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A Hermitian Curvature Flow*

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Abstract A Hermitian curvature flow on a compact Calabi-Yau manifold is proposed and a regularity result is obtained. The solution of the flow, if exists, is a balanced Hermitian-Einstein metric.

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1 Introduction

On a compact complex manifold, several Hermitian curvature flows, including the pluriclosed flow (see [13–15]), the Chern-Ricci flow (see [5, 17]) and the anomaly flow (see [9–11]), have been introduced and developed.

In this paper a new Hermitian curvature flow is proposed on a Calabi-Yau manifold. A Calabi-Yau manifold is a compact complex manifold M whose first Bott-Chern class $c_1^{BC}(M)$ vanishes as a class in the Bott-Chern cohomology:

$$H^{1,1}_{BC}(M,\mathbb{R}) = \frac{\{\phi \in \mathcal{A}^{1,1}_{\mathbb{R}}(M) \mid \mathrm{d}\phi = 0\}}{\{\mathrm{i}\partial\overline{\partial} f \mid f \in \mathcal{A}^0_{\mathbb{R}}(M)\}}.$$

Here we take such a definition as in [16] due to need for integration by parts. So if M satisfies the $\partial\overline{\partial}$-lemma, especially if M is Kähler, then it is Calabi-Yau if and only if its first Chern class $c_1(M)$ vanishes.

A class of well-known examples of non-Kähler Calabi-Yau threefolds are the complex structures on $\#_k S^3 \times S^3$ for $k \geq 2$ given by the conifold transition (see [1, 8]). They satisfy the $\partial\overline{\partial}$-lemma (see [2]). On the other hand, J. Fu, J. Li and S.-T. Yau constructed a balanced metric on $\#_k S^3 \times S^3$ (see [4]).

We are interested in the question (see [3]) whether there exists on such manifolds a balanced metric ω which is also a Hermitian-Einstein metric. Here a Hermitian metric ω is Hermitian-Einstein if its mean curvature K_ω, i.e., the second Chern Ricci curvature, of the Chern connection satisfies equation

$$K_\omega = \lambda\omega$$

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for a constant λ which is zero if $c_1^{BC}(M)=0$. A typical example is the Iwasawa manifold whose natural metric is balanced and flat, i.e., $R_\omega=0$. Hence it is a Hermitian-Einstein metric.

Motivated by the above question, we consider on a Calabi-Yau manifold $(M,\widehat{\omega})$ a Hermitian curvature flow

$$\partial_t\omega=-K_\omega+|\tau|_\omega^2\omega,\quad \omega(0)=\widehat{\omega}, \tag{1.1}$$

where $\tau=\sum g^{k\overline{j}}T_{pk\overline{j}}\mathrm{d}z^p$ is the torsion 1-form of the metric ω. The reason we consider this flow is the following.

Proposition 1.1 *Let $\widehat{\omega}$ be a Hermitian metric on a Calabi-Yau manifold of a complex dimension n. Then there exists a positive $\mathcal{T}$ such that Hermitian curvature flow* (1.1) *has a unique solution $\omega(t)$ on $[0,\mathcal{T})$. Moreover, if the long time solution exists and converges to a Hermitian metric ω_∞, then ω_∞ is a balanced Hermitian-Einstein metric.*

Proof The proof of the first conclusion is standard. As to the second conclusion, let $\omega(t)$ converge to ω_∞ along flow (1.1) if t goes to infinity. Then

$$K_{\omega_\infty}=|\tau_{\omega_\infty}|_{\omega_\infty}^2\omega_\infty.$$

Because of $c_1^{BC}(X)=0$, the Ricci curvature ρ_∞ of ω_∞ can be written as $\mathrm{i}\partial\overline{\partial}F$ for a smooth function F on X. So we have

$$\mathrm{i}\Lambda_{\omega_\infty}\partial\overline{\partial}F=\Lambda_{\omega_\infty}K_{\omega_\infty}=n|\tau_{\omega_\infty}|_{\omega_\infty}^2.$$

By the maximum principle, F is a constant and $\tau_{\omega_\infty}=0$. Hence, ω_∞ is a balanced Hermitian-Einstein metric.

In this paper, as the first step to study flow (1.1), we provide several classical estimates to its solution, including L^∞ uniform metric estimate (Theorem 3.1) in Section 3, Calabi-type estimate (Theorem 4.1) in Section 4 and Shi-type estimate (Theorem 5.1) in Section 5. We also first introduce some notations in Section 2.

2 Notations

Let ω be a Hermitian metric on M and ∇ be its Chern connection. In local holomorphic coordinates $(z^1,\cdots,z^n)$,

$$\nabla_k\Big(\frac{\partial}{\partial z^i}\Big)=\sum\Gamma_{ik}^p\frac{\partial}{\partial z^p},\quad \nabla_{\overline{l}}\Big(\frac{\partial}{\partial z^i}\Big)=0,$$

where

$$\Gamma_{ik}^p=\sum g^{p\overline{s}}\partial_k g_{i\overline{s}}.$$

The torsion of ∇ is

$$T = \partial\omega = \frac{i}{2}\sum T_{ki\overline{j}}\mathrm{d}z^k \wedge \mathrm{d}z^i \wedge \mathrm{d}\overline{z}^j,$$

where

$$T_{ki\overline{j}} = \partial_k g_{i\overline{j}} - \partial_i g_{k\overline{j}}.$$

Denote $T^p_{ki} = \sum g^{p\overline{j}} T_{ki\overline{j}}$. We have $T^p_{ki} = \Gamma^p_{ik} - \Gamma^p_{ki}$. The torsion 1-form τ of ∇ is

$$\tau = \sum T_i \mathrm{d}z^i = \sum g^{k\overline{l}} T_{ik\overline{l}} \mathrm{d}z^i.$$

Then ω is a balanced metric if $\tau = 0$.

The curvature of ∇ is

$$R_\omega = \sum R^p_{ik\overline{l}} \frac{\partial}{\partial z^p} \otimes \mathrm{d}z^i \otimes \mathrm{d}z^k \wedge \mathrm{d}\overline{z}^l$$

for

$$R^p_{ik\overline{l}} = -\sum g^{p\overline{j}} \partial_{\overline{l}} \partial_k g_{i\overline{j}} + \sum g^{p\overline{j}} g^{m\overline{n}} \partial_{\overline{l}} g_{m\overline{j}} \partial_k g_{i\overline{n}}.$$

Denote $R_{i\overline{j}k\overline{l}} = \sum g_{p\overline{j}} R^p_{ik\overline{l}}$, and denote

$$R_{i\overline{j}} = \sum g^{k\overline{l}} R_{k\overline{l}i\overline{j}}, \quad K_{i\overline{j}} = \sum g^{k\overline{l}} R_{i\overline{j}k\overline{l}}.$$

The Ricci curvature and the mean curvature of ∇ are

$$\rho_\omega = i\sum R_{i\overline{j}} \mathrm{d}z^i \wedge \mathrm{d}\overline{z}^j \quad \text{and} \quad K_\omega = i\sum K_{i\overline{j}} \mathrm{d}z^i \wedge \mathrm{d}\overline{z}^j.$$

In some references, they are called the first and second Chern-Ricci curvature of ∇ respectively.

For a Hermitian metric ω, we have the following Bianchi identities (see [12]).

$$R_{i\overline{j}k\overline{l}} - R_{k\overline{j}i\overline{l}} = \nabla_{\overline{l}} T_{ik\overline{j}}, \tag{2.1}$$

$$\nabla_p R_{i\overline{j}k\overline{l}} - \nabla_k R_{i\overline{j}p\overline{l}} = \sum R_{i\overline{j}m\overline{l}} T^m_{kp}. \tag{2.2}$$

3 L^∞ Uniform Metric Estimate

In this section, assume that the initial metric $\widehat{\omega}$ is a Chern Ricci-flat Hermitian metric (see [16, Proposition 1.1]). Similar to [15, Theorem 1.3], we consider the equation

$$\partial_t\varphi = \Delta_\omega\varphi + \mathrm{tr}_\omega\widehat{\omega} - n, \quad \varphi(0) = 0. \tag{3.1}$$

Clearly, there exists a unique solution $\varphi(t)$ to this flow in $[0, t_0)$. Our result is the following theorem.

Theorem 3.1 *Let ω be the solution of curvature flow* (3.1) *in* $[0,t_0)$. *If there exist positive constants $\widetilde{\kappa}$ and $\breve{\kappa}$ such that the equalities*

$$\varphi \le \widetilde{\kappa} \quad \text{and} \quad |\tau|^2_\omega \le \breve{\kappa} \tag{3.2}$$

hold in $[0,t_0)$, *then there exists a positive constant $\breve{\kappa}$ depending on $(M,\widehat{\omega})$, $\widetilde{\kappa}$ and κ such that the following inequalities hold.*

$$\kappa^{-1}\widehat{\omega} \le \omega \le \kappa\widehat{\omega}.$$

Proof Since $\widehat{\omega}$ is Ricci-flat, inequalities (3.2) imply

$$(\partial_t - \Delta)\log\frac{\omega^n}{\widehat{\omega}^n} = -n|\tau|^2 + \Lambda_\omega\rho_{\widehat{\omega}} \ge -n\breve{\kappa}.$$

By the maximum principle, we have

$$\frac{\omega^n}{\widehat{\omega}^n}(x,t) \le \mathrm{e}^{n\breve{\kappa}t}, \quad (x,t) \in M \times [0,t_0). \tag{3.3}$$

Then the arithmetic-geometric mean inequality implies

$$\mathrm{tr}_\omega\widehat{\omega}(x,t) \ge n\mathrm{e}^{-\breve{\kappa}t}, \quad (x,t) \in M \times [0,t_0).$$

Combined with equation (3.1), we have

$$(\partial_t - \Delta)(\varphi + n\breve{\kappa}^{-1}\mathrm{e}^{-\breve{\kappa}t} + nt) \ge 0.$$

By the maximum principle, we have

$$-nt_0 \le n\breve{\kappa}^{-1}(1-\mathrm{e}^{-\breve{\kappa}t}) - nt \le \varphi(x,t), \quad (x,t) \in M \times [0,t_0). \tag{3.4}$$

On the other hand, it is easily checked that

$$(\partial_t - \Delta)\mathrm{tr}_\omega\widehat{\omega} = -\sum \widehat{g}^{i\bar{j}}g^{k\bar{l}}g^{p\bar{q}}\nabla_{\bar{l}}\widehat{g}_{i\bar{q}}\nabla_k\widehat{g}_{p\bar{j}} + \sum g^{i\bar{j}}g^{k\bar{l}}\widehat{R}_{i\bar{j}k\bar{l}} - |\tau|^2\mathrm{tr}_\omega\widehat{\omega}.$$

So there exists a constant $\widehat{C}$ depending on $(M,\widehat{\omega})$ such that

$$\begin{aligned}(\partial_t - \Delta)\log\mathrm{tr}_\omega\widehat{\omega} &= \frac{1}{\mathrm{tr}_\omega\widehat{\omega}}\Big(-|\tau|^2\mathrm{tr}_\omega\widehat{\omega} + \sum g^{i\bar{j}}g^{k\bar{l}}\widehat{R}_{i\bar{j}k\bar{l}} \\ &\quad + \frac{|\nabla\mathrm{tr}_\omega\widehat{\omega}|^2}{\mathrm{tr}_\omega\widehat{\omega}} - \sum \widehat{g}^{i\bar{j}}g^{k\bar{l}}g^{p\bar{q}}\nabla_{\bar{l}}\widehat{g}_{i\bar{q}}\nabla_k\widehat{g}_{p\bar{j}}\Big) \\ &\le -|\tau|^2 + \widehat{C}\mathrm{tr}_\omega\widehat{\omega},\end{aligned} \tag{3.5}$$

where the inequality holds because

$$\begin{aligned}&\sum \widehat{g}^{i\bar{j}}g^{k\bar{l}}g^{p\bar{q}}\nabla_{\bar{l}}\widehat{g}_{i\bar{q}}\nabla_k\widehat{g}_{p\bar{j}} - \frac{|\nabla\mathrm{tr}_\omega\widehat{\omega}|^2}{\mathrm{tr}_\omega\widehat{\omega}} \\ &= \sum \widehat{g}^{i\bar{j}}g^{k\bar{l}}g^{p\bar{q}}\Big(\nabla_{\bar{l}}\widehat{g}_{i\bar{q}} - \widehat{g}_{i\bar{q}}\frac{\partial_{\bar{l}}\mathrm{tr}_\omega\widehat{\omega}}{\mathrm{tr}_\omega\widehat{\omega}}\Big)\Big(\nabla_k\widehat{g}_{p\bar{j}} - \widehat{g}_{p\bar{j}}\frac{\partial_k\mathrm{tr}_\omega\widehat{\omega}}{\mathrm{tr}_\omega\widehat{\omega}}\Big) \ge 0.\end{aligned}$$

Take a test function

$$G_2 = \log \mathrm{tr}_\omega \widehat{\omega} - A_2 \varphi,$$

where the constant $A_2 > 0$ will be determined later. Combining (3.1) and (3.5) implies

$$(\partial_t - \Delta) G_2 \le -(A_2 - \widehat{C}) \mathrm{tr}_\omega \widehat{\omega} + n A_2.$$

Let G_2 take the maximum at (x_2, t_2) and $t_2 > 0$. Take $A_2 = 1 + \widehat{C}$. Then we have

$$\mathrm{tr}_\omega \widehat{\omega}(x_2, t_2) \le n(1 + \widehat{C}).$$

Combined with inequality (3.4) we have

$$\mathrm{tr}_\omega \widehat{\omega}(x, t) \le n(1 + \widehat{C}) \exp((1 + \widehat{C})(\widetilde{\kappa} + n t_0)).$$

Combined with inequality (3.3), we also have

$$\begin{aligned} \mathrm{tr}_{\widehat{\omega}} \omega(x, t) &\le \frac{1}{(n-1)!} (\mathrm{tr}_\omega \widehat{\omega})^{n-1} \frac{\omega^n}{\widehat{\omega}^n}(x, t) \\ &\le (n(1 + \widehat{C}))^{n-1} \exp((n-1)(1 + \widehat{C})(\widetilde{\kappa} - n t_0) + n \check{\kappa} t_0). \end{aligned}$$

Hence the conclusion holds.

4 A Calabi-Type Estimate

Introduce the tensor Ψ whose components are $\Psi_{ik}^p = \Gamma_{ik}^p - \widehat{\Gamma}_{ik}^p$. It is easy verified that

$$|\widehat{\nabla} g|_\omega^2 = \sum g^{i\overline{j}} g^{k\overline{l}} g^{p\overline{q}} \widehat{\nabla}_p g_{i\overline{l}} \widehat{\nabla}_{\overline{q}} g_{k\overline{j}} = |\Psi|_\omega^2.$$

In this section, we deduce the Calabi-type estimate of the solution ω of flow (1.1).

By the direct computation, we have

$$\begin{aligned} \partial_t |\Psi|^2 = {} & 2\mathrm{Re}\Big(\sum g^{i\overline{j}} g^{k\overline{l}} g_{p\overline{q}} \partial_t \Psi_{ik}^p \overline{\Psi_{jl}^q} \Big) - |\tau|^2 |\Psi|^2 \\ & + \sum (K^{i\overline{j}} g^{k\overline{l}} g_{p\overline{q}} + g^{i\overline{j}} K^{k\overline{l}} g_{p\overline{q}} - g^{i\overline{j}} g^{k\overline{l}} K_{p\overline{q}}) \Psi_{ik}^p \overline{\Psi_{jl}^q}. \end{aligned} \tag{4.1}$$

Since $\partial_t \Psi_{ik}^p = \sum g^{p\overline{n}} \nabla_k (\partial_t g_{i\overline{n}})$, the first term of the right hand side is equal to

$$-2\mathrm{Re}\Big(\sum g^{i\overline{j}} g^{k\overline{l}} \nabla_k K_{i\overline{q}} \overline{\Psi_{jl}^q} \Big) + 2\mathrm{Re}\Big(\sum g^{k\overline{l}} \partial_{\overline{l}} |\tau|^2 \Psi_{ik}^i \Big).$$

On the other hand, by the direct computation we also have

$$\begin{aligned} \Delta |\Psi|^2 = {} & 2\mathrm{Re}\Big(\sum g^{i\overline{j}} g^{k\overline{l}} g_{p\overline{q}} g^{m\overline{n}} \nabla_m \nabla_{\overline{n}} \Psi_{ik}^p \overline{\Psi_{jl}^q} \Big) + |\nabla' \Psi|^2 + |\nabla'' \Psi|^2 \\ & + \sum g^{i\overline{j}} g^{k\overline{l}} g_{p\overline{q}} \Psi_{ik}^p g^{m\overline{n}} [\nabla_m, \nabla_{\overline{n}}] \overline{\Psi_{jl}^q}, \end{aligned}$$

where

$$\sum g^{m\overline{n}}[\nabla_m,\nabla_{\overline{n}}]\overline{\Psi^q_{jl}}=-\sum\overline{\Psi^s_{jl}}\overline{K^q_s}+\sum\overline{\Psi^q_{sl}}\overline{K^s_j}+\sum\overline{\Psi^q_{js}}\overline{K^s_l},$$

and by Bianchi identity (2.2),

$$\sum g^{m\overline{n}}\nabla_m\nabla_{\overline{n}}\Psi^p_{ik}=-\nabla_k K^p_i+\sum g^{m\overline{n}}(\nabla_m\widehat{R}^p_{ik\overline{n}}-R^p_{ir\overline{n}}T^r_{km}).$$

Combined all together, we have

$$\begin{aligned}(\partial_t-\Delta)|\Psi|^2=&-(|\nabla'\Psi|^2+|\nabla''\Psi|^2)-|\tau|^2|\Psi|^2+2\mathrm{Re}\Big(\sum g^{k\overline{l}}\partial_{\overline{l}}|\tau|^2\Psi^i_{ik}\Big)\\&+2\mathrm{Re}\Big(\sum g^{i\overline{j}}g^{k\overline{l}}g^{m\overline{n}}R_{i\overline{q}r\overline{n}}T^r_{km}\overline{\Psi^q_{jl}}\Big)\\&-2\mathrm{Re}\Big(\sum g^{i\overline{j}}g^{k\overline{l}}g^{m\overline{n}}g_{p\overline{q}}\nabla_m\widehat{R}^p_{ik\overline{n}}\overline{\Psi^q_{jl}}\Big).\end{aligned}\tag{4.2}$$

Proposition 4.1 *Let ω be the solution of curvature flow (1.1) in $[0,t_0)$ such that*

$$\kappa^{-1}\widehat{\omega}\le\omega\le\kappa\widehat{\omega}\tag{4.3}$$

for a positive constant κ. Then there exists a constant $\widehat{C}_\kappa$ depending on $(M,\widehat{\omega})$ and κ such that the following estimate hold.

$$(\partial_t-\Delta)|\Psi|^2\le-\frac{1}{2}(|\nabla'\Psi|^2+|\nabla''\Psi|^2)+\widehat{C}_\kappa(1+|\Psi|^2+|T|^2|\Psi|^2).$$

Proof We first deal with the third term in the right hand of (4.2). By the definition of τ, we have

$$\begin{aligned}\partial_{\overline{l}}|\tau|^2&=\sum g^{i\overline{j}}((R^p_{ip\overline{l}}-R^p_{pi\overline{l}})\overline{T_j}+T_i(\nabla_{\overline{l}}\overline{\Psi^q_{qj}}-\nabla_{\overline{l}}\overline{\Psi^q_{jq}})+\nabla_{\overline{l}}\overline{\widehat{T}_j})\\&=\sum g^{i\overline{j}}((\widehat{R}^p_{ip\overline{l}}-\widehat{R}^p_{pi\overline{l}})\overline{T_j}+(\nabla_{\overline{l}}\Psi^p_{pi}-\nabla_{\overline{l}}\Psi^p_{ip})\overline{T_j}\\&\quad+T_i(\nabla_{\overline{l}}\overline{\Psi^q_{qj}}-\nabla_{\overline{l}}\overline{\Psi^q_{jq}})+\widehat{\nabla}_{\overline{l}}\overline{\widehat{T}_j}-\overline{\widehat{T}_n}\overline{\Psi^n_{ql}}),\end{aligned}$$

and hence we can estimate

$$\begin{aligned}2\mathrm{Re}\Big(\sum g^{k\overline{l}}\partial_{\overline{l}}|\tau|^2\Psi^i_{ik}\Big)&\le\widehat{C}_\kappa(1+|\Psi|+|\nabla'\Psi|+|\nabla''\Psi|)|T||\Psi|\\&\le\frac{1}{4}(|\nabla'\Psi|^2+|\nabla''\Psi|^2)+\widehat{C}_\kappa(1+|\Psi|^2+|T|^2|\Psi|^2),\end{aligned}$$

where the constant $\widehat{C}_\kappa$ depends on $(M,\widehat{\omega})$ and κ, and is used in the generic sense.

Using the identity $R^p_{ir\overline{n}}=\widehat{R}^p_{ir\overline{n}}-\nabla_{\overline{n}}\Psi^p_{ir}$, we find that the forth term of the right hand side in (4.2) is less than

$$\frac{1}{4}(|\nabla'\Psi|^2+|\nabla''\Psi|^2)+\widehat{C}_\kappa(1+|T|^2|\Psi|^2).$$

The last term of the right hand side of (4.2) is less than

$$\widehat{C}_\kappa(1+|\Psi|^2)$$

since

$$\nabla_m \widehat{R}^p_{ik\overline{n}} = \widehat{\nabla}_m \widehat{R}^p_{ik\overline{n}} + \sum \widehat{R}^r_{ik\overline{n}} \Psi^p_{rm} - \sum \widehat{R}^p_{rk\overline{n}} \Psi^r_{im} - \sum \widehat{R}^p_{ir\overline{n}} \Psi^r_{km}.$$

Substituting the above all estimates into (4.2), we get the conclusion.

Next we compute the evolution equation of $\mathrm{tr}_{\widehat{\omega}}\omega$. Clearly,

$$\partial_t \mathrm{tr}_{\widehat{\omega}}\omega = -\sum \widehat{g}^{i\overline{j}} K_{i\overline{j}} + |\tau|^2 \mathrm{tr}_{\widehat{\omega}}\omega.$$

By the direct computation, we also have

$$\Delta \mathrm{tr}_{\widehat{\omega}}\omega = \sum \widehat{g}^{i\overline{j}} g^{k\overline{l}} g^{p\overline{q}} \widehat{\nabla}_{\overline{l}} g_{p\overline{j}} \widehat{\nabla}_k g_{i\overline{q}} - \sum \widehat{g}^{i\overline{j}} K_{i\overline{j}} + \sum g^{k\overline{l}} \widehat{g}^{p\overline{j}} g_{i\overline{j}} \widehat{R}^i_{pk\overline{l}}.$$

Hence,

$$(\partial_t - \Delta)\mathrm{tr}_{\widehat{\omega}}\omega = -\sum \widehat{g}^{i\overline{j}} g^{k\overline{l}} g^{p\overline{q}} \widehat{\nabla}_{\overline{l}} g_{p\overline{j}} \widehat{\nabla}_k g_{i\overline{q}} + |\tau|^2 \mathrm{tr}_{\widehat{\omega}}\omega - \sum g^{k\overline{l}} \widehat{g}^{p\overline{j}} g_{i\overline{j}} \widehat{R}^i_{pk\overline{l}}. \tag{4.4}$$

Proposition 4.2 *Let ω be the solution of curvature flow* (1.1) *in* $[0, t_0)$. *If inequalities* (4.3) *hold, then*

$$(\partial_t - \Delta)\mathrm{tr}_{\widehat{\omega}}\omega \le -\kappa^{-1}|\Psi|^2_\omega + \widehat{C}_\kappa + n\kappa|\tau|^2,$$

where the positive $\widehat{C}_\kappa$ depends on $(M, \widehat{\omega})$ and κ.

Proof The conclusion is directly from formula (4.4) since under condition (4.3) we clearly have $\mathrm{tr}_{\widehat{\omega}}\omega \le n\kappa$, and

$$-\sum g^{k\overline{l}} \widehat{g}^{p\overline{j}} g_{i\overline{j}} \widehat{R}^i_{pk\overline{l}} \le \widehat{C}_\kappa,$$

where $\widehat{C}_\kappa$ depends on $(X, \widehat{\omega})$ and κ, and

$$-\sum \widehat{g}^{i\overline{j}} g^{k\overline{l}} g^{p\overline{q}} \widehat{\nabla}_{\overline{l}} g_{p\overline{j}} \widehat{\nabla}_k g_{i\overline{q}} \le -\kappa^{-1}|\widehat{\nabla} g|^2_\omega = -\kappa^{-1}|\Psi|^2_\omega.$$

Now we can prove the following theorem.

Theorem 4.1 *Let ω be the solution of curvature flow* (1.1) *in* $[0, t_0)$. *If there exists a positive constant κ such that in* $[0, t_0)$,

$$\kappa^{-1}\widehat{\omega} \le \omega \le \kappa\widehat{\omega} \quad \text{and} \quad |T|^2 \le \kappa,$$

then there exists a positive constant $\widehat{C}_\kappa$ depending on $(M, \widehat{\omega})$ and κ such that

$$|\Psi|^2 \le \widehat{C}_\kappa.$$

Proof Take a test function

$$G_1 = |\Psi|^2 + A_1 \mathrm{tr}_{\widehat{\omega}}\omega,$$

where the positive constant A_1 will be determined later. By Propositions 4.1–4.2, we have

$$(\partial_t - \Delta)G_1 \le -(\kappa^{-1}A_1 - (1+\kappa)\widehat{C}_\kappa)|\Psi|^2 + (1+A_1)\widehat{C}_\kappa + n\kappa^2 A_1.$$

Assume that G_1 takes the maximum at (x_1, t_1) and without loss of generality assume $t_1 > 0$.

Take $A_1 = \kappa(1 + (1+\kappa)\widehat{C}_\kappa)$. Then at (x_1, t_1), we have

$$0 \le (\partial_t - \Delta)G_1 \le -|\Psi|^2 + \widehat{C}_\kappa.$$

Hence

$$|\Psi|^2 \le -A_1 \mathrm{tr}_{\widehat{\omega}}\omega + \widehat{C}_\kappa \le \widehat{C}_\kappa$$

for a generic constant $\widehat{C}_\kappa$.

5 Shi-Type Estimates

Proposition 5.1 *Let ω be the solution of curvature flow* (1.1) *in* $[0, t_0)$. *The evolution equation of its curvature tensor is*

$$\begin{aligned}\partial_t \nabla^k \mathrm{Rm} = {}& \Delta\nabla^k \mathrm{Rm} + \sum_{l=0}^{k} \nabla^l T * \nabla^{k+1-l}\mathrm{Rm} + \sum_{l=0}^{k} \nabla^l \mathrm{Rm} * \nabla^{k-l}\mathrm{Rm} \\ &+ \sum_{i=0}^{k}\sum_{l=0}^{i} \nabla^l T * \nabla^{i-l} T * \nabla^{k-i}\mathrm{Rm} + \sum_{l=0}^{k+2} \nabla^l T * \nabla^{k+2-l} T. \end{aligned} \tag{5.1}$$

Proof When $k = 0$, we have

$$\begin{aligned}\partial_t R_{i\bar{j}k\bar{l}} &= \sum \partial_t g_{p\bar{j}} R^p_{ik\bar{l}} - \nabla_{\bar{l}}\nabla_k(\partial_t g_{i\bar{j}}) \\ &= \nabla_{\bar{l}}\nabla_k K_{i\bar{j}} - \sum K_{p\bar{j}} R^p_{ik\bar{l}} + |\tau|^2 R_{i\bar{j}k\bar{l}} - |\tau|^2_{k\bar{l}} g_{i\bar{j}}.\end{aligned}$$

By Bianchi identities (2.1) and (2.2), we have

$$\begin{aligned}\nabla_{\bar{l}}\nabla_k K_{i\bar{j}} &= \sum g^{p\bar{q}} \nabla_{\bar{l}}(\nabla_p R_{i\bar{j}k\bar{q}} + R_{i\bar{j}m\bar{q}} T^m_{pk}) \\ &= \sum g^{p\bar{q}} \nabla_p \nabla_{\bar{q}} R_{i\bar{j}k\bar{l}} + \sum g^{p\bar{q}}(\nabla_p(R_{i\bar{j}k\bar{n}}\overline{T^n_{ql}}) + \nabla_{\bar{l}}(R_{i\bar{j}m\bar{q}} T^m_{pk})) \\ &\quad + \sum g^{p\bar{q}}[\nabla_{\bar{l}}, \nabla_p] R_{i\bar{j}k\bar{q}} \\ &= \sum g^{p\bar{q}} \nabla_p \nabla_{\bar{q}} R_{i\bar{j}k\bar{l}} + \sum g^{p\bar{q}}(\nabla_p R_{i\bar{j}k\bar{n}}\overline{T^n_{ql}} + \nabla_{\bar{l}} R_{i\bar{j}m\bar{q}} T^m_{pk}) \\ &\quad + \sum g^{p\bar{q}}(R_{m\bar{j}k\bar{q}} R^m_{ip\bar{l}} + R_{i\bar{j}m\bar{q}} R^m_{pk\bar{l}} - R_{i\bar{n}k\bar{q}}\overline{R^n_{jl\bar{p}}}) - \sum R_{i\bar{j}k\bar{q}}\overline{K^q_l}.\end{aligned}$$

Combining the above two equalities implies formula (5.1) holds when $k = 0$.

Then by the inductive proof of [14, Lemma 7.1], we can prove equality (5.1) for all k.

Proposition 5.2 *Let ω be the solution of curvature flow* (1.1) *in* $[0, t_0)$*. The evolution equation of its torsion tensor is*

$$\begin{aligned}\partial_t \nabla^k T = \Delta \nabla^k T + \sum_{l=0}^{k} \nabla^l T * \nabla^{k+1-l} T + \sum_{l=0}^{k} \nabla^l \mathrm{Rm} * \nabla^{k-l} \mathrm{Rm} \\ + \sum_{i=0}^{k} \sum_{l=0}^{i} \nabla^l T * \nabla^{i-l} T * \nabla^{k-i} T.\end{aligned} \tag{5.2}$$

Proof When $k = 0$, we have

$$\begin{aligned}\partial_t T_{pi\overline{j}} &= \partial_p(\partial_t g_{i\overline{j}}) - \partial_i(\partial_t g_{p\overline{j}}) = \nabla_p(\partial_t g_{i\overline{j}}) - \nabla_i(\partial_t g_{p\overline{j}}) - \sum \partial_t g_{k\overline{j}} T_{ip}^k \\ &= \nabla_i K_{p\overline{j}} - \nabla_p K_{i\overline{j}} + \sum K_{k\overline{j}} T_{ip}^k + |\tau|_p^2 g_{i\overline{j}} - |\tau|_i^2 g_{p\overline{j}} + |\tau|^2 T_{pi\overline{j}}.\end{aligned}$$

By Bianchi identities (2.1) and (2.2), we have

$$\begin{aligned}\nabla_i K_{p\overline{j}} - \nabla_p K_{i\overline{j}} &= \sum g^{m\overline{n}} \nabla_m (R_{p\overline{j}i\overline{n}} - R_{i\overline{j}p\overline{n}}) + \sum g^{m\overline{n}} (R_{p\overline{j}r\overline{n}} T_{mi}^r + R_{i\overline{j}r\overline{n}} T_{pm}^r) \\ &= \sum g^{m\overline{n}} \nabla_m \nabla_{\overline{n}} T_{pi\overline{j}} + \sum g^{m\overline{n}} (R_{p\overline{j}r\overline{n}} T_{mi}^r + R_{i\overline{j}r\overline{n}} T_{pm}^r).\end{aligned}$$

Combining the above two equalities implies formula (5.2) holds when $k = 0$.

Then by the inductive proof of [14, Lemma 7.2], we can prove the equality (5.2) for all k.

Now by evolution equations (5.1) and (5.2), and the proof of [14, Theorem 7.3], we have the following theorem.

Theorem 5.1 *Let M be a compact complex manifold of complex dimension n and ω be the solution of curvature flow* (1.1) *on M. Then for each $\alpha > 0$ and $k \in \mathbb{N}$, there exists a constant C_k depending only on M, k and α such that if*

$$|\mathrm{Rm}|_{\omega(t)} + |\nabla T|_{\omega(t)} + |T|_{\omega(t)}^2 \leq K$$

for all $t \in [0, \alpha K^{-1}]$, then for each $t \in (0, \alpha K^{-1}]$,

$$|\nabla^k \mathrm{Rm}|_{\omega(t)} + |\nabla^{k+1} T|_{\omega(t)} \leq C_k K t^{-\frac{k}{2}}. \tag{5.3}$$

Remark 5.1 Because of Shi-type estimates (Theorem 5.1), we can use the method in [7] to generalise Hamilton's compactness theorem (see [6]) for Ricci flow to one for flow (1.1).

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