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Boundary Regularity for Minimal Graphs of Higher Codimensions*

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Abstract In this paper, the authors derive Hölder gradient estimates for graphic functions of minimal graphs of arbitrary codimensions over bounded open sets of Euclidean space under some suitable conditions.

Keywords Boundary regularity, Minimal graphs, Higher condimension, Bernstein type theorem
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1 Introduction

Let Ω be an open set in $\mathbb{R}^n$. A minimal graph $u = (u^1, \cdots, u^m)$ in $\mathbb{R}^{n+m}$ over Ω satisfies a minimal surface system of m quasilinear elliptic equations, where m is the codimension. More precisely, we have

$$g^{ij}\partial_{ij}^2 u^\alpha = 0 \quad \text{on } \Omega, \tag{1.1}$$

where (g^{ij}) is the inverse matrix of $g_{ij} = \delta_{ij} + \sum\limits_\alpha \partial_i u^\alpha \partial_j u^\alpha$ (see [15] for more details). One of the classical problems in the field is the Dirichlet problem, that is, to find solutions to (1.1) with

$$u^\alpha = \psi^\alpha \quad \text{on } \partial\Omega \tag{1.2}$$

for some given $\psi = (\psi^1, \cdots, \psi^m)$. As it turns out, in order to obtain the existence and regularity of solutions, some conditions on the geometry of the boundary of Ω and on the boundary data are needed.

For $m = 1$, the problem is quite well understood in the classical paper [5] of Jenkins and Serrin. For higher codimension, that is, for $m > 1$, the situation is more difficult and less well studied. Compared with Theorem 13.7 in [4], one of main difficulties is that g^{ij} contains $Du^1, \cdots, Du^m$, and the moduli of continuity of g^{ij} is unknown. A counterexample due to Lawson and Osserman [10] tells us that the situation is fundamentally different from the case $m = 1$.

It now turns out that a crucial analytical step in the solution of the boundary value problems for C^2 data consists in deriving a global $C^{1,\gamma}$-estimate (for some $\gamma \in (0, 1)$). An important

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step was taken by Thorpe [1] who showed (Lemma 5.2 in [1] which is formulated for maximal spacelike graphs in Minkowski space, but also works for minimal graphs in Euclidean space) that for C^3-boundary data on a bounded smooth domain, a solution with small C^1-norm satisfies a $C^{1,\gamma}$-estimate. It is more natural, however, to assume only a bound on the C^2-norm of the boundary data.

In this paper, using a different approach we therefore derive a uniform global $C^{1,\gamma}$-estimate for any solution u to the minimal surface system as follows (see Theorem 2.1 below).

Theorem 1.1 *Let Ω be a bounded open set in $\mathbb{R}^n$ with C^2-boundary, and $\psi \in C^2(\overline{\Omega}, \mathbb{R}^m)$. For each $\gamma \in (0,1)$, let $u = (u^1, \cdots, u^m) \in C^{1,\gamma}(\overline{\Omega}, \mathbb{R}^m)$ be a smooth solution to* (1.1) *on Ω with $u^\alpha = \psi^\alpha$ on $\partial\Omega$ for each $\alpha = 1, \cdots, m$. If the 2-dilation of u satisfies $\sup\limits_{\Omega} |\Lambda^2 \mathrm{d}u| \le \sqrt{2}$, then $|u|_{1+\gamma,\Omega}$ is bounded by a constant depending only on n, m, γ, $|Du|_\Omega$, $|\psi|_{2,\Omega}$ and κ_Ω (see* (2.11) *for its definition).*

In fact, the above $C^{1,\gamma}$-estimate still holds (depending on p) if $\psi \in C^2(\overline{\Omega}, \mathbb{R}^m)$ is replaced by $\psi \in W^{2,p}(\overline{\Omega}, \mathbb{R}^m)$ with $\gamma < 1 - \frac{n}{p}$. But, the condition on the 2-dilation cannot be removed in view of the counterexample of Lawson and Osserman [10]. The proof of Theorem 2.1 relies on Bernstein type theorems for minimal graphs over half-spaces and whole spaces, and a blow-up argument that would lead to a contradiction if we have a sequence of solutions with unbounded Hölder norms for their derivatives. Here, the Bernstein type theorem over half-spaces holds only under bounded gradient and linear boundary assumptions (see Lemma 2.2).

2 A Priori Hölder Gradient Estimates for Minimal Graphs

Let $\mathbb{R}^n$ be the standard n-dimensional Euclidean space. For an open set $\Omega \subset \mathbb{R}^n$, let $u = (u^1, \cdots, u^m)$ be a C^2 (vector-valued) function on Ω. The graph of u: $\{(x, u(x)) \in \mathbb{R}^n \times \mathbb{R}^m \mid x \in \Omega\}$ is said to be minimal if and only if

$$\begin{cases} \displaystyle\sum_{i=1}^n \frac{\partial}{\partial x_i}(\sqrt{\det g_{kl}}\, g^{ij}) = 0 & \text{for } j = 1, \cdots, n, \\ \displaystyle\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\Big(\sqrt{\det g_{kl}}\, g^{ij} \frac{\partial u^\alpha}{\partial x_j}\Big) = 0 & \text{for } \alpha = 1, \cdots, m, \end{cases} \tag{2.1}$$

where $g_{ij} = \delta_{ij} + \sum\limits_{\alpha=1}^m \partial_{x_i} u^\alpha \partial_{x_j} u^\alpha$, and (g^{ij}) is the inverse matrix of (g_{ij}). Writing $U(x) = (x, u(x))$, then (2.1) is equivalent to

$$\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\Big(\sqrt{\det g_{kl}}\, g^{ij} \frac{\partial U^a}{\partial x_j}\Big) = 0 \quad \text{for } a = 1, \cdots, n+m,$$

and hence (see [10, 15]), (2.1) is also equivalent to

$$\sum_{i,j=1}^n g^{ij} \frac{\partial^2 u^\alpha}{\partial x_i \partial x_j} = 0 \quad \text{for } \alpha = 1, \cdots, m. \tag{2.2}$$

Let $\mathbb{R}^n_+$ be the half space defined by $\{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$. Let $B_r(y)$ denote the ball in $\mathbb{R}^n$ with radius $r > 0$ and centered at $y \in \mathbb{R}^n$. We define

$$P_{\rho,r} = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid |x'| < r,\ 0 < x_n < \rho r\},$$

$$S_{\rho,r}=\{(x',x_n)\in\mathbb{R}^{n-1}\times\mathbb{R}\mid |x'|<r,\ \rho r<x_n<2\rho r\}$$

for all positive constants ρ, r.

Lemma 2.1 *Let φ be a positive function in $C^2(P_{\rho,3r})$ and*

$$\Delta_a\varphi\triangleq\frac{1}{\sqrt{\det a_{kl}}}\partial_{x_i}(\sqrt{\det a_{kl}}a^{ij}\partial_{x_j}\varphi)=0\quad \text{on } P_{\rho,3r},$$

where (a^{ij}) is the inverse matrix of (a_{ij}) with the coefficients a_{ij} satisfying

$$a_{ij}=a_{ji}\le\Lambda,\qquad \inf_{\xi=(\xi_1,\cdots,\xi_n)}a_{ij}\xi_i\xi_j\ge\lambda|\xi|^2$$

on $P_{\rho,3r}$ for some constants $0<\lambda\le\Lambda<\infty$. Then, for any fixed $\rho>0$ there is a constant $C_{\rho,\lambda,\Lambda}>0$ depending only on n,ρ,λ,Λ such that

$$\sup_{S_{\rho,r}}\varphi\le C_{\rho,\lambda,\Lambda}\inf_{S_{\rho,r}}\varphi\quad\text{for each } r>0. \tag{2.3}$$

Proof Let Σ be a Riemannian manifold with the metric $a_{ij}(x)\mathrm{d}x_i\mathrm{d}x_j$ for each $x\in P_{\rho,3r}$. Then φ is a harmonic function on Σ, and the metric of Σ is bi-Lipschitz to the standard Euclidean metric on $P_{\rho,3r}$. By the famous De Giorgi-Nash-Moser iteration, we have Harnack's inequality for the harmonic function φ on Σ (see the proof of Theorem 4.3 in [2] for instance). Namely, for any ball $B_{2s}(x)\subset P_{\rho,3r}$, there is a constant $C_{\lambda,\Lambda}>0$ depending only on n,λ,Λ such that

$$\sup_{B_s(x)}\varphi\le C_{\lambda,\Lambda}\inf_{B_{s/2}(x)}\varphi.$$

By finitely covering $\overline{S_{\rho,r}}$, we complete the proof.

Now let us state a Bernstein type theorem for minimal graphs over half-spaces.

Lemma 2.2 *Let l_α be an affine linear function in $\mathbb{R}^{n-1}$ for $\alpha=1,\cdots,m$. Assume that $u=(u^1,\cdots,u^m)\in C^1(\overline{\mathbb{R}^n_+},\mathbb{R}^m)\cap C^\infty(\mathbb{R}^n_+,\mathbb{R}^m)$ is a solution of the minimal surface system*

$$\begin{cases}g^{ij}\partial^2_{ij}u^\alpha=0 & \text{on } \mathbb{R}^n_+\\ u^\alpha=l_\alpha & \text{on } \partial\mathbb{R}^n_+\end{cases}\quad\text{for } \alpha=1,\cdots,m, \tag{2.4}$$

where (g^{ij}) is the inverse matrix of $g_{ij}=\delta_{ij}+\sum\limits_\alpha\partial_iu^\alpha\partial_ju^\alpha$. If $|Du|$ is uniformly bounded in $\mathbb{R}^n_+$, then u is affine linear.

Proof The proof uses the idea of the proof of Lemma 7.47 in [12] by Lieberman. From [13–14], u is smooth in $\mathbb{R}^n_+$. From the assumption, there is a constant $\lambda\in(0,1)$ such that $I_n\le(g_{ij})\le\lambda^{-1}I_n$, where I_n is the unit $(n\times n)$-matrix. Then $\lambda I_n\le(g^{ij})\le I_n$. For any vector $\xi=(\xi_1,\cdots,\xi_n),\eta=(\eta_1,\cdots,\eta_n)\in\mathbb{R}^n$, from the Cauchy-Schwarz inequality

$$|g^{ij}\xi_i\eta_j|\le|\xi|\cdot|\eta|,$$

which implies $|g^{ij}|\le 1$ for any $i,j=1,\cdots,n$. Denote $\rho_*=\frac{\lambda}{9\sqrt{n-1}}$. For any fixed $\alpha\in\{1,\cdots,m\}$, let

$$m_r=\inf_{(x',x_n)\in P_{\rho_*,r}}x_n^{-1}(u^\alpha(x',x_n)-l_\alpha(x')),\quad M_r=\sup_{(x',x_n)\in P_{\rho_*,r}}x_n^{-1}(u^\alpha(x',x_n)-l_\alpha(x')).$$

From Lemma 2.1, there is a general constant C depending only on n, m and $|Du|$ on $\mathbb{R}^n_+$ such that

$$\begin{aligned}&\sup_{(x',x_n)\in S_{\rho_*,2r}}(u^\alpha(x',x_n)-l_\alpha(x')-m_{6r}x_n)\\ \le{}& C\inf_{(x',x_n)\in S_{\rho_*,2r}}(u^\alpha(x',x_n)-l_\alpha(x')-m_{6r}x_n)\\ \le{}& Cr\inf_{(x',x_n)\in S_{\rho_*,2r}}\frac{u^\alpha(x',x_n)-l_\alpha(x')-m_{6r}x_n}{x_n}.\end{aligned}$$

Combining this and Lemma 3.1 in the appendix, one has

$$\begin{aligned}&\sup_{(x',x_n)\in S_{\rho_*,2r}}(u^\alpha(x',x_n)-l_\alpha(x')-m_{6r}x_n)\\ \le{}& Cr\inf_{(x',x_n)\in P_{\rho_*,r}}\frac{u^\alpha(x',x_n)-l_\alpha(x')-m_{6r}x_n}{x_n}\le Cr(m_r-m_{6r}).\end{aligned}\tag{2.5}$$

Similarly,

$$\begin{aligned}&\sup_{(x',x_n)\in S_{\rho_*,2r}}(M_{6r}x_n-u^\alpha(x',x_n)+l_\alpha(x'))\\ \le{}& C\inf_{(x',x_n)\in S_{\rho_*,2r}}(M_{6r}x_n-u^\alpha(x',x_n)+l_\alpha(x'))\\ \le{}& Cr\inf_{(x',x_n)\in S_{\rho_*,2r}}\frac{M_{6r}x_n-u^\alpha(x',x_n)+l_\alpha(x')}{x_n}\\ \le{}& Cr\inf_{(x',x_n)\in P_{\rho_*,r}}\frac{M_{6r}x_n-u^\alpha(x',x_n)+l_\alpha(x')}{x_n}\le Cr(M_{6r}-M_r).\end{aligned}\tag{2.6}$$

Combining (2.5)–(2.6), we have

$$M_{6r}-m_{6r}\le C(M_{6r}-m_{6r}-M_r+m_r),\tag{2.7}$$

which implies

$$M_r-m_r\le\frac{C-1}{C}(M_{6r}-m_{6r}).\tag{2.8}$$

By iteration, there is a constant $\theta\in(0,1)$ depending only on n, m and $|Du|$ on $\mathbb{R}^n_+$ such that

$$M_r-m_r\le C\Big(\frac{r}{R}\Big)^\theta(M_R-m_R)\tag{2.9}$$

for all $0<r<R<\infty$. Since $|Du|$ is uniformly bounded in $\mathbb{R}^n_+$, from the Newton-Leibniz formula, M_R, m_R are uniformly bounded independent of $R>0$. Letting $R\to\infty$ in (2.9) implies

$$M_r-m_r=0\quad\text{for all } r>0.\tag{2.10}$$

This means that $x_n^{-1}(u^\alpha(x',x_n)-l_\alpha(x'))$ is a constant on $\mathbb{R}^n_+$, which completes the proof.

Let Ω be a bounded open set in $\mathbb{R}^n$ with C^2-boundary, and let $\kappa_{1,\Omega}(x),\cdots,\kappa_{n-1,\Omega}(x)$ be the principal curvatures of $\partial\Omega$ at each $x\in\partial\Omega$. Denote

$$\kappa_\Omega=\max_{1\le i\le n,x\in\partial\Omega}|\kappa_{i,\Omega}(x)|.\tag{2.11}$$

Let us recall the local $W^{2,p}$-estimates for elliptic differential equations (see Theorem 9.4.1 and Theorem 11.3.2 in [9] for instance).

Lemma 2.3 *Let $\mathcal{L} = a^{ij}\partial^2_{x_ix_j} + b^i\partial_{x_i} + c$. Assume $\mathcal{L}$ is uniformly elliptic with $\lambda I_n \le (a^{ij}) \le \Lambda I_n$ for some constants $\Lambda > \lambda > 0$, and there is a continuous function ω on $\mathbb{R}^+$ such that $|a^{ij}(x) - a^{ij}(y)| \le \omega(|x-y|)$. Assume $|\omega| \le c_\Omega$ on $\mathbb{R}^+$, $|b^i| + |c| \le \mu_\Omega$ on Ω for some constant $\mu_\Omega > 0$. Then for each $f \in L^p(\Omega)$ with $1 < p < \infty$, there is a unique solution $w \in W^{2,p}_0(\Omega)$ to $\mathcal{L}w = f$ a.e. on Ω. Moreover, there is a constant $c_0 > 0$ depending only on $n, p, \lambda, \Lambda, R, \kappa_\Omega$ and μ_Ω such that for any $x \in \partial\Omega$*

$$\|w\|_{W^{2,p}(\Omega\cap B_R(x))} \le c_0(\|w\|_{L^p(\Omega\cap B_{2R}(x))} + \|f\|_{L^p(\Omega\cap B_{2R}(x))}). \tag{2.12}$$

We recall the standard Hölder norms. For any $\gamma \in (0,1]$, and any (vector-valued) function f defined on Ω, we set

$$[f]_{\gamma,\Omega}(x) = \sup_{y\in\Omega\setminus\{x\}} \frac{|f(y)-f(x)|}{|y-x|^\gamma} \quad \text{for any } x \in \Omega,$$

and $[f]_{\gamma,\Omega} = \sup\limits_{x\in\Omega}[f]_{\gamma,\Omega}(x)$. Denote $|f|_\Omega = \sup\limits_{x\in\Omega}|f(x)|$. For each nonnegative integer k, each constant $\gamma \in (0,1]$ and each point $x \in \Omega$, we set

$$|f|_{k+\gamma,\Omega}(x) = \sum_{0\le i\le k} |D^i f|(x) + [D^k f]_{\gamma,\Omega}(x), \tag{2.13}$$

and $|f|_{k+\gamma,\Omega} = \sup\limits_{x\in\Omega} |f|_{k+\gamma,\Omega}(x)$.

For any vector-valued function $f = (f^1, \cdots, f^m) \in C^1(\Omega, \mathbb{R}^m)$, we define the 2-dilation of f on Ω by

$$\sup_\Omega \left|\Lambda^2 \mathrm{d}f\right| = \sup_{x\in\Omega}\left|\Lambda^2\mathrm{d}f(x)\right| = \sup_{x\in\Omega, 1\le i<j\le n} \mu_i(x)\mu_j(x),$$

where $\{\mu_k(x)\}_{k=1}^n$ are the singular values of $\mathrm{d}f(x)$ (see [3] for more results). Now we derive a priori $C^{1,\gamma}$-estimates for minimal graphs with arbitrary codimension.

Theorem 2.1 *Let Ω be a bounded open set in $\mathbb{R}^n$ with C^2-boundary, and $\psi \in C^2(\overline{\Omega}, \mathbb{R}^m)$. For each $\gamma \in (0,1)$, let $u = (u^1, \cdots, u^m) \in C^{1,\gamma}(\overline{\Omega}, \mathbb{R}^m)$ be a smooth solution of the minimal surface system*

$$\begin{cases} g^{ij}\partial^2_{ij}u^\alpha = 0 & \text{on } \Omega \\ u^\alpha = \psi^\alpha & \text{on } \partial\Omega \end{cases} \quad \text{for } \alpha = 1, \cdots, m \tag{2.14}$$

with $g_{ij} = \delta_{ij} + \sum\limits_\alpha \partial_i u^\alpha \partial_j u^\alpha$. If $\sup\limits_\Omega |\Lambda^2 \mathrm{d}u| \le \sqrt{2}$, then $|u|_{1+\gamma,\Omega}$ is bounded by a constant depending only on n, m, γ, $|Du|_\Omega$, $|\psi|_{2,\Omega}$ and κ_Ω.

Proof Without loss of generality, we assume Ω is connected. Let us prove it by contradiction. Assume there are a sequence of domains Ω_k with $\limsup\limits_k \kappa_{\Omega_k} < \infty$ and a sequence of solutions $u_k \in C^{1,\gamma}(\overline{\Omega_k}, \mathbb{R}^m)$ to (2.14) with boundary data ψ_k satisfying $\limsup\limits_k |\psi_k|_{2,\Omega_k} < \infty$ so that $\sup\limits_{\Omega_k}|Du_k| \le c$, $\sup\limits_{\Omega_k}|\Lambda^2\mathrm{d}u_k| \le \sqrt{2}$ for some $c > 0$, and $|u_k|_{1+\gamma,\Omega_k} \to \infty$ as $k \to \infty$. Thus

$$[Du_k]_{\gamma,\Omega_k} = \sup_{x,y\in\Omega_k} |x-y|^{-\gamma}|Du_k(x) - Du_k(y)| = \lambda_k^\gamma$$

for some sequence of numbers λ_k converging to ∞. There are points $z_k \in \overline{\Omega_k}$ such that

$$[Du_k]_{\gamma,\Omega_k}(z_k) \ge (1-k^{-1})^\gamma \lambda_k^\gamma.$$

Set

$$\widetilde{u_k}(x)=\lambda_k\Big(u_k\Big(\frac{x}{\lambda_k}+z_k\Big)-u_k(z_k)\Big),\quad \widetilde{\psi_k}(x)=\lambda_k\Big(\psi_k\Big(\frac{x}{\lambda_k}+z_k\Big)-\psi_k(z_k)\Big),$$

and $\widetilde{\Omega_k}=\lambda_k(\Omega_k-z_k)$. For any $\delta>0$, there are points $y_k\in\overline{\Omega_k}$ such that $|Du_k(y_k)-Du_k(z_k)|\geq(1-(1+\delta)k^{-1})^\gamma\lambda_k^\gamma|y_k-z_k|^\gamma$. So we have

$$\begin{aligned}&|D\widetilde{u_k}(\lambda_k(y_k-z_k))-D\widetilde{u_k}(0)|=|Du_k(y_k)-Du_k(z_k)|\\&\geq(1-(1+\varepsilon)k^{-1})^\gamma\lambda_k^\gamma|y_k-z_k|^\gamma=|(1-(1+\varepsilon)k^{-1})\lambda_k(y_k-z_k)|^\gamma.\end{aligned}\tag{2.15}$$

For any $\xi_k,\eta_k\in\widetilde{\Omega_k}$,

$$\begin{aligned}|D\widetilde{u_k}(\xi_k)-D\widetilde{u_k}(\eta_k)|&=\Big|Du_k\Big(\frac{\xi_k}{\lambda_k}+z_k\Big)-Du_k\Big(\frac{\eta_k}{\lambda_k}+z_k\Big)\Big|\\&\leq\lambda_k^\gamma\Big|\frac{\xi_k}{\lambda_k}-\frac{\eta_k}{\lambda_k}\Big|^\gamma=|\xi_k-\eta_k|^\gamma.\end{aligned}\tag{2.16}$$

Hence we have

$$[D\widetilde{u_k}]_{\gamma,\widetilde{\Omega_k}}(0)\geq(1-k^{-1})^\gamma$$

and

$$[D\widetilde{u_k}]_{\gamma,\widetilde{\Omega_k}}(x)\leq 1$$

for each $x\in\widetilde{\Omega_k}$. In particular, $\widetilde{u_k}$ satisfies the minimal surface system with $\widetilde{u_k}=\widetilde{\psi_k}$ on $\partial\widetilde{\Omega_k}$. It is clear that $\widetilde{\Omega_k}$ converges to a domain Ω_∞, which is $\mathbb{R}^n$ or

$$\mathbb{R}^n_{\omega,\tau}\triangleq\{x\in\mathbb{R}^n\mid\langle x,\omega\rangle<\tau\}$$

for some $(\omega,\tau)\in\mathbb{S}^{n-1}\times\mathbb{R}\subset\mathbb{R}^n\times\mathbb{R}$. Here, $\mathbb{R}^n_{\omega,\tau}$ is a half space perpendicular to the ω direction.

Denote $M_k=\mathrm{graph}_{\widetilde{u_k}}=\{(x,\widetilde{u_k}(x))\in\mathbb{R}^n\times\mathbb{R}^m\mid x\in\widetilde{\Omega_k}\}$. We use $|M_k|$ to denote the multiplicity one varifold associated with M_k, i.e., the n-rectfiable varifold with the support $\overline{M_k}$ and the multiplicity one on M_k. By the compactness of varifolds (see [11] for instance), there is a subsequence $|M_{i_k}|$ of $|M_k|$ converging to a stationary varifold T_∞ in the varifold sense, whose support can be represented as a graph over Ω_∞ with the $C^{1,\gamma}$ graphic function u_∞ such that $|Du_\infty|_{\Omega_\infty}\leq c$, $[Du_\infty]_{\gamma,\Omega_\infty}\leq 1$, $\sup\limits_{\Omega_\infty}|\Lambda^2 \mathrm{d}u_\infty|\leq\sqrt{2}$ and $u_\infty=(u_\infty^1,\cdots,u_\infty^m)$ is linear on $\partial\Omega_\infty$. By the Schauder estimates (see [4] for instance), u_∞ is smooth on Ω_∞. If $\Omega_\infty=\mathbb{R}^n_{\omega,\tau}$ for some $(\omega,\tau)\in\mathbb{S}^{n-1}\times\mathbb{R}$, then u_∞ is a linear vector-valued function according to Lemma 2.2, and $\widetilde{u_{i_k}}$ converge to u_∞ in the sense of C^1-norm. From (2.7) in [7], $\det(\delta_{ij}+\partial_i u_\infty^\alpha\partial_j u_\infty^\alpha)$ is a strictly subharmonic function on $\mathrm{graph}_{u_\infty}$. If $\Omega_\infty=\mathbb{R}^n$, then u_∞ is also a linear vector-valued function from Theorem 7.1 in [3].

Let us deduce the contradiction for the case of $\Omega_\infty=\mathbb{R}^n_{\omega,\tau}$ first. For any $R\geq 4\max\{c,\tau\}$, $\sup\limits_{\widetilde{\Omega_k}}|D\widetilde{u_k}|\leq\sup\limits_{\Omega_k}|Du_k|\leq c$ implies

$$[D\widetilde{u_k}]_{\gamma,\widetilde{\Omega_k}\cap B_R(0)}(0)=[D\widetilde{u_k}]_{\gamma,\widetilde{\Omega_k}}(0)\geq(1-k^{-1})^\gamma.$$

Since $[D\widetilde{u_k}]_{\gamma,\widetilde{\Omega_k}}\leq 1$, $|\widetilde{\psi_k}|_{2,\widetilde{\Omega_k}}$ are uniformly bounded and the maximal principal curvature $\kappa_{\widetilde{\Omega_k}}\to 0$, by Lemma 2.3 and the uniqueness theorem (see Theorem 8.1 in [4] for instance), $\widetilde{u_k}\in W^{2,p}(\widetilde{\Omega_k})$ with $p=\frac{2n}{1-\gamma}$, and $|\widetilde{u_k}|_{W^{2,p}(\widetilde{\Omega_k}\cap B_{2R}(0))}$ is bounded independent of k from (2.12).

Then the Sobolev imbedding theorem implies that there is a constant $0<\varepsilon_{\gamma,R}<1$ independent of k such that

$$[\widetilde{u_k}]_{\frac{1+\gamma}{2},\widetilde{\Omega_k}\cap B_R(0)}\le\frac{1}{\varepsilon_{\gamma,R}}.$$

Up to a choice of $\varepsilon_{\gamma,R}$, there is a sequence of points $\xi_k\in\widetilde{\Omega_k}\cap B_R(0)\setminus B_{\varepsilon_{\gamma,R}}(0)$ such that

$$|D\widetilde{u_k}(\xi_k)-D\widetilde{u_k}(0)|\ge(1-k^{-1})|\xi_k|^\gamma. \tag{2.17}$$

However, (2.17) contradicts to that $\widetilde{u_{i_k}}$ converges to a linear function in the sense of C^1-norm. Hence $\Omega_\infty\neq\mathbb{R}^n_{\omega,\tau}$.

For the case of $\Omega_\infty=\mathbb{R}^n$, we can also get the contradiction from the above argument. This suffices to complete the proof.

For any vector-valued function $f=(f^1,\cdots,f^m)\in C^2(\Omega,\mathbb{R}^m)$, set v_f deonte the slope function of f defined by $\sqrt{\det\left(\delta_{ij}+\sum\limits_\alpha\partial_if^\alpha\partial_jf^\alpha\right)}$. With the Bernstein theorem in higher codimension (see [6–8]), from the argument of the proof of Theorem 2.1, we immediately have the following result.

Corollary 2.1 *Let Ω be a bounded open set in $\mathbb{R}^n$ with C^2-boundary, and $\psi\in C^2(\overline{\Omega},\mathbb{R}^m)$. For each $\gamma\in(0,1)$, let $u=(u^1,\cdots,u^m)\in C^{1,\gamma}(\overline{\Omega},\mathbb{R}^m)$ be a smooth solution of the minimal surface system on Ω with $u=\psi$ on $\partial\Omega$. If $\sup\limits_\Omega v_u\le3$, then $|u|_{1+\gamma,\Omega}$ is bounded by a constant depending only on n, m, γ, $|Du|_\Omega$, $|\psi|_{2,\Omega}$ and κ_Ω.*

3 Appendix

Let

$$P_{\rho,r}=\{(x',x_n)\in\mathbb{R}^{n-1}\times\mathbb{R}\mid|x'|<r,\,0<x_n<\rho r\},$$
$$S_{\rho,r}=\{(x',x_n)\in\mathbb{R}^{n-1}\times\mathbb{R}\mid|x'|<r,\,\rho r<x_n<2\rho r\}$$

for all positive constants ρ,r. The following lemma is essentially the same as the elliptic version of Lemma 7.46 in [12].

Lemma 3.1 *Let L_b be an elliptic operator of the second order defined by*

$$L_b\varphi=b_{ij}\partial^2_{ij}\varphi\quad\text{on }\mathbb{R}^n_+$$

for any $\varphi\in C^2(\mathbb{R}^n_+)$ with the coefficients b_{ij} satisfying

$$b_{ij}\le\Lambda,\qquad\inf_{\xi=(\xi_1,\cdots,\xi_n)}b_{ij}\xi_i\xi_j\ge\lambda|\xi|^2$$

on $\mathbb{R}^n_+$ for some constants $0<\lambda\le\Lambda<\infty$. Suppose $L_b\varphi\le0$ with $\varphi\ge0$ on $P_{\rho_*,4r}$ with $\rho_*=\frac{1}{9\sqrt{n-1}}\frac{\lambda}{\Lambda}$. Then

$$\inf_{(x',x_n)\in S_{\rho_*,2r}}x_n^{-1}\varphi(x',x_n)\le4\inf_{(x',x_n)\in P_{\rho_*,r}}x_n^{-1}\varphi(x',x_n). \tag{3.1}$$

Proof For any fixed $r>0$, let $\phi(x',x_n)=x_n\left(1+\frac{|x'|^2}{r^2}-\frac{x_n}{2\rho_*r}\right)$ with $x'=(x_1,\cdots,x_{n-1})\in\mathbb{R}^{n-1}$ and $x_n\ge0$. Then $\phi\ge0$ on $\{x_n=0\}$ or $\{x_n=2\rho r\}$, and $\phi\ge4x_n$ on $\{|x'|=2r\}$. Let $t_*=\inf\limits_{(x',x_n)\in S_{\rho_*,2r}}x_n^{-1}\varphi(x',x_n)$. Then

$$\varphi-t_*x_n+\frac14t_*\phi\ge0\quad\text{on }\partial P_{\rho_*,2r}. \tag{3.2}$$

From the assumption, $b_{nn} \geq \lambda$. Then on $P_{\rho_*,2r}$,

$$\begin{aligned} L_b\phi &= r^{-2}b_{ij}\partial_{ij}^2(|x'|^2x_n) - \rho_*^{-1}r^{-1}b_{nn} \\ &= 2r^{-2}\sum_{i=1}^{n-1} b_{ii}x_n + 2r^{-2}\sum_{i=1}^{n-1}(b_{in}+b_{ni})x_i - \rho_*^{-1}r^{-1}b_{nn} \\ &\leq 4(n-1)r^{-1}\Lambda\rho_* + 4r^{-2}\Lambda\sum_{i=1}^{n-1}|x_i| - \rho_*^{-1}r^{-1}\lambda \\ &\leq 4(n-1)r^{-1}\Lambda\rho_* + 8\sqrt{n-1}r^{-1}\Lambda - \rho_*^{-1}r^{-1}\lambda \leq 0, \end{aligned} \tag{3.3}$$

where we have used $\rho_* = \frac{1}{9\sqrt{n-1}}\frac{\lambda}{\Lambda}$ in the last inequality. Since $L_b(\varphi - t_*x_n) \leq 0$, we get $L_b\big(\varphi - t_*x_n + \frac{1}{4}t_*\phi\big) \leq 0$ on $P_{\rho_*,2r}$. Utilizing (3.2) we have

$$\varphi - t_*x_n + \frac{1}{4}t_*\phi \geq 0 \quad \text{on } P_{\rho_*,2r} \tag{3.4}$$

from the maximum principle. With $\phi \leq 3x_n$ on $P_{\rho_*,r}$, it follows that

$$0 \leq \varphi - t_*x_n + \frac{1}{4}t_*\phi \leq \varphi - t_*x_n + \frac{3}{4}t_*x_n = \varphi - \frac{1}{4}t_*x_n \tag{3.5}$$

on $P_{\rho_*,r}$, which finishes the proof.

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