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High Speed Flight and Partial Differential Equations

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Abstract Aircraft comes out at the beginning of the last century. Accompanied by the progress of high speed flight the theory of partial differential equations has been greatly developed. This paper gives a brief review on the history of applications of partial differential equations to the study of supersonic flows arising in high speed flight.

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1 Introduction

Since the 20th century the technology of aviation and aircraft technology rapidly developed. In 1903 Wright Brothers designed a first airplane and created the history of human flight. Today people have been able to manufacture flight projectile with high speed up to more than 10 times sonic speed. The aerodynamic state near the flying projectile will determine the lift force, resistance force on the object, as well as the global state of the projectile in its flight. Therefore, clearly understand the aerodynamic state near the flight projectile is the bases of aviation and aircraft technology. Since the physical parameters of the gas near the projectile obey a system of partial differential equations, the deep understanding on the related system of partial differential equations has become the key of mature and continuously development of flight technology.

The basic systems of partial differential equations describing gas dynamics are Euler system for inviscid flow and Navier-Stokes system for viscous flow. Since the viscosity of air is quite small, it is often neglected in the study of the flight of various projectiles. The Euler system takes the following form

$$\begin{cases} \dfrac{\partial\rho}{\partial t} + \operatorname{div}(\rho\vec{v}) = 0, \\ \dfrac{\partial(\rho\vec{v})}{\partial t} + \operatorname{div}(\rho\vec{v}\otimes\vec{v}) + \nabla p = 0, \\ \dfrac{\partial(\rho E)}{\partial t} + \operatorname{div}(\rho\vec{v}E + p\vec{v}) = 0. \end{cases} \tag{1.1}$$

In three-dimensional space (with coordinates (x, y, z)) this is a system with five equations, where $\rho, \vec{v}, p, E$ represent density, velocity, pressure and energy, respectively. The three components of $\vec{v}$ are (u, v, w). All these physical parameters obey an equation of state. In the case when we

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only discuss the stable motion of projectile, all physical parameters are independent of time t, then the derivatives with respect to t in (1.1) can be omitted.

For the isentropic flow, the last equation in system (1.1) can be derived from the above four equations. In this case one usually can introduce a potential ϕ of flow, and write the velocity of the flow as the gradient of the potential: $\vec{v} = \nabla\phi$. In two-dimensional case by using the potential ϕ, system (1.1) can be written as a single equation

$$\Big(1 - \frac{u^2}{c^2}\Big)\phi_{xx} - 2\frac{uv}{c^2}\phi_{xy} + \Big(1 - \frac{v^2}{c^2}\Big)\phi_{yy} = 0, \tag{1.2}$$

where c represents the sonic speed.

For the flow with lower velocity the coefficients of $D^2\phi$ in (1.2) are near to $(1, 0, 1)$. It means that (1.2) can be approximately regarded as a Laplace equation. In the first decades of 20th century the study of various boundary value problems of Laplace equation as well as the related methods of complex analysis is widely applied to fluid dynamics. However, it is not efficient in the study of the motion of airplane with high speed.

When the velocity of the flying projectile is comparable to the sonic speed, (1.2) is far from Laplace equation. If the flow speed is less than the sonic speed c, then (1.2) is a nonlinear elliptic equation. Meanwhile, if the flow speed is larger than c, then (1.2) is a nonlinear hyperbolic equation. Before the middle of 20th century, many methods on solving nonlinear elliptic equations developed, like generalized analytic functions, quasi-conformal mapping and fixed point theory etc. Generally, the solution of (1.2) is rather smooth and can be understood in classical sense.

Since (1.2) for supersonic flow is a nonlinear hyperbolic equation, people has to develop a new way to treat it. As we well know, the nonlinear hyperbolic equation has a property that the singularity of solution can be produced inside the solution, no matter how smooth the initial data are. This property forces people to study the solution of nonlinear equation in wider functional space: The solution possibly contains various singularities like shock waves, rarefaction waves and contact discontinuities. It is known that these singularities frequently appear in gas dynamics. The solution containing these singularities is called generalized solution, and is well studied in the middle of the last century.

If the solution has discontinuity in some surface, then the value of the solution on the both sides of the surface should satisfies Rankine-Hugoniot conditions. For instance, if Σ : $\psi(t, x, y, z) = 0$ is a surface bearing discontinuity of the solution, then we have

$$\begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho E \end{bmatrix} n_t + \begin{bmatrix} \rho u \\ p + \rho u^2 \\ \rho uv \\ \rho uw \\ \rho uE + pu \end{bmatrix} n_x + \begin{bmatrix} \rho v \\ \rho uv \\ p + \rho v^2 \\ \rho vw \\ \rho vE + pv \end{bmatrix} n_y + \begin{bmatrix} \rho w \\ \rho uw \\ \rho vw \\ p + \rho w^2 \\ \rho wE + pw \end{bmatrix} n_z = 0, \tag{1.3}$$

where (n_t, n_x, n_y, n_z) is the normal direction of Σ, and the bracket means the jump of the corresponding quantity inside the bracket on the surface Σ. For steady isentropic flow (1.3) can be much simplified.

2 Supersonic Flow Past a Curved Wedge

Modern airplanes and missiles often fly in air with supersonic speed. According to the principle of relativity the motion of a flying projectile in static air is equivalent to a steady flow with the same velocity in opposite direction moves around the fixed projectile with same shape. Hence the problem on supersonic flow past a given body is fundamental in gas dynamics and attracted many people's attention (e.g. see [1, 6]). Many experiments show that for the supersonic flow past a given body there will generally appear a shock front ahead of the body. More precisely, if the body has a sharp head then the shock is attached at the head, and if the body has a blunt head then the shock is detached away from the head. Since the appearance of the shock front as well as its shape greatly influence the flow field and finally influence the design of all related flying projectile, fluid dynamicists and mathematicians paid their great effort to well understand these phenomena. Due to the complexity of the real projectiles in applications, people starts their rigorous analysis from some typical models, i.e., the supersonic flow past a wedge or a circular cone.

Consider a supersonic flow past a wedge formed by two plane with an intersection angle 2α. If the direction of the flow is parallel to the symmetric plane of the wedge then the problem is two-dimensional. On (u, v) plane of the velocity vectors, if $(q_\infty, 0)$ is the vector of an incoming supersonic flow ahead of a shock, then all possible velocity behind the shock forms a curve called shock polar (see [7, 14]). R. Courant and K. O. Friedrichs indicated that for the problem of supersonic flow past a wedge, if the vertex angle of the wedge is less than a critical value then the ray on (u, v) plane with inclination angle α intersects with the shock polar at two points. These points correspond to a weak shock and a strong shock respectively. Particularly, the weak shock gives a stable solution to the problem of supersonic flow past the plain wedge, i.e., the location of the weak shock corresponds to the attached shock, and the state of the flow behind the shock is constant, which can be determined by a set of algebraic equations, equivalent to Rankine-Hugoniot conditions.

Furthermore, for the problem of supersonic flow past a circular cone, if the velocity of the flow is parallel to the axis of the cone and the vertex angle is less than another critical value, then there is a circular conical shock attached at the tip of the cone, and the flow field between the shock front and the surface of the body can be determined by solving a boundary value problem of a system of ordinary differential equations (see [14]).

Generally, a projectile may have much more complex shape. To understand the dynamic characters of supersonic flow around a given body one has to resort to solving partial differential equations. By taking wedge and circular cone as two typical examples people gradually to increase the complexity of the shape of the body located in the supersonic flow.

In the beginning of sixties of the last century, Gu Chaohao led a group of mathematicians in Fudan University to start the research on this topic: Supersonic flow past a given body. They studied the case when the body is a curved wedge and the flow is two-dimensional stationary isentropic. The problem is reduced to a free boundary value problem of a nonlinear hyperbolic equation. The unknown shock front attached the edge of the curved wedge is a free boundary, where the solution should satisfy the Rankine-Hugoniot conditions. The local existence of the solution to a free boundary value problem was proved in [15–16]. It is the first application of the theory on discontinuous solutions of nonlinear hyperbolic equations to the problem of supersonic flow passing a given body, though people had accumulated some knowledge on discontinuous solutions to nonlinear hyperbolic equations before (see [18–19]).

In [15–16] the authors employed the method of integration along characteristics of the nonlinear hyperbolic equation. By integration along characteristics, the free boundary value problem can be reduced to a system of integral equations. Then by using iteration the local existence of the solution for the given problem can be obtained. This approach was developed by T. T. Li and W. C. Yu to more general case in 1964. In [21] they established a general local theory on initial or initial-boundary value problems of nonlinear hyperbolic system. By applying the theory T. T. Li and W. C. Yu also proved the local existence of solution with an attached shock front [22] for two-dimensional stationary non-isentropic supersonic flow past a curved wedge.

We noticed that D. G. Schaeffer also proved the similar result by using Nash-Moser iteration technique in 1976 (see [25]).

3 Supersonic Flow Past a Three-Dimensional Wing

The curved wedge can be viewed as the head of a wing with constant section of an airplane. However, the wing of airplane has variable section in practice, so that it is necessary to study the supersonic flow past a wing with variable section in three dimensional space. Because in three dimensional space the characteristics of Euler system is a surface in the space, the method of integration along characteristic line does not work. Therefore, new method based on energy estimates for boundary value problems of hyperbolic system with Majda's modification (see [24]) is introduced.

The Euler system of inviscid compressible steady flow in three dimensional space is

$$\frac{\partial}{\partial x}\begin{pmatrix}\rho u\\ p+\rho u^2\\ \rho uv\\ \rho uw\end{pmatrix}+\frac{\partial}{\partial y}\begin{pmatrix}\rho v\\ \rho uv\\ p+\rho v^2\\ \rho vw\end{pmatrix}+\frac{\partial}{\partial z}\begin{pmatrix}\rho w\\ \rho uw\\ \rho vw\\ p+\rho w^2\end{pmatrix}=0 \tag{3.1}$$

with Bernoulli relation

$$\frac{1}{2}(u^2+v^2+w^2)+\frac{\gamma p}{(\gamma-1)\rho}=\text{const.} \tag{3.2}$$

Assume $\ell : x = h(z), y = g(z)$ is a given curve in the space $Oxyz$. $h(0) = h'(0) = g(0) = g'(0) = 0$, and $\Sigma_\pm : y = f_\pm(x, z)$ is a given surface of a wing with leading edge ℓ. If the incoming flow is supersonic and the angle $\beta_\pm = \arctan\left(\frac{\partial f_\pm}{\partial x}\right) - \arctan\left(\frac{v}{u}\right)$ between the flow and the surface of the wing is not large, then a shock front $S_\pm :\ y = \phi_\pm(x, z)$ attached at the leading edge will appear, satisfying $g(z) = \phi_\pm(h(z), z)$. Restrict ourselves in the upper part of the wing and omit the subscript sign $+$, the problem of supersonic flow past the wing is to determine the solution of system (3.1)–(3.2) in the domain between Σ and S. The solution has to satisfy the boundary condition on the surface of the wing

$$u\frac{\partial f}{\partial x}-v+w\frac{\partial f}{\partial z}=0 \quad \text{on } y=f(x,z), \tag{3.3}$$

and the Rankine-Hugoniot conditions on the shock front

$$\begin{bmatrix} \rho u \\ p+\rho u^2 \\ \rho uv \\ \rho uw \end{bmatrix} \phi_x - \begin{bmatrix} \rho v \\ \rho uv \\ p+\rho v^2 \\ \rho vw \end{bmatrix} + \begin{bmatrix} \rho w \\ \rho uw \\ \rho vw \\ p+\rho w^2 \end{bmatrix} \phi_z = 0 \quad \text{on } y=\phi(x,z). \tag{3.4}$$

In the problem the function $\phi(x,z)$, representing the location of the unknown shock, will be determined with (u,v,w,p,ρ) together.

Let (u_0,v_0,w_0,p_0,ρ_0) be the flow parameters of the incoming supersonic flow at the origin with $v_0=0$, $\alpha=\arctan f_x(0,0)$. Assume

(H_1): The incoming flow satisfies (3.1)–(3.2), and it belongs to H^N; $g(z), h(z)$ and $f(x,z)$ also belong to H^N.

(H_2): $u_0 > \sqrt{\frac{\gamma p_0}{\rho_0}}$.

(H_3): Denote by θ_{ext} the critical angle of the shock polar determined by u_0,p_0,ρ_0 (see [14]), then $\alpha < \theta_{\text{ext}}$.

Then the following conclusion is proved in [4].

Theorem 3.1 *Under assumptions* (H_1)–(H_3) *with* $N \geq 5$, *there exists a neighbourhood* Ω *near the origin on the plane* xOz, *an* H^{N+1} *function* $\phi(x,z)$ *defined on* Ω, *and* H^N *functions* u,v,w,p,ρ *defined on* $G=\{(x,y,z);(x,z)\in\Omega, f(x,z)<y<\phi(x,z)\}$, *such that* (3.1)–(3.4) *are satisfied.*

Let us briefly describe the method employed in [4]. First we straighten the domain G in between the surface of the body and the unknown shock, as well as the edge of the wing by a coordinate transformation. Under such a transformation the problem to determine the solution in G becomes the following nonlinear problem

$$\begin{cases} L(U,\phi)U=0, & \text{in } \alpha>0,\ \beta>0, \\ \ell U=0, & \text{on } \alpha=0, \\ \mathbf{F}(\alpha,z,U,\phi,\nabla\phi)=0, & \text{on } \beta=0; \quad \phi|_{\alpha=0}=0, \end{cases} \tag{3.5}$$

where $\alpha=0, \beta=0$ are the image of the surface of the wing and the shock front respectively. The image of the domain G is the quadrant $\widetilde{G}:\alpha>0,\beta>0$. The third equation in (3.5) is the Rankine-Hugoniot condition in fact. The next step is to linearize the problem (3.5) as

$$\begin{cases} L(U,\phi)\delta U \triangleq A\dfrac{\partial \delta U}{\partial \alpha} + B\dfrac{\partial \delta U}{\partial \beta} + Q\dfrac{\partial \delta U}{\partial z} = f, & \text{in } \alpha>0,\ \beta>0, \\ \ell\delta U = 0, & \text{on } \alpha=0, \\ F(\delta U,\delta\phi) \triangleq p\dfrac{\partial \phi}{\partial t} + q\dfrac{\partial \delta\phi}{\partial z} + h\delta\phi + m\delta U = g, & \text{on } \beta=0, \\ \delta\phi|_{\alpha=0}=0, & \end{cases} \tag{3.6}$$

where F is the Frechet derivative of $\mathbf{F}$.

By diadic decomposition and dilation of the domain, we reduce a typical piece of $\widetilde{G}$ to a domain Σ with normal size, as well as reduce the boundary value problem on this piece to a problem on Σ. By means of the method developed in [24] and the technique to treat characteristic boundary of nonlinear symmetric hyperbolic systems, we establish the necessary estimates for the linearized problem on Σ and then on the corresponding piece of $\widetilde{G}$. Then by summarizing these estimates we establish an estimate on whole $\widetilde{G}$, which finally leads to the local existence of the solution of nonlinear problem (3.5) and the result in Theorem 3.1.

4 Supersonic Flow Past a Pointed Body

In this section we review our study on supersonic flow past a pointed body, which may represent the local shape of supersonic airplane. Based on the result given in [14] on supersonic flow past a circular cone, we assume the conical body in our study is a perturbation of a circular cone. In what follows we use the cylindrical coordinate system to describe the problem for our convenience.

Using the model of potential flow equation, the problem of supersonic flow past a pointed body can be formulated as

$$\begin{aligned}&\Big(\frac{v_1^2}{c^2}-1\Big)\phi_{x_1x_1}+\Big(\frac{v_2^2}{c^2}-1\Big)\phi_{x_2x_2}+\Big(\frac{v_3^2}{c^2}-1\Big)\phi_{x_3x_3}\\&+\frac{2v_1v_2}{c^2}\phi_{x_1x_2}+\frac{2v_1v_3}{c^2}\phi_{x_1x_3}+\frac{2v_2v_3}{c^2}\phi_{x_2x_3}=0,\end{aligned}\tag{4.1}$$

where ϕ is the potential of velocity, satisfying $\nabla\phi=(v_1,v_2,v_3)$. The potential satisfies the boundary conditions on the surface of the conical body and the shock front. Introducing the cylindrical coordinates (R,θ,z) with $r=\frac{R}{z}$ the problem takes the following form

$$\begin{aligned}&z^2a_{00}\phi_{zz}+a_{11}\phi_{rr}+a_{22}\phi_{\theta\theta}+2za_{01}\phi_{zr}+2za_{02}\phi_{z\theta}\\&+2a_{12}\phi_{r\theta}+a_1\phi_r+a_2\phi_\theta=0,\end{aligned}\tag{4.2}$$

where

$$\begin{aligned}&a_{00}=\Big(\frac{v_3}{a}\Big)^2-1,\quad a_{11}=\frac{(v_r-rv_3)^2}{a^2}-(1+r^2),\quad a_{22}=\frac{1}{r^2}\Big(\frac{v_\theta^2}{a^2}-1\Big),\\&a_{01}=\frac{v_3v_r}{a^2}-r\Big(\frac{v_3^2}{a^2}-1\Big),\quad a_{02}=\frac{v_3v_\theta}{a^2r},\quad a_{12}=\frac{v_rv_\theta}{a^2r}-\frac{v_\theta v_3}{a^2},\\&a_1=\frac{v_\theta^2}{a^2r}-\frac{1}{r}+2r\Big(\frac{v_3^2}{a^2}-1\Big)-\frac{2v_3v_r}{a^2},\quad a_2=\frac{2v_rv_\theta}{a^2r^2},\end{aligned}\tag{4.3}$$

and v_r,v_θ,v_3 are the velocity in the direction of R,θ,z respectively. The boundary condition on the surface of the body is

$$(b+zb_z)\phi_z+\frac{b_\theta}{r^2}\Big(\frac{\phi_\theta}{z}\Big)-(1+b(b+zb_z))\Big(\frac{\phi_r}{z}\Big)=0,\tag{4.4}$$

where $r=b(\theta,z)$ is the equation of the surface of the body, which is a perturbation of $r=b_0$. While the condition on shock front is

$$\Big(\Big(\frac{\phi_r}{r}\Big)^2+\frac{1}{r^2}\Big(\frac{\phi_\theta}{z}\Big)^2+\Big(\phi_z-\frac{r\phi_r}{z}-q_\infty\Big)\Big(\phi_z-\frac{r\phi_r}{z}\Big)\Big)H$$

$$-\Big(\phi_z - \frac{r\phi_r}{z} - q_\infty\Big)q_\infty\rho_\infty = 0, \tag{4.5}$$

where $\rho_\infty, p_\infty, q_\infty$ are the density, pressure and speed of the unperturbed flow respectively, H is the inverse function of $\frac{A\gamma}{\gamma-1}\rho^{\gamma-1}$, and γ is the adiabatic index, the constant A comes from the pressure-density relation $p = A\rho^\gamma$.

In [5] we proved the conclusion on the existence of the boundary problem (4.3)–(4.5) under the following assumptions

(H_1) : $q_\infty > \big(\frac{\gamma p_\infty}{\rho_\infty}\big)^{\frac{1}{2}}$.

(H_2) : $\max\limits_{z<z_0, 0\le\theta\le 2\pi} b(\theta) < b_*$, where b_* is a fixed number, determined by the shock polar corresponding to the incoming flow.

(H_3) : For a suitable integer k_1 and small number ε,

$$\|b(0,\theta) - b_0\|_{C^{k_1}} \le \varepsilon.$$

(H_4) : For a suitable integer k_2,

$$\partial_z^k b(0,\theta) = 0 \quad \text{for } 1 \le k \le k_2.$$

Then the following theorem on the local existence of the solution with an attached shock was proved.

Theorem 4.1 *Assume that the conditions* (H_1)–(H_4) *are satisfied for sufficiently small* ε, *then we can find a number* $z_0 > 0$, *such that there is a* C^2 *function* $\phi(z,r,\theta)$ *defined in* $0 \le z \le z_0$, *satisfying the following conditions*:

1) $\phi(0,r,\theta) = 0$, $\phi_r > 0$ *for* $z > 0$, *and then the equation* $\phi(z,r,\theta) = q_\infty z$ *defines a surface* $r = s(z,\theta)$.

2) $\phi(z,r,\theta)$ *satisfies* (4.2) *in* $b(z,\theta) < r < s(z,\theta)$, $0 < z < z_0$, $0 \le \theta \le 2\pi$; (4.4) *on* $r = b(z,\theta)$; (4.5) *on* $r = s(z,\theta)$.

3) $\big|\big(\frac{\phi_r}{z}\big)^2 + \frac{1}{r^2}\big(\frac{\phi_\theta}{z}\big)^2 + \big(\phi_z - \frac{r\phi_r}{z} - q_\infty\big)\big(\phi_z - \frac{r\phi_r}{z}\big)\big| < \big|\phi_z - \frac{r\phi_r}{z} - q_\infty\big|q_\infty$ *on* $r = s(z,\theta)$.

In a word, the problem (4.2), (4.4)–(4.5) *admits a weak entropy solution with a pointed shock front attached at the origin, provided* ε *is small enough.*

The assumptions (H_1)–(H_4) have their physical explanations. (H_1) means the coming flow is supersonic. (H_2) means that the conical body must be rather sharp because otherwise, the shock ahead of the body will be detached as [14] indicated. The last assumptions $(\mathrm{H}_3), (\mathrm{H}_4)$ mean that the conical body under consideration is near to a symmetric circular cone. (H_3) indicates the circular perturbation is small, while (H_4) requires the surface of the body is tangent to the corresponding straight cone at the tip in higher order.

The theorem was proved in [5]. The key point is to find an approximate solution with error $O(z^N)$ for suitable large N, then the Newton iteration technique can be applied to look for the accurate solution of the original nonlinear problem. Hence we first expand the unknown potential function $\phi(z,r,\theta)$ as a serious of z,

$$\phi(z,r,\theta)=\sum_{n=0}^{N} z^{n+1}\phi_n(r,\theta)+O(z^{N+2}), \tag{4.6}$$

and also expand the functions $b(z,\theta)$ and $s(z,\theta)$ describing the boundary and shock front as serious of z. By substituting them into (4.2), (4.4)–(4.5) and comparing the terms with same power of z, we obtain a set of boundary value problems, which may determine all terms $\phi_n(r,\theta)$ in (4.6). Such an idea was first introduced to treat symmetric version of the problem (4.2), (4.4)–(4.5) in [9].

In fact, the problem to determine $\phi_0(r,\theta)$ is nothing but the problem to determine the flow field caused by the same incoming flow past a cone formed by the tangential surface to the original curved conical body. It takes the form

$$\begin{cases} a_{11}(*)\phi_{0rr}+a_{22}(*)\phi_{0\theta\theta}+2a_{12}(*)\phi_{0r\theta}+A(\phi_0,\phi_{0r},\phi_{0\theta})=0, \\ b_0\phi_0+\dfrac{1}{b_0^2}b_{0\theta}\phi_{0\theta}-(1+b_0^2)\phi_{0r}=0 \quad \text{on } r=b_0(\theta), \\ \phi_{0r}^2+\dfrac{1}{r^2}\phi_{0\theta}^2+(\phi_0-r\phi_{0r})\rho_0=-r\phi_{0r}q_\infty\rho_\infty \quad \text{on } \phi_0=q_\infty, \end{cases} \tag{4.7}$$

where $a_{ij}(*)$ stands for the value of a_{ij} at $\left(\phi_0-r\phi_{0r},\phi_{0r},\frac{1}{r}\phi_{0\theta}\right)$, $b_0(\theta)=b(0,\theta)$. The problem (4.7) is a free boundary value problem of a nonlinear elliptic equation on (r,θ) plane. To find the solution we combine some methods incluing partial hodograph transformation, domain decomposition and nonlinear Schwarz iteration. The details can be found in [5].

The paper [10] gives a simpler proof of Theorem 4.1. But generally the assumption on the shape of the conical body is more restrictive than that in [5].

The "higher order tangency" assumption (H_4) was relieved in [11]. The condition (H_4) can be replaced by

$$M>\sqrt{1+r^2}+\frac{2(1+r^2)}{(\gamma+1)r^2}, \tag{4.8}$$

where $M=\frac{|\vec{v}|}{c}$ is the Mach number of the flow. Therefore, the conclusion of Theorem 4.1 still holds, if the assumptions (H_1)–(H_4) are replaced by (H_1)–(H_3) with (4.8). The proof of the new result relies on the careful analysis on the coefficients of auxiliary equations for $\phi_n(r,\theta)$, so that the maximal principle can be applied there.

5 Supersonic Flow Past a Delta Wing

Most modern supersonic aircraft, such as hypersonic plane or space shuttles with high Mach number, are designed as a triangle, or a delta wing. Hence the study of supersonic flow passing a delta wing is also of great practical importance. When a supersonic flow with high Mach number past a delta wing, if the angle between the incoming flow and the surface of the wing is suitable small, and the vertex angle $2\sigma_1$ of the wing is close to π $\big(\sigma = \frac{\pi}{2} - \sigma_1 \sim 0\big)$, then the shock front will be attached to the leading edge of the wing. It means that the surface of the wing is completely covered by a shock front.

In [13] we use the potential flow equation to describe the problem of supersonic flow past a delta wing. Assume that a plain triangle wing U locates on the Ox_2x_3 plane with two edges $\{x_1 = 0, x_2 = \pm x_3 \cot\sigma\}$. The expected shock front $\Lambda : x_1 = S(x_2, x_3)$ locates above the triangle wing and is attached at the two edges. Then the problem is reduced to look for a potential $\Phi(x_1, x_2, x_3)$ in the domain between U and Λ, satisfying the equation

$$\begin{aligned}&(c^2 - \Phi_{x_1}^2)\Phi_{x_1x_1} + (c^2 - \Phi_{x_2}^2)\Phi_{x_2x_2} + (c^2 - \Phi_{x_3}^2)\Phi_{x_3x_3}\\&- 2\Phi_{x_1}\Phi_{x_2}\Phi_{x_1x_2} - 2\Phi_{x_2}\Phi_{x_3}\Phi_{x_2x_3} - 2\Phi_{x_1}\Phi_{x_3}\Phi_{x_1x_3} = 0.\end{aligned} \tag{5.1}$$

Correspondingly, the boundary conditions are

$$\Phi_{x_1} = 0 \quad \text{on } x_1 = 0, \quad \Phi_{x_2} = 0 \quad \text{on } x_2 = 0, \tag{5.2}$$

and Rankine-Hugoniot conditions are

$$\Phi = \Phi_0, \quad [\rho(\nabla\Phi)\nabla\Phi]\cdot\vec{n} = 0 \quad \text{on } \Lambda, \tag{5.3}$$

where Φ_0 is the potential of the incoming flow, $\vec{n}$ is the normal vector of Λ, and the bracket means the jump of the quantity inside it. The second condition in (5.2) holds due to the symmetry of the problem, so that one can discuss the problem in $x_2 > 0$ with an additional boundary condition on $x_2 = 0$.

Since (5.1) and the boundary conditions (5.2)–(5.3) are invariant under a self-similar coordinate transformation $x_i \to sx_i$ $(i = 1, 2, 3)$, one can introduce new variables $\xi = \frac{x_1}{x_3}$, $\eta = \frac{x_2}{x_3}$ and reduce the problem (5.1)–(5.3) to a boundary value problem on (ξ, η) plane. On the (ξ, η) plane the wing U is transformed to an interval $\{\xi = 0, -\cot\sigma < \eta < \cot\sigma\}$, and the edge of the wing becomes two points $(0, \pm\cot\sigma)$. The unknown function is a reduced potential $\psi(\xi, \eta)$, such that $\Phi(x_1, x_2, x_3) = x_3\psi\big(\frac{x_1}{x_3}, \frac{x_2}{x_3}\big)$. Meanwhile, the function $\psi(\xi, \eta)$ satisfies

$$\begin{aligned}&(c^2(1 + \xi^2) - (\psi_\xi - \xi h)^2)\psi_{\xi\xi} + 2(c^2\xi\eta - (\psi_\xi - \xi h)(\psi_\eta - \eta h))\psi_{\xi\eta}\\&+ (c^2(1 + \eta^2) - (\psi_\eta - \eta h)^2)\psi_{\eta\eta} = 0,\end{aligned} \tag{5.4}$$

where $\psi_\xi = u, \psi_\eta = v\cos\omega - w\sin\omega, h = v\sin\omega + w\cos\omega$, and ω is a constant depending on σ and the velocity $\vec{v}_0$ of the incoming flow.

The discriminant of (5.4) is

$$\Delta = c^2(c^2(1 + \xi^2 + \eta^2) - (\psi_\xi - \eta h)^2 - (\psi_\eta - \xi h)^2 - (\xi\psi_\eta - \eta\psi_\xi)^2). \tag{5.5}$$

Obviously, Δ is negative on the edge of the wing and is positive at the origin. Therefore, (5.4) is hyperbolic near the edge and is elliptic near the origin. In a word, it is a nonlinear mixed type equation. By applying the property on finite propagation speed for hyperbolic equation, the flow near the edge is constant, and can be determined by Rankine-Hugoniot equations up to the degenerate line. Hence by applying an additional coordinate transformation the boundary value problem in (ξ, η) plane takes the form

$$\begin{cases} \text{Equation (5.4)}, \\ \psi = 0 \quad \text{on characteristic line}, \\ -\rho_{1\sigma}\widehat{\xi}(\widehat{c}^2 - \widehat{\xi}^2)\psi_\xi + (\rho_{1\sigma}\widehat{\xi}^2 + \rho_0\widehat{c}^2)(\eta\psi_\eta - \psi) + \vec{E}\cdot(\psi, \psi_\xi, \psi_\eta) = 0 \quad \text{on shock}, \\ -\dfrac{1}{2}(\sin 2\omega)\widehat{\xi}\psi_\xi + \psi_\eta + \dfrac{1}{2}(\sin 2\omega)\psi + w_{1\sigma}\sin\omega = 0 \quad \text{on } \eta = -\tan\omega, \\ \psi_\xi = 0 \quad \text{on } \xi = 0, \end{cases} \tag{5.6}$$

where $\widehat{\xi} = \frac{q_{1\sigma} - w_0\cos\omega}{u_0}$, E is a given functions of ψ and $\nabla\psi$, whose expressions is omitted here.

Inside the domain bounded by the characteristic line, the shock, the surface of the wing and the symmetric line (5.4) is elliptic, but it is also degenerate on the characteristic line. Therefore, problem (5.6) is a boundary value problem of a nonlinear degenerate elliptic equation.

The existence of solution to problem (5.6) was established via a delicate approximate process. The usual iteration is to substitute an approximate solution into the coefficients of the nonlinear equation (or boundary conditions, if necessary), then by solving the reduced linear problem to obtain a new approximate solution. However, due to the degeneracy of the problem such a substitution may let the boundary become non-characteristic, so that the regular iteration process is unsustainable. Therefore, in each step we are looking for an approximate solution with assigned singularity at the degenerate boundary, so that the boundary is always characteristics of the reduced approximate equation in the whole iterative process. Hence the whole approximate process is split to several sub-process, in each one only a part of nonlinearity is linearized. The approach is first employed by [3] to treat the problem of a supersonic flow attacking a ramp. By applying their technique we obtained the existence of the solution of (5.6). The details can be found in [13].

6 Global Existence

In all cases discussed in above sections the flow behind the attached shock is still supersonic. Then to establish the global existence of solution to the problem of supersonic flow past a sharp body is possible. Notice that the solution of nonlinear hyperbolic equations can produce singularities due to collection of characteristics, then we have to assume the surface of the body is not far from a circular cone globally. In [12] the authors obtained the global solution with a shock attached at the tip for supersonic flow past a symmetric perturbed cone. The solution between the shock and the body is smooth and tends to a self-similar solution at infinity.

When the conical body is axisymmetric, the assumption (H_3) is Section 4 automatically holds. With an additional assumption on asymptotic behavior of the shape of the body at infinity, the following conclusion for polytropic gas is proven in [12].

Theorem 6.1 *Assume that a curved and symmetric cone is given by $r = b(z)$, which satisfies*

$$b(0) = 0, \quad b'(0) = b_0, \quad b^{(k)}(0) = 0, \quad 2 \le k \le k_1, \tag{6.1}$$

$$\left| z^{k-1} \frac{d^k}{dz^k} (b(z) - b_0 z) \right| \le \varepsilon \quad \text{for } 0 \le k \le k_2,\ z > 0, \tag{6.2}$$

where k_1, k_2 are suitable integers. Suppose that a supersonic flow parallel to the z-axis comes from infinity with velocity q_∞, density ρ_∞ satisfying $q_\infty > c_\infty = \sqrt{\gamma}\rho_\infty^{\frac{\gamma-1}{2}}$. Additionally, $0 < b_0 < \sqrt{2} - 1$ is assumed to be less than the critical value determined by q_∞ and ρ_∞. Then for sufficiently large q_∞ and sufficiently small ε depending on $q_\infty, \rho_\infty, b_0, k_1, k_2$, the problem (6.1) *admits a global weak entropy solution with a pointed shock front attached at the origin. Moreover, the location of the shock front and the flow field between the shock and the surface of the body tend to the corresponding ones for the flow past the unperturbed circular cone $r = b_0 z$ with the rate $z^{-\frac{1}{4}}$.*

Later, [20, 26] and some other works extended the result to non-symmetric perturbed cone. Meanwhile, based on Glimm's scheme method, the authors of [23] also studied the global stability of supersonic flow past a conical body.

7 Remark on Transonic Shocks

When a given supersonic flow runs across an oblique shock front, if the turning angle of the flow at the shock is given, then one can determine the location of the shock and the flow behind the shock. However, as Courant and Friedrichs indicated in [14], if the turning angle is less than a critical value, then there are two possibilities on the location of the shock and the state of downward flow behind the shock. Two possible shocks have different strength: Weak one and strong one. For the weak shock, the downward flow behind the shock is still supersonic, while for the strong shock the downward flow behind the shock is subsonic. All attached shocks discussed in preceding sections of this paper are weak ones, which and the flow behind it are stable. In [14] the authors put forward such a problem: If the attached shock is a strong shock, then are the shock and the downward flow running across the shock still stable?

In the discussion on supersonic flow passing a wedge, we studied the possibility of appearance of the stronger attached shock and its stability (see [8]). It is found that if one add an additional restriction to the state of the downward flow at infinity (i.e., the asymptotic behaviour of the downward flow), then the strong attached shock and the subsonic flow can also be stable. The strong shock connecting the upward supersonic flow and the downward subsonic flow is called transonic shock. In this sense we say that the strong transonic flow is conditionally stable. Next let us briefly introduce the proof of conditional stability of the transonic shock given in [8].

We again restrict ourselves to the case of potential flow with two variables. Let ϕ be the velocity potential, then the equation is (as (1.2))

$$(c^2-\phi_{x_1}^2)\phi_{x_1x_1}-2\phi_{x_1}\phi_{x_2}\phi_{x_1x_2}+(c^2-\phi_{x_2})\phi_{x_2x_2}=0. \tag{7.1}$$

To simplify notations we take the coordinate system so that the upper surface of wedge with vertex angle $2\alpha_0$ is $\{W: x_2=0\}$, the velocity of the unperturbed incoming flow parallel to the symmetric axis of the wedge is $(q_\ell\cos\alpha_0, -q_\ell\sin\alpha_0)$, the velocity of the unperturbed flow field behind the shock is $(q_r,0)$ and the unperturbed strong attached shock is $x_2=kx_1$. Then the unperturbed flow potential is

$$\begin{cases}\phi_0^\ell(x_1,x_2)=x_1q_\ell\cos\alpha_0-x_2q_\ell\sin\alpha_0,\\ \phi_0^r(x_1,x_2)=x_1q_r.\end{cases} \tag{7.2}$$

The continuity of the potential on $x_2=kx_1$ yields

$$q_\ell(\cos\alpha_0-k\sin\alpha_0)=q_r. \tag{7.3}$$

Besides, Rankine-Hugoniot condition implies

$$(\rho(q_r)q_r-\rho(q_\ell)q_\ell\cos\alpha_0)(q_r-q_\ell\cos\alpha_0)+\rho(q_\ell)(q_\ell\sin\alpha_0)^2=0, \tag{7.4}$$

where

$$\rho=\Big(1-\frac{\gamma-1}{2}q^2\Big)^{\frac{1}{\gamma-1}}.$$

Since the flow behind the transonic shock is subsonic, (7.1) in this region is elliptic. Hence a new restriction on the downstream part is required. The restriction is represented by the functional space to which the potential belongs.

For an unbounded domain D in R^2 we denote $r_x=(x_1^2+x_2^2)^{\frac12}, r_{x,y}=\min(r_x,r_y)$, and define the weighted Hölder norms

$$\begin{aligned}
[u]^{(k,\ell)}_{m,0;\mathsf{D}} &= \sum_{|\beta|=m}\Big(\sup_{x\in\mathsf{D},0<r_x<1}|r_x^{k+m}D^\beta u(x)|+\sup_{x\in\mathsf{D},r_x\ge1}|r_x^{\ell+m}D^\beta u(x)|\Big),\\
[u]^{(k,\ell)}_{m,\alpha;\mathsf{D}} &= \sum_{|\beta|=m}\Big(\sup_{x,y\in\mathsf{D},0<r_{x,y}<1} r_{x,y}^{k+m+\alpha}\frac{|D^\beta u(x)-D^\beta u(y)|}{|x-y|^\alpha}\\
&\qquad+\sup_{x,y\in\mathsf{D},r_{x,y}\ge1} r_{x,y}^{\ell+m+\alpha}\frac{|D^\beta u(x)-D^\beta u(y)|}{|x-y|^\alpha}\Big),\\
\|u\|^{(k,\ell)}_{m,0,\mathsf{D}} &= \sum_{j=0}^m [u]^{(k,\ell)}_{j,0;\mathsf{D}},\\
\|u\|^{(k,\ell)}_{m,\alpha,\mathsf{D}} &= \|u\|^{(k,\ell)}_{m,0,\mathsf{D}}+[u]^{(k,\ell)}_{m,\alpha,\mathsf{D}},
\end{aligned}$$

and corresponding space

$$H^{(k,\ell)}_{m,\alpha}=\{u\in C^{m,\alpha}_{\mathrm{loc}}(\mathsf{D}\setminus\{0\}); \|u\|^{(k,\ell)}_{m,\alpha;\mathsf{D}}<+\infty\}.$$

Then we consider the solution of the related problem in this weighted Hölder space. The unperturbed flow field ahead of the shock is given by the potential $\phi^\ell(x,y)$, it is the perturbation of $\phi_0^\ell(x,y)$.

Denote by S the expected transonic shock. It is a perturbation of the plain unperturbed transonic shock S_0. Denote by Ω_1^r the domain between S and W. To obtain the solution of the perturbed supersonic flow is to determine the location of S and the potential $\phi^+(x_1,x_2)$ in the domain Ω_1^r satisfying

$$\begin{cases} \sum_{i,j=1}^{2} a_{ij}(D\phi^r)\partial_{ij}\phi^r = 0 & \text{in } \Omega_1^r, \\ \phi^\ell(x_1,x_2) = \phi^r(x_1,x_2) & \text{on } S, \\ \nu_1(\rho_r\partial_{x_1}\phi^r - \rho_\ell\partial_{x_1}\phi^\ell) + \nu_2(\rho_r\partial_{x_2}\phi^r - \rho_\ell\partial_{x_2}\phi^\ell) = 0 & \text{on } S, \\ \partial_{x_2}\phi^r = 0 & \text{on } W, \\ \phi^r(0,0) = 0, \quad \lim_{|x|\to\infty} |D\phi^r| \text{ exists}, \end{cases} \tag{7.5}$$

where $\nu = (\nu_1,\nu_2)$ is the normal direction of S, pointed from Ω_1^ℓ to Ω_1^r, $\rho_r = \rho(|D\phi^r|)$, $\rho_\ell = \rho(|D\phi^\ell|)$.

The following conclusion is proven in [8].

Theorem 7.1 *Suppose that* $0 < \alpha < 1, q_- > c_*, q_+ < c_*$, *and* q_-, q_+, α_0 *satisfy* (7.4) *and*

$$\mu_0 \equiv \rho(q_r)\Big(1 - \frac{q_r^2}{c_r^2}\Big)(q_\ell\cos\alpha_0 - q_r)^2 - \rho(q_\ell)(q_\ell\sin\alpha_0)^2 > 0. \tag{7.6}$$

Then there exist constants $\delta_0, \delta_\infty \in (0,1)$ *and* $\sigma_0 > 0$, *such that for any* $\sigma \in (0,\sigma_0)$, *if*

$$\|\phi^\ell - \phi_0^\ell\|_{3,\alpha,\Omega_{10}^\ell}^{(-1-\delta_0,-1+\delta_\infty)} \le \sigma, \tag{7.7}$$

then the boundary value problem (7.5) *admits a unique solution* $\phi^r \in C^3(\overline{\Omega^r}\backslash\{0\})$, *satisfying estimate*

$$\|\phi^r - \phi_0^r\|_{3,\alpha,\Omega_1^r}^{(-1-\delta_0,-1+\delta_\infty)} \le C_1\sigma. \tag{7.8}$$

In [17] the author also proved the conditional stability of the transonic shock in the problem for non-isentropic supersonic flow past a curved wedge, where the potential equation is replaced by the full Euler system. Furthermore, by applying similar approach the result was extended to the problem of supersonic flow past a conical body in [2, 27]. In that case the transonic shock attached at the tip of the conical cone is also conditionally stable, if the downward flow behind the transonic shock is suitably fixed at infinity.

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