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Complete λ-Hypersurfaces in Euclidean Spaces*

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Abstract In this paper, the authors give a survey about λ-hypersurfaces in Euclidean spaces. Especially, they focus on examples and rigidity of λ-hypersurfaces in Euclidean spaces.

Keywords Self-shrinker, λ-Hypersurface, Mean curvature flow, Weighted volume, Rigidity theorem
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1 Introduction

Let $X : M \to \mathbb{R}^{n+1}$ be an n-dimensional hypersurface in the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. Let $X(t) : M \to \mathbb{R}^{n+1}$, $t \in (-\varepsilon, \varepsilon)$ with $X(0) = X$ be a variation of X. The weighted area functional is defined by $A : (-\varepsilon, \varepsilon) \to \mathbb{R}$ by

$$A(t) = \int_M \mathrm{e}^{-\frac{|X(t)|^2}{2}} \mathrm{d}\mu_t,$$

where $\mathrm{d}\mu_t$ is the area element of M in the metric induced by $X(t)$. In 2014, Cheng and Wei [17] introduced a definition of the weighted volume of M. The weighted volume function $V : (-\varepsilon, \varepsilon) \to \mathbb{R}$ of M is defined by

$$V(t) = \int_M \langle X(t), N\rangle \mathrm{e}^{-\frac{|X|^2}{2}} \mathrm{d}\mu.$$

Cheng and Wei [17] studied a new type of mean curvature flow

$$\frac{\partial X(t)}{\partial t} = (-\alpha(t)N(t) + \mathbf{H}(t))$$

with special function $\alpha(t)$ and introduced the λ-hypersurface, where $N(t)$, $\mathbf{H}(t)$ and $H(t)$ are the unit normal vector, mean curvature vector and mean curvature of $X(t)$, respectively.

Definition 1.1 *An n-dimensional hypersurface $X : M \to \mathbb{R}^{n+1}$ in the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$ is called a λ-hypersurface if it satisfies*

$$\langle X, N\rangle + H = \lambda, \tag{1.1}$$

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where N, H denote the unit normal vector and mean curvature of X, respectively. λ is a constant.

On the other hand, a variation $X(t)$ of X is called a weighted volume-preserving normal variation if $V(t) = V(0)$ for all t and $\frac{\partial X(t)}{\partial t}|_{t=0} = fN$. Cheng and Wei [17] proved that $X : M \to \mathbb{R}^{n+1}$ is a critical point of the weighted area functional $A(t)$ for all weighted volume preserving variations if and only if $X : M \to \mathbb{R}^{n+1}$ satisfies $\langle X, N\rangle + H = \lambda$, that is, λ-hypersurface. Moreover, λ-hypersurface $X : M \to \mathbb{R}^{n+1}$ is equivalent to a hypersurface with constant weighted mean curvature $H_w = \mathrm{e}^{-\frac{|X|^2}{2n}} H = \lambda$ in $\mathbb{R}^{n+1}$ equipped with the metric $g_{AB} = \mathrm{e}^{-\frac{|X|^2}{n}} \delta_{AB}$.

Remark 1.1 The equation (1.1) also arises in the Gaussian isoperimetric problem. Borell [7] proved that the half space minimizes the weighted boundary area (see also [55]).

Remark 1.2 In the probability theory, the equation (1.1) is natural in the study of sets minimizing Gaussian surface area since the equation (1.1) holds if and only if M is a critical point of the Gaussian surface area (see [41]).

Remark 1.3 If $\lambda = 0$, $\langle X, N\rangle + H = \lambda = 0$, then $X : M \to \mathbb{R}^{n+1}$ is a self-shrinkers. Hence, one can consider that the notation of λ-hypersurfaces is a natural generalization of self-shrinkers of mean curvature flow, which plays an important role for study on singularities of the mean curvature flow.

2 Preliminaries

Let $X : M^n \to \mathbb{R}^{n+1}$ be an n-dimensional connected hypersurface of the (n+1)-dimensional Euclidean space $\mathbb{R}^{n+1}$. We choose a local orthonormal frame field $\{e_A\}_{A=1}^{n+1}$ in $\mathbb{R}^{n+1}$ with dual coframe field $\{\omega_A\}_{A=1}^{n+1}$, such that, restricted to M^n, $e_1, \cdots, e_n$ are tangent to M^n. The following conventions on the ranges of indices are used in this paper $1 \le i, j, k, l \le n$. Then we have

$$\mathrm{d}X = \sum_i \omega_i e_i, \quad \mathrm{d}e_i = \sum_j \omega_{ij} e_j + \omega_{in+1} e_{n+1}$$

and

$$\mathrm{d}e_{n+1} = \sum_i \omega_{n+1i} e_i.$$

When these forms are restricted to M^n, we have

$$\omega_{n+1} = 0 \tag{2.1}$$

and the induced Riemannian metric of M^n is written as $\mathrm{d}s_M^2 = \sum\limits_i \omega_i^2$. From (2.1), we get

$$\omega_{in+1} = \sum_j h_{ij}\omega_j, \quad h_{ij} = h_{ji}.$$

The induced structure equations of M^n are given by

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} = -\omega_{ji},$$
$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l,$$

where

$$R_{ijkl} = h_{ik}h_{jl} - h_{il}h_{jk} \tag{2.2}$$

denotes components of the curvature tensor of M^n. The second fundamental form and the mean curvature vector field of M^n are given by

$$A = \sum_{i,j} h_{ij} \omega_i \otimes \omega_j \otimes e_{n+1}$$

and

$$\mathbf{H} = He_{n+1} = \sum_i h_{ii} e_{n+1},$$

respectively. Let $S = \sum_{i,j}(h_{ij})^2$ be the squared norm of the second fundamental form and $H = |\mathbf{H}|$ denotes the mean curvature of M^n. From (2.2), components of the Ricci curvature of M^n are given by

$$R_{ik} = Hh_{ik} - \sum_j h_{ij}h_{jk}. \tag{2.3}$$

Defining the covariant derivative of h_{ij} by

$$\sum_k h_{ijk}\omega_k = dh_{ij} + \sum_k h_{ik}\omega_{kj} + \sum_k h_{kj}\omega_{ki}, \tag{2.4}$$

we obtain the Codazzi equations

$$h_{ijk} = h_{ikj}. \tag{2.5}$$

By taking exterior differentiation of (2.4), and defining

$$\sum_l h_{ijkl}\omega_l = dh_{ijk} + \sum_l h_{ljk}\omega_{li} + \sum_l h_{ilk}\omega_{lj} + \sum_l h_{ijl}\omega_{lk}, \tag{2.6}$$

we have the following Ricci identities

$$h_{ijkl} - h_{ijlk} = \sum_m h_{mj}R_{mikl} + \sum_m h_{im}R_{mjkl}. \tag{2.7}$$

Let f be a smooth function on M^n, we define the covariant derivatives f_i, f_{ij}, and the Laplacian of f as follows

$$df = \sum_i f_i\omega_i, \quad \sum_j f_{ij}\omega_j = df_i + \sum_j f_j\omega_{ji}, \quad \Delta f = \sum_i f_{ii}.$$

The following elliptic operator $\mathcal{L}$ introduced by Colding and Minicozzi in [26] will play a very important role for studying complete λ-hypersurfaces,

$$\mathcal{L}f = \Delta f - \langle X, \nabla f \rangle, \tag{2.8}$$

where Δ and ∇ denote the Laplacian and the gradient operator on the λ-hypersurface, respectively and $\langle \cdot, \cdot \rangle$ denotes the standard inner product of $\mathbb{R}^{n+1}$. By a direct computation, we have the following equations for λ-hypersurfaces in $\mathbb{R}^{n+1}$,

$$\frac{1}{2}\mathcal{L}|X|^2 = n - |X|^2 + \lambda\langle X, N \rangle, \tag{2.9}$$

$$\mathcal{L}H = H + S(\lambda - H), \tag{2.10}$$

$$\frac{1}{2}\mathcal{L}S = \sum_{i,j,k,\alpha} (h^{\alpha}_{ijk})^2 + S(1 - S) + \lambda f_3, \tag{2.11}$$

where $f_3 = \sum_{i,j,k} h_{ij}h_{jk}h_{ki}$.

3 Examples of λ-Hypersurfaces

Besides the standard examples of λ-hypersurfaces in $\mathbb{R}^{n+1}$: The n-dimensional Euclidean space $\mathbb{R}^n$, the n-dimensional sphere $S^n(r)$ with $\lambda = \frac{n}{r} - r$ and the n-dimensional cylinder $S^k(r) \times \mathbb{R}^{n-k}$ with $\lambda = \frac{k}{r} - r$, we give some non-standard examples.

3.1 0-hypersurfaces

0-hypersurfaces are just self-shrinkers. In 1989, by using the shooting method for geodesics, Angenent [5] constructs compact embedded rotational 0-hypersurface, called “Angenent torus”, whose profile curve intersects symmetry axis perpendicularly. In 1994, Chopp [24] finds several new 0-hypersurfaces. Later, Drugan, Lee and Nguyen [31], Drugan and Kleene [32] construct an infinite number of complete, immersed and non-embedded rotational 0-hypersurfaces for each of the topological types: The sphere, the plane, the cylinder and the torus. These examples whose profile curves also intersect symmetry axis perpendicularly. Recently, Cheng and Wei [20] numerically compute and find many interesting compact immersed rotational 0-hypersurfaces whose profile curves do not intersect symmetry axis perpendicularly (see Figure 1).

In addition, Kapouleas, Kleene and Møller [47] (also see [48, 56]) and Nguyen [57–59] construct complete embedded 0-hypersurfaces with higher genus in $\mathbb{R}^3$.

3.2 λ-hypersurfaces with $\lambda \neq 0$

Some of them are embedded, some of them are immersed.

3.2.1 λ-curves

There are no closed embedded 0-curves of mean curvature flow except circle with radius 1. But for λ-curves, their behaviors are different. For some $\lambda < 0$, we can prove that there exist closed embedded λ-curves Γ_λ in $\mathbb{R}^2$, which is not circle (also see [11]). Hence, for any positive

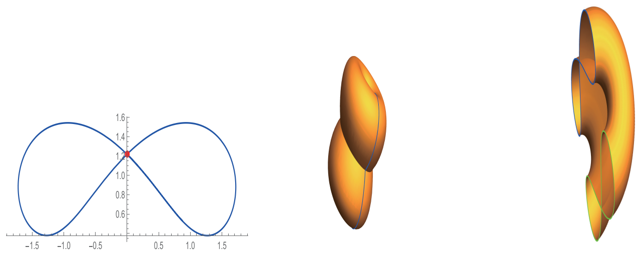

Figure 1 The graph of profile curve of compact 0-hypersurface, 0-hypersurface and half of 0-hypersurface, here $n = 2$.

integer n, there exist complete embedded λ-hypersurfaces, which are given by $\Gamma_\lambda \times \mathbb{R}^{n-1}$ in $\mathbb{R}^{n+1}$.

3.2.2 λ-torus

In 2015, Cheng and Wei [20] proved the following theorem.

Theorem 3.1 *For $n \geq 2$ and $\lambda \geq 0$, there exists embedding revolution λ-hypersurface $X : S^1 \times S^{n-1} \to \mathbb{R}^{n+1}$ in $\mathbb{R}^{n+1}$.*

Let $(x(s), r(s))$, $s \in (a, b)$ be a curve in the xr-plane with $r > 0$ and $S^{n-1}(1)$ denote the standard unit sphere of dimension $n - 1$. Then we consider

$$X : (a, b) \times S^{n-1}(1) \to \mathbb{R}^{n+1}$$

defined by $X(s, \alpha) = (x(s), r(s)\alpha)$, $s \in (a, b)$, $\alpha \in S^{n-1}(1)$. Namely, X is obtained by rotating the plane curve $(x(s), r(s))$ around x axis, where the plane curve $(x(s), r(s))$ is called the profile curves (see Figure 2).

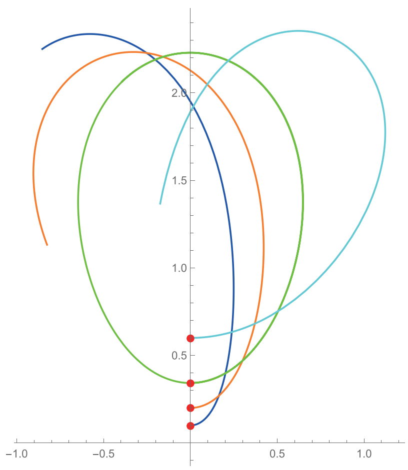

Figure 2 The profile curves of λ-hypersurfaces, here $n = 2$, $\lambda = 0.1$.

Lemma 3.1 *$X:(a,b)\times S^{n-1}(1)\to\mathbb{R}^{n+1}$ is a λ-hypersurface if and only if (x,r) satisfies*

$$\begin{cases}(x')^2+(r')^2=1,\\ -\dfrac{x''}{r'}=xr'+\Big(\dfrac{n-1}{r}-r\Big)x'+\lambda.\end{cases}$$

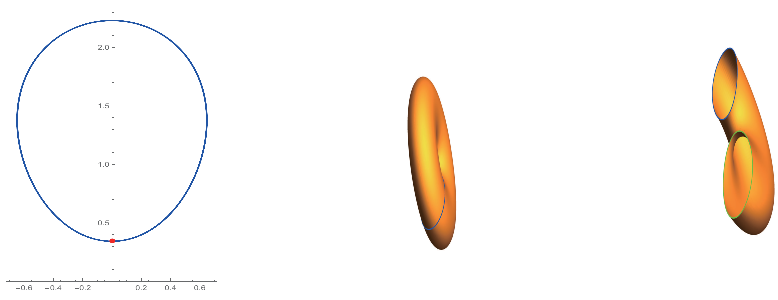

Figure 3 The graph of profile curve of λ-torus, λ-torus and half of λ-torus, here $n=2$, $\lambda=0.1$ and $r_0\approx 0.343$.

In the same paper, Cheng and Wei [20] also proved the following theorem.

Theorem 3.2 *For $n\geq 2$ and small λ, there are many compact immersed λ-hypersurfaces in $\mathbb{R}^{n+1}$.*

Additional details on the behavior of the profile curves needed to be discussed and established. Here are some numerical approximation of profile curves and λ-hypersurfaces. The horizontal axis is the axis of rotation. For small λ, compact immersed λ-hypersurfaces can be given by rotating a closed curve in the upper half plane around the horizontal axis; see Figure 4.

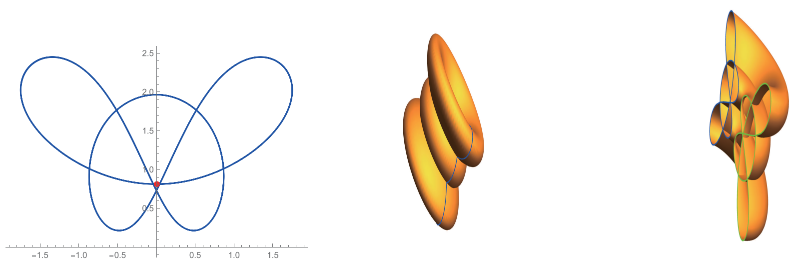

Figure 4 The graph of profile curve of compact λ-hypersurface, λ-hypersurface and half of λ-hypersurface, here $n=2$, $\lambda=0.1$ and $r_0\approx 0.811$.

Moreover, we also found many compact immersed rotational λ-hypersurfaces whose profile curves do not intersect r-axis perpendicularly; see Figure 5.

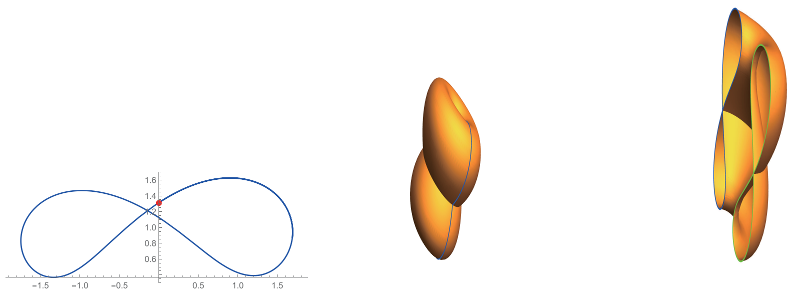

Figure 5 The graph of profile curve of compact λ-hypersurface, λ-hypersurface and half of λ-hypersurface, here $n = 2$, $\lambda = 0.1$ and $r_0 \approx 0.811$.

3.2.3 Some other λ-hypersurfaces

In 2017, Ross [60] constructed closed, embedded λ-hypersurfaces by using a "shooting method". He proved the following theorem.

Theorem 3.3 *Let $n > 1$ and $\lambda < 0$, then there exists a λ-hypersurface $X : M^{2n+1} \to \mathbb{R}^{2n+2}$ which is diffeomorphic to $S^n \times S^n \times S^1$ and exhibits a $O(n) \times O(n)$ rotational symmetry.*

In 2018, Li and Wei [54] proved the following theorem.

Theorem 3.4 *Let $n > 1$ and small $\lambda < 0$, then there exists an immersed, non-embedded S^n λ-hypersurface $X : M^n \to \mathbb{R}^{n+1}$.*

4 Rigidity Results of λ-Hypersurfaces

4.1 Rigidity results of 0-hypersurfaces

For complete 0-curves, Abresch and Langer [1] gave a complete classification about closed 0-curves and showed that the round circle is the only embedded 0-hypersurfaces.

For complete 0-hypersurfaces with dimension $n \geq 2$, the classification of smooth embedded 0-hypersurfaces $X : M \to \mathbb{R}^{n+1}$ in $\mathbb{R}^{n+1}$ with mean curvature $H \geq 0$ began with [44], where Huisken proved that round spheres are only compact ones. In [45], Huisken showed that the generalized cylinders $S^m(\sqrt{m}) \times \mathbb{R}^{n-m}$ are only open ones with polynomial volume growth if the squared norm S of the second fundamental form is bounded. Colding and Minicozzi [26] completed the classification by removing the condition that S is bounded, they proved the following theorem.

Theorem 4.1 *If $X : M \to \mathbb{R}^{n+1}$ is an n-dimensional complete embedded 0-hypersurface in $\mathbb{R}^{n+1}$ with mean curvature $H \geq 0$ and with polynomial volume growth, then $X : M \to \mathbb{R}^{n+1}$ is the generalized cylinder $S^m(\sqrt{m}) \times \mathbb{R}^{n-m}$, $0 \leq m \leq n$.*

Remark 4.1 From (2.10) and maximum principle, we know that $H \equiv 0$ or $H > 0$ in M. In order to prove the theorem, one needs to compute $\mathcal{L}\frac{S}{H^2}$, that is,

$$\frac{1}{2}\mathcal{L}\frac{S}{H^2} = \frac{1}{H^4}\sum_{i,j,k}(h_{ij}\nabla_k H - h_{ijk}H)^2 - \frac{1}{H}\Big\langle \nabla H, \nabla \frac{S}{H^2}\Big\rangle.$$

Furthermore, one wants to use Stokes formula. But in the case that $X : M \to \mathbb{R}^{n+1}$ is complete and non-compact, Stokes formula does not hold in general. If $X : M \to \mathbb{R}^{n+1}$ has polynomial volume growth, for functions S, $\log H$ and $S^{\frac{1}{2}}$, namely, the following formulas

$$-\int_M \langle \nabla S, \nabla \log H\rangle \mathrm{e}^{-\frac{|X|^2}{2}} \mathrm{d}v = \int_M S\mathcal{L}\log H \mathrm{e}^{-\frac{|X|^2}{2}} \mathrm{d}v$$

and

$$-\int_M \langle \nabla S^{\frac{1}{2}}, \nabla S^{\frac{1}{2}}\rangle \mathrm{e}^{-\frac{|X|^2}{2}} \mathrm{d}v = \int_M S^{\frac{1}{2}}\mathcal{L}S^{\frac{1}{2}} \mathrm{e}^{-\frac{|X|^2}{2}} \mathrm{d}v$$

are true. By making use of the above formulas, one can prove the theorem.

For 0-hypersurfaces, the gap phenomenon for the squared norm of the second fundamental form is interesting.

Le and Sesum [50] got the first gap theorem and proved that if $X : M \to \mathbb{R}^{n+1}$ is an n-dimensional complete embedded 0-hypersurface in $\mathbb{R}^{n+1}$ with polynomial volume growth and with $S < 1$, then $X : M \to \mathbb{R}^{n+1}$ is $\mathbb{R}^n$.

Cao and Li [10] proved the following theorem.

Theorem 4.2 *Let $X : M \to \mathbb{R}^{n+1}$ be an n-dimensional complete 0-hypersurface with polynomial volume growth in Euclidean space $\mathbb{R}^{n+1}$. If the squared norm S of the second fundamental form satisfies $S \leq 1$, then $X : M \to \mathbb{R}^{n+1}$ is one of the followings*:

(1) *$S = 0$ and $X : M \to \mathbb{R}^{n+1}$ is a hyperplane in $\mathbb{R}^{n+1}$,*

(2) *$S = 1$ and $X : M \to \mathbb{R}^{n+1}$ is either a round sphere $S^n(\sqrt{n})$ in $\mathbb{R}^{n+1}$ or a cylinder $S^m(\sqrt{m}) \times \mathbb{R}^{n-m}$, $1 \leq m \leq n-1$ in $\mathbb{R}^{n+1}$.*

About the second pinching theorem, Ding and Xin [30] proved the following theorem.

Theorem 4.3 *Let $X : M \to \mathbb{R}^{n+1}$ be an n-dimensional complete 0-hypersurface with polynomial volume growth in Euclidean space $\mathbb{R}^{n+1}$, there exists a positive number $\delta = 0.022$ such that if $1 \leq S \leq 1 + 0.022$, then $S = 1$.*

Recently, Lei, Xu and Xu [52] proved the following theorem.

Theorem 4.4 *Let $X : M \to \mathbb{R}^{n+1}$ be an n-dimensional complete 0-hypersurface with polynomial volume growth in Euclidean space $\mathbb{R}^{n+1}$, there exists a positive number $\delta = 0.022$ such that if $1 \leq S \leq 1 + \frac{1}{18}$, then $S = 1$.*

Cheng and Wei [16] considered the second gap for the squared norm of the second fundamental form and proved the following gap theorem for 0-hypersurfaces.

Theorem 4.5 *Let $X : M \to \mathbb{R}^{n+1}$ be an n-dimensional complete 0-hypersurface with polynomial volume growth in $\mathbb{R}^{n+1}$. If the squared norm S of the second fundamental form is constant and satisfies $S \le 1 + \frac{3}{7}$, then $X : M \to \mathbb{R}^{n+1}$ is one of the followings*:

(1) *The n-dimensional hyperplane $\mathbb{R}^n$,*

(2) *the cylinder $\mathbb{R}^{n-m} \times S^m(\sqrt{m})$ for $1 \le m \le n-1$,*

(3) *the round sphere $S^n(\sqrt{n})$.*

Since the subject of 0-hypersurfaces in the Euclidean space are closely related with the theory of minimal hypersurfaces in the sphere. For minimal hypersurfaces in a unit sphere, there is the following famous Chern conjecture.

Chern conjecture Let M be a compact minimal hypersurface in the unit sphere $S^{n+1}(1)$. If M has constant squared norm of the second fundamental form, then the possible values of squared norm of the second fundamental form of M form a discrete set.

Hence, it is nature to consider the similar problems for 0-hypersurfaces.

Conjecture 1 Let $X : M \to \mathbb{R}^{n+1}$ be an n-dimensional complete 0-hypersurface in $\mathbb{R}^{n+1}$. If the squared norm S of the second fundamental form is constant, then $X : M \to \mathbb{R}^{n+1}$ is one of the followings:

(1) the n-dimensional hyperplane $\mathbb{R}^n$,

(2) the cylinder $\mathbb{R}^{n-m} \times S^m(\sqrt{m})$ for $1 \le m \le n-1$,

(3) the round sphere $S^n(\sqrt{n})$.

For $n = 2$, Ding and Xin [30] studied 2-dimensional complete 0-hypersurfaces with polynomial volume growth and with constant squared norm S of the second fundamental form. They proved the following theorem.

Theorem 4.6 *A 2-dimensional complete 0-hypersurface $X : M \to \mathbb{R}^3$ with polynomial volume growth and S constant is one of the followings*: (1) $\mathbb{R}^2$, (2) $S^1(1) \times \mathbb{R}$, (3) $S^2(\sqrt{2})$.

On the other hand, Halldorsson in [39] proved that there exist complete 0-curves Γ in $\mathbb{R}^2$, which are contained in an annulus around the origin and whose images are dense in the annulus. Furthermore, Ding and Xin [29], Cheng and Zhou [23] proved the following theorem.

Theorem 4.7 *A complete 0-hypersurface $X : M \to \mathbb{R}^{n+1}$ has polynomial volume growth if and only if it is proper.*

Thus, the condition on polynomial volume growth in [45] and [26] is essential since these complete 0-curves Γ of Halldorsson [39] are not proper and for any integer $n \ge 1$, $\Gamma \times \mathbb{R}^{n-1}$ is a complete 0-hypersurface without polynomial volume growth in $\mathbb{R}^{n+1}$.

In order to study complete 0-hypersurfaces with polynomial volume growth, one often uses an elliptic operator $\mathcal{L}$ introduced by Colding and Minicozzi in [26],

$$\mathcal{L}f = \Delta f - \langle X, \nabla f\rangle = \mathrm{e}^{\frac{|X|^2}{2}} \mathrm{div}(\mathrm{e}^{-\frac{|X|^2}{2}} \nabla f) \tag{4.1}$$

and the following integral formula.

Corollary 4.1 *Let $X : M \to \mathbb{R}^{n+p}$ be a complete hypersurface. If u, v are C^2 functions satisfying*

$$\int_M (|u\nabla v| + |\nabla u||\nabla v| + |u\mathcal{L}v|)\mathrm{e}^{-\frac{|X|^2}{2}}\mathrm{d}\mu < +\infty, \tag{4.2}$$

then

$$\int_M u(\mathcal{L}v)\mathrm{e}^{-\frac{|X|^2}{2}}\mathrm{d}\mu = -\int_M \langle \nabla u, \nabla v\rangle \mathrm{e}^{-\frac{|X|^2}{2}}\mathrm{d}\mu \tag{4.3}$$

holds, where Δ and ∇ denote the Laplacian and the gradient operator, respectively.

If one does not assume the condition polynomial volume growth for complete 0-hypersurfaces, the following generalized maximum principle for $\mathcal{L}$-operator on 0-hypersurfaces which were proven by Cheng and Peng in [15] plays a very important role.

Lemma 4.1 (*Generalized maximum principle for $\mathcal{L}$-operator*) *Let $X : M^n \to \mathbb{R}^{n+1}$ be a complete 0-hypersurface with Ricci curvature bounded from below. Let f be any C^2-function bounded from above on this 0-hypersurface. Then, there exists a sequence of points $\{p_m\} \subset M^n$, such that*

$$\lim_{m\to\infty} f(X(p_m)) = \sup f, \quad \lim_{m\to\infty} |\nabla f|(X(p_m)) = 0, \quad \limsup_{m\to\infty} \mathcal{L}f(X(p_m)) \le 0.$$

In [15], without the assumption of polynomial volume growth about 0-hypersurfaces, Cheng and Peng proved the following theorem.

Theorem 4.8 *For an n-dimensional complete 0-hypersurface $X : M^n \to \mathbb{R}^{n+1}$ with $\inf H^2 > 0$, if the squared norm S of the second fundamental form is constant, then M^n is one of the followings*: (1) $S^n(\sqrt{n})$, (2) $S^m(\sqrt{m}) \times \mathbb{R}^{n-m} \subset \mathbb{R}^{n+1}$.

Cheng and Ogata [13] removed both the assumption on polynomial volume growth in the above theorem of Ding and Xin [30] and the assumption $\inf H^2 > 0$ in the theorem of Cheng and Peng [15] for $n = 2$. Cheng and Ogata [13] got the following theorem.

Theorem 4.9 *Let $X : M \to \mathbb{R}^3$ be a 2-dimensional complete 0-hypersurface in Euclidean space $\mathbb{R}^3$. If the squared norm S of the second fundamental form is constant, then $X : M \to \mathbb{R}^3$ is one of the followings*: (1) $\mathbb{R}^2$, (2) *the round sphere* $S^2(\sqrt{2})$, (3) *the cylinder* $S^1(1) \times \mathbb{R}$.

Remark 4.2 According to the results of Cheng and Ogata [13], one knows that the conjecture 1 was solved affirmatively for $n = 2$.

Recently, Cheng, Li and Wei [12], under the assumption f_4 constant, we solved this conjecture 1 for $n = 3$.

Theorem 4.10 *Let $X : M^3 \to \mathbb{R}^4$ be a 3-dimensional complete 0-hypersurface in $\mathbb{R}^4$. If the squared norm S of the second fundamental form and $f_4 = \sum_i \lambda_i^4$ are constant, then $X : M^3 \to \mathbb{R}^4$ is isometric to one of the followings*:

(1) $\mathbb{R}^3$,

(2) $S^1(1) \times \mathbb{R}^2$,

(3) $S^2(\sqrt{2}) \times \mathbb{R}^1$,
(4) $S^3(\sqrt{3})$.

In the proof of the above theorem, we need to compute $\nabla_m \nabla_l f_3$, $\nabla_m \nabla_l \nabla_k f_4$ and $\nabla_n \nabla_m \nabla_l \nabla_k f_4$, where f_3 and f_4 are defined by $f_3 = \sum\limits_{i,j,k} h_{ij} h_{jk} h_{ki}$ and $f_4 = \sum\limits_{i,j,k,l} h_{ij} h_{jk} h_{kl} h_{li}$.

4.2 Rigidity results of λ-hypersurfaces with $\lambda \neq 0$

In 2014, for λ-curve, Guang [35] proved the following theorem.

Theorem 4.11 *Any smooth complete embedded λ-curve in $\mathbb{R}^2$ with $\lambda \geq 0$ must either be a line or a round circle.*

For entire graph, Guang [35] proved the following theorem.

Theorem 4.12 *If a λ-hypersurface $X : M \to \mathbb{R}^{n+1}$ is an entire graph with polynomial volume growth, then $X : M \to \mathbb{R}^{n+1}$ is a hyperplane.*

In the paper [36], Guang expected that one may remove the condition of polynomial volume growth (see Remark 1.6). In 2015, Cheng and Wei [18] solved Guang's problem and proved the following theorem.

Theorem 4.13 *Let $X : M \to \mathbb{R}^{n+1}$ be an n-dimensional entire graphic λ-hypersurface in the Euclidean space $\mathbb{R}^{n+1}$. Then $X : M \to \mathbb{R}^{n+1}$ is a hyperplane $\mathbb{R}^n$.*

Remark 4.3 In the case of 0-hypersurfaces, Ecker and Huisken [33] proved that $X : M \to \mathbb{R}^{n+1}$ is a hyperplane if it is an entire graphic 0-hypersurface with polynomial volume growth in $\mathbb{R}^{n+1}$. Recently, Wang [63] removed the assumption of polynomial volume growth (see also Ding and Wang [28]).

In [35], Guang also proved some rigidity theorems for complete embedded λ-hypersurfaces in terms of the norm of the second fundamental form.

In 2014, for complete λ-hypersurfaces, Cheng and Wei [17] proved the following theorem.

Theorem 4.14 *Let $X : M \to \mathbb{R}^{n+1}$ be an n-dimensional complete embedded λ-hypersurface with polynomial area growth in $\mathbb{R}^{n+1}$. If $H - \lambda \geq 0$ and*

$$\lambda(f_3(H - \lambda) - S) \geq 0,$$

then $X : M \to \mathbb{R}^{n+1}$ is isometric to one of the followings:
(1) $S^n(r)$ *with* $\lambda = \frac{n}{r} - r$,
(2) $\mathbb{R}^n$,
(3) $S^k(r) \times \mathbb{R}^{n-k}$, $0 < k < n$,
where $S = \sum\limits_{i,j} h_{ij}^2$ is the squared norm of the second fundamental form and $f_3 = \sum\limits_{i,j,k} h_{ij} h_{jk} h_{ki}$.

Remark 4.4 The condition

$$\lambda(f_3(H - \lambda) - S) \geq 0$$

is essential. In fact, for any positive integer n, complete embedded λ-hypersurfaces $\Gamma_\lambda \times \mathbb{R}^{n-1}$ in $\mathbb{R}^{n+1}$ do not satisfy this condition, where Γ_λ is a closed embedded λ-curve in $\mathbb{R}^2$.

Later, Peng and Wei [64] proved the following rigidity result.

Theorem 4.15 *If $X : M \to \mathbb{R}^{n+1}$ is an n-dimensional complete λ-hypersurface with polynomial area growth and bounded S satisfies*

$$H(H-\lambda)S \leq H^2, \tag{4.4}$$

then M is one of the followings:

(1) *a round sphere $S^n(r)$,*

(2) *a cylinder $S^k(r) \times \mathbb{R}^{n-k}$, $1 \leq k \leq n-1$,*

(3) *a hyperplane $\mathbb{R}^n$,*

where H is the mean curvature of M, S is the norm square of the second fundamental form of M.

Remark 4.5 The theorem is a general generalization of Cao and Li [10] and Le and Sesum [50].

In [62], Wang, Xu and Zhao proved that if the L^n-norm of the second fundamental form of the λ-hypersurface $X : M \to \mathbb{R}^{n+1}$ with $n \geq 3$ is less than an explicit positive constant, then M is a hyperplane.

In [67], Zhu, Fang and Chen considered the volume comparison theorem of complete bounded λ-hypersurfaces with bounded S and got some applications of the volume comparison theorem. They also got some estimates for the intrinsic diameter and the extrinsic radius.

In particular, for λ-surfaces, Guang [34] obtained the following theorem.

Theorem 4.16 *Let $X : M^2 \to \mathbb{R}^3$ be a 2-dimensional compact λ-surface in $\mathbb{R}^3$ with $\lambda \geq 0$. If the squared norm of the second fundamental form S is constant, then $X : M^2 \to \mathbb{R}^3$ is a round sphere.*

The proof of his theorem has two ingredients. The first ingredient is to consider the point where the norm of the position vector $|x|$ achieves its minimum. This will give that the genus is 0. The second ingredient is an interesting result from [40] that any smooth closed special W-surface of genus 0 is a round sphere.

In [34], Guang proposed the following conjecture (see [34, Page 74], also see [35]).

Conjecture 2 *Any complete λ-surface in $\mathbb{R}^3$ with constant S=constant is either $\mathbb{R}^2$, or $S^1(r_1) \times \mathbb{R}$, or $S^2(r_2)$ for some positive constants r_1 and r_2.*

By using of the generalized maximum principle introduced by Cheng-Ogata-Wei [14], Cheng and Wei [21] confirmed the conjecture of Guang [34]. More precisely, we proved the following.

Theorem 4.17 *Let $X : M^2 \to \mathbb{R}^3$ be a 2-dimensional complete λ-surface in $\mathbb{R}^3$. If the squared norm S of the second fundamental form is constant, then either $S = 0$, or $S = \frac{2+\lambda^2+\lambda\sqrt{\lambda^2+4}}{2}$, or $S = \frac{4+\lambda^2+\lambda\sqrt{\lambda^2+8}}{4}$ and $X : M^2 \to \mathbb{R}^3$ is isometric to one of*

(1) $\mathbb{R}^2$,

(2) $S^1(\frac{-\lambda+\sqrt{\lambda^2+4}}{2}) \times \mathbb{R}$,

(3) $S^2(\frac{-\lambda+\sqrt{\lambda^2+8}}{2})$.

Proof In the proof, we should use the following lemma and theorems.

Lemma 4.2 *Let $X : M^2 \to \mathbb{R}^3$ be a 2-dimensional λ-surface in $\mathbb{R}^3$. If S is constant, we have*

$$\begin{aligned}\frac{1}{2}\mathcal{L}\sum_{i,j,k}(h_{ijk})^2 = \sum_{i,j,k,l}(h_{ijkl})^2 + (2-S)\sum_{i,j,k}(h_{ijk})^2 + 6\sum_{i,j,k,l,p} h_{ijk}h_{il}h_{jp}h_{klp} \\ - 3\sum_{i,j,k,l,p} h_{ijk}h_{ijl}h_{kp}h_{lp} + 3\lambda\sum_{i,j,k,l} h_{ijk}h_{ijl}h_{kl}\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}&\frac{1}{2}\mathcal{L}\sum_{i,j,k}(h_{ijk})^2 \\ &= \frac{3}{2}\lambda H|\nabla H|^2 + \frac{3}{4}\lambda H^3 + \frac{3}{4}\lambda H^2 S(\lambda - H) - \frac{3}{4}\lambda SH - \frac{3}{4}\lambda S^2(\lambda - H).\end{aligned} \tag{4.6}$$

Theorem 4.18 *For a 2-dimensional complete λ-surface $X : M^2 \to \mathbb{R}^3$ with constant squared norm S of the second fundamental form, we have either*

(1) $\lambda^2 S = (S-1)^2$ *and* $\sup H^2 = S$, *or*

(2) $\lambda^2 S = 2(S-1)^2$ *and* $\sup H^2 = 2S$, *or*

(3) $\lambda^2 S = \frac{2(1+S)^2}{9}$ *and* $\sup H^2 = 2S$.

Theorem 4.19 *Let $X : M^2 \to \mathbb{R}^3$ be a 2-dimensional λ-surface. If either $\lambda^2 S = (S-1)^2$, or $\lambda^2 S = 2(S-1)^2$, or $9\lambda^2 S = 2(S-1)^2$, then the mean curvature H satisfies $H \neq 0$ on M^2.*

Theorem 4.20 *Let $X : M^2 \to \mathbb{R}^3$ be a 2-dimensional complete λ-surface with constant squared norm S of the second fundamental form. Then either $\lambda^2 S = (S-1)^2$ and $\inf H^2 = S$, or $\lambda^2 S = 2(S-1)^2$ and $\inf H^2 = 2S$.*

If $\lambda \neq 0$, from Theorem 4.20, we know that $\lambda^2 S = (S-1)^2$ or $\lambda^2 S = 2(S-1)^2$. It is easy to check that $\lambda^2 S = (S-1)^2$ and $\lambda^2 S = 2(S-1)^2$ do not hold simultaneously. If $\lambda^2 S = (S-1)^2$, we have $\inf H^2 = S = \sup H^2$ from Theorem 4.18. Hence, H is constant. If $\lambda^2 S = (S-1)^2$, we have $\inf H^2 = 2S = \sup H^2$ from Theorem 4.18, H is also constant. Thus, we conclude that $X : M^2 \to \mathbb{R}^3$ is an isoparametric surface. By a classification theorem due to Lawson [49], $X : M^2 \to \mathbb{R}^3$ is $S^k(r) \times \mathbb{R}^{2-k}$, $k = 1, 2$. By a direct calculation, we conclude $X : M^2 \to \mathbb{R}^3$ is either $S^1\big(\frac{-\lambda+\sqrt{\lambda^2+4}}{2}\big) \times \mathbb{R}^1$, or $S^2\big(\frac{-\lambda+\sqrt{\lambda^2+8}}{2}\big)$.

There are some other rigidity results about λ-hypersurfaces (see [2–4, 6, 8–9, 19, 22, 25, 27, 37–38, 41–43, 46, 51, 53, 61, 65–66]).

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