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Long-time Asymptotic Behavior for the Derivative Schrödinger Equation with Finite Density Type Initial Data*

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Abstract In this paper, the authors apply $\overline{\partial}$ steepest descent method to study the Cauchy problem for the derivative nonlinear Schrödinger equation with finite density type initial data

$$\begin{aligned}&\mathrm{i}q_t + q_{xx} + \mathrm{i}(|q|^2 q)_x = 0,\\ &q(x,0) = q_0(x),\end{aligned}$$

where $\lim_{x\to\pm\infty} q_0(x) = q_\pm$ and $|q_\pm| = 1$. Based on the spectral analysis of the Lax pair, they express the solution of the derivative Schrödinger equation in terms of solutions of a Riemann-Hilbert problem. They compute the long time asymptotic expansion of the solution $q(x,t)$ in different space-time regions. For the region $\xi = \frac{x}{t}$ with $|\xi + 2| < 1$, the long time asymptotic is given by

$$q(x,t) = T(\infty)^{-2} q_\Lambda^r(x,t) + \mathcal{O}(t^{-\frac{3}{4}}),$$

in which the leading term is $N(I)$ solitons, the second term is a residual error from a $\overline{\partial}$ equation. For the region $|\xi + 2| > 1$, the long time asymptotic is given by

$$q(x,t) = T(\infty)^{-2} q_\Lambda^r(x,t) - t^{-\frac{1}{2}} \mathrm{i} f_{11} + \mathcal{O}(t^{-\frac{3}{4}}),$$

in which the leading term is $N(I)$ solitons, the second $t^{-\frac{1}{2}}$ order term is soliton-radiation interactions and the third term is a residual error from a $\overline{\partial}$ equation. These results are verification of the soliton resolution conjecture for the derivative Schrödinger equation. In their case of finite density type initial data, the phase function $\theta(z)$ is more complicated that in finite mass initial data. Moreover, two triangular decompositions of the jump matrix are used to open jump lines on the whole real axis and imaginary axis, respectively.

Keywords Derivative Schrödinger equation, Riemann-Hilbert problem, $\overline{\partial}$ steepest descent method, Long-time asymptotics, Soliton resolution, Asymptotic stability

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1 Introduction

The study on the long-time behavior of nonlinear wave equations which is solvable by the inverse scattering method was first carried out by Manakov in 1974 (see [1]). By using this

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method, Zakharov and Manakov gave the first result for large-time asymptotic of solutions for the NLS equation with decaying initial data (see [2]). The inverse scattering method also worked for long-time behavior of integrable systems such as KdV, Landau-Lifshitz and the reduced Maxwell-Bloch system (see [3–5]). In 1993, Deift and Zhou developed a nonlinear steepest descent method to rigorously analyze the long-time asymptotic behavior of the solution for the mKdV equation by deforming the original Riemann-Hilbert (RH for short) problem to a model one whose solution is calculated in terms of parabolic cylinder functions (see [6]). Since then this method has been widely applied to the focusing NLS equation, KdV equation, Fokas-Lenells equation, short-pulse equation and Camassa-Holm equation etc. (see [7–12]).

In recent years, McLaughlin and Miller further presented a $\overline{\partial}$ steepest descent method which combines steepest descent with $\overline{\partial}$-problem rather than the asymptotic analysis of singular integrals on contours to analyze asymptotic of orthogonal polynomials with non-analytic weights (see [13–14]). When it was applied to integrable systems, the $\overline{\partial}$ steepest descent method also has displayed some advantages, such as avoiding delicate estimates involving L^p estimates of the Cauchy projection operators, and leading the non-analyticity in the RH problem reductions to a $\overline{\partial}$-problem in some sectors of the complex plane which can be solved by being recast into an integral equation and by using Neumann series. Dieng and McLaughin used it to study the defocusing NLS equation under essentially minimal regularity assumptions on finite mass initial data (see [15]). This $\overline{\partial}$ steepest descent method was also successfully applied to prove asymptotic stability of N-soliton solutions to focusing NLS equation (see [16]). Jenkins et al. studied soliton resolution for the derivative NLS equation for generic initial data in a weighted Sobolev space (see [17]). Their work provided the soliton resolution property for derivative NLS equation, which decomposes the solution into the sum of a finite number of separated solitons and a radiative parts when $t \to \infty$. Its dispersive part contains two components, one coming from the continuous spectrum and another from the interaction of the discrete and continuous spectrum. For finite density initial data, Cuccagna and Jenkins studied the defocusing NLS equation (see [18]). Recently, we further extended this method to obtain the long-time asymptotics and the soliton resolution conjecture for some integrable systems (see [19–23]).

In this paper, we study the long time asymptotic behavior for the derivative nonlinear Schrödinger (DNLS for short) equation with finite density initial data

$$\mathrm{i}q_t + q_{xx} + \mathrm{i}\sigma(|q|^2 q)_x = 0, \tag{1.1}$$

$$q(x, 0) = q_0(x), \tag{1.2}$$

where $\lim_{x\to\pm\infty} q_0(x) = q_\pm,\ |q_\pm| = 1$. Since the solution space of (1.1) with $\sigma = 1$ and $\sigma = -1$ is equivalently by the simple mapping $q(x,t) \to q(-x,t)$, we only need to consider the case $\sigma = -1$ in our paper. The DNLS equation as a completely integrable system was first proposed by Kaup and Newell [24]. The DNLS equation is often used to describe various nonlinear waves. For instance, DNLS equation governs the evolution of small but finite amplitude nonlinear Alfvén waves which propagates quasi-parallel to the magnetic field in space plasma physics (see [25–29]), sub-picosecond pulses in single mode optical fibers (see [30–32]). Moreover, DNLS equation also describe weak nonlinear electromagnetic waves in ferromagnetic (see [33]), dielectric (see [34]) and anti-ferromagnetic systems under external magnetic fields (see [35]). Either zero boundary conditions or nonzero boundary conditions for the DNLS equation have well physical significance. For problems of nonlinear Alfvén waves, weak nonlinear electromagnetic waves in magnetic and dielectric media, waves propagating strictly parallel to the ambient magnetic fields are modeled by zero boundary conditions, while those oblique waves are modeled by the nonzero boundary conditions. In optical fibers, pulses under bright background waves are modeled by the zero boundary conditions.

Much work was done on the N-soliton solutions for the DNLS equation with zero/nonzero boundary conditions on discrete spectrum by using inverse scattering transform (see [36–41]). Tsutsumi and Fukuda established the local existence of the DNLS equation for initial value $q_0 \in H^s(\mathbb{R}), s > 3$ by using a parabolic regularization (see [42]). Later, they used the first five conserved quantities of the DNLS equation to establish the global existence of solutions for $q_0 \in H^2(\mathbb{R})$ with small initial data in $H^1(\mathbb{R})$ (see [43]). Hayashi proved local and global existence of solutions to the DNLS equation for $q_0 \in H^1(\mathbb{R})$ with small initial data in $L^2(\mathbb{R})$ (see [44]). For Schwartz initial value $q_0(x) \in \mathcal{S}(\mathbb{R})$, we first used Deift-Zhou steepest descent method to derive the long-time asymptotic for the DNLS equation (1.1) in soliton-free region (see [45])

$$q(x,t) = t^{-\frac{1}{2}}\alpha(\lambda_0)\mathrm{e}^{\frac{\mathrm{i}x^2}{4t} - \mathrm{i}\nu(\lambda_0)\log t} + \mathcal{O}(t^{-1}\log t). \tag{1.3}$$

Later we further investigated the long-time asymptotic for the DNLS equation (1.1) with step-like initial data (see [46]). Pelinovsky and Shimabukuro studied the existence of global solutions to the DNLS equation with the inverse scattering transform method (see [47]). Recently, generic initial data in a weighted Sobolev space defined by

$$H^{2,2}(\mathbb{R}) = \{f(x) \in L^2(\mathbb{R}) \mid (1+|x|^2)\partial^j f(x) \in L^2(\mathbb{R}) \text{ for } j = 1, \cdots, 2\},$$

applying $\overline{\partial}$ steepest descent method, Jenkins et al. obtained the following asymptotics for the DNLS equation

$$q(x,t) = q_{\mathrm{sol}}(x,t;D_I) + t^{-\frac{1}{2}} f(x,t) + \mathcal{O}(t^{-\frac{3}{4}}), \tag{1.4}$$

where $q_{\mathrm{sol}}(x,t;D_I)$ is the soliton solutions of (1.1) with modulating reflectionless scattering data (1.1) (see [17, 48]).

In our present paper, for finite density initial data $q_0(x) - q_\pm \in H^{2,2}(\mathbb{R})$, we apply $\overline{\partial}$ steepest descent method to obtain the following long-time asymptotic of the DNLS equation (1.1).

For $\xi = \frac{x}{t}$ with $|\xi + 2| < 1$:

$$q(x,t) = T(\infty)^{-2} q_\Lambda^r(x,t) + \mathcal{O}(t^{-\frac{3}{4}}). \tag{1.5}$$

For $|\xi + 2| > 1$:

$$q(x,t) = T(\infty)^{-2} q_\Lambda^r(x,t) - t^{-\frac{1}{2}}\mathrm{i} f_{11} + \mathcal{O}(t^{-\frac{3}{4}}), \tag{1.6}$$

where meanings of the notations $q_\Lambda^r(x,t)$, $T(z)$ and f_{11} are shown in Proposition 3.1, Corollary 6.2 and (8.14), respectively. Our work is different from those [17, 45] in the following three aspects. Firstly, for our case with finite density initial data, the corresponding phase function and its phase points are more complicated. On the jump contour $\mathrm{i}\mathbb{R}$ and $\mathbb{R}$, there does not always exist phase point. And in the case that phase point absences on $\mathrm{i}\mathbb{R}$ (or $\mathbb{R}$), unlike usual steepest descent method to open jump contour at phase points, we open the jump contour $\mathrm{i}\mathbb{R}$ (or $\mathbb{R}$) at $z = 0$. And under this method, jump contour will decay to zero, and its non-analytical component is transformed into a $\overline{\partial}$ equation. So we do not need to consider usual parabolic-cylinder model. Secondly, from characteristics of two triangular decompositions of jump matrix in the case of non-zero boundary conditions, one decomposition is used to open jump line on the whole real axis, another is used to open jump line on the whole imaginary axis. Thirdly, in the case of non-vanishing initial data, to avoid multi-valued function, we need to introduce uniformization variables. This also results in extra singularities on two branch cut points $\pm\mathrm{i}$, which leads to some adjustments in the structure of standard matrix factorizations.

This paper is arranged as follows. In Section 2, we recall some main results on the construction process of the RH problem with respect to the initial problem of the DNLS equation (1.1) obtained in [38, 41], which will be used to analyze long-time asymptotics of the DNLS equation in our paper. In Section 3, we introduce a function $T(z)$ to define a new RH problem for $M^{(1)}(z)$, which admits a regular discrete spectrum and two triangular decompositions of the jump matrix. In Section 4, by introducing a matrix-valued function $R(z)$, we obtain a mixed $\overline{\partial}$-RH problem for $M^{(2)}(z)$ by continuous extension of $M^{(1)}(z)$. In Section 5, we decompose $M^{(2)}(z)$ into a model RH problem for $M^{R}(z)$ and a pure $\overline{\partial}$-problem for $M^{(3)}(z)$. The $M^{R}(z)$ can be obtained via a modified reflectionless RH problem $M^{(r)}$ (see Section 6), local RH problem $M^{\mathrm{lo}}(z)$ (see Section 7) and error function $E(z)$ (see Section 8). In Section 9, we analyze the $\overline{\partial}$-problem for $M^{(3)}(z)$. Finally, in Section 10, based on the result obtained above, a relation formula is found

$$M(z) = T(\infty)^{\sigma_3} M^{(3)}(z) M^{R}(z) R^{(2)}(z)^{-1} T(z)^{-\sigma_3},$$

from which we then obtain the long-time asymptotic behavior for the DNLS equation (1.1) via a reconstruction formula.

2 The Spectral Analysis and a RH Problem

The DNLS equation (1.1) admits the Lax pair (see [24])

$$\Phi_x = X\Phi, \quad \Phi_t = T\Phi, \tag{2.1}$$

where

$$\begin{aligned} X &= \mathrm{i}k^2\sigma_3 + kQ, \\ T &= -(2k^2 + Q^2)X - \mathrm{i}kQ_x\sigma_3 \end{aligned}$$

with $k \in \mathbb{C}$ being a spectral parameter and

$$Q = \begin{pmatrix} 0 & q \\ -\overline{q} & 0 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

By using the boundary condition (1.1), Lax pair (2.1) becomes

$$\Phi_{\pm,x} \sim X_{\pm}\Phi_{\pm}, \quad \Phi_{\pm,t} \sim T_{\pm}\Phi_{\pm}, \quad x \to \pm\infty, \tag{2.2}$$

where

$$X_\pm = \mathrm{i}k^2\sigma_3 + kQ_\pm, \quad T_\pm = -(2k^2-1)X_\pm \tag{2.3}$$

and

$$Q_\pm = \begin{pmatrix} 0 & q_\pm \\ -\overline{q}_\pm & 0 \end{pmatrix}.$$

The eigenvalues of the matrix $X_\pm$ are $\pm\mathrm{i}k\lambda$, which admit

$$\lambda^2 = k^2 + 1. \tag{2.4}$$

To avoid multi-valued case of eigenvalue λ, we introduce a uniformization variable

$$z = k + \lambda, \tag{2.5}$$

and obtain two single-valued functions

$$k(z) = \frac{1}{2}\Big(z - \frac{1}{z}\Big), \quad \lambda(z) = \frac{1}{2}\Big(z + \frac{1}{z}\Big). \tag{2.6}$$

Define two domains D^+, D^- and their boundary Σ on z-plane by

$$\begin{aligned} D^- &= \{z : \operatorname{Re} z \operatorname{Im} z < 0\}, \quad D^+ = \{z : \operatorname{Re} z \operatorname{Im} z > 0\}, \\ \Sigma &= \{z : \operatorname{Re} z \operatorname{Im} z = 0\} = \mathbb{R} \cup \mathrm{i}\mathbb{R}\backslash\{0\}, \end{aligned}$$

which are shown in Figure 1.

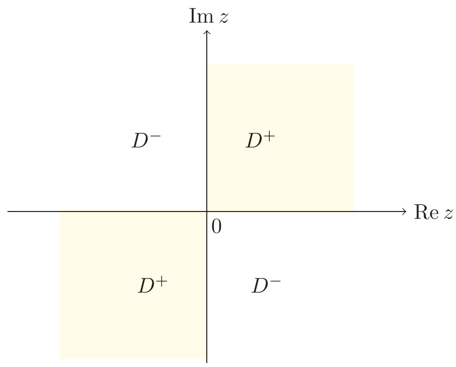


Figure 1 The domains D^-, D^+ and boundary $\Sigma = \mathbb{R} \cup \mathrm{i}\mathbb{R}\backslash\{0\}$.

We derive the solution of the asymptotic spectral problem (2.2),

$$\Phi_\pm \sim Y_\pm \mathrm{e}^{\mathrm{i}k(z)\lambda(z)x\sigma_3}, \tag{2.7}$$

where

$$Y_\pm = \begin{pmatrix} 1 & \frac{\mathrm{i}q_\pm}{z} \\ \frac{\mathrm{i}\overline{q}_\pm}{z} & 1 \end{pmatrix}.$$

By making transformation

$$\mu_\pm = \Phi_\pm \mathrm{e}^{-\mathrm{i}k\lambda x\sigma_3}, \tag{2.8}$$

we further have

$$\begin{aligned} &\mu_\pm \sim Y_\pm, \quad x \to \pm\infty, \\ &\det[\Phi_\pm] = \det[\mu_\pm] = \det[Y_\pm] = 1 + z^{-2}. \end{aligned}$$

$\mu_\pm$ satisfy the Volterra integral equations

$$\mu_\pm(z) = Y_\pm + \int_{\pm\infty}^{x} Y_\pm \mathrm{e}^{\mathrm{i}k\lambda(x-y)\widehat{\sigma}_3}[Y_\pm^{-1}\Delta X_\pm \mu_\pm(z)]\mathrm{d}y, \quad z \neq \pm\mathrm{i}, \tag{2.9}$$

$$\mu_\pm(z) = Y_\pm + \int_{\pm\infty}^{x} [I + (x-y)X_\pm(z)]\Delta X_\pm \mu_\pm(z)\mathrm{d}y, \quad z = \pm\mathrm{i}, \tag{2.10}$$

where $\Delta X_\pm = k(Q - Q_\pm)$. It can be shown that the eigenfunctions $\mu_\pm$ admit symmetries (see [41]).

Proposition 2.1 *Jost functions admit two reduction conditions on the z-plane*:
The first symmetry reduction:

$$\mu_\pm(z) = \sigma_2 \overline{\mu_\pm(\overline{z})} \sigma_2 = \sigma_1 \overline{\mu_\pm(-\overline{z})} \sigma_1. \tag{2.11}$$

The second symmetry reduction:

$$\mu_\pm(z) = \frac{\mathrm{i}}{z}\mu_\pm(-z^{-1})\sigma_3 Q_\pm. \tag{2.12}$$

For $z \in \Sigma^0 = \Sigma \setminus \{\pm\mathrm{i}\}$, there exists scattering matrix which is a linear relation between Φ_+ and Φ_-,

$$\Phi_+(x,t,z) = \Phi_-(x,t,z)S(z), \tag{2.13}$$

where

$$S(z) = \begin{pmatrix} a(z) & -\overline{b(\overline{z})} \\ b(z) & \overline{a(\overline{z})} \end{pmatrix}, \quad \det[S(z)] = 1 \tag{2.14}$$

with symmetry reduction:

$$S(z) = \sigma_1 \overline{S(-\overline{z})} \sigma_1 = (\sigma_3 Q_-)^{-1} S(-z^{-1}) \sigma_3 Q_+. \tag{2.15}$$

And the reflection coefficients are defined by

$$\rho(z) = \frac{b(z)}{a(z)}, \quad \widetilde{\rho}(z) = -\overline{\rho(\overline{z})} \tag{2.16}$$

with symmetry reduction:

$$\rho(z) = \overline{\widetilde{\rho}(-\overline{z})} = \frac{\overline{q}_-}{q_-}\widetilde{\rho}(-z^{-1}). \tag{2.17}$$

(2.13) also gives

$$a(z)=\frac{\mathrm{Wr}(\Phi_+^1,\Phi_-^2)}{1+z^{-2}},\quad b(z)=\frac{\mathrm{Wr}(\Phi_-^1,\Phi_+^1)}{1+z^{-2}}. \tag{2.18}$$

Although $a(z)$ and $b(z)$ has singularities at points $\pm\mathrm{i}$, $|\rho(\pm\mathrm{i})|=1$. The uniqueness and existences of Lax pair come from [41].

Proposition 2.2 *If $q(x)-q_\pm\in H^{2,2}(\mathbb{R}_\pm)$, the fundamental eigenfunctions $\mu_\pm$ defined by (2.9)–(2.10) exist uniquely. Define $\mu_\pm=(\mu_\pm^1,\mu_\pm^2)$ with $\mu_\pm^1$ and $\mu_\pm^2$ denoting the first and second column of $\mu_\pm$, respectively. Then μ_+^1 and μ_-^2 are analytic on the D^+, and continuous in $\overline{D^+}$; μ_-^1 and μ_+^2 are analytic on the D^-, and continuous in $\overline{D^-}$. Moreover, from (2.18), $a(z)$ is analytic on the D^+, and continuous in $\overline{D^+}\setminus\{\pm\mathrm{i}\}$. Further, $\lambda a(z)$ is analytic on the D^+, and continuous in $\overline{D^+}$. $b(z)$ and $\lambda b(z)$ are continuous in Σ^0 and Σ, respectively.*

Proof We give the detail of the first column of $Y_+^{-1}\mu_+$. The other case can be proved in the same way. The analyticity and continuity of $a(z)$ and $b(z)$ come from (2.18). For brevity, we denote it as $w(z;x,t)$. It satisfies the Volterra integral equation:

$$w(z;x,t)=\begin{pmatrix}1\\0\end{pmatrix}+\frac{z(z^2-1)}{2(z^2+1)}\int_{+\infty}^{x}G(z;x,y,t)w(z;y,t)\mathrm{d}y. \tag{2.19}$$

Here

$$G(z;x,y,t)=\begin{pmatrix}G_1(z;y,t) & G_2(z;y,t)\\ -\mathrm{e}^{2\mathrm{i}k\lambda(x-y)}\overline{G_2(\overline{z};y,t)} & -\mathrm{e}^{2\mathrm{i}k\lambda(x-y)}G_1(z;y,t)\end{pmatrix}, \tag{2.20}$$

$$G_1(z;y,t)=\frac{\mathrm{i}}{z}(q_+\overline{q}(y,t)+q(y,t)\overline{q}_+-2), \tag{2.21}$$

$$G_2(z;y,t)=q(y,t)-q_+-\frac{1}{z^2}q_+^2\overline{q(y,t)-q_+}. \tag{2.22}$$

We define a series of recursive functions for $n=0,1,\cdots$, by

$$w^{(0)}=\begin{pmatrix}1\\0\end{pmatrix},\quad w^{(n+1)}(z;x,t)=\frac{z(z^2-1)}{2(z^2+1)}\int_{+\infty}^{x}G(z;x,y,t)w^{(n)}(z;y,t)\mathrm{d}y. \tag{2.23}$$

By using (2.23), we construct a Neumann series

$$w(z;x,t)=\sum_{n=0}^{+\infty}w^{(n)}(z;x,t), \tag{2.24}$$

and show that $w(z;x,t)$ is a uniquely vector-type analytical solution of (2.9) in D^+. It is simply obtained by $q-q_\pm\in L^1(\mathbb{R})$ that $w(z;x,t)$ exists for any $z\in D^+$. Then we complete the proof from two aspects. Firstly, we prove that for any compact subset $\mathbb{D}_0$ of D^+, $\sum_{n=0}^{N}w^{(n)}(z;x,t)$ converges uniformly to $w(z;x,t)$. For compact subset $\mathbb{D}_0$, $\left|\frac{z(z^2-1)}{2(z^2+1)}\right|$ is bounded with a constant C_1. For any z in $\mathbb{D}_0$,

$$\|w^{(n+1)}(z;x,t)\|\le\left|\frac{z(z^2-1)}{2(z^2+1)}\right|\int_{+\infty}^{x}\|G(z;x,y,t)\|\|w^{(n)}(z;y,t)\|\mathrm{d}y. \tag{2.25}$$

In the above inequality, $|e^{2ik\lambda(x-y)}| < 1$ for $z \in D^+$, so there also has

$$\|G(z;x,y,t)\| \le C_2|q(y,t) - q_+|.$$

Thus,

$$\|w^{(n+1)}(z;x,t)\| \le C_1C_2 \int_{+\infty}^{x} |q(y,t) - q_+|\|w^{(n)}(z;y,t)\|\mathrm{d}y. \tag{2.26}$$

Through mathematical induction we deduce that,

$$\|w^{(n+1)}(z;x,t)\| \le \frac{\left(C_{z_0}C_0 \int_{+\infty}^{x} |q(y,t) - q_+|\mathrm{d}y\right)^n}{n!}, \tag{2.27}$$

which means the integral in (2.23) converges and is finite, then from analyticity of $w^{(0)}$ and $G(z;x,y,t)$, we obtain $w^{(n)}$, $n \ge 1$ are analytic by mathematical induction. Noting that for all $x \in \mathbb{R}^+$ and $z \in \mathbb{D}_0$, we have

$$\|w^{(n)}\| \le \frac{\left(C_1C_2 \int_{+\infty}^{0} |q(y,t) - q_+|\mathrm{d}y\right)^n}{n!}. \tag{2.28}$$

Therefore $w(z;x,t)$ defined by (2.24) absolutely and uniformly converges with respect to $x \in \mathbb{R}^+$ for $z \in \mathbb{D}_0$. So $w(z;x,t)$ is holomorphic in D^+. Next we prove that the solution of the Volterra integral equation is unique. By using (2.23)–(2.24), we have

$$\begin{aligned} w(z;x,t) &= w^{(0)} + \sum_{n=1}^{+\infty} w^{(n)} = w^{(0)} + \frac{z(z^2-1)}{2(z^2+1)} \int_{+\infty}^{x} G(z;x,y,t) \sum_{n=0}^{+\infty} w^{(n)}(z;y,t)\mathrm{d}y \\ &= w^{(0)} + \frac{z(z^2-1)}{2(z^2+1)} \int_{+\infty}^{x} G(z;x,y,t)w(z;y,t)\mathrm{d}y, \end{aligned} \tag{2.29}$$

which means $w(z;x,t)$ defined by (2.24) is a solution of (2.25). In addition, $w(z;x,t)$ is analytical in D^+. Next we give the uniqueness of $w(z;x,t)$. If there has an another solution $h(z;x,t)$ of (2.25), let $H(z;x,t) = w(z;x,t) - h(z;x,t)$. Then it admits

$$H(z;x,t) = \frac{z(z^2-1)}{2(z^2+1)} \int_{+\infty}^{x} G(z;x,y,t)H(z;y,t)\mathrm{d}y. \tag{2.30}$$

In a similar way, we deduce that

$$\|H(z;x,t)\| \le C_1C_2 \int_{+\infty}^{x} |q(y,t) - q_+|\|H(z;y,t)\|\mathrm{d}y. \tag{2.31}$$

Then by Bellmann inequality we have $H \equiv 0$, which means $w(z;x,t)$ defined by (2.24) is the unique solution of (2.25).

The zeros of $a(z)$ on Σ are known to occur and they correspond to spectral singularities. They are excluded from our analysis in this paper. Let Z_N be a subset of $H^{2,2}(\mathbb{R})$ such that $a(z)$ has N simple zeros in the first quadrant. Zeros of $a(z)$ are assumed to be simple in order to simplify our presentation. This is not a restricted assumption because the union $\bigcup_{N=0}^{\infty} Z_N$ is dense in space $H^{2,2}(\mathbb{R})$ thanks to the classical results of Beals and Coifman [49]. Therefore to deal with our following work and make $\rho(z)$ have smoothness and decaying property, we assume our initial data satisfy this assumption.

Assumption 2.1 The initial data $q(x) - q_\pm \in H^{2,2}(\mathbb{R})$ and it generates generic scattering data satisfying that

(1) $a(z)$ has no zero in Σ.

(2) $a(z)$ only has finite number of simple zeros.

Suppose that $a(z)$ has N_1 simple zeros $z_1, \cdots, z_{N_1}$ on $D^+ \cap \{z \in \mathbb{C} : \operatorname{Im} z > 0, |z| > 1\}$, and N_2 simple zeros $w_1, \cdots, w_{N_2}$ on the circle $\left\{z = \mathrm{e}^{\mathrm{i}\varphi} : 0 < \varphi < \frac{\pi}{2}\right\}$. The symmetry (2.15) implies that

$$\begin{aligned}&a(\pm z_n) = 0 \Leftrightarrow \overline{a(\pm \overline{z}_n)} = 0 \Leftrightarrow \overline{a(\pm z_n^{-1})} = 0\\&\Leftrightarrow a(\pm \overline{z}_n^{-1}) = 0, \quad n = 1, \cdots, N_1,\end{aligned}$$

and on the circle

$$a(\pm w_m) = 0 \Leftrightarrow \overline{a(\pm \overline{w}_m)} = 0, \quad m = 1, \cdots, N_2.$$

So the zeros of $a(z)$ come in pairs. It is convenient to define $\zeta_n = z_n$, $\zeta_{n+N_1} = -z_n$, $\zeta_{n+2N_1} = \overline{z}_n^{-1}$ and $\zeta_{n+3N_1} = -\overline{z}_n^{-1}$ for $n = 1, \cdots, N_1$; $\zeta_{m+4N_1} = w_m$ and $\zeta_{m+4N_1+N_2} = -w_m$ for $m = 1, \cdots, N_2$. Therefore, the discrete spectrum is

$$\mathcal{Z} = \{\zeta_n, \ \overline{\zeta}_n\}_{n=1}^{4N_1+2N_2} \tag{2.32}$$

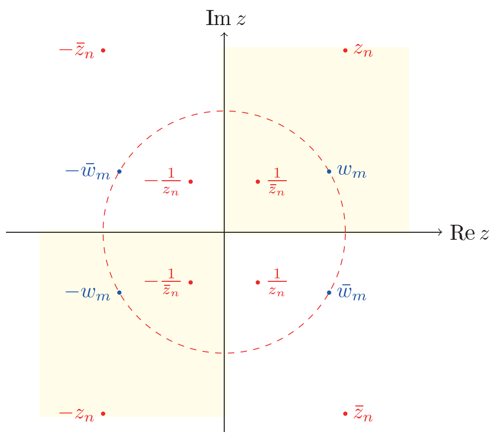


Figure 2 Distribution of the discrete spectrum $\mathcal{Z}$. The red curve is unit circle.

with $\zeta_n \in D^+$ and $\overline{\zeta}_n \in D^-$. The distribution of $\mathcal{Z}$ on the z-plane is shown in Figure 2. As shown in [41], the zero z_n gives the breather solution of the DNLS equation with nonzero boundary conditions (NZBCs), while the zero w_m gives the soliton solution. As shown in [38], there exists a constant b_n such that

$$\mu_+^1(z_n) = b_n \mathrm{e}^{-2\mathrm{i}k(z_n)\lambda(z_n)x} \mu_-^2(z_n). \tag{2.33}$$

Denote norming constant $c_n = \frac{b_n}{a'(z_n)}$. Then we have residue conditions as

$$\operatorname*{Res}_{z=\pm z_n}\Big[\frac{\mu_+^1(z)}{a(z)}\Big] = c_n \mathrm{e}^{-2\mathrm{i}k(\pm z_n)\lambda(\pm z_n)x}\mu_-^2(\pm z_n), \tag{2.34}$$

$$\operatorname*{Res}_{z=\pm \overline{z}_n^{-1}}\Big[\frac{\mu_+^1(z)}{a(z)}\Big] = \pm\frac{\overline{q}_-}{q_-}\overline{z}_n^{-2}\overline{c}_n \mathrm{e}^{-2\mathrm{i}k(\pm \overline{z}_n^{-1})\lambda(\pm \overline{z}_n^{-1})x}\mu_-^2(\pm \overline{z}_n^{-1}), \tag{2.35}$$

$$\operatorname*{Res}_{z=\pm \overline{z}_n}\Big[\frac{\mu_+^2(z)}{\overline{a(\overline{z})}}\Big] = -\overline{c}_n \mathrm{e}^{2\mathrm{i}k(\pm \overline{z}_n)\lambda(\pm \overline{z}_n)x}\mu_-^1(\pm \overline{z}_n), \tag{2.36}$$

$$\operatorname*{Res}_{z=\pm z_n^{-1}}\Big[\frac{\mu_+^2(z)}{\overline{a(\overline{z})}}\Big] = \pm\frac{q_-}{\overline{q}_-}z_n^{-2}c_n \mathrm{e}^{-2\mathrm{i}k(\pm \overline{z}_n^{-1})\lambda(\pm z_n^{-1})x}\mu_-^2(\pm z_n^{-1}). \tag{2.37}$$

For $m = 1, \cdots, N_2$, we also have $c_{N_1+m} = \frac{b_{N_1+m}}{a'(w_m)}$ and

$$\operatorname*{Res}_{z=\pm w_m}\Big[\frac{\mu_+^1(z)}{a(z)}\Big] = c_{N_1+m} \mathrm{e}^{-2\mathrm{i}k(\pm w_m)\lambda(\pm w_m)x}\mu_-^2(\pm w_m), \tag{2.38}$$

$$\operatorname*{Res}_{z=\pm \overline{w}_m}\Big[\frac{\mu_+^2(z)}{\overline{a(\overline{z})}}\Big] = -\overline{c}_{N_1+m} \mathrm{e}^{2\mathrm{i}k(\pm \overline{w}_m)\lambda(\pm \overline{w}_m)x}\mu_-^1(\pm \overline{w}_m). \tag{2.39}$$

For brevity, we introduce a new constant C_n as: For $n = 1, \cdots, N_1$, $C_n = C_{n+N_1} = c_n$, $C_{n+2N_1} = -C_{n+3N_1} = \frac{\overline{q}_-}{q_-}\overline{z}_n^{-2}\overline{c}_n$; for $m = 1, \cdots, N_2$, $C_{m+4N_1} = C_{m+4N_1+N_2} = c_{m+N_1}$, and the collection $\sigma_d = \{\zeta_n, C_n\}_{n=1}^{4N_1+2N_2}$ is called the scattering data.

Now we are going to take into account the time evolution of scattering data. If q also depends on t (i.e., $q = q(x,t)$), we can obtain the functions a and b as above for all times $t \in \mathbb{R}$. Taking account of the t-part in (2.1) and (2.33), the t-derivative of a, b and b_n come into

$$a_t(z;t) = 0, \quad b_t(z;t) = -(2k^2-1)k\lambda b(z;t), \tag{2.40}$$

$$b_n(t) = -(2k^2-1)k\lambda b_n(0). \tag{2.41}$$

Then we can obtain time dependence of scattering data which can be expressed as the following replacement

$$C(\zeta_n) \to C(t,\zeta_n) = c(0,\zeta_n)\mathrm{e}^{-(2k(\zeta_n)^2-1)k(\zeta_n)\lambda(\zeta_n)t}, \tag{2.42}$$

$$\rho(z) \to \rho(t,z) = \rho(0,z)\mathrm{e}^{-(2k^2-1)k\lambda t}. \tag{2.43}$$

In particular, if at time $t = 0$ the initial function $q(x,0)$ produces $4N_1 + 2N_2$ simple zeros $\zeta_1, \cdots, \zeta_{4N_1+2N_2}$ of $a(z;0)$ and if q evolves accordingly to (1.1), then $q(x,t)$ will produce exactly the same $4N_1 + 2N_2$ simple zeros at any other time $t \in \mathbb{R}$. The scattering data with time t is given by

$$\{\mathrm{e}^{-(2k^2-1)k\lambda t}\rho(z), \{\zeta_n, \mathrm{e}^{-(2k(\zeta_n)^2-1)k(\zeta_n)\lambda(\zeta_n)t}C_n\}_{n=1}^{4N_1+2N_2}\},$$

where $\{\rho(z), \{\zeta_n, C_n\}_{n=1}^{4N_1+2N_2}\}$ are obtained from the initial data $q(x,0) = q_0(x)$. Denote the phase function

$$\theta(z) = k(z)\lambda(z)\Big[\frac{x}{t} - (2k(z)^2-1)\Big] = \frac{z^2 - \frac{1}{z^2}}{4}\Big[\frac{x}{t} + 1 - \frac{\big(z-\frac{1}{z}\big)^2}{2}\Big] \tag{2.44}$$

and for convenience we denote $\theta_n = \theta(\zeta_n)$.

To solve the matrix RH problem in the following inverse problem, we finally give the asymptotic behaviors of the modified Jost solutions and scattering matrix as $z \to \infty$ and $z \to 0$ in [41].

Proposition 2.3 *The Jost solutions possess the following asymptotic behaviors*

$$\mu_\pm(z) = \mathrm{e}^{\mathrm{i}\nu_\pm(x,t;q)\sigma_3} + O(z^{-1}), \quad z \to \infty, \tag{2.45}$$

$$\mu_\pm(z) = \frac{\mathrm{i}}{z}\mathrm{e}^{\mathrm{i}\nu_\pm(x,t;q)\sigma_3}\sigma_3 Q_\pm + O(1), \quad z \to 0, \tag{2.46}$$

where

$$\nu_\pm(x,t;q) = \frac{1}{2}\int_{\pm\infty}^{x}(|q|^2 - 1)\mathrm{d}y. \tag{2.47}$$

The scattering matrices admit asymptotic behaviors

$$S(z) = \mathrm{e}^{-\mathrm{i}\nu_0(t;q)\sigma_3} + O(z^{-1}), \quad z \to \infty, \tag{2.48}$$

$$S(z) = \mathrm{diag}\Big(\frac{q_-}{q_+}, \frac{q_+}{q_-}\Big)\mathrm{e}^{\mathrm{i}\nu_0(t;q)\sigma_3} + O(z), \quad z \to 0, \tag{2.49}$$

where

$$\nu_0(t;q) = \frac{1}{2}\int_{-\infty}^{+\infty}(|q|^2 - 1)\mathrm{d}y. \tag{2.50}$$

Further we have $\rho(0) = \widetilde{\rho}(0) = 0$.

Moreover, the trace formula gives that

$$a(z) = \mathrm{e}^{-\mathrm{i}\nu_0(t;q)}\prod_{j=1}^{4N_1+2N_2}\frac{z-\zeta_j}{z-\overline{\zeta}_j}\exp\Big\{-\frac{1}{2\pi\mathrm{i}}\int_\Sigma\frac{\log(1-\rho(s)\widetilde{\rho}(s))}{s-z}\mathrm{d}s\Big\}. \tag{2.51}$$

Proposition 2.4 *If the initial valve* $q_0 - q_\pm \in H^{2,2}(\mathbb{R}^\pm)$, *then the reflection coefficients* $\rho(z)$ *and* $\widetilde{\rho}(z)$ *belong in* $H^1(\mathbb{R}\cup\mathrm{i}\mathbb{R})$.

Proof We take $\rho(z) \in H^1(\mathbb{R})$ as an example. The proof of $\widetilde{\rho}(z)$ and on $\mathrm{i}\mathbb{R}$ is similar. To prove this, we just need to prove $a(z)$, $b(z)$, $a'(z)$ and $b'(z)$ are bounded on $\mathbb{R}$, and $b(z)$ and $b'(z)$ are in $L^{2,1}(\mathbb{R})$. In fact, although points $z = \pm\mathrm{i}$ are a singularity of $a(z)$ and $b(z)$ as shown in (2.18), they are not the singularity of $\rho(z)$ and $\widetilde{\rho}(z)$. So when considering at $z = \pm\mathrm{i}$, on their neighborhood we just need to calculate

$$\rho(z) = \frac{\mathrm{Wr}(\Phi_-^1, \Phi_+^1)}{\mathrm{Wr}(\Phi_+^1, \Phi_-^2)}.$$

Thus the analysis on $\mathrm{i}\mathbb{R}$ is similar as it on $\mathbb{R}$. The symmetry reduction $\rho(z) = \frac{\overline{q}_-}{q_-}\widetilde{\rho}(-z^{-1})$ implies us to divide $\mathbb{R}$ to two part: $(-\infty,-1]\cup[1,\infty)$ and $(-1,1)$. Then we only need to compute on $(-\infty,-1]\cup[1,\infty)$. For convenience, denote $I_0 = (-\infty,-1]\cup[1,\infty)$. (2.18) inspires us to analyze in the same way as Proposition 2.2. Recall the Volterra integral equation of the first column of $Y_+^{-1}\mu_+$ with $t = 0$ and denote $h(x,z) = (h_1(x,z), h_2(x,z))^T = w(z;x) - e_1$:

$$h(z;x) = [Te_1](z;x) + [Th](z;x), \tag{2.52}$$

$$\frac{\partial}{\partial z}h(z;x) = [T_z e_1](z;x) + [T_z h](z;x) + \left[T\frac{\partial}{\partial z}h\right](z;x), \tag{2.53}$$

where T, T_z are integral operators with

$$[Tf](z;x) = \frac{z(z^2-1)}{2(z^2+1)}\int_{+\infty}^{x} G(z;x,y)f(z;y)\mathrm{d}y, \tag{2.54}$$

$$\begin{aligned}[T_z f](z;x) = {} & \frac{z^4-1+4z^2}{2(z^2+1)^2}\int_{+\infty}^{x} G(z;x,y)f(z;y)\mathrm{d}y \\ & + \frac{z(z^2-1)}{2(z^2+1)}\int_{+\infty}^{x} \frac{\partial}{\partial z}G(z;x,y)f(z;y)\mathrm{d}y,\end{aligned} \tag{2.55}$$

and $G(z;x,y)$ is defined in (2.20). Then h and $\frac{\partial}{\partial z}h$ can be written as the Neumann series:

$$h(z;x) = \sum_{n=0}^{+\infty} h^{(n)}(z;x), \quad \frac{\partial}{\partial z}h(z;x) = \sum_{n=0}^{+\infty} h_z^{(n)}(z;x) \tag{2.56}$$

with

$$h^{(0)}(z;x) = [Te_1](z;x), \quad h^{(n+1)}(z;x) = Th^{(n)}(z;x), \tag{2.57}$$

$$h_z^{(0)}(z;x) = [T_z e_1](z;x) + [T_z h](z;x), \quad h_z^{(n+1)}(z;x) = Th_z^{(n)}(z;x). \tag{2.58}$$

By integration by parts we obtain:

$$\begin{aligned}[h^{(0)}]_2(z;x) = {} & \frac{i(z^2-1)}{z(z^2+1)}\int_{+\infty}^{x} \mathrm{e}^{\mathrm{i}\frac{(z^2-z^{-2})(x-y)}{2}} \frac{\partial}{\partial y}\overline{q_0(y)-q_+}\mathrm{d}y \\ & - \frac{z^2-1}{2z^3(z^2+1)}\int_{+\infty}^{x} \mathrm{e}^{\mathrm{i}\frac{(z^2-z^{-2})(x-y)}{2}} \overline{q_0(y)-q_+}\mathrm{d}y \\ & + \frac{z^2-1}{2z^5(z^2+1)}\int_{+\infty}^{x} \mathrm{e}^{\mathrm{i}\frac{(z^2-z^{-2})(x-y)}{2}} \overline{q}_+^2(q_0(y)-q_+)\mathrm{d}y \\ & - \frac{(z^2-1)\mathrm{i}}{2z^3(z^2+1)}\int_{+\infty}^{x} \mathrm{e}^{\mathrm{i}\frac{(z^2-z^{-2})(x-y)}{2}} \overline{q}_+^2\frac{\partial}{\partial y}(q_0(y)-q_+)\mathrm{d}y \\ = {} & \frac{2(z^2-1)}{z^3(z^2+1)}\int_{+\infty}^{x} \mathrm{e}^{\mathrm{i}\frac{(z^2-z^{-2})(x-y)}{2}} \frac{\partial^2}{\partial y^2}\overline{q_0(y)-q_+}\mathrm{d}y \\ & + \frac{i(z^2-1)}{z^5(z^2+1)}\int_{+\infty}^{x} \mathrm{e}^{\mathrm{i}\frac{(z^2-z^{-2})(x-y)}{2}} \frac{\partial}{\partial y}\overline{q_0(y)-q_+}\mathrm{d}y \\ & - \frac{z^2-1}{2z^3(z^2+1)}\int_{+\infty}^{x} \mathrm{e}^{\mathrm{i}\frac{(z^2-z^{-2})(x-y)}{2}} \overline{q_0(y)-q_+}\mathrm{d}y \\ & + \frac{z^2-1}{2z^5(z^2+1)}\int_{+\infty}^{x} \mathrm{e}^{\mathrm{i}\frac{(z^2-z^{-2})(x-y)}{2}} \overline{q}_+^2(q_0(y)-q_+)\mathrm{d}y \\ & - \frac{(z^2-1)\mathrm{i}}{2z^3(z^2+1)}\int_{+\infty}^{x} \mathrm{e}^{\mathrm{i}\frac{(z^2-z^{-2})(x-y)}{2}} \overline{q}_+^2\frac{\partial}{\partial y}(q_0(y)-q_+)\mathrm{d}y.\end{aligned} \tag{2.59}$$

Thus, $[h^{(0)}]_2(z;x) \in C^0(\mathbb{R}_x^+, L^{2,1}(I_0)) \cap L^\infty(I_0 \times \mathbb{R}_x^+)$ when $q_0(x) - q_+ \in H^{2,2}(\mathbb{R}^+)$. Simple calculations give that $[h^{(0)}]_1(z;x) \in L^\infty(I_0 \times \mathbb{R}_x^+)$. In fact, similarly through mathematical induction as in Proposition 2.2, we further have that for any $n \geq 1$, $f_n(x) \in \{q_0(x) - q_+, \overline{q_0(x) - q_+}\}$,

$$\left\| f_1(y_1)\int_{+\infty}^{y_1} f_2(y_2)\cdots\int_{+\infty}^{y_1} f_n(y_n)\mathrm{d}y_n\cdots\mathrm{d}y_2\right\|_{W^{1,2}(\mathbb{R}^+)} \lesssim \frac{\|q_0(x)-q_+\|_{W^{1,2}(\mathbb{R}^+)}^n}{n!},$$

which implies that

$$|T^n e_1| \lesssim \frac{\|q_0(x)-q_+\|^n_{W^{1,2}(\mathbb{R}^+)}}{n!},$$

$$\|[T^n e_1]_2\|_{C^0(\mathbb{R}^+_x, L^{2,1}(I_0))} \lesssim \frac{\|q_0(x)-q_+\|^n_{W^{1,2}(\mathbb{R}^+)}}{n!}.$$

So $h_1(z;x) \in L^\infty(I_0 \times \mathbb{R}^+_x)$ and $h_2(z;x) \in C^0(\mathbb{R}^+_x, L^{2,1}(I_0)) \cap L^\infty(I_0 \times \mathbb{R}^+_x)$ for $x \geq 0$. Similarly, note that

$$\frac{\partial}{\partial z}G(z;x,y) = \begin{pmatrix} \frac{\partial}{\partial z}G_1(z;y) & \frac{\partial}{\partial z}G_2(z;y) \\ G_3(z;y) & G_4(z;y) \end{pmatrix},$$

$$\frac{\partial}{\partial z}G_1(z;y) = -\frac{\mathrm{i}}{z^2}(q_+\overline{q}_0(y) + q_0(y)\overline{q}_+ - 2), \quad \frac{\partial}{\partial z}G_2(z;y) = \frac{2}{z^3}q_+^2\overline{q_0(y)-q_+},$$

$$G_3(z;y) = -\mathrm{e}^{2\mathrm{i}k\lambda(x-y)}\frac{2}{z^3}\overline{q_+^2}(q_0(y)-q_+) - \mathrm{i}\Big(z+\frac{1}{z^3}\Big)(x-y)\mathrm{e}^{2\mathrm{i}k\lambda(x-y)}\overline{G_2(\overline{z};y)},$$

$$G_4(z;y) = -\mathrm{e}^{2\mathrm{i}k\lambda(x-y)}\frac{\partial}{\partial z}G_1(z;y) - \mathrm{i}\Big(z+\frac{1}{z^3}\Big)(x-y)\mathrm{e}^{2\mathrm{i}k\lambda(x-y)}G_1(z;y).$$

Therefore, there also have $[T_z e_1]_1(z;x) + [T_z h]_1(z;x) \in L^\infty(I_0 \times \mathbb{R}^+_x)$ and $[T_z e_1]_2(z;x) + [T_z h]_2(z;x) \in C^0(\mathbb{R}^+_x, L^{2,1}(I_0)) \cap L^\infty(I_0 \times \mathbb{R}^+_x)$ for $x \geq 0$. It also gives that $\frac{\partial}{\partial z}h(z;x) \in L^\infty(I_0 \times \mathbb{R}^+_x) \times (C^0(\mathbb{R}^+_x, L^{2,1}(I_0)) \cap L^\infty(I_0 \times \mathbb{R}^+_x))$. Substitute these results and the asymptotic behaviors as $z \to \infty$ in Proposition 2.3 into (2.18), we obtain our result.

To obtain a model Riemann-Hilbert problem, we define a sectionally meromorphic matrix

$$\mathrm{e}^{\mathrm{i}\nu_-(x,t;q)\sigma_3}M(z;x,t) = \begin{cases} (a(z)^{-1}\mu_+^1,\ \mu_-^2) & \text{as } z \in D^+, \\ (\mu_-^1,\ \overline{a(\overline{z})}^{-1}\mu_+^2) & \text{as } z \in D^-, \end{cases} \tag{2.60}$$

which solves the following RH problem.

RHP 0 Find a matrix-valued function $M(z)$ which satisfies:

- ▶ Analyticity: $M(z)$ is meromorphic in $\mathbb{C} \setminus \Sigma$ and has single poles $\mathcal{Z}$.
- ▶ Symmetry: $M(z) = \sigma_2\overline{M(\overline{z})}\sigma_2 = \sigma_1\overline{M(-\overline{z})}\sigma_1 = \frac{\mathrm{i}}{z}M\big(-\frac{1}{z}\big)\sigma_3 Q_-$.
- ▶ Jump condition: $M(z)$ has continuous boundary values $M_\pm(z)$ on Σ and

$$M_+(z) = M_-(z)V(z), \quad z \in \Sigma, \tag{2.61}$$

where

$$V(z) = \begin{pmatrix} 1-\widetilde{\rho}(z)\rho(z) & -\mathrm{e}^{2\mathrm{i}t\theta}\widetilde{\rho}(z) \\ \mathrm{e}^{-2\mathrm{i}t\theta}\rho(z) & 1 \end{pmatrix}. \tag{2.62}$$

- ▶ Asymptotic behaviors:

$$M(z) = I + \mathcal{O}(z^{-1}), \quad z \to \infty, \tag{2.63}$$

$$M(z) = \frac{\mathrm{i}}{z}\sigma_3 Q_- + \mathcal{O}(1), \quad z \to 0. \tag{2.64}$$

- ▶ Residue conditions: M has simple poles at each point in $\mathcal{Z} \cup \overline{\mathcal{Z}}$ with:

$$\operatorname*{Res}_{z=\zeta_n} M(z) = \lim_{z\to\zeta_n} M(z)\begin{pmatrix} 0 & 0 \\ C_n\mathrm{e}^{-2\mathrm{i}t\theta_n} & 0 \end{pmatrix}, \tag{2.65}$$

$$\operatorname*{Res}_{z=\overline{\zeta}_n} M(z) = \lim_{z\to\overline{\zeta}_n} M(z)\begin{pmatrix} 0 & -\overline{C}_n \mathrm{e}^{2\mathrm{i}t\overline{\theta}_n} \\ 0 & 0 \end{pmatrix}. \tag{2.66}$$

From the asymptotic behavior in Proposition 2.2, the reconstruction formula of $q(x,t)$ is given by

$$q(x,t) = -\mathrm{i}\lim_{z\to\infty}[zM]_{12}. \tag{2.67}$$

3 Deformation to a Mixed $\overline{\partial}$-RH Problem

The long-time asymptotic of RHP 0 is affected by the growth and decay of the exponential function $\mathrm{e}^{\pm 2\mathrm{i}t\theta}$ appearing in both the jump relation and the residue conditions. Therefore, in this section, we introduce a new transform $M(z)\to M^{(1)}(z)$, which make that the $M^{(1)}(z)$ is well behaved as $t\to\infty$ along any characteristic line.

Let $\xi=\frac{x}{t}$. Consider the derivative of $\theta(z)$ defined in (2.44),

$$\frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{\xi+2}{2}\Big(z+\frac{1}{z^3}\Big) - \frac{1}{2}\Big(z^3+\frac{1}{z^5}\Big).$$

Then $\frac{\mathrm{d}\theta}{\mathrm{d}z}=0 \Leftrightarrow z^8-(\xi+2)z^6-(\xi+2)z^2+1=0$. Obviously, $z=0$ is not the solution of this equation. It is naturally to consider the range of real function $\frac{s^4+1}{s^3+s}$. Simple calculation gives that $\big|\frac{s^4+1}{s^3+s}\big|\in(1,+\infty)$ for $s\in\mathbb{R}$. So we deduce that $\frac{\mathrm{d}\theta}{\mathrm{d}z}=0$ has solution on $\mathbb{R}\cup\mathrm{i}\mathbb{R}$ if and only if $|\xi+2|\geq 1$. In fact, when the stationary phase point is not on jump line, as $t\to\infty$, these stationary phase points have no effect. In addition, consider the quartic equation:

$$s^4-(\xi+2)s^3-(\xi+2)s+1=0.$$

By simple calculation and the symmetry of the solution of above equation, we can easily deduce that when $\xi+2<-1$, this equation has two negative solution, and when $\xi+2>1$, this equation has two positive solution. Namely, in the case $\xi+2<-1$, there are four stationary phase points $\xi_j=\widetilde{\xi}_j\mathrm{i}$, $j=1,2,3,4$, on $\mathrm{i}\mathbb{R}$ (see Figure 3(a)) with $\widetilde{\xi}_1>1$, $\widetilde{\xi}_1=\frac{1}{\widetilde{\xi}_2}=-\frac{1}{\widetilde{\xi}_3}=-\widetilde{\xi}_4$. In the case $\xi+2>1$, there are four stationary phase points ξ_j, $j=1,2,3,4$, on $\mathbb{R}$ (see Figure 3(f)) with $\xi_1>1$, $\xi_1=\frac{1}{\xi_2}=-\frac{1}{\xi_3}=-\xi_4$. For brevity, we denote

$$n(\xi)=\begin{cases} 0 & \text{as } |\xi+2|<1, \\ 4 & \text{as } |\xi+2|>1 \end{cases} \tag{3.1}$$

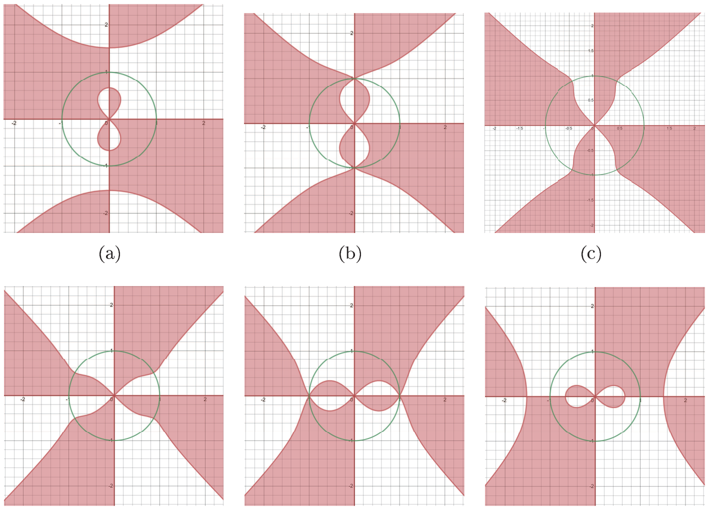
as the number of stationary phase points. To obtain asymptotic behavior of $\mathrm{e}^{2\mathrm{i}t\theta}$ as $t\to\infty$, we consider the real part of $2\mathrm{i}t\theta$:

$$\mathrm{Re}(2\mathrm{i}t\theta) = -t\,\mathrm{Im}\,z\,\mathrm{Re}\,z[(\xi+2)(1+|z|^{-4})-(\mathrm{Re}^2 z-\mathrm{Im}^2 z)(1+|z|^{-8})]. \tag{3.2}$$

The signature of $\mathrm{Im}\,\theta$ is shown in Figure 3.

We introduce some notations with respect to subscripts

$$\begin{aligned} &\mathcal{N}\triangleq\{1,\cdots,4N_1+2N_2\}, \quad \nabla=\{n\in\mathcal{N}:\ \mathrm{Im}\,\theta_n<0\}, \\ &\Delta=\{n\in\mathcal{N}:\ \mathrm{Im}\,\theta_n>0\}. \end{aligned} \tag{3.3}$$



(a) (b) (c)

(d) (e) (f)

Figure 3 In these figure we take $\xi = -4, -3, -2.6, -1.5, -1, 0$, respectively to show all type of $\operatorname{Im}\theta$. The green curve is unit circle. In the red region, $\operatorname{Im}\theta > 0$ while $\operatorname{Im}\theta = 0$ on the red curve. And $\operatorname{Im}\theta < 0$ in the white region.

For $n \in \Delta$, the residue of $M(z)$ at ζ_n in (2.65) are unbounded as $t \to \infty$. Similarly, for $n \in \nabla$, the residue at ζ_n approaches to be zero as $t \to \infty$. To distinguish different type of zeros, we further give

$$\nabla_1 = \{j \in \{1, \cdots, N_1\} : \operatorname{Im}\theta(z_j) < 0\}, \quad \Delta_1 = \{j \in \{1, \cdots, N_1\} : \operatorname{Im}\theta(z_j) > 0\},$$
$$\nabla_2 = \{i \in \{1, \cdots, N_2\} : \operatorname{Im}\theta(w_i) < 0\}, \quad \Delta_2 = \{i \in \{1, \cdots, N_2\} : \operatorname{Im}\theta(w_i) > 0\},$$

For the poles ζ_n with $n \in \Delta \cup \nabla$, we want to trap them for jumps along small closed circles enclosing themselves respectively. The jump matrix in (2.62) also needs to be restricted. Recall the well known factorizations of $V(z)$:

$$\begin{aligned} V(z) &= \begin{pmatrix} 1 & -\widetilde{\rho}(z)\mathrm{e}^{2\mathrm{i}t\theta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \rho(z)\mathrm{e}^{-2\mathrm{i}t\theta} & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ \frac{\rho(z)\mathrm{e}^{-2\mathrm{i}t\theta}}{1-\rho(z)\widetilde{\rho}(z)} & 1 \end{pmatrix} (1-\rho(z)\widetilde{\rho}(z))^{\sigma_3} \begin{pmatrix} 1 & -\frac{\widetilde{\rho}(z)\mathrm{e}^{2\mathrm{i}t\theta}}{1-\rho(z)\widetilde{\rho}(z)} \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{3.4}$$

We will use these factorizations to deform the jump contours so that exponentials $\mathrm{e}^{\pm 2\mathrm{i}t\theta}$ are decaying in corresponding regions respectively. Define

$$I(\xi) = \begin{cases} (-\infty\mathrm{i}, 0\mathrm{i}) \cup (+\infty\mathrm{i}, 0\mathrm{i}) \cup (-\infty, \xi_4) \cup (\xi_3, 0) \cup (\xi_2, \xi_1) & \text{as } -1 < \xi, \\ (\widetilde{\xi}_4\mathrm{i}, \widetilde{\xi}_3\mathrm{i}) \cup (\widetilde{\xi}_2\mathrm{i}, 0) \cup (+\infty\mathrm{i}, \widetilde{\xi}_1\mathrm{i}) & \text{as } \xi < -3, \\ (-\infty\mathrm{i}, 0\mathrm{i}) \cup (+\infty\mathrm{i}, 0\mathrm{i}) & \text{as } -3 < \xi < -1 \end{cases} \tag{3.5}$$

as integral curve of functions:

$$\nu(z) = \frac{1}{2\pi} \log(1 - \rho(z)\widetilde{\rho}(z)); \tag{3.6}$$

$$\delta(z)=\exp\Big(\mathrm{i}\int_{I(\xi)}\Big(\frac{1}{s-z}-\frac{1}{2s}\Big)\nu(s)\mathrm{d}s\Big); \tag{3.7}$$

$$\begin{aligned}T(z)=T(z,\xi)&=\prod_{n\in\Delta}\frac{z-\zeta_n}{\overline{\zeta}_n^{-1}z-1}\delta(z)\\&=\prod_{j\in\Delta_1}\frac{z^2-z_j^2}{\overline{z}_j^{-2}z^2-1}\frac{z^2-\overline{z}_j^{-2}}{z_j^2z^2-1}\prod_{i\in\Delta_2}\frac{z^2-w_i^2}{w_i^2z^2-1}\delta(z).\end{aligned} \tag{3.8}$$

In the above formulae, we choose the principal branch $(0,-\infty)$ of power and logarithm functions. Additionally, introduce a positive constant $\widetilde{\varrho}=\frac{1}{6}\min_{j\neq i\in\mathcal{N}}|\zeta_i-\zeta_j|$ and a set of characteristic functions $\mathcal{X}(z;\xi,j)$ on the interval $\eta(\xi,j)\xi_j-\widetilde{\varrho}<\eta(\xi,j)z<\eta(\xi,j)\xi_j$ when $-1<\xi$ and on the interval $\eta(\xi,j)\widetilde{\xi}_j-\widetilde{\varrho}<\eta(\xi,j)\frac{z}{i}<\eta(\xi,j)\widetilde{\xi}_j$ when $-3>\xi$ for $j=1,\cdots,4$, respectively. And $\eta(\xi,j)$ is a constant depend on ξ and j:

$$\eta(\xi,j)=\begin{cases}(-1)^{j+1} & \text{as } -1<\xi;\\(-1)^j & \text{as } \xi<-3.\end{cases} \tag{3.9}$$

Proposition 3.1 *The function defined by* (3.8) *has the following properties*:

(a) *T is meromorphic in $\mathbb{C}\setminus\mathrm{i}\mathbb{R}$, and for each $n\in\Delta$, $T(z)$ has simple zeros ζ_n and simple poles $\overline{\zeta}_n$;*

(b) $T(z)=\overline{T^{-1}(\overline{z})}=T^{-1}(z^{-1})$;

(c) *for $z\in\mathrm{i}\mathbb{R}$, as z approaching the $I(\xi)$, T has boundary values $T_\pm$, which satisfy*:

$$T_+(z)=T_-(z)(1-\rho(z)\widetilde{\rho}(z)),\quad z\in I(\xi); \tag{3.10}$$

(d) $\lim_{z\to\infty}T(z)\triangleq T(\infty)$, *where*

$$T(\infty)=\prod_{j\in\Delta_1}\overline{z}_j^2z_j^{-2}\prod_{i\in\Delta_2}\overline{w}_i^2\exp\Big(\frac{1}{4\pi i}\int_{I(\xi)}s^{-1}\log(1-\rho(s)\widetilde{\rho}(s))\mathrm{d}s\Big) \tag{3.11}$$

with $|T(\infty)|=1$;

(e) *as $|z|\to\infty$ with $|\arg(z)|\leq c<\pi$,*

$$T(z)=T(\infty)\Big(1+z^{-1}\frac{1}{2\pi\mathrm{i}}\int_{I(\xi)}\log(1-\rho(s)\widetilde{\rho}(s))\mathrm{d}s+\mathcal{O}(z^{-2})\Big); \tag{3.12}$$

(f) *$T(z)$ is continuous at $z=0$, and*

$$\lim_{z\to0}T(z)=T(0)=T(\infty)^{-1}; \tag{3.13}$$

(g) *$\frac{a(z)}{T(z)}$ is holomorphic in D^+. And its absolute value is bounded in $D^+\cap\{z\in\mathbb{C}\mid\mathrm{Re}\,z>0\}$. Additionally, the ratio extends as a continuous function on $\mathrm{i}\mathbb{R}$;*

(h) *when $|\xi+2|>1$, as $z\to\xi_j$ along any ray $\xi_j+\mathrm{e}^{\mathrm{i}\phi}\mathbb{R}^+$ with $|\phi|<\pi$,*

$$|T(z,\xi)-T_j(\xi)[\eta(\xi,j)(z-\xi_j)]^{-\mathrm{i}\eta(\xi,j)\nu(\xi_j)}|\lesssim\|r\|_{H^{1,1}(\mathbb{R})}|z-\xi_j|^{\frac{1}{2}}, \tag{3.14}$$

where $T_j(\xi)$ is the complex unit

$$T_j(\xi)=\prod_{n\in\Delta}\frac{\zeta_j-\zeta_n}{\overline{\zeta}_n^{-1}\zeta_j-1}\mathrm{e}^{-\mathrm{i}\beta(\xi_j,\xi)} \tag{3.15}$$

for $j=1,\cdots,4$. *In above function, when* $\xi>-1$,

$$\beta_j(z,\xi)=\int_{I(\xi)}\frac{\nu(s)-\mathcal{X}(z;\xi,j)\nu(\xi_j)}{s-z}\mathrm{d}s-\eta(\xi,j)\nu(\xi_j)\log(\eta(\xi,j)(z-\xi_j+\widetilde{\varrho})), \tag{3.16}$$

and when $\xi<-3$,

$$\beta_j(z,\xi)=\int_{I(\xi)}\frac{\nu(s)-\mathcal{X}(z;\xi,j)\nu(\xi_j)}{s-z}\mathrm{d}s-\eta(\xi,j)\nu(\xi_j)\log(\eta(\xi,j)(z-\xi_j+\widetilde{\varrho}\mathrm{i})). \tag{3.17}$$

Proof Properties (a), (b), (d) and (f) can be obtained by simple calculation. Specially, to prove $T(z)=\overline{T^{-1}(\overline{z})}$ in (b), it is imperative to change variable of the integral in $\overline{\delta(\overline{z})}$ as $s=\mathrm{i}\eta$, and noting that $\rho(-\mathrm{i}\eta)\widetilde{\rho}(-\mathrm{i}\eta)=\rho(\mathrm{i}\eta)\widetilde{\rho}(\mathrm{i}\eta)$, then

$$\begin{aligned}\overline{\delta(\overline{z})}&=\exp\Big(\frac{1}{2\pi}\Big(\int_0^{+\infty}+\int_0^{-\infty}\Big)\Big(\frac{1}{-\mathrm{i}\eta-z}-\frac{1}{-2\mathrm{i}\eta}\Big)\log(1-\rho(\mathrm{i}\eta)\widetilde{\rho}(\mathrm{i}\eta))\mathrm{d}\eta\Big)\\&=\exp\Big(-\frac{1}{2\pi}\Big(\int_0^{+\infty}+\int_0^{-\infty}\Big)\Big(\frac{1}{\mathrm{i}\eta-z}-\frac{1}{2\mathrm{i}\eta}\Big)\log(1-\rho(\mathrm{i}\eta)\widetilde{\rho}(\mathrm{i}\eta))\mathrm{d}\eta\Big)\\&=\delta(z)^{-1}.\end{aligned} \tag{3.18}$$

(c) follows from the Plemelj formula. By the Laurent expansion (e) immediately. For brevity, we omit calculation. For (g), for brief we only give the details for the $|\xi+2|<1$ case, from (2.51) we have

$$\frac{a(z)}{T(z)}=T(\infty)^{-1}\prod_{j\in\nabla_1}\frac{z^2-z_j^2}{\overline{z}_j^{-2}z^2-1}\frac{z^2-\overline{z}_j^{-2}}{z_j^2z^2-1}\prod_{i\in\nabla_2}\frac{z^2-w_i^2}{w_i^2z^2-1}\exp\Big\{-\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}}\frac{\log(1-\rho(s)\widetilde{\rho}(s))}{s-z}\Big\}.$$

So $\frac{a(z)}{T(z)}$ is holomorphic in D^+. In above expression, all factors except the last integral is bounded for $z\in D^+$. From (2.16)–(2.17), $1-\rho(s)\widetilde{\rho}(s)=1+|\rho(s)|^2$. Let $z=x+y\mathrm{i}$, then the real part of the exponential is $-\frac{y}{2\pi}\int_{\mathbb{R}}\frac{\log(1+|\rho(s)|^2)}{|s-z|^2}\mathrm{d}s$ which can be bounded as follows:

$$\begin{aligned}\Big|\frac{y}{2\pi}\int_{\mathbb{R}}\frac{\log(1+|\rho(s)|^2)}{|s-z|^2}\mathrm{d}s\Big|&\le\frac{1}{2\pi}\|\log(1+|\rho(s)|^2)\|_{L^\infty(\mathbb{R})}\Big\|\frac{y}{(s-x)^2+y^2}\Big\|_{L^1(\mathbb{R})}\\&\lesssim\|\rho(s)\|_{L^\infty(\mathbb{R})}.\end{aligned}$$

And the proof of (h) is same as [16, Proposition 3.1].

Additionally, define three subsets of $\mathcal{N}$:

$$\Lambda=\{n\in\mathcal{N}:\ |\mathrm{Im}\,\theta_n|=0\},\quad\Lambda_1=\{j_0\in\{1,\cdots,N_1\}:\ |\mathrm{Im}\,\theta(z_{j_0})|=0\}, \tag{3.19}$$

$$\Lambda_2=\{i_0\in\{1,\cdots,N_2\}:\ |\mathrm{Im}\,\theta(w_{i_0})|=0\}. \tag{3.20}$$

And let ϱ be a positive constant satisfying

$$\varrho=\frac{1}{2}\min\Big\{\min_{j\neq i\in\mathcal{N}}|\zeta_i-\zeta_j|,\min_{j\in\mathcal{N}\setminus\Lambda,\mathrm{Im}\,\theta(z)=0}|\zeta_j-z|\Big\}. \tag{3.21}$$

By above definition, for every $n\in\mathcal{N}$, we define disks $\mathbb{D}(\zeta_n,\varrho)$, such that they pairwise disjoint, also disjoint with Σ. Introduce a piecewise matrix function

$$G(z)=\begin{cases}\begin{pmatrix}1 & 0\\ -C_n(z-\zeta_n)^{-1}\mathrm{e}^{-2\mathrm{i}t\theta_n} & 1\end{pmatrix} & \text{as } z\in\mathbb{D}(\zeta_n,\varrho), n\in\nabla;\\ \begin{pmatrix}1 & -C_n^{-1}(z-\zeta_n)\mathrm{e}^{2\mathrm{i}t\theta_n}\\ 0 & 1\end{pmatrix} & \text{as } z\in\mathbb{D}(\zeta_n,\varrho), n\in\Delta;\\ \begin{pmatrix}1 & \overline{C}_n(z-\overline{\zeta}_n)^{-1}\mathrm{e}^{2\mathrm{i}t\overline{\theta}_n}\\ 0 & 1\end{pmatrix} & \text{as } z\in\mathbb{D}(\overline{\zeta}_n,\varrho), n\in\nabla;\\ \begin{pmatrix}1 & 0\\ \overline{C}_n^{-1}(z-\overline{\zeta}_n)\mathrm{e}^{-2\mathrm{i}t\overline{\theta}_n} & 1\end{pmatrix} & \text{as } z\in\mathbb{D}(\overline{\zeta}_n,\varrho), n\in\Delta;\\ I & \text{as } z \text{ in elsewhere.}\end{cases} \tag{3.22}$$

Now we use $T(z)$ and $G(z)$ to define a new matrix-valued function $M^{(1)}(z)$:

$$M^{(1)}(z)=T(\infty)^{-\sigma_3}M(z)G(z)T(z)^{\sigma_3}, \tag{3.23}$$

which is solution the following RH problem.

RHP 1 Find a matrix-valued function $M^{(1)}(z)$ which satisfies:

▶ Analyticity: $M^{(1)}(z)$ is meromorphic in $\mathbb{C}\setminus\Sigma^{(1)}$, where

$$\Sigma^{(1)}=\mathbb{R}\cup\mathrm{i}\mathbb{R}\cup[\cup_{n\in\mathcal{N}\setminus\Lambda}(\partial\mathbb{D}(\overline{\zeta}_n,\varrho)\cup\partial\mathbb{D}(\zeta_n,\varrho))], \tag{3.24}$$

is shown in Figure 4.

▶ Symmetry: $M^{(1)}(z)=\sigma_2\overline{M^{(1)}(\overline{z})}\sigma_2=\sigma_1\overline{M^{(1)}(-\overline{z})}\sigma_1=\frac{\mathrm{i}}{z}M^{(1)}\big(-\frac{1}{z}\big)\sigma_3Q_-$.

▶ Jump condition: $M^{(1)}(z)$ has continuous boundary values $M_\pm^{(1)}$ on $\Sigma^{(1)}$ and

$$M_+^{(1)}(z)=M_-^{(1)}(z)V^{(1)}(z),\quad z\in\Sigma^{(1)}, \tag{3.25}$$

where

$$V^{(1)}(z)=\begin{cases}\begin{pmatrix}1 & -\mathrm{e}^{2\mathrm{i}t\theta}\widetilde{\rho}(z)T^{-2}(z)\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \mathrm{e}^{-2\mathrm{i}t\theta}\rho(z)T^2(z) & 1\end{pmatrix} & \text{as } z\in\Sigma\setminus I(\xi);\\ \begin{pmatrix}1 & 0\\ \frac{\mathrm{e}^{-2\mathrm{i}t\theta}\rho(z)T_-^2(z)}{1-\widetilde{\rho}(z)\rho(z)} & 1\end{pmatrix}\begin{pmatrix}1 & -\frac{\mathrm{e}^{2\mathrm{i}t\theta}\widetilde{\rho}(z)T_+^{-2}(z)}{1-\widetilde{\rho}(z)\rho(z)}\\ 0 & 1\end{pmatrix} & \text{as } z\in I(\xi);\\ \begin{pmatrix}1 & 0\\ -C_n(z-\zeta_n)^{-1}T^2(z)\mathrm{e}^{-2\mathrm{i}t\theta_n} & 1\end{pmatrix} & \text{as } z\in\partial\mathbb{D}(\zeta_n,\varrho), n\in\nabla;\\ \begin{pmatrix}1 & -C_n^{-1}(z-\zeta_n)T^{-2}(z)\mathrm{e}^{2\mathrm{i}t\theta_n}\\ 0 & 1\end{pmatrix} & \text{as } z\in\partial\mathbb{D}(\zeta_n,\varrho), n\in\Delta;\\ \begin{pmatrix}1 & \overline{C}_n(z-\overline{\zeta}_n)^{-1}T^{-2}(z)\mathrm{e}^{2\mathrm{i}t\overline{\theta}_n}\\ 0 & 1\end{pmatrix} & \text{as } z\in\partial\mathbb{D}(\overline{\zeta}_n,\varrho), n\in\nabla;\\ \begin{pmatrix}1 & 0\\ \overline{C}_n^{-1}(z-\overline{\zeta}_n)\mathrm{e}^{-2\mathrm{i}t\overline{\theta}_n}T^2(z) & 1\end{pmatrix} & \text{as } z\in\partial\mathbb{D}(\overline{\zeta}_n,\varrho), n\in\Delta.\end{cases} \tag{3.26}$$

▶ Asymptotic behaviors:

$$M^{(1)}(z)=I+\mathcal{O}(z^{-1}),\quad z\to\infty, \tag{3.27}$$

$$M^{(1)}(z)=\frac{\mathrm{i}}{z}\sigma_3Q_-+\mathcal{O}(1),\quad z\to 0. \tag{3.28}$$

► Residue conditions: $M^{(1)}(z)$ has simple poles at each point ζ_n and $\overline{\zeta}_n$ for $n \in \Lambda$ with

$$\operatorname*{Res}_{z=\zeta_n} M^{(1)}(z) = \lim_{z\to\zeta_n} M^{(1)}(z) \begin{pmatrix} 0 & 0 \\ C_n \mathrm{e}^{-2\mathrm{i}t\theta_n} T^2(\zeta_n) & 0 \end{pmatrix}, \tag{3.29}$$

$$\operatorname*{Res}_{z=\overline{\zeta}_n} M^{(1)}(z) = \lim_{z\to\overline{\zeta}_n} M^{(1)}(z) \begin{pmatrix} 0 & -\overline{C}_n T^{-2}(\overline{\zeta}_n) \mathrm{e}^{2\mathrm{i}t\overline{\theta}_n} \\ 0 & 0 \end{pmatrix}. \tag{3.30}$$

Proof Note that the triangular factors (3.22) trap poles ζ_n and $\overline{\zeta}_n$ to jumps on the disk boundaries $\partial\mathbb{D}(\zeta_n, \varrho)$ and $\partial\mathbb{D}(\overline{\zeta}_n, \varrho)$, respectively for $n \in \mathcal{N} \setminus \Lambda$. Then by simple calculation we can obtain the residues condition and jump condition from (2.62), (2.65)–(2.66) and (3.22)–(3.23). The analyticity and symmetry of $M^{(1)}(z)$ is directly from its definition, Proposition 3.1, (3.22) and the properties of $M(z)$. As for asymptotic behaviors, from $\lim_{z\to0} G(z) = \lim_{z\to\infty} G(z) = I$ and Proposition 3.1(f), we obtain that $M^{(1)}(z)$ has same asymptotic behaviors as $M(z)$.

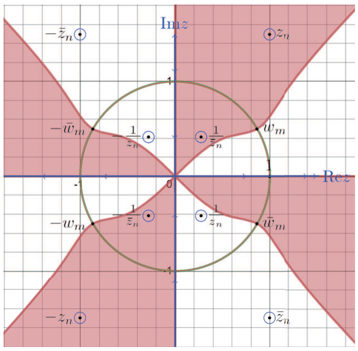


Figure 4 The blue curve, including $\mathbb{R}$, i$\mathbb{R}$ and the small circles constitute $\Sigma^{(1)}$. For $\zeta_n \in \mathcal{Z} \setminus \Lambda$, we change it to jump on $\partial\mathbb{D}(\zeta_n, \varrho)$. In this figure, we take w_m as the pole point which satisfies $\operatorname{Im}\theta(w_m) = 0$ as an example, while take z_n as the pole point which satisfies $\operatorname{Im}\theta(z_n) \neq 0$ as an example.

4 Mixed $\overline{\partial}$-RH Problem

In this section, we make continuous extension for the jump matrix $V^{(1)}$ to remove the jump from Σ. Besides, the new problem is hoped to take advantage of the decay/growth of $\mathrm{e}^{2\mathrm{i}t\theta(z)}$ for $z \notin \Sigma$. For this purpose, we introduce some new regions and contours relayed on ξ.

4.1 For the region $\xi \in (-3, -1)$

We define

$$\Omega_{2n+1} = \left\{ z \in \mathbb{C} \,\middle|\, \frac{n\pi}{2} \le \arg z \le \frac{n\pi}{2} + \varphi \right\}, \tag{4.1}$$

$$\Omega_{2n+2} = \left\{ z \in \mathbb{C} \,\middle|\, (n+1)\frac{\pi}{2} - \varphi \le \arg z \le (n+1)\frac{\pi}{2} \right\}, \tag{4.2}$$

where $n = 0, 1, 2, 3$ and $\varphi > 0$ is a fixed sufficiently small angle achieving following conditions:

(1) $\frac{2|\xi+2|}{|\xi+2|+1} < \cos 2\varphi < 1$;

(2) Each Ω_i doesn't intersect any of $\mathbb{D}(\zeta_n, \varrho)$ or $\mathbb{D}(\overline{\zeta}_n, \varrho)$.

Define new contours as follow:

$$\Sigma_k = \mathrm{e}^{(k-1)\mathrm{i}\frac{\pi}{4}+\varphi}R_+, \quad k=1,3,5,7, \tag{4.3}$$

$$\Sigma_k = \mathrm{e}^{k\mathrm{i}\frac{\pi}{4}-\varphi}R_+, \quad k=2,4,6,8, \tag{4.4}$$

$$\widetilde{\Sigma} = \Sigma_1 \cup \Sigma_2 \cup \cdots \cup \Sigma_8, \tag{4.5}$$

which is the boundary of Ω_k, respectively. In addition, let

$$\Omega = \Omega_1 \cup \cdots \cup \Omega_8, \tag{4.6}$$

$$\Sigma^{(2)} = \bigcup_{n\in\mathcal{N}\setminus\Lambda} (\partial\mathbb{D}(\overline{\zeta}_n,\varrho)\cup\partial\mathbb{D}(\zeta_n,\varrho)), \tag{4.7}$$

which are shown in Figure 5.

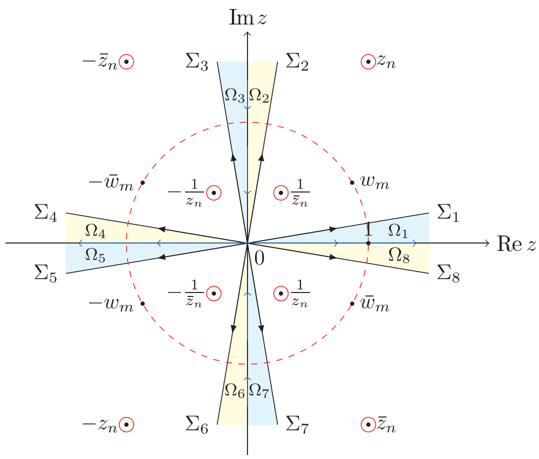


Figure 5 The yellow and blue region is Ω. The red circles constitute $\Sigma^{(2)}$ together. Similarly in this figure we suppose that $\operatorname{Im}\theta(w_m)=0$ while $\operatorname{Im}\theta(z_n)=0$.

Lemma 4.1 *Let $\xi=\frac{x}{t}\in(-3,-1)$. $F(r)=r^2+\frac{1}{r^2}$ is a real-valued function. Then for $z=r\mathrm{e}^{\mathrm{i}\phi}$, the imaginary part of phase function* (3.2) *satisfies*

$$\operatorname{Im}\theta(z) \le \frac{1}{16}|\sin 2\phi|(|\xi+2|-1)F(r)^2 \quad \text{as } z\in\Omega_1,\Omega_3,\Omega_5,\Omega_7; \tag{4.8}$$

$$\operatorname{Im}\theta(z) \ge \frac{1}{16}|\sin 2\phi|(1-|\xi+2|)F(r)^2 \quad \text{as } z\in\Omega_2,\Omega_4,\Omega_6,\Omega_8. \tag{4.9}$$

Proof We only prove the case $z\in\Omega_1$, and the other regions are similarly. From (3.2) we have

$$\begin{aligned}\operatorname{Im}\theta(z) &= \frac{1}{2}\operatorname{Im}z\operatorname{Re}z[(\xi+2)(1+|z|^{-4})-(\operatorname{Re}^2z-\operatorname{Im}^2z)(1+|z|^{-8})]\\ &= \frac{1}{4}r^2\sin 2\phi[(\xi+2)(1+r^{-4})-r^2\cos 2\phi(1+r^{-8})]\\ &= \frac{1}{4}\sin 2\phi[(\xi+2)F(r)-\cos 2\phi(F(r)^2-2)].\end{aligned} \tag{4.10}$$

$F(r) \geq 2$ leads to $2 \leq \frac{F(r)^2}{2}$. For $z \in \Omega_1$, $\frac{2|\xi+2|}{|\xi+2|+1} < \cos 2\varphi < \cos 2\phi$, then we have

$$\frac{|\xi+2|}{\cos 2\phi} F(r) \leq \frac{|\xi+2|+1}{4} F(r)^2. \tag{4.11}$$

Substitute above inequality into (4.10) we obtain the consequence immediately. Introduce a small enough constant $1 > \varepsilon_0 > 0$ with $(1-\varepsilon_0)\cos\varphi > \frac{1}{2}$. Let $X_1 \in C_0^\infty(\mathbb{R}, [0,1])$, which is support in $(1-\varepsilon_0, 1+\varepsilon_0)$. X_0 has support in $(-\varepsilon_0, \varepsilon_0)$ with $X_0(z) = X_1(1+z)$. In addition, we denote following functions:

$$p_1(z) = p_5(z) = \rho(z), \quad p_2(z) = p_6(z) = \frac{\widetilde{\rho}(z)}{1-\rho(z)\widetilde{\rho}(z)}, \tag{4.12}$$

$$p_3(z) = p_7(z) = \frac{\rho(z)}{1-\rho(z)\widetilde{\rho}(z)}, \quad p_4(z) = p_8(z) = \widetilde{\rho}(z). \tag{4.13}$$

Then the next step is to construct a matrix function $R^{(2)}(z)$. We need to remove jump on $\mathbb{R}$ and $\mathrm{i}\mathbb{R}$, and have some mild control on $\overline{\partial} R^{(2)}(z)$ sufficient to ensure that the $\overline{\partial}$-contribution to the long-time asymptotics of $q(x,t)$ is negligible. So we choose $R^{(2)}(z)$ as

$$R^{(2)}(z) = \begin{cases} \begin{pmatrix} 1 & R_j(z)\mathrm{e}^{2\mathrm{i}t\theta} \\ 0 & 1 \end{pmatrix}, & z \in \Omega_j, \ j = 2,4,6,8; \\ \\ \begin{pmatrix} 1 & 0 \\ R_j(z)\mathrm{e}^{-2\mathrm{i}t\theta} & 1 \end{pmatrix}, & z \in \Omega_j, \ j = 1,3,5,7; \\ \\ I, & \text{elsewhere}, \end{cases} \tag{4.14}$$

where the functions R_j, $j = 1, 2, \cdots, 8$, is defined in following Proposition.

Proposition 4.1 *R_j: $\overline{\Omega}_j \to C$, $j = 1, 2, \cdots, 8$ have boundary values as follows*:

$$R_1(z) = \begin{cases} -\rho(z)T(z)^2, & z \in \mathbb{R}^+, \\ 0, & z \in \Sigma_1, \end{cases} \qquad R_2(z) = \begin{cases} 0, & z \in \Sigma_2, \\ \dfrac{\widetilde{\rho}(z), T_+(z)^2}{1-\rho(z)\widetilde{\rho}(z)} & z \in \mathrm{i}\mathbb{R}^+, \end{cases} \tag{4.15}$$

$$R_3(z) = \begin{cases} \dfrac{\rho(z)T_-(z)^2}{1-\rho(z)\widetilde{\rho}(z)}, & z \in \mathrm{i}\mathbb{R}^+, \\ 0, & z \in \Sigma_3, \end{cases} \qquad R_4(z) = \begin{cases} 0, & z \in \Sigma_4, \\ -\widetilde{\rho}(z)T(z)^{-2}, & z \in \mathbb{R}^-, \end{cases} \tag{4.16}$$

$$R_5(z) = \begin{cases} -\rho(z)T(z)^2, & z \in \mathbb{R}^-, \\ 0, & z \in \Sigma_5, \end{cases} \qquad R_6(z) = \begin{cases} 0, & z \in \Sigma_6, \\ \dfrac{\widetilde{\rho}(z)T_+(z)^2}{1-\rho(z)\widetilde{\rho}(z)}, & z \in \mathrm{i}\mathbb{R}^-, \end{cases} \tag{4.17}$$

$$R_7(z) = \begin{cases} \dfrac{\rho(z)T_-(z)^2}{1-\rho(z)\widetilde{\rho}(z)}, & z \in \mathrm{i}\mathbb{R}^-, \\ 0, & z \in \Sigma_7, \end{cases} \qquad R_8(z) = \begin{cases} 0, & z \in \Sigma_8, \\ -\widetilde{\rho}(z)T(z)^{-2}, & z \in \mathbb{R}^+. \end{cases} \tag{4.18}$$

R_j have the following properties: *For $j = 1, 5, 4, 8$,*

$$|\overline{\partial} R_j(z)| \lesssim |p_j'(|z|)| + |z|^{-\frac{1}{2}} \quad \textit{for all } z \in \Omega_j; \tag{4.19}$$

and for $j = 2, 3, 6, 7$,

$$|\overline{\partial} R_j(z)| \lesssim |z \mp \mathrm{i}| \quad \textit{for all } z \in \Omega_j \textit{ in a small fixed neighborhood of } \pm\mathrm{i}, \tag{4.20}$$

$$|\overline{\partial}R_j(z)| \lesssim |p_j'(\mathrm{i}|z|)| + |z|^{-\frac{1}{2}} + |\overline{\partial}X_1(|z|)| \quad \textit{for all } z \in \Omega_j. \tag{4.21}$$

And

$$\overline{\partial}R_j(z) = 0 \quad \textit{if } z \in \textit{elsewhere}. \tag{4.22}$$

Proof Case I $z \in \overline{\Omega}_j,\ j = 1, 5, 4, 8.$

Take $R_1(z)$ as an example with extensions

$$R_1(z) = p_1(|z|)T^2(z)\cos(k_0 \arg z), \quad k_0 = \frac{2\pi}{\varphi}. \tag{4.23}$$

The other cases are easily inferred. $p_1(|z|) = \rho(|z|)$ is bounded. Denote $z = r\mathrm{e}^{\mathrm{i}\phi}$, then we have $\overline{\partial} = \frac{\mathrm{e}^{\mathrm{i}\phi}}{2}\big(\partial_r + \frac{\mathrm{i}}{r}\partial_\phi\big)$. So

$$\overline{\partial}R_1(z) = \frac{\mathrm{e}^{\mathrm{i}\phi}}{2}T^2(z)\Big(p_1'(r)\cos(k_0\phi) - \frac{\mathrm{i}}{r}p_1(r)k_0\sin(k_0\phi)\Big). \tag{4.24}$$

To bound second term we use Cauchy-Schwarz inequality and obtain

$$|p_1(r)| = |\rho(r)| = |\rho(r) - \rho(0)| = \Big|\int_0^r \rho'(s)\mathrm{d}s\Big| \le \|\rho'(s)\|_{L^2} r^{\frac{1}{2}}. \tag{4.25}$$

Note that $T(z)$ is a bounded function in $\overline{\Omega}_1$. Then the boundedness of (4.19) follows immediately.

Case II $z \in \overline{\Omega}_j$, $j = 2, 3, 6, 7$.

The details of the proof are only given for R_2. Unlike the vanishing boundary condition case in [16], the determinant of $M(z)$ is $1 + z^{-2}$. So to bound the $\overline{\partial}$-derivative construct by $R^{(2)}$ in the following section, the property of $R^{(2)}$ at $\pm\mathrm{i}$ needs to be control. For this purpose, we make small adjustments to the extensions of R_2 as

$$R_2(z) = R_{21}(z) + R_{22}(z) \tag{4.26}$$

with a constant δ_0 satisfying $\varphi > \delta_0\varepsilon_0$ and

$$R_{21}(z) = [1 - X_1(|z|)]p_2(\mathrm{i}|z|)T^{-2}(z)\cos\Big[k_0\Big(\frac{\pi}{2} - \arg z\Big)\Big], \tag{4.27}$$

$$\begin{aligned} R_{22}(z) = {} & f(|z|)g(z)\cos\Big[k_0\Big(\frac{\pi}{2} - \arg z\Big)\Big] \\ & - \frac{\mathrm{i}|z|}{k_0}X_0\Big(\frac{\arg z}{\delta_0}\Big)f'(|z|)g(z)\sin\Big[k_0\Big(\frac{\pi}{2} - \arg z\Big)\Big]. \end{aligned} \tag{4.28}$$

Among above functions,

$$f(z) = X_1(z)\frac{\overline{b}(z)}{a(z)}, \quad g(z) = \Big(\frac{a(z)}{T(z)}\Big)^2. \tag{4.29}$$

Then $f(z) \in W^{2,\infty}$. Obviously, $R_{21}(z) \equiv 0$ with $|z|$ in the support of X_1 and $R_{22}(z) \equiv 0$ out the support of X_1. Note that

$$|p_2(z)| = \Big|\frac{\widetilde{\rho}(z)}{1 - \rho(z)\widetilde{\rho}(z)}\Big| = \Big|\frac{\widetilde{\rho}(z)}{1 - |\rho(z)|^2}\Big| \lesssim |\rho(z)| \quad \text{for } z \text{ out of } \mathrm{supp}(X_1). \tag{4.30}$$

Similarly in Case I, $R_{21}(z)$ can be bounded as

$$|\overline{\partial}R_{21}(z)| \lesssim (1 - X_1(|z|))(|p_2'(\mathrm{i}|z|)| + |z|^{-\frac{1}{2}}) + |\overline{\partial}X_1(|z|)|. \tag{4.31}$$

As for $R_{22}(z)$, $z = r\mathrm{e}^{\mathrm{i}\phi}$,

$$\begin{aligned}\overline{\partial}R_{22}(z) = {} & \frac{\mathrm{e}^{\mathrm{i}\phi}}{2}g(z)\cos\Big[k_0\Big(\frac{\pi}{2} - \varphi\Big)\Big]f'(\mathrm{i}r)\Big(1 - X_0\Big(\frac{\varphi}{\delta_0}\Big)\Big)\\ & + \sin\Big[k_0\Big(\frac{\pi}{2} - \varphi\Big)\Big]\Big[\frac{\mathrm{i}k_0}{r}f(\mathrm{i}r) + \frac{1}{\delta_0 k_0}X_0'\Big(\frac{\varphi}{\delta_0}\Big)f'(\mathrm{i}r)\Big]\\ & - \frac{\mathrm{i}}{k_0}\sin\Big[k_0\Big(\frac{\pi}{2} - \varphi\Big)\Big]X_0\Big(\frac{\arg z}{\delta_0}\Big)(rf'(\mathrm{i}r))'.\end{aligned} \tag{4.32}$$

So $|\overline{\partial}R_{22}(z)|$ is bounded, and we can write $|\overline{\partial}R_{22}(z)| \lesssim X_1(z)|z|^{-\frac{1}{2}}$. So (4.20) is obtained. In addition, for $z \sim i$,

$$|\overline{\partial}R_{22}(z)| \lesssim \Big|\sin\Big[k_0\Big(\frac{\pi}{2} - \varphi\Big)\Big]\Big| + \Big|1 - X_0\Big(\frac{\varphi}{\delta_0}\Big)\Big| = \mathcal{O}(\varphi), \tag{4.33}$$

from which (4.20) follows immediately.

4.2 For the region $|\xi + 2| > 1$

We define deformation contours and domains:

For $j = 2, 3$,

$$\Sigma_{jk}(\xi) = \begin{cases}\xi_j + \mathrm{e}^{\mathrm{i}[-\varphi + \frac{k+1}{2\pi}]}l, & \xi > -1,\\ \xi_j + \mathrm{e}^{\mathrm{i}[-\varphi + \frac{k+1}{2\pi}]}\mathrm{i}l, & \xi < -3,\end{cases} \qquad k = 1, 3,\ l \in \Big(0, \frac{|\xi_{j+(-1)^{\frac{k}{2}+1}} - \xi_j|}{4\cos\varphi}\Big),$$

$$\Sigma_{jk}(\xi) = \begin{cases}\xi_j + \mathrm{e}^{\mathrm{i}[(\frac{k}{2})\pi + \varphi]}l, & \xi > -1,\\ \xi_j + \mathrm{e}^{\mathrm{i}[(\frac{k}{2})\pi + \varphi]}\mathrm{i}l, & \xi < -3,\end{cases} \qquad k = 2, 4,\ l \in \Big(0, \frac{|\xi_{j+(-1)^{\frac{k}{2}}} - \xi_j|}{4\cos\varphi}\Big).$$

And for $j = 1, 4$,

$$\Sigma_{jk}(\xi) = \begin{cases}\xi_j + \mathrm{e}^{(-1)^{k-1}\varphi\mathrm{i}}l_j, & \xi > -1,\\ \xi_j + \mathrm{e}^{(-1)^{k-1}\varphi\mathrm{i}}\mathrm{i}l_j, & \xi < -3,\end{cases} \qquad l_1 \in \mathbb{R}^+,\ l_4 \in \Big(0, \frac{|\xi_4 - \xi_3|}{4\cos\varphi}\Big),\ k = 1, 2,$$

$$\Sigma_{jk}(\xi) = \begin{cases}\xi_j + \mathrm{e}^{\pi i + (-1)^{j+1}\mathrm{i}\varphi}l_j, & \xi > -1,\\ \xi_j + \mathrm{e}^{\pi i + (-1)^{j+1}\mathrm{i}\varphi}\mathrm{i}l_j, & \xi < -3,\end{cases} \qquad l_4 \in \mathbb{R}^+,\ l_1 \in \Big(0, \frac{|\xi_1 - \xi_2|}{4\cos\varphi}\Big),\ k = 3, 4.$$

In addition, for $j = 1, 2, 3$, Σ_{jk}' is a straight line which connects the end of Σ_{jk} and the intermediate transit point $\frac{\xi_j + \xi_{j+1}}{2}$ or $\frac{\xi_j + \xi_{j-1}}{2}$ (see Figure 7). And

$$\Sigma_k = \begin{cases}\mathrm{i}\mathrm{e}^{\mathrm{i}[\pi\frac{k-1}{2} + \varphi]}\mathbb{R}^+, & \xi > -1,\\ \mathrm{e}^{\mathrm{i}[\pi\frac{k-1}{2} + \varphi]}\mathbb{R}^+, & \xi < -3,\end{cases} \qquad k = 1, 3,$$

$$\Sigma_k = \begin{cases}\mathrm{i}\mathrm{e}^{\mathrm{i}[\pi(\frac{k}{2} - 1) - \varphi]}\mathbb{R}^+, & \xi > -1,\\ \mathrm{e}^{\mathrm{i}[\pi(\frac{k}{2} - 1) - \varphi]}\mathbb{R}^+, & \xi < -3,\end{cases} \qquad k = 2, 4.$$

Besides, denote $\xi_0 = +\infty$, $\xi_5 = -\infty$, and introduce some intervals when $j = 1, \cdots, 4$, for $\xi > -1$,

$$I_{j1} = I_{j2} = \begin{cases} \Big(\dfrac{\xi_j + \xi_{j+1}}{2}, \xi_j\Big), & j \text{ is odd number}, \\ \Big(\xi_j, \dfrac{\xi_j + \xi_{j-1}}{2}\Big), & j \text{ is even number}, \end{cases} \tag{4.34}$$

$$I_{j3} = I_{j4} = \begin{cases} \Big(\xi_j, \dfrac{\xi_j + \xi_{j-1}}{2}\Big), & j \text{ is odd number}, \\ \Big(\dfrac{\xi_j + \xi_{j+1}}{2}, \xi_j\Big), & j \text{ is even number}, \end{cases} \tag{4.35}$$

and for $\xi < -3$,

$$I_{j1} = I_{j2} = \begin{cases} \mathrm{i}\Big(\xi_j, \dfrac{\xi_j + \xi_{j-1}}{2}\Big), & j \text{ is odd number}, \\ \mathrm{i}\Big(\dfrac{\xi_j + \xi_{j+1}}{2}, \xi_j\Big), & j \text{ is even number}, \end{cases} \tag{4.36}$$

$$I_{j3} = I_{j4} = \begin{cases} \mathrm{i}\Big(\dfrac{\xi_j + \xi_{j+1}}{2}, \xi_j\Big), & j \text{ is odd number}, \\ \mathrm{i}\Big(\xi_j, \dfrac{\xi_j + \xi_{j-1}}{2}\Big), & j \text{ is even number}. \end{cases} \tag{4.37}$$

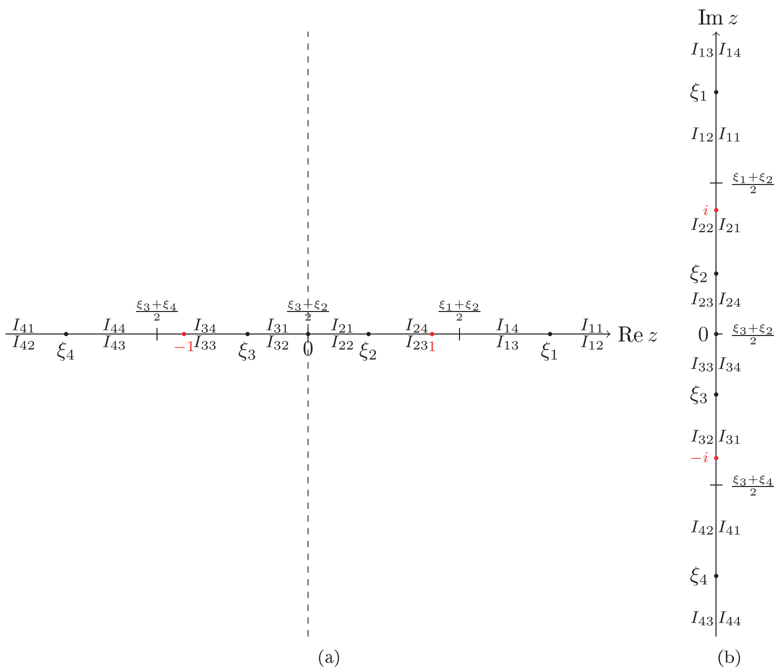


Figure 6 Figure (a) and (b) are corresponding to the $\xi > -1$ and $\xi < -3$, respectively. There are four stationary phase points $\xi_1, \cdots, \xi_4$ with $\xi_1 = -\xi_4 = \frac{1}{\xi_2} = -\frac{1}{\xi_3}$.

These intervals are shown in Figure 6. Then Σ_{jk}, Σ'_{jk} and I_{jk} common constitute the region Ω_{jk} as boundary. And Σ_k together with $\mathrm{i}\mathbb{R}$ constitute the region Ω_k as boundary when $\xi > -1$

while Σ_k together with $\mathbb{R}$ constitute the region Ω_k as boundary when $\xi < -3$. These contours separate complex plane $\mathbb{C}$ into sectors. In addition, let

$$\widetilde{\Sigma}(\xi) = \Big(\bigcup_{\substack{k=1,\cdots,4, \\ j=1,\cdots,4}} \Sigma_{jk} \Big) \cup \Big(\bigcup_{\substack{k=1,\cdots,4, \\ j=1,\cdots,3}} \Sigma'_{jk} \Big),$$

$$\Sigma^{(2)}(\xi) = \widetilde{\Sigma}(\xi) \bigcup_{n \in \mathcal{N} \setminus \Lambda} \Big(\partial \overline{\mathbb{D}}_n \cup \partial \mathbb{D}_n \Big),$$

$$\Omega(\xi) = \bigcup_{k,j=1,\cdots,4} \Omega_{jk} \cup \Big(\bigcup_{k=1,\cdots,4,} \Omega_k \Big), \quad \Omega_{\pm}(\xi) = D^{\pm} \setminus \Omega,$$

which are shown in Figure 7. And $\frac{\pi}{8} > \varphi > 0$ is a fixed sufficiently small angle achieving the following conditions. Firstly, each Ω_i does not intersect $\{z \in \mathbb{C}; \operatorname{Im} \theta(z) = 0\}$ and any of $\mathbb{D}_n$ or $\overline{\mathbb{D}}_n$. This condition is to guarantee the uniformity of the sign of $\operatorname{Im} \theta(z)$. For bounded region Ω_{jk}, obviously, there must exist sufficiently small φ does not intersect the curve $\{z \in \mathbb{C}; \operatorname{Im}\ \theta(z) = 0\}$. But for the unbounded region origin from ξ_1 or ξ_4, the existence of angle φ may not be obviously. To illustrate it, we give the following stronger lemma.

Lemma 4.2 *There exists a sufficiently small φ and a constant $c(\xi) > 0$ relied on $|\xi + 2| > 1$ such that the imaginary part of phase function* (3.2), $\operatorname{Im} \theta(z)$ *have the following estimation for $i = 1, \cdots, 4$:*

$$\operatorname{Im} \theta(z) \leq -c(\xi) \operatorname{Im} z (\operatorname{Re} z - \xi_i) \quad \textit{as } z \in \Omega_{i1}, \Omega_{i3}; \tag{4.38}$$

$$\operatorname{Im} \theta(z) \geq c(\xi) \operatorname{Im} z (\operatorname{Re} z - \xi_i) \quad \textit{as } z \in \Omega_{i2}, \Omega_{i4}. \tag{4.39}$$

Proof We only give the details of the case $\xi > -1$, and take $z \in \Omega_{11}$ as an example, because the proof in the other regions are similar. Denote $\operatorname{Re} z = u, \operatorname{Im} z = v$ with $u - \xi_1, v \in \mathbb{R}^+$, thus,

$$\begin{aligned} &\operatorname{Im} \theta(z) \leq -c(\xi) \operatorname{Im} z (\operatorname{Re} z - \xi_i) \\ \Leftrightarrow\ & u[(\xi + 2)(1 + (u^2 + v^2)^{-2}) - (u^2 - v^2)(1 + (u^2 + v^2)^{-4})] \leq -2c(u - \xi_1). \end{aligned}$$

Consider the following real value function:

$$f(u, v) = (\xi + 2)(1 + (u^2 + v^2)^{-2}) - (u^2 - v^2)(1 + (u^2 + v^2)^{-4}). \tag{4.40}$$

Obviously, $f(u, v)$ is smooth at the point $(\xi_1, 0)$ and

$$\frac{\partial f}{\partial u}(\xi_1, 0) = \frac{2}{\xi_1^7 (\xi_1^4 + 1)} (\xi_1^4 - 1)(-\xi_1^8 - 4\xi_1^4 - 1) < 0, \tag{4.41}$$

$$\frac{\partial f}{\partial v}(\xi_1, 0) = 0. \tag{4.42}$$

Therefore a constant $c(\xi) > 0$ and a small neighbourhood $D((u, v), \delta) \cap \Omega_{11}$, $\delta > 0$ of $(\xi_1, 0)$ exist which satisfy

$$|f(u, v) - f(\xi_1, 0)| \leq -c|u - \xi_1|, \quad (u, v) \in D((u, v), \delta) \cap \Omega_{11}. \tag{4.43}$$

Therefore, when z is near ξ_1, (4.38) is proved. Next we consider the case that $|z|$ is efficient large, (4.38) is equivalent to

$$\frac{u}{(u - \xi_1)} [(\xi + 2)(1 + (u^2 + v^2)^{-2}) - (u^2 - v^2)(1 + (u^2 + v^2)^{-4})] \leq -2c. \tag{4.44}$$

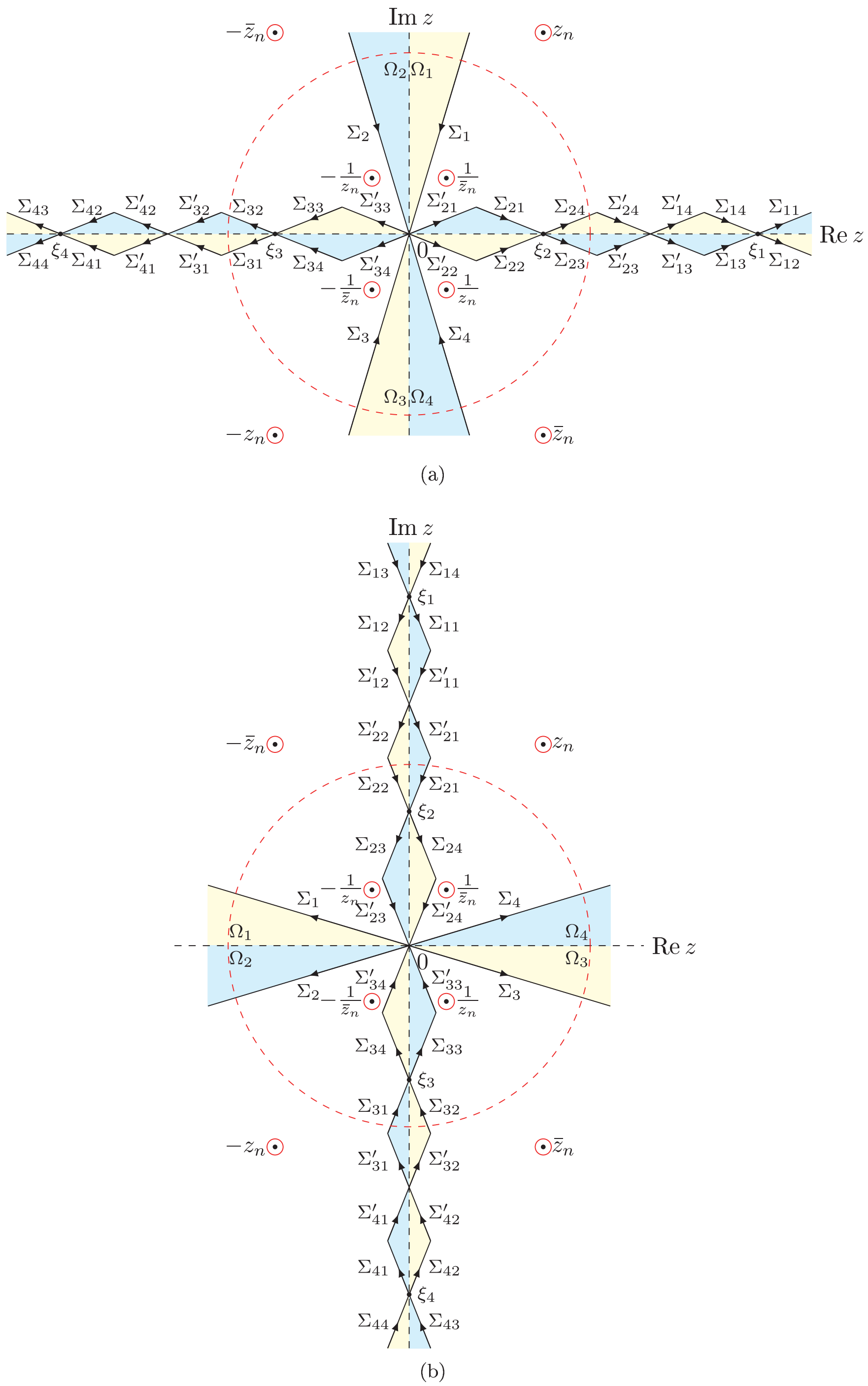


Figure 7 The yellow and blue region is Ω. The red circle around the poles and Σ_{11} constitute $\Sigma^{(2)}$ together.

Obviously, the left hand of the above inequation is going to $-\infty$. Finally, when z is in the remaining compact subset of Ω_{11}, let

$$g(x)=\frac{1+x^8}{x^2(1+x^4)}. \tag{4.45}$$

Then

$$\begin{aligned}&\frac{u}{(u-\xi_1)}[(\xi+2)(1+(u^2+v^2)^{-2})-(u^2-v^2)(1+(u^2+v^2)^{-4})]\\&=\frac{u(1+|z|^4)}{(u-\xi_1)|z|^4}\Big[g(\xi_1)-g(|z|)+\frac{2v^2}{|z|^2}g(|z|)\Big]\\&\le\frac{u(1+|z|^4)}{(u-\xi_1)|z|^4}[g(\xi_1)-g(|z|)+2\sin^2\varphi g(|z|)].\end{aligned} \tag{4.46}$$

Because z is not in $D(\xi_1,\delta)$ and $g(u)$ is monotonic increasing,

$$g(|z|)\ge g(\xi_1+\delta)>g(\xi_1). \tag{4.47}$$

Therefore, the exists a efficient small positive φ, such that

$$g(\xi_1)-g(|z|)+2\sin^2\varphi g(|z|)<0. \tag{4.48}$$

When z in the compact subset of Ω_{11},

$$\frac{u}{(u-\xi_1)}[(\xi+2)(1+(u^2+v^2)^{-2})-(u^2-v^2)(1+(u^2+v^2)^{-4})]<0,$$

implies existing positive constant c that satisfies

$$\frac{u}{(u-\xi_1)}[(\xi+2)(1+(u^2+v^2)^{-2})-(u^2-v^2)(1+(u^2+v^2)^{-4})]<-2c.$$

In fact, for the bounded region Ω_{jk}, the existence of $c(\xi)$ and φ only need to discuss near stationary phase point and in the remaining compact set case. Therefore, we complete the proof of (4.38).

Thus, we can give the sign of the imaginary part of phase function (3.2) in each sector. Introduce a new unknown function

$$R^{(2)}(z,\xi)=\begin{cases}\begin{pmatrix}1 & R_{kj}(z,\xi)\mathrm{e}^{2\mathrm{i}t\theta}\\0 & 1\end{pmatrix}, & z\in\Omega_{kj},\ k=1,\cdots,4,\ j=2,4;\\ \begin{pmatrix}1 & 0\\R_{kj}(z,\xi)\mathrm{e}^{-2\mathrm{i}t\theta} & 1\end{pmatrix}, & z\in\Omega_{kj},\ k=1,\cdots,4,\ j=1,3;\\ \begin{pmatrix}1 & R_k(z,\xi)\mathrm{e}^{2\mathrm{i}t\theta}\\0 & 1\end{pmatrix}, & z\in\Omega_k,\ k=1,3(\xi>-1),\ k=2,4(\xi<-3);\\ \begin{pmatrix}1 & 0\\R_k(z,\xi)\mathrm{e}^{-2\mathrm{i}t\theta} & 1\end{pmatrix}, & z\in\Omega_k,\ k=2,4(\xi>-1),\ k=1,3(\xi<-3);\\ I, & \text{elsewhere};\end{cases} \tag{4.49}$$

where the functions $R_{kj}(z,\xi)$, $R_k(z,\xi)$, $k,j=1,2,3,4$ are defined in the following proposition.

Proposition 4.2 *As for $|\xi+2|>1$, the functions R_{kj}: $\overline{\Omega}_{kj}\to\mathbb{C}$, $k,j=1,2,3,4$ have boundary values as follow*:

$$R_{k1}(z,\xi)=\begin{cases}p_{k1}(z,\xi)T(z)^2, & z\in I_{k1},\\ p_{k1}(\xi_k,\xi)T_k(\xi)^2(\eta(\xi,k)(z-\xi_k))^{-2i\eta(\xi,k)\nu(\xi_k)}, & z\in\Sigma_{k1},\end{cases}\tag{4.50}$$

$$R_{k2}(z,\xi)=\begin{cases}p_{k2}(\xi_k,\xi)T_k(\xi)^{-2}(\eta(\xi,k)(z-\xi_k))^{2i\eta(\xi,k)\nu(\xi_k)}, & z\in\Sigma_{k2},\\ p_{k2}(z,\xi)T(z)^{-2}, & z\in I_{k2},\end{cases}\tag{4.51}$$

$$R_{k3}(z,\xi)=\begin{cases}p_{k3}(z,\xi)T_-(z)^2, & z\in I_{k3},\\ p_{k3}(\xi_k,\xi)T_k(\xi)^2(\eta(\xi,k)(z-\xi_k))^{-2i\eta(\xi,k)\nu(\xi_k)}, & z\in\Sigma_{k3},\end{cases}\tag{4.52}$$

$$R_{k4}(z,\xi)=\begin{cases}p_{k4}(\xi_k,\xi)T_k(\xi)^{-2}(\eta(\xi,k)(z-\xi_k))^{\eta(\xi,k)2i\nu(\xi_k)}, & z\in\Sigma_{k4},\\ p_{k4}(z,\xi)T_+(z)^{-2}, & z\in I_{k4},\end{cases}\tag{4.53}$$

where I_{kj} are specified in (4.34)–(4.37) *and*

$$p_{j1}(z,\xi)=-\rho(z),\quad p_{j3}(z,\xi)=\frac{\rho(z)}{1-\widetilde{\rho}(z)\rho(z)},\tag{4.54}$$

$$p_{j2}(z,\xi)=-\widetilde{\rho}(z),\quad p_{j4}(z,\xi)=\frac{\widetilde{\rho}(z)}{1-\widetilde{\rho}(z)\rho(z)}.\tag{4.55}$$

The functions R_{kj} have following properties:

$$|R_{kj}(z,\xi)|\lesssim\sin^2(k_0\arg(z-\xi_k))+(1+\mathrm{Re}(z)^2)^{-\frac{1}{2}}\quad\textit{for all } z\in\Omega_{kj},\tag{4.56}$$

$$|\overline{\partial}R_{kj}(z,\xi)|\lesssim|p_{kj}'(\mathrm{Re}\,z)|+|z-\xi_k|^{-\frac{1}{2}}\quad\textit{for all } z\in\Omega_{kj}.\tag{4.57}$$

$$|\overline{\partial}R_{kj}(z)|\lesssim|z\mp i|\quad\textit{for all } z\in\Omega_{kj}\textit{ in a small fixed neighborhood of }\pm i,\tag{4.58}$$

$$\overline{\partial}R_{kj}(z,\xi)=0\quad\textit{if } z,\textit{at elsewhere}.\tag{4.59}$$

The matrix functions $R_j:\overline{\Omega}_k\to\mathbb{C}$, $k=1,2,3,4$ have boundary values as follows:
(1) *When $\xi>-1$*,

$$R_1(z)=\begin{cases}0, & z\in\Sigma_2,\\ \dfrac{\widetilde{\rho}(z)T_+(z)^2}{1-\rho(z)\widetilde{\rho}(z)}, & z\in i\mathbb{R}^+,\end{cases}\qquad R_2(z)=\begin{cases}\dfrac{\rho(z)T_-(z)^2}{1-\rho(z)\widetilde{\rho}(z)}, & z\in i\mathbb{R}^+,\\ 0, & z\in\Sigma_2,\end{cases}$$

$$R_3(z)=\begin{cases}0, & z\in\Sigma_3,\\ \dfrac{\widetilde{\rho}(z)T_+(z)^2}{1-\rho(z)\widetilde{\rho}(z)}, & z\in i\mathbb{R}^-,\end{cases}\qquad R_4(z)=\begin{cases}\dfrac{\rho(z)T_-(z)^2}{1-\rho(z)\widetilde{\rho}(z)}, & z\in i\mathbb{R}^-,\\ 0, & z\in\Sigma_4.\end{cases}$$

R_j have the following properties: *For $j=1,2,3,4$*,

$$|\overline{\partial}R_j(z)|\lesssim|z\mp i|\quad\textit{for all } z\in\Omega_j\textit{ in a small fixed neighborhood of }\pm i,\tag{4.60}$$

$$|\overline{\partial}R_j(z)|\lesssim|p_j'(i|z|)|+|z|^{-\frac{1}{2}}+|\overline{\partial}X_1(|z|)|\quad\textit{for all } z\in\Omega_j.\tag{4.61}$$

(2) *when $\xi<-3$*,

$$R_1(z)=\begin{cases}-\widetilde{\rho}(z)T(z)^{-2}, & z\in\mathbb{R}^-,\\ 0, & z\in\Sigma_1,\end{cases}\qquad R_2(z)=\begin{cases}0, & z\in\Sigma_2,\\ -\rho(z)T(z)^2, & z\in\mathbb{R}^-,\end{cases}$$

$$R_3(z)=\begin{cases}-\widetilde{\rho}(z)T(z)^{-2}, & z\in\mathbb{R}^+,\\ 0, & z\in\Sigma_3,\end{cases}\qquad R_4(z)=\begin{cases}0, & z\in\Sigma_4,\\ -\rho(z)T(z)^2, & z\in\mathbb{R}^+.\end{cases}$$

R_j have the following properties:

$$|\overline{\partial}R_j(z)|\lesssim|p_j'(|z|)|+|z|^{-\frac{1}{2}}\quad\textit{for all } z\in\Omega_j.\tag{4.62}$$

Proof We give the details for R_{11} when $\xi \in [0,2)$ only. The other cases are easily inferred. Using the constants $T_k(\xi)$ defined in Proposition 3.1, we give the extension of $R_{11}(z,\xi)$ on Ω_{11}:

$$\begin{aligned} R_{11}(z,\xi) = {} & p_{11}(\xi_1,\xi)T_1(\xi)^2(z-\xi_1)^{-2\mathrm{i}\nu(\xi_1)}[1-\cos(k_0\arg(z-\xi_1))] \\ & + \cos(k_0\arg(z-\xi_1))p_{11}(\operatorname{Re} z,\xi)T(z)^2. \end{aligned} \tag{4.63}$$

Let $z-\xi_1 = l\mathrm{e}^{\mathrm{i}\psi} = u+v\mathrm{i}$ with $l,\psi,u,v \in \mathbb{R}$. From $r \in H^{1,1}(\mathbb{R})$, which means $p_{11} \in H^{1,1}(R)$ we have $|p_{11}(u)| \lesssim (1+u^2)^{-\frac{1}{2}}$. Then we have (4.56). Since

$$\overline{\partial} = \frac{1}{2}(\partial_u + \mathrm{i}\partial_v) = \frac{\mathrm{e}^{\mathrm{i}\psi}}{2}(\partial_l + \mathrm{i}l^{-1}\partial_\psi),$$

we have

$$\begin{aligned} \overline{\partial}R_{11} = {} & (p_{11}(u,\xi)T(z)^2 - p_{11}(\xi_1,\xi)T_1(\xi)^2(z-\xi_1)^{-2\mathrm{i}\nu(\xi_1)})\overline{\partial}\cos(k_0\psi) \\ & + \frac{1}{2}T(z)^2 p'_{11}(u,\xi)\cos(k_0\psi). \end{aligned} \tag{4.64}$$

Substitute (3.14) into above equation, (4.57) comes immediately. And the proof of the properties for R_k, $k=1,2,3,4$ is similar in Proposition 4.1.

In addition, from Proposition 2.1, $R^{(2)}$ achieves the symmetry:

$$R^{(2)}(z) = \sigma_2\overline{R^{(2)}(\overline{z})}\sigma_2 = \sigma_1\overline{R^{(2)}(-\overline{z})}\sigma_1 = \sigma_3 Q_- R^{(2)}\Big(-\frac{1}{z}\Big)\sigma_3 Q_-. \tag{4.65}$$

We now use $R^{(2)}$ to define the new transformation

$$M^{(2)}(z) = M^{(1)}(z)R^{(2)}(z), \tag{4.66}$$

which satisfies the following mixed $\overline{\partial}$-RH problem.

RHP 2 Find a matrix valued function $M^{(2)}(z;x,t)$ with the following properties:

▶ Analyticity: $M^{(2)}(z;x,t)$ is continuous in $\mathbb{C}\setminus\Sigma^{(2)}$, sectionally continuous first partial derivatives in $\mathbb{C}\setminus(\Sigma^{(2)}\cup\{\zeta_n,\overline{\zeta}_n\}_{n\in\Lambda})$ and meromorphic out $\overline{\Omega}$.

▶ Symmetry: $M^{(2)}(z) = \sigma_2\overline{M^{(2)}(\overline{z})}\sigma_2 = \sigma_1\overline{M^{(2)}(-\overline{z})}\sigma_1 = \frac{\mathrm{i}}{z}M^{(2)}\big(-\frac{1}{z}\big)\sigma_3 Q_-$.

▶ Jump condition: $M^{(2)}$ has continuous boundary values $M^{(2)}_\pm$ on $\Sigma^{(2)}$ and

$$M^{(2)}_+(z;x,t) = M^{(2)}_-(z;x,t)V^{(2)}(z), \quad z \in \Sigma^{(2)}, \tag{4.67}$$

where when $|\xi+2|<1$,

$$V^{(2)}(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ -C_n(z-\zeta_n)^{-1}T^2(z)\mathrm{e}^{-2\mathrm{i}t\theta_n} & 1 \end{pmatrix} & \text{as } z \in \partial\mathbb{D}(\zeta_n,\varrho),\ n\in\nabla; \\ \begin{pmatrix} 1 & -C_n^{-1}(z-\zeta_n)T^{-2}(z)\mathrm{e}^{2\mathrm{i}t\theta_n} \\ 0 & 1 \end{pmatrix} & \text{as } z \in \partial\mathbb{D}(\zeta_n,\varrho),\ n\in\Delta; \\ \begin{pmatrix} 1 & \overline{C}_n(z-\overline{\zeta}_n)^{-1}T^{-2}(z)\mathrm{e}^{2\mathrm{i}t\overline{\theta}_n} \\ 0 & 1 \end{pmatrix} & \text{as } z \in \partial\mathbb{D}(\overline{\zeta}_n,\varrho),\ n\in\nabla; \\ \begin{pmatrix} 1 & 0 \\ \overline{C}_n^{-1}(z-\overline{\zeta}_n)\mathrm{e}^{-2\mathrm{i}t\overline{\theta}_n}T^2(z) & 1 \end{pmatrix} & \text{as } z \in \partial\mathbb{D}(\overline{\zeta}_n,\varrho),\ n\in\Delta; \end{cases} \tag{4.68}$$

and when $|\xi+2|>1$,

$$V^{(2)}(z)=\begin{cases} R^{(2)}(z)^{-1}|_{\Sigma_{k1}\cup\Sigma_{k4}} & \text{as } z\in\Sigma_{k1}\cup\Sigma_{k4};\\ R^{(2)}(z)|_{\Sigma_{k2}\cup\Sigma_{k3}} & \text{as } z\in\Sigma_{k2}\cup\Sigma_{k3};\\ R^{(2)}(z)^{-1}|_{\Sigma'_{k1}\cup\Sigma'_{k4}} & \text{as } z\in\Sigma'_{k1}\cup\Sigma'_{k4};\\ R^{(2)}(z)|_{\Sigma'_{k2}\cup\Sigma'_{k3}} & \text{as } z\in\Sigma'_{k2}\cup\Sigma'_{k3};\\ \begin{pmatrix} 1 & 0\\ -C_n(z-\zeta_n)^{-1}T^2(z)\mathrm{e}^{-2\mathrm{i}t\theta_n} & 1\end{pmatrix} & \text{as } z\in\partial\mathbb{D}_n,\ n\in\nabla;\\ \begin{pmatrix} 1 & -C_n^{-1}(z-\zeta_n)T^{-2}(z)\mathrm{e}^{2\mathrm{i}t\theta_n}\\ 0 & 1\end{pmatrix} & \text{as } z\in\partial\mathbb{D}_n,\ n\in\Delta;\\ \begin{pmatrix} 1 & \overline{C}_n(z-\overline{\zeta}_n)^{-1}T^{-2}(z)\mathrm{e}^{2\mathrm{i}t\overline{\theta}_n}\\ 0 & 1\end{pmatrix} & \text{as } z\in\partial\overline{\mathbb{D}}_n,\ n\in\nabla;\\ \begin{pmatrix} 1 & 0\\ \overline{C}_n^{-1}(z-\overline{\zeta}_n)\mathrm{e}^{-2\mathrm{i}t\overline{\theta}_n}T^2(z) & 1\end{pmatrix} & \text{as } z\in\partial\overline{\mathbb{D}}_n,\ n\in\Delta.\end{cases} \tag{4.69}$$

► Asymptotic behaviors:

$$M^{(2)}(z)=I+\mathcal{O}(z^{-1}),\quad z\to\infty, \tag{4.70}$$

$$M^{(2)}(z)=\frac{\mathrm{i}}{z}\sigma_3Q_-+\mathcal{O}(1),\quad z\to 0. \tag{4.71}$$

► $\overline{\partial}$-Derivative: For $z\in\mathbb{C}$ we have

$$\overline{\partial}M^{(2)}=M^{(2)}\overline{\partial}R^{(2)}, \tag{4.72}$$

where when $|\xi+2|<1$,

$$\overline{\partial}R^{(2)}=\begin{cases}\begin{pmatrix}0 & \overline{\partial}R_j(z)\mathrm{e}^{2\mathrm{i}t\theta}\\ 0 & 0\end{pmatrix}, & z\in\Omega_j,\ j=1,3,5,7,\\ \begin{pmatrix}0 & 0\\ \overline{\partial}R_j(z)\mathrm{e}^{-2\mathrm{i}t\theta} & 0\end{pmatrix}, & z\in\Omega_j,\ j=2,4,6,8,\\ 0, & \text{elsewhere},\end{cases} \tag{4.73}$$

and when $|\xi+2|>1$,

$$\overline{\partial}R^{(2)}(z,\xi)=\begin{cases}\begin{pmatrix}0 & \overline{\partial}R_{kj}(z,\xi)\mathrm{e}^{2\mathrm{i}t\theta}\\ 0 & 0\end{pmatrix}, & z\in\Omega_{kj},\ k=1,\cdots,4,\ j=2,4;\\ \begin{pmatrix}0 & 0\\ \overline{\partial}R_{kj}(z,\xi)\mathrm{e}^{-2\mathrm{i}t\theta} & 0\end{pmatrix}, & z\in\Omega_{kj},\ k=1,\cdots,4,\ j=1,3;\\ \begin{pmatrix}0 & \overline{\partial}R_k(z,\xi)\mathrm{e}^{2\mathrm{i}t\theta}\\ 0 & 0\end{pmatrix}, & z\in\Omega_k,\ k=1,3\ (\xi>-1),\ k=2,4\ (\xi<-3);\\ \begin{pmatrix}0 & 0\\ \overline{\partial}R_k(z,\xi)\mathrm{e}^{-2\mathrm{i}t\theta} & 0\end{pmatrix}, & z\in\Omega_k,\ k=2,4\ (\xi>-1),\ k=1,3\ (\xi<-3);\\ 0, & \text{elsewhere}.\end{cases} \tag{4.74}$$

▶ Residue conditions: $M^{(2)}$ has simple poles at each point ζ_n and $\overline{\zeta}_n$ for $n \in \Lambda$ with

$$\operatorname*{Res}_{z=\zeta_n} M^{(2)}(z) = \lim_{z\to\zeta_n} M^{(2)}(z) \begin{pmatrix} 0 & 0 \\ C_n \mathrm{e}^{-2\mathrm{i}t\theta_n} T^2(\zeta_n) & 0 \end{pmatrix}, \tag{4.75}$$

$$\operatorname*{Res}_{z=\overline{\zeta}_n} M^{(2)}(z) = \lim_{z\to\overline{\zeta}_n} M^{(2)}(z) \begin{pmatrix} 0 & -\overline{C}_n T^{-2}(\overline{\zeta}_n) \mathrm{e}^{2\mathrm{i}t\overline{\theta}_n} \\ 0 & 0 \end{pmatrix}. \tag{4.76}$$

5 Decomposition of the Mixed $\overline{\partial}$-RH Problem

To solve RHP 2, we decompose it into a model RH problem for $M^R(z)$ with $\overline{\partial}R^{(2)} \equiv 0$ and a pure $\overline{\partial}$-Problem with nonzero $\overline{\partial}$-derivatives. For the first step, we establish a RH problem for the $M^R(z)$ as follows.

RHP 3 Find a matrix-valued function $M^R(z)$ with the following properties:
▶ Analyticity: $M^R(z)$ is meromorphic in $\mathbb{C} \setminus \Sigma^{(2)}$.
▶ Jump condition: M^R has continuous boundary values $M^R_{\pm}$ on $\Sigma^{(2)}$ and

$$M^R_+(z) = M^R_-(z) V^{(2)}(z), \quad z \in \Sigma^{(2)}. \tag{5.1}$$

▶ Symmetry: $M^R(z) = \sigma_2 \overline{M^R(\overline{z})} \sigma_2 = \sigma_1 \overline{M^R(-\overline{z})} \sigma_1 = \frac{\mathrm{i}}{z} M^R\big(-\frac{1}{z}\big) \sigma_3 Q_-$.
▶ $\overline{\partial}$-Derivative: $\overline{\partial}R^{(2)} = 0$ for $z \in \mathbb{C}$.
▶ Asymptotic behaviors:

$$M^R(z) = I + \mathcal{O}(z^{-1}), \quad z \to \infty, \tag{5.2}$$

$$M^R(z) = \frac{\mathrm{i}}{z} \sigma_3 Q_- + \mathcal{O}(1), \quad z \to 0. \tag{5.3}$$

▶ Residue conditions: M^R has simple poles at each point ζ_n and $\overline{\zeta}_n$ for $n \in \Lambda$ with:

$$\operatorname*{Res}_{z=\zeta_n} M^R(z) = \lim_{z\to\zeta_n} M^R(z) \begin{pmatrix} 0 & 0 \\ C_n \mathrm{e}^{-2\mathrm{i}t\theta_n} T^2(\zeta_n) & 0 \end{pmatrix}, \tag{5.4}$$

$$\operatorname*{Res}_{z=\overline{\zeta}_n} M^R(z) = \lim_{z\to\overline{\zeta}_n} M^R(z) \begin{pmatrix} 0 & -\overline{C}_n T^{-2}(\overline{\zeta}_n) \mathrm{e}^{2\mathrm{i}t\overline{\theta}_n} \\ 0 & 0 \end{pmatrix}. \tag{5.5}$$

We now use $M^R(z)$ to construct a new matrix function

$$M^{(3)}(z) = M^{(2)}(z) M^R(z)^{-1}, \tag{5.6}$$

which removes analytic component of $M^R(z)$ to get a pure $\overline{\partial}$-problem.

$\overline{\partial}$-Problem 4 Find a matrix-valued function $M^{(3)}(z)$ with the following properties:
▶ Analyticity: $M^{(3)}(z)$ is continuous and has sectionally continuous first partial derivatives in $\mathbb{C}$.
▶ Asymptotic behavior:

$$M^{(3)}(z) \sim I + \mathcal{O}(z^{-1}), \quad z \to \infty. \tag{5.7}$$

▶ $\overline{\partial}$-Derivative: We have

$$\overline{\partial} M^{(3)}(z) = M^{(3)}(z) W(z), \quad z \in \mathbb{C},$$

where

$$W(z)=M^R(z)\overline{\partial}R^{(2)}(z)M^R(z)^{-1}. \tag{5.8}$$

Proof By using properties of the solutions $M^{(2)}(z)$ and $M^R(z)$ for RHP 3 and $\overline{\partial}$-Problem 4, the analyticity is obtained immediately. For its asymptotic behavior, from $M^R(z)^{-1}=(1+z^{-2})\sigma_2M^R(z)^T\sigma_2$ we have

$$\begin{aligned}\lim_{z\to0}M^{(3)}(z)&=\lim_{z\to0}\frac{(zM^{(2)}(z))\sigma_2(zM^R(z)^T)\sigma_2}{1+z^2}\\&=\mathrm{i}\sigma_3Q_-\sigma_2(\mathrm{i}\sigma_3Q_-)^T\sigma_2=I.\end{aligned} \tag{5.9}$$

Since $M^{(2)}(z)$ and $M^R(z)$ achieve same jump matrix, we have

$$\begin{aligned}M_-^{(3)}(z)^{-1}M_+^{(3)}(z)&=M_-^{(2)}(z)^{-1}M_-^{(r)}(z)M_+^{(r)}(z)^{-1}M_+^{(2)}(z)\\&=M_-^{(2)}(z)^{-1}V^{(2)}(z)^{-1}M_+^{(2)}(z)=I,\end{aligned}$$

which implies $M^{(3)}(z)$ has no jumps and is continuous on $\mathbb{C}$. We can also show that $M^{(3)}(z)$ has no pole. For $\lambda\in\{\zeta_n,\overline{\zeta}_n\}_{n\in\Lambda}$, let $\mathcal{N}_\lambda$ denote the nilpotent matrix which appears in the left side of the corresponding residue condition of $\overline{\partial}$-problem 4 and RHP 5, we have the Laurent expansions in $z-\lambda$,

$$\begin{aligned}M^{(2)}(z)&=a(\lambda)\Big[\frac{\mathcal{N}_\lambda}{z-\lambda}+I\Big]+\mathcal{O}(z-\lambda),\\M^R(z)&=A(\lambda)\Big[\frac{\mathcal{N}_\lambda}{z-\lambda}+I\Big]+\mathcal{O}(z-\lambda),\end{aligned}$$

where $a(\lambda)$ and $A(\lambda)$ are the constant matrices in their respective expansions. Then

$$\begin{aligned}M^{(3)}(z)&=\Big\{a(\lambda)\Big[\frac{\mathcal{N}_\lambda}{z-\lambda}+I\Big]\Big\}\Big\{\Big[\frac{-\mathcal{N}_\lambda}{z-\lambda}+I\Big]\sigma_2A(\lambda)^T\sigma_2\Big\}+\mathcal{O}(z-\lambda)\\&=\mathcal{O}(1),\end{aligned} \tag{5.10}$$

which implies that $M^{(3)}(z)$ has removable singularities at λ. The $\overline{\partial}$-derivative of $M^{(3)}(z)$ only comes from $M^{(2)}(z)$, because of the analyticity of $M^R(z)$. In addition, unlike the zero boundary case, we must check its property at $\pm\mathrm{i}$. The symmetries of $M^{(2)}(z)$ and $M^R(z)$ imply that

$$M^{(2)}(z)=\begin{pmatrix}\gamma & \pm q_-\gamma\\ \pm\overline{q}_-\overline{\gamma} & \overline{\gamma}\end{pmatrix}+\mathcal{O}(z\mp\mathrm{i}), \tag{5.11}$$

$$M^R(z)=\frac{\pm\mathrm{i}}{2(z\mp\mathrm{i})}\begin{pmatrix}\overline{\iota} & \mp q_-\iota\\ \mp\overline{q}_-\overline{\iota} & \iota\end{pmatrix}+\mathcal{O}(1) \tag{5.12}$$

for two constants γ and ι. Then the singular part of $M^{(3)}(z)$ vanishes at $z=\pm\mathrm{i}$ by simple calculation immediately.

The unique existence and asymptotic of $M^{(3)}(z)$ will be shown in Section 8. Compared with $|\xi+1|<1$, it can be found that its jump matrix $V^{(2)}$ has additional portion on Σ_{jk} and $\Sigma_{j\pm}$ in the case of $|\xi+1|>1$. So this case is more difficult to be dealt with. Denote $U(n(\xi))$ as the union set of neighborhood of ξ_j for $j=1,\cdots,4$,

$$U(n(\xi))=\underset{k=1,\cdots,n(\xi)}{\cup}U_{\xi_k},\ U_{\xi_k}=\Big\{z:|z-\xi_j|\le\min\Big\{\varrho,\frac{1}{3}\min_{j\neq i\in\mathcal{N}}|\zeta_i-\zeta_j|\Big\}\Big\}, \tag{5.13}$$

where $n(\xi)=0,4$, correspond to three cases $|\xi+1|>1$ and $|\xi+1|<1$, respectively. The case $n(\xi)=0$ implies that there is no phase point and $U(n(\xi))=\emptyset$. Then jump matrix $V^{(2)}(z)$ outside of $U(n(\xi))$ has the following estimates.

Proposition 5.1 *For $1\le p\le+\infty$, there exists a positive constant K_p relied on p satisfying that the jump matrix $V^{(2)}(z)$ defined in* (4.67) *admits*

$$\|V^{(2)}(z)-I\|_{L^p(\Sigma_{kj}\setminus U_{\xi_k})}=\mathcal{O}(\mathrm{e}^{-K_pt}),\quad t\to\infty \tag{5.14}$$

for $j,k=1,\cdots,4$. And there also exists a positive constant K_p' relied on p satisfying that the jump matrix $V^{(2)}$ admits for $j,k=1,\cdots,4$,

$$\|V^{(2)}(z)-I\|_{L^p(\Sigma_{kj}')}=\mathcal{O}(\mathrm{e}^{-K_p't}),\quad t\to\infty. \tag{5.15}$$

Proof It can be proved simply by using definition of $V^{(2)}(z)$ and Lemma 4.2.

This proposition means that the jump matrix $V^{(2)}(z)$ uniformly goes to I on $\widetilde{\Sigma}\setminus U(n(\xi))$. So outside the $U(n(\xi))$ there is only exponentially small error (in t) by completely ignoring the jump condition of $M^R(z)$. This proposition enlightens us to construct the solution $M^R(z)$ as follows:

$$M^R(z)=\begin{cases}E(z,\xi)M^{(r)}(z), & z\notin U(n(\xi)),\\ E(z,\xi)M^{(r)}(z)M^{\mathrm{lo}}(z), & z\in U(n(\xi)).\end{cases} \tag{5.16}$$

Note that, for regions $|\xi+2|<1$, $M^{(r)}(z)$ has no jump except the circle around poles not in Λ, and it has no phase point, which implies that $M^R(z)=M^{(r)}(z)$. For the region $|\xi+2|>1$, we decompose $M^R(z)$ into two part: $M^{(r)}(z)$ solves the pure RH problem obtained by ignoring the jump conditions of RHP 3, which is shown in Section 6; $M^{\mathrm{lo}}(z)$ is a localized model to match parabolic cylinder functions in a neighborhood of each critical point ξ_j, and further error function $E(z,\xi)$ is computed by using a small-norm RH problem. These results will be shown in Section 7.

6 Asymptotic of $\mathcal{N}(\Lambda)$-Soliton/Breathers Solutions

The propose of this section is to show that $M^{(r)}(z)$ as solution of the RHP 3 with scattering data $\{\rho(z),\{\zeta_n,C_n\}_{n\in\mathcal{N}}\}$ given above can approximate with renormalization reflectionless original RHP 0.

Proposition 6.1 *The solution $M^{(r)}(z;D)$ of the RHP* 3 *with scattering data $D=\{\rho(z),\{\zeta_n,C_n\}_{n\in\mathcal{N}}\}$ exists and is unique. By an explicit transformation, $M^{(r)}(z;D)$ can be regarded as a reflectionless solution of the original RHP* 0 *with modified scattering data $\widehat{D}=\{0,\{\zeta_n,\mathring{c}_n\}_{n\in\mathcal{N}}\}$, where*

$$\mathring{c}_n(x,t)=C_n\exp\Big\{-\frac{1}{\mathrm{i}\pi}\int_{I(\xi)}\log(1-|\rho(s)|^2)\Big(\frac{1}{s-\zeta_n}-\frac{1}{2s}\Big)\Big\}. \tag{6.1}$$

Proof To transform $M^{(r)}(z;D)$ to the soliton-solution of RHP 0, the jumps and poles need to be restored. We reverses the triangularity effected in (3.23):

$$N(z;D)=\Big(\prod_{n\in\Delta}\overline{\zeta}_n\Big)^{\sigma_3}M^{(r)}(z)T^{-\widehat{\sigma}_3}G^{-1}(z)\Big(\prod_{n\in\Delta}\frac{z-\zeta_n}{\overline{\zeta}_n^{-1}z-1}\Big)^{-\sigma_3} \tag{6.2}$$

with $G(z)$ defined in (3.22). First we verify $N(z)$ satisfying RHP 0. This transformation to $N(z)$ preserves the normalization conditions at the origin and infinity obviously. Comparing with (3.23), this transformation restores the jump on $\mathbb{D}(\overline{\zeta}_n,\varrho)$ and $\mathbb{D}(\zeta_n,\varrho)$ to residue for $n\notin\Lambda$. As for $n\in\Lambda$, take ζ_n as an example. Substitute (5.5) into the transformation:

$$\begin{aligned}\operatorname*{Res}_{z=\zeta_n}N(z)&=\Big(\prod_{n\in\Delta}\overline{\zeta}_n\Big)^{\sigma_3}\operatorname*{Res}_{z=\zeta_n}M^{(r)}(z)T^{-\widehat{\sigma}_3}G(z)^{-1}\Big(\prod_{n\in\Delta}\frac{z-\zeta_n}{\overline{\zeta}_n^{-1}z-1}\Big)^{-\sigma_3}\\&=\lim_{z\to\zeta_n}-\Big(\prod_{n\in\Delta}\overline{\zeta}_n\Big)^{\sigma_3}M^{(r)}(z)\begin{pmatrix}0&0\\C_n\mathrm{e}^{-2\mathrm{i}t\theta_n}T^2(\zeta_n)&0\end{pmatrix}\Big(\prod_{n\in\Delta}\frac{z-\zeta_n}{\overline{\zeta}_n^{-1}z-1}\Big)^{-\sigma_3}\\&=\lim_{z\to\zeta_n}N(z)\begin{pmatrix}0&0\\\mathring{c}_n\mathrm{e}^{-2\mathrm{i}t\theta_n}&0\end{pmatrix}.\end{aligned}\tag{6.3}$$

Its analyticity and symmetry follow from the proposition of $M^{(r)}(z)$, $T(z)$ and $G(z)$ immediately. Thus, $N(z)$ is the solution of RHP 0 with absence of reflection, whose uniquely exact solution exists and can be obtained as described similarly in [18]. So $M^{(r)}(z)$ unique exists.

Denote $q(x,t;\widehat{D})$ as the solution of (1.1) with modified scattering data $\widehat{D}$, namely,

$$q(x,t;\widehat{D})=\lim_{z\to\infty}[zN]_{12}.\tag{6.4}$$

In addition, let

$$q^{(r)}(x,t;D)=\lim_{z\to\infty}[zM^{(r)}]_{12}.\tag{6.5}$$

Then (6.2) gives

$$q^{(r)}(x,t;D)=\Big(\prod_{n\in\Delta}\overline{\zeta}_n^{2}\Big)q(x,t;\widehat{D}).\tag{6.6}$$

The jump matrix $V^{(2)}$ uniformly goes to identity and doesn't contribute to the asymptotic behavior of the solution. Define

$$\rho_0=\min_{n\in\Delta\cup\nabla\setminus\Lambda}|\operatorname{Im}\theta_n|\neq 0.\tag{6.7}$$

Lemma 6.1 *The jump matrix $V^{(2)}(z)$ in (4.69) satisfies*

$$\|V^{(2)}(z)-I\|_{L^\infty(\Sigma^{(2)})}=\mathcal{O}(\mathrm{e}^{-2\rho_0 t}),\tag{6.8}$$

where ρ_0 is defined by (6.7).

Proof Take $z\in\partial\mathbb{D}(\zeta_n,\varrho)$, $n\in\nabla$ as an example.

$$\begin{aligned}\|V^{(2)}(z)-I\|_{L^\infty(\partial\mathbb{D}(\zeta_n,\varrho))}&=|C_n(z-\zeta_n)^{-1}T^2(z)\mathrm{e}^{-2\mathrm{i}t\theta_n}|\\&\lesssim\varrho^{-1}\mathrm{e}^{-\mathrm{Re}(2\mathrm{i}t\theta_n)}\lesssim\mathrm{e}^{2t\mathrm{Im}(\theta_n)}\leq\mathrm{e}^{-2\rho_0 t}.\end{aligned}\tag{6.9}$$

The last step follows from that for $n\in\nabla$, $\operatorname{Im}\theta_n<0$.

Corollary 6.1 *For $1\leq p\leq+\infty$, the jump matrix $V^{(2)}(z)$ satisfies*

$$\|V^{(2)}(z)-I\|_{L^p(\Sigma^{(2)})}\leq K_p\mathrm{e}^{-2\rho_0 t}\tag{6.10}$$

for some constant $K_p\geq 0$ depending on p.

This estimation of $V^{(2)}(z)$ inspires us to completely ignore the jump condition on $M^{(r)}(z)$, because there is only exponentially small error (in t). We decompose $M^{(r)}(z)$ as

$$M^{(r)}(z) = \widetilde{E}(z) M^{(r)}_{\Lambda}(z), \tag{6.11}$$

where $\widetilde{E}(z)$ is a error function and is a solution of a small-norm RH problem. We discuss it in the following subsection. $M^{(r)}_{\Lambda}(z)$ solves RHP 3 with $V^{(2)}(z) \equiv I$.

RHP 5 Find a matrix-valued function $M^{(r)}_{\Lambda}(z)$ with the following properties:

▶ Analyticity: $M^{(r)}_{\Lambda}(z)$ is analytic in $\mathbb{C} \setminus \{\zeta_n, \overline{\zeta}_n\}_{n\in\Lambda}$.

▶ Symmetry: $M^{(r)}_{\Lambda}(z) = \sigma_2 \overline{M^{(r)}_{\Lambda}(\overline{z})} \sigma_2 = \sigma_1 \overline{M^{(r)}_{\Lambda}(-\overline{z})} \sigma_1 = \frac{\mathrm{i}}{z} M^{(r)}_{\Lambda}\big(-\frac{1}{z}\big)\sigma_3 Q_-$.

▶ Asymptotic behaviors:

$$M^{(r)}_{\Lambda}(z) = I + \mathcal{O}(z^{-1}), \quad z \to \infty, \tag{6.12}$$

$$M^{(r)}_{\Lambda}(z) = \frac{\mathrm{i}}{z}\sigma_3 Q_- + \mathcal{O}(1), \quad z \to 0. \tag{6.13}$$

▶ Residue conditions: $M^{(r)}_{\Lambda}(z)$ has simple poles at each point ζ_n and $\overline{\zeta}_n$ for $n \in \Lambda$ with:

$$\operatorname*{Res}_{z=\zeta_n} M^{(r)}_{\Lambda}(z) = \lim_{z\to\zeta_n} M^{(r)}_{\Lambda}(z) \begin{pmatrix} 0 & 0 \\ C_n \mathrm{e}^{-2\mathrm{i}t\theta_n} T^2(\zeta_n) & 0 \end{pmatrix}, \tag{6.14}$$

$$\operatorname*{Res}_{z=\overline{\zeta}_n} M^{(r)}_{\Lambda}(z) = \lim_{z\to\overline{\zeta}_n} M^{(r)}_{\Lambda}(z) \begin{pmatrix} 0 & -\overline{C}_n T^{-2}(\overline{\zeta}_n) \mathrm{e}^{2\mathrm{i}t\overline{\theta}_n} \\ 0 & 0 \end{pmatrix}. \tag{6.15}$$

Proposition 6.2 *The RHP* 5 *exists a unique solution.*

Proof The uniqueness of solution follows from the Liouville's theorem. Its exact solution exists and can be obtained as described similarly in [18].

Case I If $\Lambda = \varnothing$, then

$$M^{(r)}_{\Lambda}(z) = I + \frac{\mathrm{i}}{z}\sigma_3 Q_-. \tag{6.16}$$

Case II If $\Lambda \neq \varnothing$ with $\Lambda_1 = \{z_{j_k}\}_{k=1}^{n_1}$ and $\Lambda_2 = \{w_{i_s}\}_{s=1}^{n_2}$, then

$$\begin{aligned} M^{(r)}_{\Lambda}(z) = {} & I + \frac{\mathrm{i}}{z}\sigma_3 Q_- \\ & + \sum_{s=1}^{n_2} \left[\begin{pmatrix} \frac{\alpha_s}{z-w_{i_s}} & \frac{\overline{\kappa_s}}{z-\overline{w}_{i_s}} \\ \frac{\kappa_s}{z-w_{i_s}} & \frac{\overline{\alpha_s}}{z-\overline{w}_{i_s}} \end{pmatrix} + \begin{pmatrix} -\frac{\alpha_s}{z+w_{i_s}} & \frac{\overline{\kappa_s}}{z+\overline{w}_{i_s}} \\ \frac{\kappa_s}{z+w_{i_s}} & -\frac{\overline{\alpha_s}}{z+\overline{w}_{i_s}} \end{pmatrix} \right] \\ & + \sum_{k=1}^{n_1} \left[\begin{pmatrix} \frac{\beta_k}{z-z_{j_k}} & \frac{\overline{\varsigma_k}}{z-\overline{z}_{j_k}} \\ \frac{\varsigma_k}{z-z_{j_k}} & \frac{\overline{\beta_k}}{z-\overline{z}_{j_k}} \end{pmatrix} + \begin{pmatrix} -\frac{\beta_k}{z+z_{j_k}} & \frac{\overline{\varsigma_k}}{z+\overline{z}_{j_k}} \\ \frac{\varsigma_k}{z+z_{j_k}} & -\frac{\overline{\beta_k}}{z+\overline{z}_{j_k}} \end{pmatrix} \right] \\ & + \sum_{k=1}^{n_1} i \left[\begin{pmatrix} \frac{-\overline{q_-\beta_k}}{\overline{z}_{j_k} z-1} & \frac{-q_-\varsigma_k}{z_{j_k} z-1} \\ \frac{-\overline{q_-\varsigma_k}}{\overline{z}_{j_k} z-1} & \frac{q_-\beta_k}{z_{j_k} z-1} \end{pmatrix} + \begin{pmatrix} \frac{\overline{q_-\beta_k}}{\overline{z}_{j_k} z+1} & \frac{-q_-\varsigma_k}{z_{j_k} z+1} \\ \frac{-\overline{q_-\varsigma_k}}{\overline{z}_{j_k} z+1} & \frac{-q_-\beta_k}{z_{j_k} z+1} \end{pmatrix} \right], \end{aligned} \tag{6.17}$$

where $\beta_k = \beta_k(x,t)$, $\varsigma_k = \varsigma_k(x,t)$, $\alpha_s = \alpha_s(x,t)$ and $\kappa_s = \kappa_s(x,t)$ can be obtained with linearly dependant equations:

$$c_{j_k}^{-1} T(z_{j_k})^{-2} \mathrm{e}^{-2\mathrm{i}\theta(z_{j_k})t} \beta_k = \frac{\mathrm{i}}{z_{j_k}} q_- + \sum_{h=1}^{n_2} \Big(\frac{\overline{\kappa_h}}{z_{j_k} - \overline{w}_{i_h}} + \frac{\overline{\kappa_h}}{z_{j_k} + \overline{w}_{i_h}} \Big)$$

$$+\sum_{l=1}^{n_1}\Big(\frac{\overline{\varsigma_l}}{z_{j_k}-\overline{z}_{j_l}}+\frac{\overline{\varsigma_l}}{z_{j_k}+\overline{z}_{j_l}}-\frac{\mathrm{i}q_-\varsigma_l}{z_{j_l}z_{j_k}-1}-\frac{\mathrm{i}q_-\varsigma_l}{z_{j_l}z_{j_k}+1}\Big), \tag{6.18}$$

$$c_{j_k}^{-1}T(z_{j_k})^{-2}\mathrm{e}^{-2\mathrm{i}\theta(z_{j_k})t}\varsigma_k=\frac{\mathrm{i}}{z_{j_k}}\overline{q}_-+\sum_{h=1}^{n_2}\Big(\frac{\overline{\alpha_h}}{z_{j_k}-\overline{w}_{i_h}}-\frac{\overline{\alpha_h}}{z_{j_k}+\overline{w}_{i_h}}\Big)$$
$$+\sum_{l=1}^{n_1}\Big(\frac{\overline{\beta_l}}{z_{j_k}-\overline{z}_{j_l}}-\frac{\overline{\beta_l}}{z_{j_k}+\overline{z}_{j_l}}+\frac{\mathrm{i}q_-\beta_l}{z_{j_l}z_{j_k}-1}-\frac{\mathrm{i}q_-\beta_l}{z_{j_l}z_{j_k}+1}\Big) \tag{6.19}$$

and

$$c_{i_s+N_1}^{-1}T(w_{i_s})^{-2}\mathrm{e}^{-2\mathrm{i}\theta(w_{i_s})t}\alpha_s=\frac{\mathrm{i}}{w_{i_s}}q_-+\sum_{h=1}^{n_2}\Big(\frac{\overline{\kappa_h}}{w_{i_s}-\overline{w}_{i_h}}+\frac{\overline{\kappa_h}}{w_{i_s}+\overline{w}_{i_h}}\Big)$$
$$+\sum_{l=1}^{n_1}\Big(\frac{\overline{\varsigma_l}}{w_{i_s}-\overline{z}_{j_l}}+\frac{\overline{\varsigma_l}}{w_{i_s}+\overline{z}_{j_l}}-\frac{\mathrm{i}q_-\varsigma_l}{z_{j_l}w_{i_s}-1}-\frac{\mathrm{i}q_-\varsigma_l}{z_{j_l}w_{i_s}+1}\Big),$$
$$c_{i_s+N_1}^{-1}T(w_{i_s})^{-2}\mathrm{e}^{-2\mathrm{i}\theta(w_{i_s})t}\kappa_s=\frac{\mathrm{i}}{w_{i_s}}\overline{q}_-+\sum_{h=1}^{n_2}\Big(\frac{\overline{\alpha_h}}{w_{i_s}-\overline{w}_{i_h}}-\frac{\overline{\alpha_h}}{w_{i_s}+\overline{w}_{i_h}}\Big)$$
$$+\sum_{l=1}^{n_1}\Big(\frac{\overline{\beta_l}}{w_{i_s}-\overline{z}_{j_l}}-\frac{\overline{\beta_l}}{w_{i_s}+\overline{z}_{j_l}}+\frac{\mathrm{i}q_-\beta_l}{z_{j_l}w_{i_s}-1}-\frac{\mathrm{i}q_-\beta_l}{z_{j_l}w_{i_s}+1}\Big)$$

for $k=1,\cdots,n_1$, $s=1,\cdots,n_2$, respectively.

Corollary 6.2 *Denote $q^r_\Lambda(x,t)$ to be the $\mathcal{N}(\Lambda)$-solution with scattering data*

$$\widehat{D}=\{0,\{\zeta_n,\mathring{c}_n\}_{n\in\Lambda}\}.$$

By the reconstruction formula (2.67)*, the solution $q^r_\Lambda(x,t)$ of* (1.1) *with scattering data* $\{0,\{\zeta_n,\mathring{c}_n\}_{n\in\Lambda}\}$ *is given by*

$$q^r_\Lambda(x,t)=-\mathrm{i}\lim_{z\to\infty}z[M^{(r)}_\Lambda(z)]_{12}. \tag{6.20}$$

Then in Case I,

$$q^r_\Lambda(x,t)=q_-. \tag{6.21}$$

And in Case II,

$$q^r_\Lambda(x,t)=-\mathrm{i}\lim_{z\to\infty}z[M^{(r)}_\Lambda(z)]_{12}$$
$$=q_--\mathrm{i}2\sum_{s=1}^{n_2}\overline{\kappa}_k-2\mathrm{i}\sum_{k=1}^{n_1}\Big(\overline{\varsigma}_k-\mathrm{i}q_-\frac{\varsigma_k}{z_{j_k}}\Big). \tag{6.22}$$

Remark 6.1 When $\rho(s)\equiv 0$, the scattering matrices $S(z)\equiv I$, which means $q_-=q_+\mathrm{e}^{2\mathrm{i}\nu_0(\widehat{D})}$. Then in case I, $|q^r_\Lambda(x,t)|=|q_-|=1$ implies that $\mathrm{e}^{2\mathrm{i}\nu_0(\widehat{D})}=1$, so $q_-=q_+$.

Remark 6.2 The $\mathcal{N}(\Lambda)$-solution is not $\mathcal{N}(\Lambda)$-soliton solution. Because when the discrete spectrum ζ_n is not on unit circle, it corresponds to breather solution, while when the discrete spectrum ζ_n is on unit circle, it corresponds to soliton solution. Suppose that the discrete spectrum only distributes on unit circle, then it corresponds to pure soliton solution. We will show that under this assumption, through Beal-Cofiman theorem, the N-soliton can be expressed asymptotically as a sum (adjusted for boundary conditions) of N simple solitons.

Consider the original scattering data $D^{\mathrm{sol}} = \{\rho(z), \{\zeta_n^{\mathrm{sol}}, C_n\}_{n=1}^{2N_2}\}$ of RHP 0 under this case with $\zeta_m^{\mathrm{sol}} = w_m$ and $\zeta_{m+N_2}^{\mathrm{sol}} = -w_m$ for $m = 1, \cdots, N_2$. For convenience, we assume that only $\operatorname{Im}\theta(w_1) = 0$. And the solution $M^{(r)}(z; D^{\mathrm{sol}})$ of the RHP 3 with scattering data D^{sol} can also be regarded as a reflectionless soliton solution $N^{\mathrm{sol}}(z) = N(z; \widehat{D}^{\mathrm{sol}})$ of the original RHP 0 with modified scattering data $\widehat{D}^{\mathrm{sol}} = \{0, \{\zeta_n^{\mathrm{sol}}, \mathring{c}_n\}_{n=1}^{2N_2}\}$. Further, there also has

$$q^{(r)}(x,t;D^{\mathrm{sol}}) = \Big(\prod_{n\in\Delta_2} \overline{\zeta}_n^{-2}\Big) q(x,t;\widehat{D}^{\mathrm{sol}}). \tag{6.23}$$

And $N^{\mathrm{sol}}(z)$ is also given by

$$N^{\mathrm{sol}}(z) = I + \frac{\mathrm{i}}{z}\sigma_3 Q_- + \sum_{l=1}^{N_2}\Big[\begin{pmatrix} \frac{\alpha_l^{\mathrm{sol}}}{z-w_l} & \frac{\overline{\kappa_l^{\mathrm{sol}}}}{z-\overline{w}_l} \\ \frac{\kappa_l^{\mathrm{sol}}}{z-w_l} & \frac{\overline{\alpha_l^{\mathrm{sol}}}}{z-\overline{w}_l}\end{pmatrix} + \begin{pmatrix} -\frac{\alpha_l^{\mathrm{sol}}}{z+w_l} & \frac{\overline{\kappa_l^{\mathrm{sol}}}}{z+\overline{w}_l} \\ \frac{\kappa_l^{\mathrm{sol}}}{z+w_l} & -\frac{\overline{\alpha_l^{\mathrm{sol}}}}{z+\overline{w}_l}\end{pmatrix}\Big] \tag{6.24}$$

with

$$\alpha_l^{\mathrm{sol}} = \mathrm{i}\overline{q}_- w_l \overline{\kappa_l^{\mathrm{sol}}}. \tag{6.25}$$

Here, κ_l^{sol} is solution of the following equation set:

$$\kappa_l^{\mathrm{sol}} = \mathring{c}_l \mathrm{e}^{2\mathrm{i}t\theta(w_l)}\Big[1 - \sum_{j\neq l, j=1}^{N_2} \mathrm{i}q_- \overline{w}_j \kappa_j^{\mathrm{sol}}\Big(\frac{1}{w_l - \overline{w}_j} - \frac{1}{w_l + \overline{w}_j}\Big)\Big] \tag{6.26}$$

for $l = 1, \cdots, N_2$. Moreover,

$$\lim_{z\to\infty} z[N^{\mathrm{sol}}(z)]_{12} = \mathrm{i}q_- + 2\sum_{l=1}^{N_2}\overline{\kappa_l^{\mathrm{sol}}}. \tag{6.27}$$

Consider

$$\begin{aligned}\widetilde{N}^{\mathrm{sol}}(z) &= N^{\mathrm{sol}}(z) - \frac{\mathrm{i}}{z}\sigma_3 Q_- \\ &= I + \sum_{l=1}^{N_2}\Big[\begin{pmatrix} \frac{\alpha_l^{\mathrm{sol}}}{z-w_l} & \frac{\overline{\kappa_l^{\mathrm{sol}}}}{z-\overline{w}_l} \\ \frac{\kappa_l^{\mathrm{sol}}}{z-w_l} & \frac{\overline{\alpha_l^{\mathrm{sol}}}}{z-\overline{w}_l}\end{pmatrix} + \begin{pmatrix} -\frac{\alpha_l^{\mathrm{sol}}}{z+w_l} & \frac{\overline{\kappa_l^{\mathrm{sol}}}}{z+\overline{w}_l} \\ \frac{\kappa_l^{\mathrm{sol}}}{z+w_l} & -\frac{\overline{\alpha_l^{\mathrm{sol}}}}{z+\overline{w}_l}\end{pmatrix}\Big].\end{aligned} \tag{6.28}$$

It is a solution of the original RHP 0 with modified scattering data $\widehat{D}^{\mathrm{sol}}$ and the absence of pole at $z = 0$. Like (3.23), all the residue at discrete spectrum ζ_n^{sol}(including $n = 1$) is transformed to jump at a small circle around it through

$$\widetilde{M}^{\mathrm{sol}}(z) = T^{\mathrm{sol}}(\infty)^{-\sigma_3}\widetilde{N}^{\mathrm{sol}}(z) G^{\mathrm{sol}}(z) T^{\mathrm{sol}}(z)^{\sigma_3}, \tag{6.29}$$

where like in (3.8) and (3.22),

$$T^{\mathrm{sol}}(z) = \prod_{l\in\Delta_2}\frac{z^2 - w_l^2}{w_l^2 z^2 - 1}, \quad T^{\mathrm{sol}}(\infty) = \prod_{l\in\Delta_2}\overline{w}_l^2$$

and

$$G^{\mathrm{sol}}(z)=\begin{cases}\begin{pmatrix}1 & 0\\ -\mathring{c}_n(z-\zeta_n)^{-1}\mathrm{e}^{-2\mathrm{i}t\theta_n} & 1\end{pmatrix} & \text{as } z\in\mathbb{D}(\zeta_n^{\mathrm{sol}},\varrho),\ n\in\nabla_2\cup\Lambda_2;\\ \begin{pmatrix}1 & -\mathring{c}_n^{-1}(z-\zeta_n)\mathrm{e}^{2\mathrm{i}t\theta_n}\\ 0 & 1\end{pmatrix} & \text{as } z\in\mathbb{D}(\zeta_n^{\mathrm{sol}},\varrho),\ n\in\Delta_2;\\ \begin{pmatrix}1 & \bar{\mathring{c}}_n(z-\overline{\zeta}_n)^{-1}\mathrm{e}^{2\mathrm{i}t\overline{\theta}_n}\\ 0 & 1\end{pmatrix} & \text{as } z\in\mathbb{D}(\overline{\zeta}_n^{\mathrm{sol}},\varrho),\ n\in\nabla_2\cup\Lambda_2;\\ \begin{pmatrix}1 & 0\\ \bar{\mathring{c}}_n^{-1}(z-\overline{\zeta}_n)\mathrm{e}^{-2\mathrm{i}t\overline{\theta}_n} & 1\end{pmatrix} & \text{as } z\in\mathbb{D}(\overline{\zeta}_n^{\mathrm{sol}},\varrho),\ n\in\Delta_2;\\ I & \text{as } z \text{ in elsewhere.}\end{cases}$$

Denote $\partial\mathbb{D}^{\mathrm{sol}}=\bigcup_{n=1}^{2N_2}\partial(\mathbb{D}(\zeta_n^{\mathrm{sol}},\varrho)\cup\mathbb{D}(\overline{\zeta}_n^{\mathrm{sol}},\varrho))$ and $\partial\mathbb{D}_l^{\mathrm{sol}}=\partial\mathbb{D}(\zeta_l^{\mathrm{sol}},\varrho)\cup\partial\mathbb{D}(\overline{\zeta}_l^{\mathrm{sol}},\varrho)\cup\partial\mathbb{D}(\zeta_{l+N_2}^{\mathrm{sol}},\varrho)\cup\partial\mathbb{D}(\overline{\zeta}_{l+N_2}^{\mathrm{sol}},\varrho)$. Thus, $\widetilde{M}^{\mathrm{sol}}(z)$ admits the following properties:

► Analyticity: $\widetilde{M}^{\mathrm{sol}}(z)$ is analytic in $\mathbb{C}\setminus\partial\mathbb{D}_N^{\mathrm{sol}}$.

► Jump condition: $\widetilde{M}^{\mathrm{sol}}$ has continuous boundary values $\widetilde{M}_\pm^{\mathrm{sol}}$ on $\partial\mathbb{D}(\zeta_n^{\mathrm{sol}},\varrho)$ and $\partial\mathbb{D}(\overline{\zeta}_n^{\mathrm{sol}},\varrho)$ and

$$\widetilde{M}_+^{\mathrm{sol}}(z;x,t)=\widetilde{M}_-^{\mathrm{sol}}(z;x,t)V^{\mathrm{sol}}(z), \tag{6.30}$$

where

$$V^{\mathrm{sol}}(z)=\begin{cases}\begin{pmatrix}1 & 0\\ -\mathring{c}_n(z-\zeta_n)^{-1}\mathrm{e}^{-2\mathrm{i}t\theta_n} & 1\end{pmatrix} & \text{as } z\in\partial\mathbb{D}(\zeta_n^{\mathrm{sol}},\varrho),\ n\in\nabla_2;\\ \begin{pmatrix}1 & -\mathring{c}_n^{-1}(z-\zeta_n)\mathrm{e}^{2\mathrm{i}t\theta_n}\\ 0 & 1\end{pmatrix} & \text{as } z\in\partial\mathbb{D}(\zeta_n^{\mathrm{sol}},\varrho),\ n\in\Delta_2;\\ \begin{pmatrix}1 & \bar{\mathring{c}}_n(z-\overline{\zeta}_n)^{-1}\mathrm{e}^{2\mathrm{i}t\overline{\theta}_n}\\ 0 & 1\end{pmatrix} & \text{as } z\in\partial\mathbb{D}(\overline{\zeta}_n^{\mathrm{sol}},\varrho),\ n\in\nabla_2;\\ \begin{pmatrix}1 & 0\\ \bar{\mathring{c}}_n^{-1}(z-\overline{\zeta}_n)\mathrm{e}^{-2\mathrm{i}t\overline{\theta}_n} & 1\end{pmatrix} & \text{as } z\in\partial\mathbb{D}(\overline{\zeta}_n^{\mathrm{sol}},\varrho),\ n\in\Delta_2.\end{cases} \tag{6.31}$$

► Asymptotic behaviors:

$$\widetilde{M}^{\mathrm{sol}}(z)=I+\mathcal{O}(z^{-1}),\quad z\to\infty. \tag{6.32}$$

Then

$$\lim_{z\to\infty}z[\widetilde{M}^{\mathrm{sol}}(z)]_{12}=\prod_{l\in\Delta_2}w_l^4\Big(\sum_{j=1}^{N_2}2\overline{\kappa_j^{\mathrm{sol}}}\Big). \tag{6.33}$$

By Beal-Cofiman theorem,

$$\widetilde{M}^{\mathrm{sol}}(z)=I+\frac{1}{2\pi\mathrm{i}}\int_{\partial\mathbb{D}^{\mathrm{sol}}}\frac{[(1-C_\omega)^{-1}I](s)\omega(s)}{s-z}\mathrm{d}s,$$

where

$$\omega(s)=V^{\mathrm{sol}}-I,\quad C_\omega f=C_-(f\omega),$$

$$C_-(f)(s) = \lim_{z\to\Sigma_-^{(2)}} \frac{1}{2\pi \mathrm{i}} \int_{\partial\mathbb{D}^{\mathrm{sol}}} \frac{f(s)}{s-z}\mathrm{d}s$$

for any $f \in L^2(\partial\mathbb{D}^{\mathrm{sol}})$. Following the idea of [6], we write

$$\omega = \sum_{l=1}^{N_2} \omega_l, \tag{6.34}$$

where $\omega_l(z) = \omega(z)$ for $z \in \partial\mathbb{D}_l^{\mathrm{sol}}$, and $\omega_l(z) = 0$ for $z \in \partial\mathbb{D}_n^{\mathrm{sol}}$, $n \neq l$. For $l \neq 1$, $|\omega_l| = \mathcal{O}(\mathrm{e}^{-2\rho_0 t})$, then

$$\|C_{\omega_l}\|_{L^2(\partial\mathbb{D}_l^{\mathrm{sol}})\to L^2(\partial\mathbb{D}^{\mathrm{sol}})} = \mathcal{O}(\mathrm{e}^{-2\rho_0 t}), \quad \|C_{\omega_l}\|_{L\infty(\partial\mathbb{D}^{\mathrm{sol}})\to L^2(\partial\mathbb{D}^{\mathrm{sol}})} = \mathcal{O}(\mathrm{e}^{-2\rho_0 t}).$$

Therefore, although C_{ω_1} and ω_1 do not decay as $t\to\infty$, the interaction of between the operator C_{ω_1} and C_{ω_l} for $l = 1, \cdots, N_2$ decay exponentially. Thus,

$$\int_{\partial\mathbb{D}^{\mathrm{sol}}} [(1-C_\omega)^{-1}I](s)\omega(s)\mathrm{d}s = \sum_{l=1}^{N_2} \int_{\partial\mathbb{D}_l^{\mathrm{sol}}} [(1-C_\omega)^{-1}I](s)\omega_l(s)\mathrm{d}s + \mathcal{O}(\mathrm{e}^{-2\rho_0 t}).$$

So it is reasonable to separate out the contributions of each jump on $\partial\mathbb{D}_l^{\mathrm{sol}}$. For each $l = 1, \cdots, N_2$, consider

$$\widetilde{M}_l^{\mathrm{sol}}(z) = I + \frac{1}{2\pi\mathrm{i}} \int_{\partial\mathbb{D}_l^{\mathrm{sol}}} \frac{[(1-C_{\omega_l})^{-1}I](s)\omega_l(s)}{s-z}\mathrm{d}s,$$

which only has jump on $\partial\mathbb{D}_l^{\mathrm{sol}}$ with

$$\lim_{z\to\infty} z[\widetilde{M}_l^{\mathrm{sol}}(z)]_{12} = -\frac{1}{2\pi\mathrm{i}} \int_{\partial\mathbb{D}_l^{\mathrm{sol}}} [[(1-C_{\omega_l})^{-1}I](s)\omega_l(s)]_{12}\mathrm{d}s. \tag{6.35}$$

Similar to (6.2), we restore the jump to poles:

$$\widetilde{N}_l^{\mathrm{sol}}(z) = T_l^{\mathrm{sol}}(\infty)^{\sigma_3} \widetilde{M}_l^{\mathrm{sol}}(z) G_l^{\mathrm{sol}}(z)^{-1} T_l^{\mathrm{sol}}(z)^{-\sigma_3}, \tag{6.36}$$

where when $l \in \nabla_2 \cup \Lambda_2$, $T_l^{\mathrm{sol}}(\infty) = T_l^{\mathrm{sol}}(z) = 1$ and when $l \in \Delta_2$,

$$T_l^{\mathrm{sol}}(z) = \frac{z^2 - w_l^2}{w_l^2 z^2 - 1}, \quad T_l^{\mathrm{sol}}(\infty) = \overline{w}_l^2 \tag{6.37}$$

and

$$G_l^{\mathrm{sol}}(z) = \begin{cases} G^{\mathrm{sol}}(z) & \text{as } z \in \mathbb{D}_l^{\mathrm{sol}}; \\ I & \text{as } z \text{ in elsewhere.} \end{cases}$$

$N_l^{\mathrm{sol}}(z) = \widetilde{N}_l^{\mathrm{sol}}(z) + \frac{\mathrm{i}}{z}\sigma_3 Q_- \lim_{z\to\infty} \widetilde{N}_l^{\mathrm{sol}}(z)$ corresponds to the soliton solution of (1.1) with scattering data $D_l^{\mathrm{sol}} = \{0, \{w_l, \mathring{c}_l\}, \{-w_l, \mathring{c}_{l+N_2}\}\}$, and

$$q(x,t;D_l^{\mathrm{sol}}) = -\mathrm{i} \lim_{z\to\infty} [zN_l^{\mathrm{sol}}]_{12}. \tag{6.38}$$

Combining above formulae, we deduce that:

$$\begin{aligned} q^{(r)}(x,t;D) = {} & q_- \prod_{l=1}^{N_2} w_l^4 + \sum_{l\in\Delta_2} w_l^4 (q(x,t;D_l^{\mathrm{sol}}) - q_-) \\ & + \sum_{l\in\nabla_2\cup\Lambda_2} (q(x,t;D_l^{\mathrm{sol}}) - q_-) + \mathcal{O}(\mathrm{e}^{-2\rho_0 t}). \end{aligned} \tag{6.39}$$

6.1 Error estimate between $M^{(r)}(z)$ and $M_\Lambda^{(r)}(z)$

In this subsection, we consider the error matrix-function $\widetilde{E}(z)$ and show that for large times, the error function $\widetilde{E}(z)$ solves a small norm RH problem which can be expanded asymptotically. From the definition (6.11), we can obtain a RH problem for the matrix function $\widetilde{E}(z)$.

RHP 6 Find a matrix-valued function $\widetilde{E}(z)$ with the following properties:
▶ Analyticity: $\widetilde{E}(z)$ is analytic in $\mathbb{C} \setminus \Sigma^{(2)}$.
▶ Asymptotic behaviors:

$$\widetilde{E}(z) \sim I + \mathcal{O}(z^{-1}), \quad |z| \to \infty. \tag{6.40}$$

▶ Jump condition: $\widetilde{E}$ has continuous boundary values $\widetilde{E}_\pm$ on $\Sigma^{(2)}$ satisfying

$$\widetilde{E}_+(z) = \widetilde{E}_-(z)V^{\widetilde{E}},$$

where the jump matrix $V^{\widetilde{E}}$ is given by

$$V^{\widetilde{E}}(z) = M_\Lambda^{(r)}(z)V^{(2)}(z)M_\Lambda^{(r)}(z)^{-1}. \tag{6.41}$$

Proposition 6.2 implies that $M_\Lambda^{(r)}(z)$ is bounded on $\Sigma^{(2)}$. By using Lemma 6.1 and Corollary 6.1, we have the following estimates

$$\|V^{\widetilde{E}}(z) - I\|_{L^p} \lesssim \|V^{(2)}(z) - I\|_{L^p(\Sigma^{(2)} \setminus \widetilde{\Sigma})} = \mathcal{O}(\mathrm{e}^{-2\rho_0 t}) \tag{6.42}$$

for $1 \le p \le +\infty$. This uniformly vanishing bound $\|V^{\widetilde{E}}(z) - I\|$ establishes RHP 6 as a small-norm RH problem. Therefore, the existence and uniqueness of the RHP 6 can be shown by using a small-norm RH problem

$$\widetilde{E}(z) = I + \frac{1}{2\pi \mathrm{i}} \int_{\Sigma^{(2)}} \frac{(I + \eta(s))(V^{\widetilde{E}}(s) - I)}{s - z} \mathrm{d}s, \tag{6.43}$$

where the $\widetilde{\eta} \in L^2(\Sigma^{(2)} \setminus \widetilde{\Sigma})$ is the unique solution of the following equation

$$(1 - C_{\widetilde{E}})\widetilde{\eta} = C_{\widetilde{E}}(I), \tag{6.44}$$

here $C_{\widetilde{E}} : L^2(\Sigma^{(2)} \setminus \widetilde{\Sigma}) \to L^2(\Sigma^{(2)} \setminus \widetilde{\Sigma})$ is a integral operator defined by

$$C_{\widetilde{E}}(f)(z) = C_-(f(V^{\widetilde{E}}(z) - I)), \tag{6.45}$$

the Cauchy projection operator C_- on $\Sigma^{(2)} \setminus \widetilde{\Sigma}$. Then by (6.41) we have

$$\|C_{\widetilde{E}}\| \le \|C_-\| \|V^{\widetilde{E}}(z) - I\|_{L^\infty} \lesssim \mathcal{O}(\mathrm{e}^{-2\rho_0 t}), \tag{6.46}$$

which means $\|C_{\widetilde{E}}\| < 1$ for sufficiently large t, therefore $1 - C_{\widetilde{E}}$ is invertible, and $\widetilde{\eta}$ exists and is unique. Moreover,

$$\|\widetilde{\eta}\|_{L^2(\Sigma^{(2)} \setminus \widetilde{\Sigma})} \lesssim \frac{\|C_{\widetilde{E}}\|}{1 - \|C_{\widetilde{E}}\|} \lesssim \mathcal{O}(\mathrm{e}^{-2\rho_0 t}). \tag{6.47}$$

Then we have the existence and boundedness of $\widetilde{E}(z)$.

Proposition 6.3 *For $\widetilde{E}(z)$ defined in* (6.43), *it satisfies*

$$|\widetilde{E}(z) - I| \lesssim \mathcal{O}(\mathrm{e}^{-2\rho_0 t}). \tag{6.48}$$

As $z \to \infty$, the large z expansion of $\widetilde{E}$ is

$$\widetilde{E}(z) = I + \widetilde{E}_1 z^{-1} + \mathcal{O}(z^{-2}), \tag{6.49}$$

where

$$\widetilde{E}_1 = -\frac{1}{2\pi\mathrm{i}} \int_{\Sigma^{(2)}} (I + \widetilde{\eta}(s))(V^{\widetilde{E}}(z) - I)\mathrm{d}s, \tag{6.50}$$

satisfying long time asymptotic behavior condition

$$\widetilde{E}_1 \lesssim \mathcal{O}(\mathrm{e}^{-2\rho_0 t}). \tag{6.51}$$

Proof By combining (6.47) and (6.42), we obtain

$$|\widetilde{E}(z) - I| \leq |(1 - C_{\widetilde{E}})(\widetilde{\eta})| + |C\widetilde{E}(\widetilde{\eta})| \lesssim \mathcal{O}(\mathrm{e}^{-2\rho_0 t}). \tag{6.52}$$

As $z \to \infty$, geometrically expanding $(s - z)^{-1}$ for z large in (6.43) leads to (6.49). Finally for $\widetilde{E}_1$,

$$|\widetilde{E}_1| \lesssim \|V^{\widetilde{E}}(z) - I\|_{L^1} + \|\widetilde{\eta}\|_{L^2} 2 \|V^{\widetilde{E}}(z) - I\|_{L^2} \lesssim \mathcal{O}(\mathrm{e}^{-2\rho_0 t}). \tag{6.53}$$

7 Localized RH Problem near Phase Points

When $|\xi+2| > 1$, it is necessary to consider the effect of stationary phase points. Proposition 5.1 gives that out of $U(n(\xi))$, the jumps are exponentially close to the identity. Hence we only need to continue our investigation near the stationary phase points in this section. Denote a new contour $\Sigma^{(0)} = \big(\bigcup\limits_{k,j=1,2,3,4} \Sigma_{kj}\big) \cap U(n(\xi))$ shown in Figure 8. Consider the following RH problem.

RHP 7 Find a matrix-valued function $M^{\mathrm{lo}}(z)$ with the following properties:

▶ Analyticity: $M^{\mathrm{lo}}(z)$ is analytical in $\mathbb{C} \setminus \Sigma^{(0)}$.

▶ Symmetry: $M^{\mathrm{lo}}(z) = \sigma_2 \overline{M^{\mathrm{lo}}(\overline{z})}\sigma_2 = \sigma_1 \overline{M^{\mathrm{lo}}(-\overline{z})}\sigma_1 = \frac{\mathrm{i}}{z} M^{\mathrm{lo}}\big(-\frac{1}{z}\big)\sigma_3 Q_-$.

▶ Jump condition: $M^{\mathrm{lo}}(z)$ has continuous boundary values $M^{\mathrm{lo}}_\pm(z)$ on $\Sigma^{(0)}$ and

$$M^{\mathrm{lo}}_+(z) = M^{\mathrm{lo}}_-(z) V^{(2)}(z), \quad z \in \Sigma^{(0)}. \tag{7.1}$$

▶ Asymptotic behaviors:

$$M^{\mathrm{lo}}(z) = I + \mathcal{O}(z^{-1}), \quad z \to \infty. \tag{7.2}$$

This RH problem only has jump conditions without poles. The jump matrix $V^{(2)}(z)$ is a upper/lower matrix with entry on the diagonal. For $k = 1, \cdots, 4$, we denote

$$w_{kj}(z) = \begin{cases} \begin{pmatrix} 0 & \mathrm{i}^j R_{kj}(z,\xi)\mathrm{e}^{2\mathrm{i}t\theta} \\ 0 & 0 \end{pmatrix}, & z \in \Sigma_{kj}, \ j = 2, 4, \\ \begin{pmatrix} 0 & 0 \\ \mathrm{i}^{j-1} R_{kj}(z,\xi)\mathrm{e}^{-2\mathrm{i}t\theta} & 0 \end{pmatrix}, & z \in \Sigma_{kj}, \ j = 1, 3. \end{cases} \tag{7.3}$$

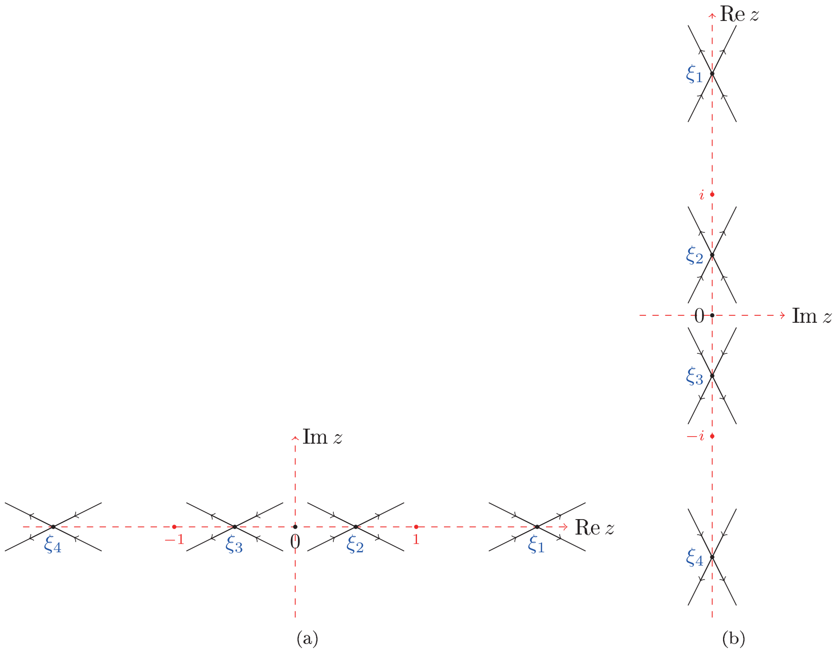


Figure 8 Jump contours $\Sigma^{(0)}$ of $M^{\mathrm{lo}}(z)$. The figures (a) and (b) are corresponding to the cases $-1<\xi$ and $\xi<-3$, respectively.

Then $V^{(2)}(z)=I-w_{kj}(z)$ for $z\in\Sigma_{kj}$. Besides, let

$$\Sigma_k^{(0)}=\bigcup_{j=1,\cdots,4}\Sigma_{kj},\quad w_k(z)=\sum_{j=1,\cdots,4}w_{kj}(z), \tag{7.4}$$

$$w_{kj}^{\pm}(z)=w_{kj}(z)|_{\mathbb{C}^{\pm}},\quad w_k^{\pm}(z)=w_k(z)|_{\mathbb{C}^{\pm}},\quad w^{\pm}(z)=w(z)|_{\mathbb{C}^{\pm}}. \tag{7.5}$$

Recall the Cauchy projection operator $C_{\pm}$ on $\Sigma^{(2)}$,

$$C_{\pm}(f)(s)=\lim_{z\to\Sigma_{\pm}^{(2)}}\frac{1}{2\pi\mathrm{i}}\int_{\Sigma^{(2)}}\frac{f(s)}{s-z}\mathrm{d}s, \tag{7.6}$$

by which, we further define operator

$$C_w(f)=C_+(fw^-)+C_-(fw^+),\quad C_{w_k}(f)=C_+(fw_k^-)+C_-(fw_k^+). \tag{7.7}$$

Then $C_w=\sum_{k=1}^{4}C_{w_k}$.

Lemma 7.1 *The matrix functions w_{kj} defined above admit the following estimation:*

$$\|w_{kj}\|_{L^p(\Sigma_{kj})}=\mathcal{O}(t^{-\frac{1}{2}}),\quad 1\le p<+\infty. \tag{7.8}$$

This lemma can be obtained by simple calculation. And it implies that $I-C_w$ and $I-C_{w_k}$ are reversible. So the solution of above RHP 7 exists uniquely, and it can be written as

$$M^{\mathrm{lo}}(z)=I+\frac{1}{2\pi\mathrm{i}}\int_{\Sigma^{(0)}}\frac{(I-C_w)^{-1}Iw}{s-z}\mathrm{d}s. \tag{7.9}$$

Next, we show the contributions of every crosses $\Sigma_k^{(0)}$ can be separated out.

Corollary 7.1 *As* $t \to +\infty$,

$$\|C_{w_k} C_{w_j}\|_{B(L^2(\Sigma^{(0)}))} \lesssim t^{-1}, \quad \|C_{w_k} C_{w_j}\|_{L^\infty(\Sigma^{(0)}) \to L^2(\Sigma^{(0)})} \lesssim t^{-1}. \tag{7.10}$$

Direct calculation establishes that

$$(I - C_w)\Big(I + \sum_{k=1}^{n(\xi)} C_{w_k}(I - C_{w_k})^{-1}\Big) = I - \sum_{1 \le k \ne j \le n(\xi)} C_{w_j} C_{w_k} (I - C_{w_k})^{-1},$$

$$\Big(I + \sum_{k=1}^{n(\xi)} C_{w_k}(I - C_{w_k})^{-1}\Big)(I - C_w) = I - \sum_{1 \le k \ne j \le n(\xi)} (I - C_{w_k})^{-1} C_{w_k} C_{w_j}.$$

Then following the step of [6], we derive the following proposition.

Proposition 7.1 *As* $t \to +\infty$,

$$\int_{\Sigma^{(0)}} \frac{(I - C_w)^{-1} I w}{s - z} \mathrm{d}s = \sum_{k=1}^{n(\xi)} \int_{\Sigma_k^{(0)}} \frac{(I - C_{w_k})^{-1} I w_k}{s - z} \mathrm{d}s + \mathcal{O}(t^{-\frac{3}{2}}). \tag{7.11}$$

So, as $t \to +\infty$, we consider to reduce the RHP 7 to a model RHP whose solution can be given explicitly in terms of parabolic cylinder functions on every contour $\Sigma_k^{(0)}$, respectively. And we only give the details of $\Sigma_1^{(0)}$, the model of other critical point can be constructed similarly. We denote $\widehat{\Sigma}_1^{(0)}$ as the contour $\{z = \xi_1 + l\mathrm{e}^{\pm\varphi \mathrm{i}},\ l \in \mathbb{R}\}$ oriented from $\Sigma_1^{(0)}$, and $\widehat{\Sigma}_{1j}$ is the extension of Σ_{1j} respectively. For z near ξ_1, rewrite phase function as

$$\theta(z) = \theta(\xi_1) + (z - \xi_1)^2 \theta'' \frac{(\xi_1)}{2} + \mathcal{O}((z - \xi_1)^3). \tag{7.12}$$

When $\xi > -1$, $\theta''(\xi_1) < 0$ and when $\xi < -3$, $\theta''(\xi_1) > 0$. It is naturally to consider the following local RH problem.

RHP 8 Find a matrix-valued function $M^{\mathrm{lo},1}(z)$ with the following properties:

► Analyticity: $M^{\mathrm{lo},1}(z)$ is analytical in $\mathbb{C} \setminus \widehat{\Sigma}_1$.

► Jump condition: $M^{\mathrm{lo},1}(z)$ has continuous boundary values $M_\pm^{\mathrm{lo},1}$ on $\widehat{\Sigma}_1$ and

$$M_+^{\mathrm{lo},1}(z) = M_-^{\mathrm{lo},1}(z) V^{\mathrm{lo},1}(z), \quad z \in \widehat{\Sigma}_1^{(0)}, \tag{7.13}$$

where

$$V^{\mathrm{lo},1}(z) = V^{(2)}(z), \quad z \in \widehat{\Sigma}_1^{(0)}. \tag{7.14}$$

► Asymptotic behaviors:

$$M^{\mathrm{lo},1}(z) = I + \mathcal{O}(z^{-1}), \quad z \to \infty. \tag{7.15}$$

RHP 8 does not possess the symmetry condition shared by preceding RHP, because it is a local model and will only be used for bounded values of z. In order to motivate the model, let $\zeta = \zeta(z, \xi)$ denote the rescaled local variable

$$\zeta(z, \xi) = \begin{cases} t^{\frac{1}{2}} \sqrt{-2\theta''(\xi_1)}(z - \xi_1), & \xi > -1; \\ t^{\frac{1}{2}} i \sqrt{2\theta''(\xi_1)}(z - \xi_1), & \xi < -3. \end{cases} \tag{7.16}$$

In fact, although when $\xi < -3$, the jump line $\widehat{\Sigma}_1^{(0)}$ is around the imaginary axis, when we trans it to the model pc RHP, the expression of $\zeta(z,\xi)$ is the same as the case of $\xi > -1$. This change of variable maps U_{ξ_1} to an expanding neighborhood of $\zeta = 0$. Additionally, let

$$\rho_{\xi_1} = \begin{cases} \rho(\xi_1)T_1(\xi)^2 \mathrm{e}^{-2\mathrm{i}t\theta(\xi_1)} \mathrm{e}^{-\mathrm{i}\nu(\xi_1)\log(-2t\theta''(\xi_1))}, & \xi > -1; \\ \rho(\xi_1)T_1(\xi)^2 \mathrm{e}^{\mathrm{i}\nu(\xi_1)\log(2t\theta''(\xi_1))}, & \xi < -3 \end{cases} \tag{7.17}$$

with $|\rho_{\xi_1}| = |\rho(\xi_1)|$. In the above expression, the complex powers are defined by choosing the branch of the logarithm with $-\pi < \arg\zeta < \pi$ in the cases $\xi > -1$, and the branch of the logarithm with $0 < \arg\zeta < 2\pi$ in the case $\xi < -3$. Through this change of variable, the jump $V^{\mathrm{lo},1}(z)$ approximates to the jump of two parabolic cylinder model problem (depending on ξ) as follows: RHP pc: Find a matrix-valued function $M^{\mathrm{pc}}(\zeta;\xi)$ with the following properties:

▶ Analyticity: $M^{\mathrm{pc}}(\zeta;\xi)$ is analytical in $\mathbb{C} \setminus \Sigma^{\mathrm{pc}}$ with $\Sigma^{\mathrm{pc}} = \{\mathbb{R}\mathrm{e}^{\varphi \mathrm{i}}\} \cup \{\mathbb{R}\mathrm{e}^{(\pi-\varphi)\mathrm{i}}\}$ shown in Figure 9.

▶ Jump condition: M^{pc} has continuous boundary values $M^{\mathrm{pc}}_\pm$ on Σ^{pc} and

$$M^{\mathrm{pc}}_+(\zeta;\xi) = M^{\mathrm{pc}}_-(\zeta;\xi)V^{\mathrm{pc}}(\zeta), \quad \zeta \in \Sigma^{\zeta}, \tag{7.18}$$

where in the case $\xi > -1$,

$$V^{\mathrm{pc}}(\zeta;\xi) = \begin{cases} \begin{pmatrix} 1 & 0 \\ r_{\xi_1}\zeta^{-2\mathrm{i}\nu(\xi_1)}\mathrm{e}^{\frac{\mathrm{i}}{2}\zeta^2} & 1 \end{pmatrix}, & \zeta \in \mathbb{R}^+\mathrm{e}^{\varphi \mathrm{i}}, \\ \begin{pmatrix} 1 & \overline{r}_{\xi_1}\zeta^{2\mathrm{i}\nu(\xi_1)}\mathrm{e}^{-\frac{\mathrm{i}}{2}\zeta^2} \\ 0 & 1 \end{pmatrix}, & \zeta \in \mathbb{R}^+\mathrm{e}^{-\varphi \mathrm{i}}, \\ \begin{pmatrix} 1 & 0 \\ \frac{r_{\xi_1}}{1+|r_{\xi_1}|^2}\zeta^{-2\mathrm{i}\nu(\xi_1)}\mathrm{e}^{\frac{\mathrm{i}}{2}\zeta^2} & 1 \end{pmatrix}, & \zeta \in \mathbb{R}^+\mathrm{e}^{(-\pi+\varphi)\mathrm{i}}, \\ \begin{pmatrix} 1 & \frac{\overline{r}_{\xi_1}}{1+|r_{\xi_1}|^2}\zeta^{2\mathrm{i}\nu(\xi_1)}\mathrm{e}^{-\frac{\mathrm{i}}{2}\zeta^2} \\ 0 & 1 \end{pmatrix}, & \zeta \in \mathbb{R}^+\mathrm{e}^{(\pi-\varphi)\mathrm{i}}, \end{cases} \tag{7.19}$$

and in the case $\xi < -3$,

$$V^{\mathrm{pc}}(\zeta;\xi) = \begin{cases} \begin{pmatrix} 1 & -\frac{\overline{r}_{\xi_1}}{1-|r_{\xi_1}|^2}\zeta^{-2\mathrm{i}\nu(\xi_1)}\mathrm{e}^{-\frac{\mathrm{i}}{2}\zeta^2} \\ 0 & 1 \end{pmatrix}, & \zeta \in \mathbb{R}^+\mathrm{e}^{\varphi \mathrm{i}}, \\ \begin{pmatrix} 1 & 0 \\ \frac{r_{\xi_1}}{1-|r_{\xi_1}|^2}\zeta^{2\mathrm{i}\nu(\xi_1)}\mathrm{e}^{\frac{\mathrm{i}}{2}\zeta^2} & 1 \end{pmatrix}, & \zeta \in \mathbb{R}^+\mathrm{e}^{(2\pi-\varphi)\mathrm{i}}, \\ \begin{pmatrix} 1 & -\overline{r}_{\xi_1}\zeta^{-2\mathrm{i}\nu(\xi_1)}\mathrm{e}^{-\frac{\mathrm{i}}{2}\zeta^2} \\ 0 & 1 \end{pmatrix}, & \zeta \in \mathbb{R}^+\mathrm{e}^{(\pi+\varphi)\mathrm{i}}, \\ \begin{pmatrix} 1 & 0 \\ r_{\xi_1}\zeta^{2\mathrm{i}\nu(\xi_1)}\mathrm{e}^{\frac{\mathrm{i}}{2}\zeta^2} & 1 \end{pmatrix}, & \zeta \in \mathbb{R}^+\mathrm{e}^{(\pi-\varphi)\mathrm{i}}. \end{cases} \tag{7.20}$$

▶ Asymptotic behaviors:

$$M^{\mathrm{pc}}(\zeta;\xi) = I + M^{\mathrm{pc}}_1\zeta^{-1} + \mathcal{O}(\zeta^{-2}), \quad \zeta \to \infty. \tag{7.21}$$

Then [16, Theorems A.1–A.6] proved that as $t \to \infty$,

$$M^{\mathrm{lo},1}(z) = I + \frac{1}{\zeta(z,t)}\begin{pmatrix} 0 & [M^{\mathrm{pc}}_1]_{12} \\ [M^{\mathrm{pc}}_1]_{21} & 0 \end{pmatrix} + \mathcal{O}(t^{-1}). \tag{7.22}$$

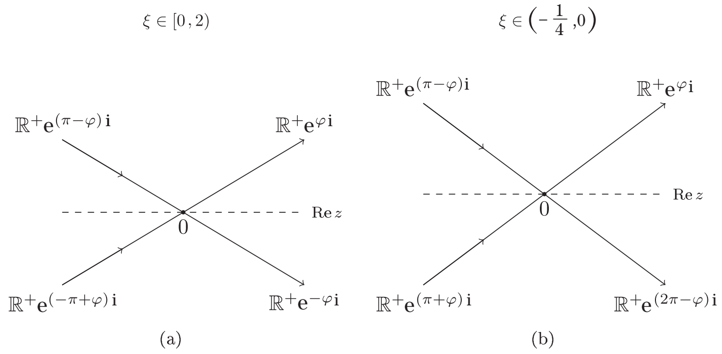


Figure 9 The contour $\Sigma^{\rm pc}$ in case $\xi > -1$ and $\xi < -3$, respectively.

Although the angle φ of the jump line is not $\frac{\pi}{4}$, as we assuming previous, $\varphi < \frac{\pi}{4}$, by a simple transformation we can trans the jump to $\mathbb{R}$ then restore it on $\frac{\pi}{4}$. So it can also match the classical parabolic-cylinder model. Namely, RHP pc has an explicit solution $M^{\rm pc}(\zeta,\xi)$, which is expressed in terms of solutions of the parabolic-cylinder equation similarly as [16],

$$\Big(\frac{\partial^2}{\partial z^2} + \Big(\frac{1}{2} - \frac{z^2}{2} + a\Big)\Big) D_a(z) = 0.$$

Note that the difference of jump matrices in $\xi > -1$ and $\xi < -3$ lead to two parabolic-cylinder models. The difference of this two pc models is the branch cut of the logarithmic function. When $\xi > -1$, it matches the pc model in [16] while when $\xi < -3$, it matches the pc model in [15]. A derivation of pc model is given in [6], and a direct verification of this two pc models is given in [17]. For brevity, denote $\widetilde{\beta}^1_{12} = -{\rm i}[M_1^{\rm pc}]_{12}$ and $\widetilde{\beta}^1 = {\rm i}[M_1^{\rm pc}]_{21}$. Therefore, when $\xi > -1$,,

$$\widetilde{\beta}^1_{12} = \frac{\sqrt{2\pi}{\rm e}^{{\rm i}\frac{\pi}{4}}{\rm e}^{-\pi\nu\frac{\xi_1}{2}}}{r_{\xi_1}\Gamma(-{\rm i}\nu(\xi_1))}, \quad \widetilde{\beta}^1_{21} = \frac{\nu(\xi_1)}{\widetilde{\beta}^1_{12}}. \tag{7.23}$$

Here, $\Gamma(z)$ is the Gamma function. And when $\xi < -3$,

$$\widetilde{\beta}^1_{12} = \frac{\sqrt{2\pi}{\rm e}^{{\rm i}\frac{\pi}{4}}{\rm e}^{\pi\nu\frac{\xi_1}{2}}}{r_{\xi_1}\Gamma({\rm i}\nu(\xi_1))}, \quad \widetilde{\beta}^1_{21} = \frac{-\nu(\xi_1)}{\widetilde{\beta}^1_{12}}. \tag{7.24}$$

Substitute above consequences into (7.22) and obtain:

$$M^{{\rm lo},1}(z) = I + \frac{t^{-\frac{1}{2}}(-2\theta''(\xi_1))^{-\frac{1}{2}}}{z-\xi_1}\begin{pmatrix} 0 & -{\rm i}\widetilde{\beta}^1_{12} \\ {\rm i}\widetilde{\beta}^1_{21} & 0 \end{pmatrix} + \mathcal{O}(t^{-1}). \tag{7.25}$$

For the model around other stationary phase points, it also admits

$$M^{{\rm lo},k}(z) = I + \frac{t^{-\frac{1}{2}}((-1)^k 2\theta''(\xi_1))^{-\frac{1}{2}}}{z-\xi_1}\begin{pmatrix} 0 & -{\rm i}\widetilde{\beta}^k_{12} \\ {\rm i}\widetilde{\beta}^k_{21} & 0 \end{pmatrix} + \mathcal{O}(t^{-1}) \tag{7.26}$$

for $k = 2,3,4$. Here, by the symmetry of $M^{\rm lo}(z)$, we obtain that for $z \in U_{\xi_1}$,

$$M^{{\rm lo},1}(z) = \frac{\rm i}{z}\sigma_3 M^{{\rm lo},2}\Big(\frac{1}{z}\Big)Q_- = \frac{\rm i}{z}M^{{\rm lo},3}\Big(-\frac{1}{z}\Big)\sigma_3 Q_- = \sigma_3 M^{{\rm lo},4}(-z)\sigma_3,$$

which also lead to the relationship of $\widetilde{\beta}_{12}^k$ with $\widetilde{\beta}_{12}^1$. Then together with Proposition 7.1, we final obtain the following proposition.

Proposition 7.2 *As* $t \to +\infty$,

$$M^{\mathrm{lo}}(z) = I + t^{-\frac{1}{2}} \sum_{k=1}^{n(\xi)} \frac{A_k(\xi)}{z-\xi_k} + \mathcal{O}(t^{-1}), \tag{7.27}$$

where

$$A_k(\xi) = ((-1)^k 2\theta''(\xi_1))^{-\frac{1}{2}} \begin{pmatrix} 0 & -\mathrm{i}\widetilde{\beta}_{12}^k \\ \mathrm{i}\widetilde{\beta}_{21}^k & 0 \end{pmatrix}. \tag{7.28}$$

8 The Small Norm RH Problem for Error Function

In this section, we consider the error matrix-function $E(z;\xi)$. When $|\xi+2|<1$ the definition (5.16) implies that $E(z;\xi)\equiv I$, so only the case $|\xi+2|>1$ needs to be investigate. And we can obtain a RH problem for the matrix function $E(z;\xi)$ for $|\xi+2|>1$.

RHP 10 Find a matrix-valued function $E(z;\xi)$ with the following properties:
▶ Analyticity: $E(z;\xi)$ is analytical in $\mathbb{C}\setminus\Sigma^{(E)}$, where

$$\Sigma^{(E)} = \partial U(\xi) \cup (\Sigma^{(2)} \setminus U(\xi)).$$

▶ Asymptotic behaviors:

$$E(z;\xi) \sim I + \mathcal{O}(z^{-1}), \quad |z|\to\infty. \tag{8.1}$$

▶ Jump condition: $E(z;\xi)$ has continuous boundary values $E_\pm(z;\xi)$ on $\Sigma^{(E)}$ satisfying

$$E_+(z;\xi) = E_-(z;\xi) V^{(E)}(z),$$

where the jump matrix $V^{(E)}(z)$ is given by

$$V^{(E)}(z) = \begin{cases} M^{(r)}(z)V^{(2)}(z)M^{(r)}(z)^{-1}, & z\in\Sigma^{(2)}\setminus U(\xi), \\ M^{(r)}(z)M^{\mathrm{lo}}(z)M^{(r)}(z)^{-1}, & z\in\partial U(\xi), \end{cases} \tag{8.2}$$

which is shown in Figure 10. We will show that for large times, the error function $E(z;\xi)$ solves following small norm RH problem. By using Proposition 5.1, we have the following estimates of $V^{(E)}$:

$$\|V^{(E)}(z) - I\|_p \lesssim \begin{cases} \exp\{-tK_p\}, & z\in\Sigma_{kj}\setminus U(\xi), \\ \exp\{-tK_p'\}, & z\in\Sigma_{kj}'. \end{cases} \tag{8.3}$$

For $z\in\partial U(\xi)$, $M^{(r)}(z)$ is bounded, so by using (7.2), we find that

$$|V^{(E)}(z) - I| = |M^{(r)}(z)^{-1}(M^{\mathrm{lo}}(z)-I)M^{(r)}(z)| = \mathcal{O}(t^{-\frac{1}{2}}). \tag{8.4}$$

Therefore, the existence and uniqueness of the RHP 10 can be shown by using a small-norm RH problem (see [7–8]). Moreover, according to Beal-Coifman theory, the solution of the RHP 10 can be given by

$$E(z;\xi) = I + \frac{1}{2\pi\mathrm{i}} \int_{\Sigma^{(E)}} \frac{(I+\varpi(s))(V^{(E)}(s)-I)}{s-z} \mathrm{d}s, \tag{8.5}$$

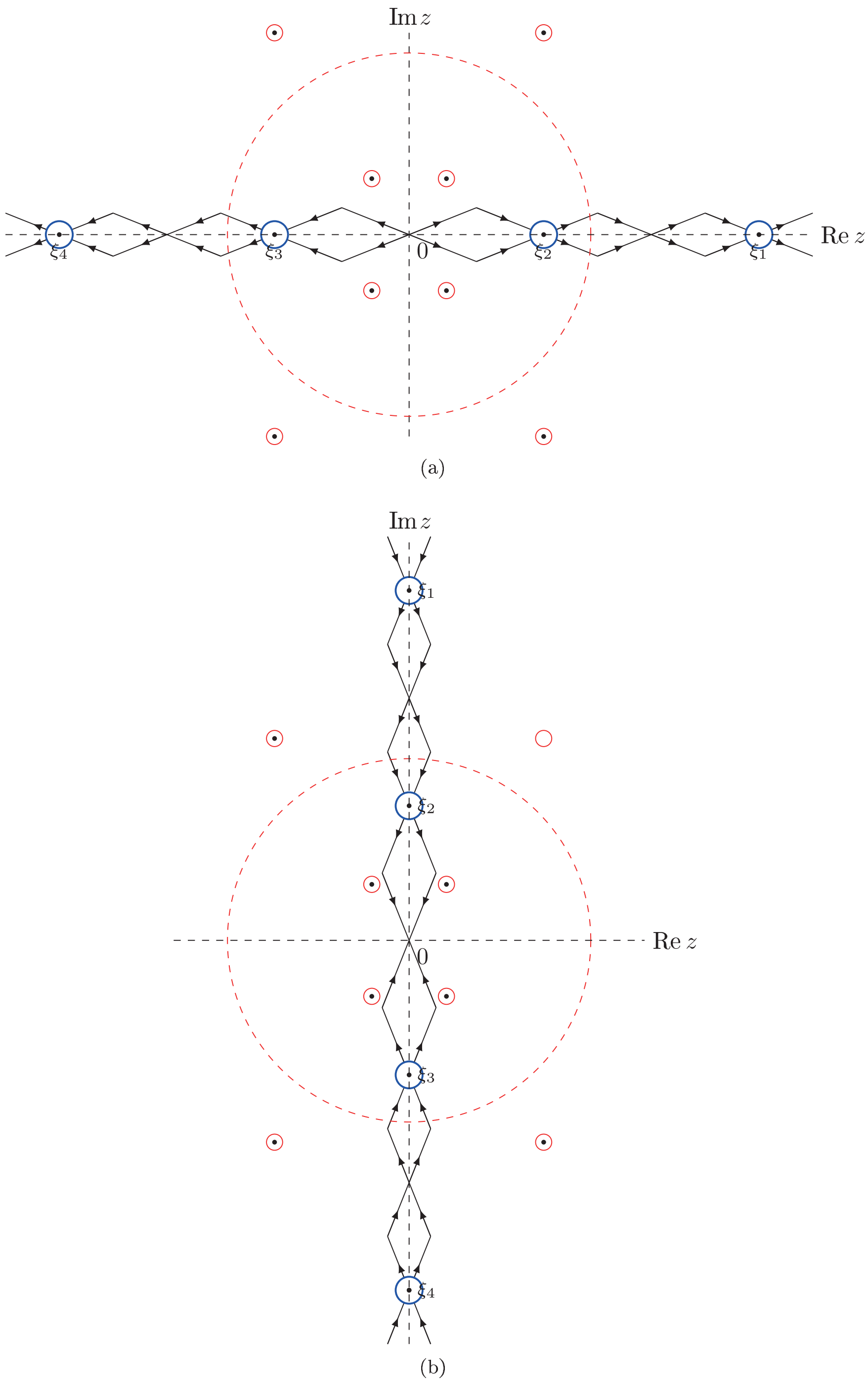


Figure 10 The jump contour $\Sigma^{(E)}$ for the $E(z;\xi)$. The blue circles are $U(\xi)$.

where the $\varpi \in L^\infty(\Sigma^{(E)})$ is the unique solution of the following equation

$$(1 - C_E)\varpi = C_E(I). \tag{8.6}$$

C_E is a integral operator: $L^\infty(\Sigma^{(E)}) \to L^2(\Sigma^{(E)})$ defined by

$$C_E(f)(z) = C_-(f(V^{(E)}(z) - I)), \tag{8.7}$$

where the C_- is the usual Cauchy projection operator on $\Sigma^{(E)}$. By (8.4), we have

$$\|C_E\| \le \|C_-\|\|V^{(E)}(z) - I\|_2 \lesssim \mathcal{O}(t^{-\frac{1}{2}}), \tag{8.8}$$

which implies that $1-C_E$ is invertible for sufficiently large t. So ϖ exists and is unique. Besides,

$$\|\varpi\|_{L^\infty(\Sigma^{(E)})} \lesssim \frac{\|C_E\|}{1 - \|C_E\|} \lesssim t^{-\frac{1}{2}}. \tag{8.9}$$

In order to reconstruct the solution $q(x, t)$ of (1.1), we need the asymptotic behavior of $E(z; \xi)$ as $z \to \infty$ and the long time asymptotic behavior of its coefficient of $\frac{1}{z}$ term in its expansion as $z \to \infty$. Note that when we estimate its asymptotic behavior, from (8.3) and (8.5) we only need to consider the calculation on $\partial U(\xi)$ because the others approach zero exponentially on other boundary.

Proposition 8.1 *As $z \to \infty$, we have*

$$E(z; \xi) = I + \frac{E_1}{z} + \mathcal{O}(z^{-2}), \tag{8.10}$$

where

$$E_1 = -\frac{1}{2\pi \mathrm{i}} \int_{\Sigma^{(E)}} (I + \varpi(s))(V^{(E)} - I)\mathrm{d}s, \tag{8.11}$$

satisfying long time asymptotic behavior condition

$$E_1 = t^{-\frac{1}{2}} H^{(1)} + \mathcal{O}(t^{-1}), \tag{8.12}$$

in which

$$\begin{aligned} H^{(1)} &= -\sum_{k=1}^{4} \frac{1}{2\pi \mathrm{i}} \int_{\partial U_{\xi_k}} \frac{M^{(r)}(s)^{-1} A_k(\xi) M^{(r)}(s)}{(s - \xi_k)} \mathrm{d}s \\ &= -\sum_{k=1}^{4} M^{(r)}(\xi_k)^{-1} A_k(\xi) M^{(r)}(\xi_k). \end{aligned} \tag{8.13}$$

In order to facilitate calculation, denote

$$f_{11} = [H^{(1)}]_{12}. \tag{8.14}$$

9 Analysis on the Pure $\overline{\partial}$-Problem

Now we consider the asymptotics behavior of $M^{(3)}(z)$. The $\overline{\partial}$-problem 4 of $M^{(3)}(z)$ is equivalent to the integral equation

$$M^{(3)}(z) = I + \frac{1}{\pi} \int_{\mathbb{C}} \frac{M^{(3)}(s)W(s)}{z - s} \mathrm{d}m(s), \tag{9.1}$$

where $m(s)$ is the Lebesgue measure on the $\mathbb{C}$. Denote C_z as the left Cauchy-Green integral operator with

$$fC_z(z) = \frac{1}{\pi}\int_C \frac{f(s)W(s)}{z-s}\mathrm{d}m(s).$$

Then (9.1) can be rewritten as

$$M^{(3)}(z) = I \cdot (I - C_z)^{-1}. \tag{9.2}$$

The existence of operator $(I - C_z)^{-1}$ is given by the following lemma.

Lemma 9.1 *The norm of the integral operator C_z decays to zero as $t \to \infty$:*

$$\|C_z\|_{L^\infty \to L^\infty} \lesssim t^{-\frac{1}{2}}, \tag{9.3}$$

which implies that $(I - C_z)^{-1}$ exists.

Proof For any $f \in L^\infty$,

$$\|fC_z\|_{L^\infty} \le \|f\|_{L^\infty} \frac{1}{\pi}\int_C \frac{|W(s)|}{|z-s|}\mathrm{d}m(s),$$

where $W(s) = M^{(r)}(s)\overline{\partial}R^{(2)}(s)M^{(r)}(s)^{-1}$. So we only need to estimate the integral

$$\frac{1}{\pi}\int_C \frac{|W(s)|}{|z-s|}\mathrm{d}m(s).$$

Since $W(s) \equiv 0$ out of $\overline{\Omega}$, we only need to focus on the estimation of

$$\frac{1}{\pi}\int_\Omega \frac{|W(s)|}{|z-s|}\mathrm{d}m(s).$$

Unlike the zero boundary case in [16], here $\det M^{(r)}(z) = 1 + z^{-2}$, and Proposition 6.1 implies that $|M^{(r)}(z)| \lesssim \sqrt{1+|z|^{-2}}$. So

$$\frac{1}{\pi}\int_\Omega \frac{|W(s)|}{|z-s|}\mathrm{d}m(s) \lesssim \frac{1}{\pi}\int_\Omega \frac{|\overline{\partial}R^{(2)}(s)|}{|z-s|}\frac{1+|s|^{-2}}{|1+s^{-2}|}\mathrm{d}m(s). \tag{9.4}$$

Note that, in the case $|\xi+2| > 1$ and the case $|\xi+2| < 1$, there are different kinds of Ω and $\overline{\partial}R^{(2)}$. So it needs to be discussed separately.

(1) Case $|\xi+2| < 1$.

For $j = 1, 4, 5, 8$, $|M^{(r)}(z)|$ is bounded in Ω_j. But when $z \in \Omega_j$ for $j = 2, 3, 6, 7$, the singularity at $z = \pm \mathrm{i}$ needs to be treated more carefully. So in the following calculation, we take Ω_2 in the second case as an example, because it is more elaborate than Ω_j for $j = 1, 4, 5, 8$. Denote three sub-regions of Ω_2 as

$$\begin{aligned} &D_1 = \mathbb{D}(0, 1-\varepsilon_0) \cap \Omega_2, \quad D_2 = \mathbb{D}(0, 1+\varepsilon_0) \setminus \mathbb{D}(0, 1-\varepsilon_0) \cap \Omega_2, \\ &D_3 = \Omega_2 \setminus \mathbb{D}(0, 1+\varepsilon_0). \end{aligned} \tag{9.5}$$

Then the integral $\int_{\Omega_2} \frac{|W(s)|}{|z-s|}\mathrm{d}m(s)$ is divide to three parts:

$$I_i = \int_{D_i} \frac{|\overline{\partial}R^{(2)}(s)|}{|z-s|}\frac{1+|s|^{-2}}{|1+s^{-2}|}\mathrm{d}m(s) \quad \text{for } i = 1, 2, 3. \tag{9.6}$$

Let $s=u+v\mathrm{i}=r\mathrm{e}^{\mathrm{i}\vartheta}$, $z=\zeta+\mathrm{i}\eta$. In the following calculation, we will use the inequality

$$\||s-z|^{-1}\|_{L^q(\mathbb{R}^+)}=\Big\{\int_0^{+\infty}\Big[\Big(\frac{v-\eta}{u-\zeta}\Big)^2+1\Big]^{-\frac{q}{2}}\mathrm{d}\Big(\frac{v-\eta}{|u-\zeta|}\Big)\Big\}^{\frac{1}{q}}|u-\zeta|^{-\frac{1}{p}}\lesssim|u-\zeta|^{-\frac{1}{p}} \tag{9.7}$$

with $1\le q<+\infty$ and $\frac{1}{p}+\frac{1}{q}=1$. For $s\in D_3$, $|s|>1+\varepsilon_0$, then

$$\frac{1+|s|^{-2}}{|1+s^{-2}|}<\frac{1+|s|^2}{|s|^2-1}<1+\frac{2}{\varepsilon_0^2+2\varepsilon_0}<\infty. \tag{9.8}$$

Then together with (4.14), we have

$$I_3\lesssim\int_{\Omega_2}\frac{|\overline{\partial}R^{(2)}(s)|}{|z-s|}\mathrm{d}m(s)=\int_{\Omega_2}\frac{|\overline{\partial}R_2(s)\mathrm{e}^{2\mathrm{i}t\theta}|}{|z-s|}\mathrm{d}m(s). \tag{9.9}$$

Moreover, by Lemma 4.1,

$$|\mathrm{e}^{2\mathrm{i}t\theta}|\le\mathrm{e}^{-c\sin 2\vartheta F(r)^2}\le\mathrm{e}^{-2cuv}\le\mathrm{e}^{-2cu}, \tag{9.10}$$

where c is a positive constant, and the last step follows form

$$v\ge\max\Big\{1+\varepsilon_0,\frac{u}{\tan\varphi}\Big\}\ge1+\varepsilon_0>1.$$

Substitute (4.19) and above inequality into (9.9) and obtain:

$$\begin{aligned}I_3\lesssim&\int_0^{+\infty}\int_{\frac{u}{\tan\varphi}}^{+\infty}\frac{|p_2'(\mathrm{i}r)|\mathrm{e}^{-4cut}}{|z-s|}\mathrm{d}v\mathrm{d}u+\int_0^{+\infty}\int_{\frac{u}{\tan\varphi}}^{+\infty}\frac{|r|^{-\frac{1}{2}}\mathrm{e}^{-2cut}}{|z-s|}\mathrm{d}v\mathrm{d}u\\&+\int_0^{+\infty}\int_{\frac{u}{\tan\varphi}}^{+\infty}\frac{|\overline{\partial}X_1(r)|\mathrm{e}^{-4cut}}{|z-s|}\mathrm{d}v\mathrm{d}u.\end{aligned}$$

By Cauchy-Schwarz inequality, the first term has

$$\begin{aligned}&\int_0^{+\infty}\int_{\frac{u}{\tan\varphi}}^{+\infty}\frac{|p_2'(\mathrm{i}r)|\mathrm{e}^{-4cut}}{|z-s|}\mathrm{d}v\mathrm{d}u\\&\le\int_0^{+\infty}\|\widetilde{\rho}'\|_{L^2(\mathrm{i}\mathbb{R})}\||s-z|^{-1}\|_{L^2(\mathbb{R}^+)}\mathrm{e}^{-2cut}\mathrm{d}u\\&\le\int_0^{+\infty}\mathrm{e}^{-2cut}|u-x|^{-\frac{1}{2}}\mathrm{d}u\lesssim t^{-\frac{1}{2}}.\end{aligned} \tag{9.11}$$

So does the last term. Before we estimate the second term, we consider for $p>2$,

$$\Big(\int_{\frac{u}{\tan\varphi}}^{+\infty}|u^2+v^2|^{-\frac{p}{2}}\mathrm{d}v\Big)^{\frac{1}{p}}=\Big(\int_{\frac{u}{\sin\varphi}}^{+\infty}|r|^{-\frac{p}{2}+1}v^{-1}\mathrm{d}r\Big)^{\frac{1}{p}}\lesssim u^{-\frac{1}{2}+\frac{1}{p}}. \tag{9.12}$$

Then

$$\begin{aligned}\int_0^{+\infty}\int_{\frac{u}{\tan\varphi}}^{+\infty}\frac{|r|^{-\frac{1}{2}}\mathrm{e}^{-2cut}}{|z-s|}\mathrm{d}v\mathrm{d}u&\le\int_0^{+\infty}\|r\|_{L^p_v(\frac{u}{\tan\varphi},+\infty)}\||s-z|^{-1}\|_{L^q(\mathbb{R}^+)}\mathrm{e}^{-2cut}\mathrm{d}u\\&\le\int_0^{+\infty}\mathrm{e}^{-2cut}|u-x|^{-\frac{1}{p}}u^{-\frac{1}{2}+\frac{1}{p}}\mathrm{d}u\lesssim t^{-\frac{1}{2}}.\end{aligned} \tag{9.13}$$

Combining above inequalities we finally have $I_3 \lesssim t^{-\frac{1}{2}}$. As for I_2, the singularity at i can be balanced by (4.20), and recall that $1 > \varepsilon_0 > 0$ with $(1-\varepsilon_0)\cos\varphi > \frac{1}{2}$,

$$\begin{aligned} I_2 &\lesssim \int_0^2 \int_{\frac{1}{2}}^2 \frac{\mathrm{e}^{-2cut}}{|z-s|} \frac{1+|s|^2}{|s+\mathrm{i}|} \mathrm{d}v\mathrm{d}u \lesssim \int_0^2 \int_{\frac{1}{2}}^2 \frac{\mathrm{e}^{-2cut}}{|z-s|} \mathrm{d}v\mathrm{d}u \\ &\lesssim \int_0^2 |u-x|^{-\frac{1}{2}} \mathrm{e}^{-2cut} \mathrm{d}u \lesssim t^{-\frac{1}{2}}. \end{aligned} \tag{9.14}$$

Finally, consider I_1, similarly we have

$$I_1 \lesssim \int_0^{1-\varepsilon_0} \int_u^{1-\varepsilon_0} (|p_2'(\mathrm{i}r)| + |r|^{-\frac{1}{2}} + |\overline{\partial} X_1(r)|) \frac{\mathrm{e}^{-2cut}}{|z-s|} \mathrm{d}v\mathrm{d}u, \tag{9.15}$$

which can be estimated same as I_3.

(2) Case $|\xi+2| > 1$.

The proof in region Ω_{jk} containing $z = \pm\mathrm{i}$ is similar as above case $|\xi+2| < 1$. By Lemma 4.2 and (4.57), the other region can be estimated similarly as [16]. So the proof is completed.

As $z \to \infty$, $M^{(3)}(z)$ has asymptotic expansion:

$$M^{(3)}(z) = I - \frac{M_1^{(3)}(x,t)}{z} + \mathcal{O}(z^{-2}), \tag{9.16}$$

where $M_1^{(3)}$ is a z-independent coefficient. The asymptotic behavior of $M_1^{(3)}$ is given by the following proposition.

Proposition 9.1 *As $z \to \infty$, the expansion above holds with*

$$M_1^{(3)}(x,t) = \frac{1}{\pi} \int_C M^{(3)}(s) W^{(3)}(s) \mathrm{d}m(s). \tag{9.17}$$

There exist constants T_1, such that for all $t > T_1$, $M_1^{(3)}(x,t)$ satisfies

$$|M_1^{(3)}(x,t)| \lesssim t^{-\frac{3}{4}}. \tag{9.18}$$

Proof Lemma 9.1 and (9.2) imply that for large t, $\|M^{(3)}(z)\|_{L^\infty} \lesssim 1$. The proof proceeds along the same lines as the proof of above proposition. Like in the above proposition, the two cases need to be discussed separately. When $|\xi+2| < 1$, for the same reason, we only estimate the integral on Ω_2.

$$\frac{1}{\pi} \int_{\Omega_2} M^{(3)}(s) W^{(3)}(s) \mathrm{d}m(s) \lesssim \frac{1}{\pi} \int_{\Omega_2} |\overline{\partial} R_2(s) \mathrm{e}^{2\mathrm{i}t\theta}| \frac{1+|s|^{-2}}{|1+s^{-2}|} \mathrm{d}m(s). \tag{9.19}$$

Let $s = u + vi = r\mathrm{e}^{\mathrm{i}\vartheta}$. And we also divide right integral of above inequality to three parts

$$I_{i+3} = \frac{1}{\pi} \int_{D_i} |\overline{\partial} R_2(s) \mathrm{e}^{2\mathrm{i}t\theta}| \frac{1+|s|^{-2}}{|1+s^{-2}|} \mathrm{d}m(s). \tag{9.20}$$

For I_4, $\frac{1+|s|^{-2}}{|1+s^{-2}|} < \infty$, so

$$I_4 \lesssim \int_0^{+\infty} \int_{\frac{u}{\tan\varphi}}^{+\infty} |p_2'(\mathrm{i}r)| \mathrm{e}^{-2cuvt} \mathrm{d}v\mathrm{d}u + \int_0^{+\infty} \int_{\frac{u}{\tan\varphi}}^{+\infty} |r|^{-\frac{1}{2}} \mathrm{e}^{-2cuvt} \mathrm{d}v\mathrm{d}u$$

$$+\int_0^{+\infty}\int_{\frac{u}{\tan\varphi}}^{+\infty}|\overline{\partial}X_1(r)|\mathrm{e}^{-2cuvt}\mathrm{d}v\mathrm{d}u. \tag{9.21}$$

Note that

$$\begin{aligned}\Big(\int_{\frac{u}{\tan\varphi}}^{+\infty}\mathrm{e}^{-2cuvtq}\mathrm{d}v\Big)^{\frac{1}{q}}&=\Big(\int_{\frac{u}{\tan\varphi}}^{+\infty}\mathrm{e}^{-2cuvtq}\mathrm{d}(2cuvtq)\Big)^{\frac{1}{q}}(2cutq)^{-\frac{1}{q}}\\&\lesssim \mathrm{e}^{-c'u^2t}(ut)^{-\frac{1}{q}},\end{aligned} \tag{9.22}$$

where c' is a positive constant. Then the first integral in (9.21) have

$$\int_0^{+\infty}\int_{\frac{u}{\tan\varphi}}^{+\infty}|p_2'(\mathrm{i}r)|\mathrm{e}^{-2cuvt}\mathrm{d}v\mathrm{d}u \ \lesssim t^{-\frac{1}{2}}\int_0^{+\infty}\|\widetilde{\rho}'\|_{L^2(\mathrm{i}\mathbb{R})}u^{-\frac{1}{q}}\mathrm{e}^{-c'u^2t}\mathrm{d}u\lesssim t^{-\frac{3}{4}}.$$

The last integral can be bounded in the same way. To estimate the second term, we also use Cauchy-Schwarz inequality for $4>p>2$ and $\frac{1}{q}+\frac{1}{p}=1$,

$$\int_0^{+\infty}\int_{\frac{u}{\tan\varphi}}^{+\infty}|r|^{-\frac{1}{2}}\mathrm{e}^{-2cuvt}\mathrm{d}v\mathrm{d}u\lesssim t^{-\frac{1}{q}}\int_0^{+\infty}u^{\frac{2}{p}-\frac{3}{2}}\mathrm{e}^{-c'u^2t}\mathrm{d}u\lesssim t^{-\frac{3}{4}}. \tag{9.23}$$

The bound for I_4 follows in the same manner as for I_6. Turning to I_5, we also use $|\overline{\partial}R_2(z)|\lesssim|z-\mathrm{i}|$ and obtain

$$\begin{aligned}I_5&\lesssim\int_{D_2}\frac{\mathrm{e}^{-2cut}}{|z-s|}\frac{1+|s|^2}{|s+\mathrm{i}|}\mathrm{d}m(s)\lesssim\int_{\frac{1}{2}}^{2}\int_u^2\mathrm{e}^{-2cuvt}\mathrm{d}v\mathrm{d}u\\&=\int_{\frac{1}{2}}^{2}(cut)^{-1}(\mathrm{e}^{-2cu^2t}-\mathrm{e}^{-4cut})\mathrm{d}u\lesssim t^{-1}.\end{aligned} \tag{9.24}$$

This estimate is strong enough to obtain the result. And when $|\xi+2|>1$, the regions containing $z=\pm\mathrm{i}$ are estimated in the same way as above case. Lemma 4.2 and (4.57) give that the other region can be estimated similarly as [16].

10 Asymptotic for the DNLS Equation

Now we begin to construct the long time asymptotics of the DNLS equation (1.1). Inverting the sequence of transformations (3.23), (4.66), (5.6) and (6.11), we have

$$M(z)=T(\infty)^{\sigma_3}M^{(3)}(z)M^{(r)}(z)R^{(2)}(z)^{-1}T(z)^{-\sigma_3}. \tag{10.1}$$

To reconstruct the solution $q(x,t)$ through (2.67), we take $z\to\infty$ out of $\overline{\Omega}$ in above equation. In this case, $R^{(2)}(z)=I$. When $|\xi+2|<1$, using Propositions 3.1, 6.3 and 9.1, we deduce that

$$\begin{aligned}M(z)&=T(\infty)^{\sigma_3}(I+M_1^{(3)}(z)z^{-1})\widetilde{E}(z)M_\Lambda^{(r)}(z)\\&\quad\cdot T(\infty)^{-\sigma_3}\Big(1+z^{-1}\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}}\log(1-\rho(s)\widetilde{\rho}(s))\mathrm{d}s\Big)^{-\sigma_3}+\mathcal{O}(z^{-2}),\end{aligned} \tag{10.2}$$

which admits long time asymptotics

$$M(z)=T(\infty)^{\sigma_3}M_\Lambda^{(r)}(z)T(\infty)^{-\sigma_3}\Big(1+z^{-1}\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}}\log(1-\rho(s)\widetilde{\rho}(s))\mathrm{d}s\Big)^{-\sigma_3}$$

$$+\mathcal{O}(z^{-2})+\mathcal{O}(t^{-\frac{3}{4}}).$$

From (2.67),

$$\begin{aligned} q(x,t) &= -\mathrm{i}\lim_{z\to\infty}[zM]_{12} \\ &= T(\infty)^{-2}q_\Lambda^r(x,t)+\mathcal{O}(t^{-\frac{3}{4}}), \end{aligned} \tag{10.3}$$

where $q_\Lambda^r(x,t)$ is given in Corollary 6.2. And when $|\xi+2|>1$, we also have

$$\begin{aligned} M(z) &= T(\infty)^{\sigma_3}(I+M_1^{(3)}(z)z^{-1})E(z)\widetilde{E}(z)M_\Lambda^{(r)}(z) \\ &\quad T(\infty)^{-\sigma_3}\Big(1+z^{-1}\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}}\log(1-\rho(s)\widetilde{\rho}(s))\mathrm{d}s\Big)^{-\sigma_3}+\mathcal{O}(z^{-2}). \end{aligned} \tag{10.4}$$

Propositions 3.1, 6.3, 8.1 and 9.1 give that as $t\to\infty$,

$$\begin{aligned} M(z) &= T(\infty)^{\sigma_3}\Big(I+t^{-\frac{1}{2}}\frac{H^{(1)}}{z}\Big)M_\Lambda^{(r)}(z)T(\infty)^{-\sigma_3} \\ &\quad \Big(1+z^{-1}\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}}\log(1-\rho(s)\widetilde{\rho}(s))\mathrm{d}s\Big)^{-\sigma_3}+\mathcal{O}(z^{-2})+\mathcal{O}(t^{-\frac{3}{4}}). \end{aligned}$$

From (2.67),

$$\begin{aligned} q(x,t) &= -\mathrm{i}\lim_{z\to\infty}[zM]_{12} \\ &= T(\infty)^{-2}q_\Lambda^r(x,t)-t^{-\frac{1}{2}}if_{11}+\mathcal{O}(t^{-\frac{3}{4}}) \end{aligned} \tag{10.5}$$

with f_{11} defined in (8.14). Therefore, we achieve main result of this paper.

Theorem 10.1 *Let $q(x,t)$ be the solution for the initial-value problem* (1.1) *with generic data $u_0(x)$ admitting Assumption* 2.1 *and scatting data $\{r(z),\{\zeta_n,C_n\}_{n=1}^{4N_1+2N_2}\}$. Let $\xi=\frac{x}{t}$. Denote $q_\Lambda^r(x,t)$ to be the $\mathcal{N}(\Lambda)$-solution corresponding to scattering data $\{0,\{\zeta_n,\mathring{c}_n\}_{n\in\Lambda}\}$ shown in Corollary* 6.2. *And Λ is defined in* (3.3). *There exists a large constant $T_1=T_1(\xi)$ for all $T_1<t\to\infty$,*

(1) *when $|\xi+2|<1$,*

$$q(x,t)=T(\infty)^{-2}q_\Lambda^r(x,t)+\mathcal{O}(t^{-\frac{3}{4}}); \tag{10.6}$$

(2) *when $|\xi+2|>1$,*

$$q(x,t)=T(\infty)^{-2}q_\Lambda^r(x,t)-t^{-\frac{1}{2}}\mathrm{i}f_{11}+\mathcal{O}(t^{-\frac{3}{4}}), \tag{10.7}$$

where $q_\Lambda^r(x,t)$, $T(z)$ and f_{11} are shown in Propositions 3.1, *Corollary* 6.2 *and* (8.14), *respectively.*

Corollary 10.1 *Suppose that the simple poles only distribute on unit circle, there exists a large constant $T_1=T_1(\xi)$ for all $T_1<t\to\infty$,*

(1) *when $|\xi+2|<1$,*

$$q(x,t)=T(\infty)^{-2}\Big(q_-\prod_{l=1}^{N_2}w_l^4+\sum_{l\in\Delta_2}w_l^4(q(x,t;D_l^{\mathrm{sol}})-q_-)$$

$$+\sum_{l\in\nabla_2\cup\Lambda_2}\left(q(x,t;D_l^{\mathrm{sol}})-q_-\right)\Big)+\mathcal{O}(t^{-\frac{3}{4}}), \tag{10.8}$$

(2) *when* $|\xi+2|>1$,

$$q(x,t)=T(\infty)^{-2}\Big(q_-\prod_{l=1}^{N_2}w_l^4+\sum_{l\in\Delta_2}w_l^4(q(x,t;D_l^{\mathrm{sol}})-q_-)$$

$$+\sum_{l\in\nabla_2\cup\Lambda_2}\left(q(x,t;D_l^{\mathrm{sol}})-q_-\right)\Big)-t^{-\frac{1}{2}}\mathrm{i}f_{11}+\mathcal{O}(t^{-\frac{3}{4}}), \tag{10.9}$$

where $q(x,t;D_l^{\mathrm{sol}})$ *is the soliton solution defined in Corollary* 6.2 *with scattering data* $D_l^{\mathrm{sol}}=\{0,\{w_l,\mathring{c}_l\},\{-w_l,\mathring{c}_{l+N_2}\}\}$.

The long time asymptotic expansion (10.8)–(10.9) shows the soliton resolution of for the initial value problem of the derivative nonlinear schrödinger equation, which can be characterized with an $\mathcal{N}(\Lambda)$-solution whose parameters are modulated by a sum of localized soliton-soliton interactions. Our results also show that the poles on curve soliton solutions of the derivative Schrödinger equation has dominant contribution to the solution as $t\to\infty$.

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