{3}{5}\ln 2 + \ln\frac{h_2h_5 - h_{11}h_{12}}{h_{11}h_{12}}, \\ &\widetilde{u}_3 = -\frac{1}{5}\ln 2 + \ln\frac{h_3h_4 - h_2h_5 + h_{11}h_{12}}{h_2h_5 - h_{11}h_{12}} \end{aligned} \tag{4.34}$$

is a solution of (1.4).

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