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Local Well-posedness of the Derivative Schrödinger Equation in Higher Dimension for Any Large Data

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Abstract In this paper, the authors consider the local well-posedness for the derivative Schrödinger equation in higher dimension

$$u_t - \mathrm{i}\triangle u + |u|^2(\overrightarrow{\gamma}\cdot\nabla u) + u^2(\overrightarrow{\lambda}\cdot\nabla\overline{u}) = 0, \quad (x,t)\in\mathbb{R}^n\times\mathbb{R},\ \overrightarrow{\gamma},\overrightarrow{\lambda}\in\mathbb{R}^n;\ n\geq 2.$$

It is shown that the Cauchy problem of the derivative Schrödinger equation in higher dimension is locally well-posed in $H^s(\mathbb{R}^n)$ $(s>\frac{n}{2})$ for any large initial data. Thus this result can compare with that in one dimension except for the endpoint space $H^{\frac{n}{2}}$.

Keywords Well-posedness, Derivative Schrödinger equation in higher dimension, Short-time $X_{s,b}$, Large initial data
2000 MR Subject Classification 35E15, 35Q55

1 Introduction

The aim in this work is to study the well-posedness for the Cauchy problem of the Schrödinger equation with derivative in higher dimension (DNLS):

$$\begin{cases} u_t - \mathrm{i}\triangle u + |u|^2(\overrightarrow{\gamma}\cdot\nabla u) + u^2(\overrightarrow{\lambda}\cdot\nabla\overline{u}) = 0, & (x,t)\in\mathbb{R}^n\times\mathbb{R},\ n\geq 2,\\ u(x,0)=u_0(x)\in H^s(\mathbb{R}^n), \end{cases} \tag{1.1}$$

where $\overline{u}(x,t)$ is the complex conjugate of $u(x,t)$, $\overrightarrow{\gamma}$ and $\overrightarrow{\lambda}$ are real vectors; $\nabla=(\partial_{x_1},\partial_{x_2},\cdots,\partial_{x_n})$, $\triangle$ is the Laplacian in $\mathbb{R}^n$. Notice that DNLS has a conserved quantity

$$\|u(t)\|_{L^2}=\|u_0\|_{H^s}.$$

There exists a generalized form of the DNLS equation

$$\begin{cases} u_t - \mathrm{i}\triangle u = F(u,\overline{u},\nabla u,\nabla\overline{u}), & (x,t)\in\mathbb{R}^n\times\mathbb{R},\\ u(x,0)=u_0(x)\in H^s(\mathbb{R}^n), \end{cases} \tag{1.2}$$

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where $F:\mathbb{C}^{2n+2}\to\mathbb{C}$ is a polynomial having no constant or linear terms,

$$F(z)=F(z_1,z_2,\cdots,z_{2n+2})=\sum_{3\le|\rho|<\infty}C_\rho z^\rho, \tag{1.3}$$

here $C_\rho\in\mathbb{C},\ \rho=(\rho_1,\rho_2,\cdots,\rho_{2n+2})\in\mathbb{Z}_+^{2n+2}$. We assume that every term in F contains at least one derivative. An example of F satisfying (1.3) is given by

$$F(u,\overline{u},\nabla u,\nabla\overline{u})=|u|^2(\overrightarrow{\gamma}\cdot\nabla u)+(u)^2(\overrightarrow{\lambda}\cdot\nabla\overline{u}). \tag{1.4}$$

In one dimension, Takaoka [13] showed that the Cauchy problem (1.1) is locally well-posed in H^s for $s\ge\frac{1}{2}$, where he used the Gauge transformation to remove the derivative in u. In [14], Takaoka showed that this result is sharp in the sense that the data map fails to be C^3 or uniformly C^0 for $s<\frac{1}{2}$.

In higher dimension, the local and global well posedness of (DNLS) (1.2) have been extensively studied. For example: Ozawa and Zhai [12] showed that if F is smooth function vanishing of third order at the origin and satisfies an energy structure condition, then the Cauchy problem (DNLS) (1.2) is globally wellposed in $H^s(\mathbb{R}^n)\big(n\ge3,s>2+\frac{n}{2}\big)$ with small data. By using the local smooth effects for the solutions of the Schrödinger equation, Kenig, Ponce and Vega [8, 10] were able to deal with the non-elliptical case and they established the local well posedness of equation (1.2) in H^s with $s\gg\frac{n}{2}$ for large data (H^s with $s\ge n+2+\frac{1}{2}$ for small data in [8]); the local well posedness results have been generalized to the quasi-linear (ultrahyperbolic) Schrödinger equations, see [6, 11].

From above known results, it follows that there exists an open problem: How about solution of (1.1) in high dimension with large data? For large data, Kenig, Ponce and Vega [10] obtained local well posedness of the generalized DNLS equation (1.2) in H^s with $s\gg\frac{n}{2}$. Moreover, Bienaimé [1] considered the generalized DNLS equation (1.2), and proved the local existence, the uniqueness and the smoothing effect given any $u_0\in H^s$ with $s>\frac{n}{2}+3$; his proof followed the same plan as that of Kenig, Ponce and Vega [10]. In this paper, we will consider the solution of (1.1) for large data, DNLS equation (1.1) is simpler than DNLS equation (1.2). In fact, we can also use the stand energy method in [4] to show that the Cauchy problem (1.1) is locally well-posed in H^s with $s>\frac{n}{2}+1$ for any large initial data. From the result in one dimension, there exists a gap between $H^{\frac{n}{2}}$ and $H^{\frac{n}{2}+1}$ in high dimension. In this paper, we will solve this gap except for the endpoint space $H^{\frac{n}{2}}$. That is, the Cauchy problem (1.1) is locally well-posed in H^s with $s>\frac{n}{2}$ for any large initial data. Moreover, from the following proof, it follows that the flow map defines a continuous map on H^s with $s>\frac{n}{2}$, not a uniformly continuous map. But basing on the method in this paper, we will show that the Cauchy problem (1.2) is locally well-posed in H^s with $s>\frac{n}{2}+1$ for any large initial data in the future.

We usually use its integral equivalent formulation to study problem (1.1),

$$u(t)=S(t)u_0+\int_0^t S(t-t')(|u|^2(\overrightarrow{\gamma}\cdot\nabla u)+u^2(\overrightarrow{\lambda}\cdot\nabla\overline{u}))(t')\mathrm{d}t',$$

where $S(t)=\mathcal{F}_x^{-1}\mathrm{e}^{\mathrm{i}t|\xi|^2}\mathcal{F}_x$ is the group of the equation (1.1). From the previous arguments, we guess that the first step is to choose the suitable spaces $\mathbf{F}^s$ and $\mathbf{N}^s$, then the key is to show the following holds for some s and $\theta>0$,

$$\||u|^2(\overrightarrow{\gamma}\cdot\nabla u)+u^2(\overrightarrow{\lambda}\cdot\nabla\overline{u})\|_{\mathbf{N}^s(T)}\lesssim T^\theta\|u\|^3_{\mathbf{F}^s(T)}. \tag{1.5}$$

In fact, if $\theta = 0$, (1.5) holds for the Bourgain's space $\mathbf{F}^s = X_{s,\frac{1}{2}}$ and dual space $\mathbf{N}^s = X_{s,-\frac{1}{2}}$; that is, we can easily obtain local well-posedness with small initial data.

Next we can define some suitable spaces $\mathbf{F}^s(T)$, $\mathbf{N}^s(T)$ and $\mathbf{E}^s(T)$, where the definitions of $\mathbf{F}^s(T)$, and $\mathbf{N}^s(T)$ in Section 2 are different from these of [5]. Following some ideas in [5], we will show that if $u(t)$ is a solution of the Cauchy problem (1.1) in H^∞ for some $T > 0$, $s > \frac{n}{2}$ and some $\theta > 0$,

$$\begin{cases} \|u\|_{\mathbf{F}^s(T)} \lesssim \|u\|_{\mathbf{E}^s(T)} + \||u|^2(\overrightarrow{\gamma} \cdot \nabla u) + u^2(\overrightarrow{\lambda} \cdot \nabla \overline{u})\|_{\mathbf{N}^s(T)}, \\ \||u|^2(\overrightarrow{\gamma} \cdot \nabla u) + u^2(\overrightarrow{\lambda} \cdot \nabla \overline{u})\|_{\mathbf{N}^s(T)} \le CT^\theta \|u\|^3_{\mathbf{F}^s(T)}, \\ \|u\|^2_{\mathbf{E}^s(T)} \le \|u_0\|^2_{H^s} + T^\theta \|u\|^4_{\mathbf{F}^s(T)}, \end{cases} \tag{1.6}$$

where the constant C is independent of $\|u_0\|_{H^s}$ and T depends on $\|u_0\|_{H^s}$. Notice that the first inequality in (1.6) is the analogue of the linear estimates, the second inequality in (1.6) is the analogue of the trilinear estimate estimate, the third inequality in (1.6) is the analogue of the energy-type estimate. From this, we can obtain the existence of the solution in H^s. To prove the uniqueness and continuity of solution in H^s, we need to consider the difference equation about (1.1). But the symmetries of the difference equation are not as good as the symmetries of (1.1), which causes difficulties in the proofs of suitable energy estimates, which can be found in Section 7. In fact, if $u(x,t)$ and $v(x,t)$ are solutions of (1.1) with initial data $u_0, v_0 \in H^s$ with $s > \frac{n}{2}$, then we can show that for some small $\theta > 0$,

$$\begin{aligned} \sup_{t\in(0,T)} \|w(t)\|_{H^s} \le \|w_0\|_{H^s} + T^\theta \|u\|^2_{\mathbf{F}^s} \|w\|_{\mathbf{F}^s} \|w\|_{\mathbf{F}^s} + T^\theta \|u\|_{\mathbf{F}^{s+\widetilde{\theta}}} \|u\|_{\mathbf{F}^s} \|w\|_{\mathbf{F}^s} \|w\|_{\mathbf{F}^{s-\widetilde{\theta}}} \\ + T^\theta \|u\|_{\mathbf{F}^s} \|w\|^3_{\mathbf{F}^s} + T^\theta \|w\|^4_{\mathbf{F}^s}, \end{aligned} \tag{1.7}$$

where $\widetilde{\theta} > 0$ is small, and will be chosen later. In this paper, we mainly prove (1.7). In fact, we will use some dispersive properties in Section 3 to prove (1.7).

The main result of the paper is listed as below.

Theorem 1.1 (a) *Let $n \ge 2$ and $s > \frac{n}{2}$. Assume $u_0 \in H^\infty$ for any large data. Then there exist some $T = T(\|u_0\|_{H^s})$ and a unique local solution*

$$u = (\mathcal{A}_T^\infty(u_0))(t) \in C([0,T]; H^\infty)$$

of the Cauchy problem (1.1). (*Denote the solution operator $\mathcal{A}_T^s$ by $(\mathcal{A}_T^s(u_0))(t) = u(t)$, where $u(t)$ is the solution of* (1.1) *with initial data $u_0 \in H^s$.*) *Moreover, we have*

$$\|(\mathcal{A}_T^\infty(u_0))(t)\|_{H^s} \le C(T, s, \|u_0\|_{H^s}) \quad \text{for any } T \in \mathbb{R},\ s > \frac{n}{2}.$$

(b) *Let $u_0 \in H^s$. Then the mapping*

$$\mathcal{A}_T^\infty : H^\infty \to C([-T,T] : H^\infty)$$

extends uniquely to a continuous mapping

$$\mathcal{A}_T^s : H^s \to C([-T,T] : H^s).$$

2 Definitions and Notations

Define

$$x = (\widetilde{x}, \widetilde{x}') \in \mathbb{R} \times \mathbb{R}^{n-1}, \quad (\widetilde{x}, \widetilde{x}') \to (\widetilde{\xi}, \widetilde{\xi}') = \xi \in \mathbb{R} \times \mathbb{R}^{n-1}.$$

The norm standard space $X_{s,b}$ for the Schrödinger equation is (see defined [2, 7, 9]),

$$\|u\|_{X_{s,b}} = \|\langle\xi\rangle^s \langle\tau - \phi(\xi)\rangle^b \widehat{u}(\xi,\tau)\|_{L^2_{\xi\in\mathbb{R}^n} L^2_{\tau\in\mathbb{R}}}. \tag{2.1}$$

Define the norm of dyadic $X_{s,b}$ spaces as follows

$$\|u\|_{F_{s,\frac{1}{2}}} = \|S(-t)u\|_{H^s_x B^{\frac{1}{2}}_{(2,1)}} = \|\langle\xi\rangle^s \langle\tau - \phi(\xi)\rangle^{\frac{1}{2}} \widehat{u}(\xi,\tau)\|_{L^2_\xi L^2_\tau}, \tag{2.2}$$

$$\|u\|_{F_{s,-\frac{1}{2}}} = \|S(-t)u\|_{H^s_x B^{-\frac{1}{2}}_{(2,1)}} = \|\langle\xi\rangle^s \langle\tau - \phi(\xi)\rangle^{-\frac{1}{2}} \widehat{u}(\xi,\tau)\|_{L^2_\xi L^2_\tau}. \tag{2.3}$$

Denote $\widehat{u}(\xi,\tau) = \mathcal{F}u(x,t)$ by the Fourier transform in t and x of u and $\mathcal{F}_{(\cdot)}u$ by the Fourier transform in the $(\cdot)$ variable. Let the phase function $\phi(\xi) = |\xi|^2$ or $-|\xi|^2$ (sometimes, we also let $\phi(\xi) = |\xi|^2$ and $\overline{\phi}(\xi) = -|\xi|^2$).

Then, we give definitions of some dyadic spaces, the properties of these spaces can be found in [5]. First, we define the dyadic decomposition. Let $\eta : \mathbb{R}^n \to [0,1]$ denote an even smooth function supported in $\left[-\frac{8}{5}, \frac{8}{5}\right]$ and equal to 1 in $\left[-\frac{5}{4}, \frac{5}{4}\right]$. For $j \in \mathbb{Z}$, let $\chi_j(|\xi|) = \eta\left(\frac{|\xi|}{2^j}\right) - \eta\left(\frac{|\xi|}{2^{j-1}}\right)$, and

$$\chi_{[j_1,j_2]} = \sum_{j=j_1}^{j_2} \chi_j \quad \text{and} \quad \chi_{\le j_2} = \sum_{j=0}^{j_2} \chi_j.$$

For simplicity of notation, let $\eta_j = \chi_j$ if $j \ge 1$ and $\eta_0 = 1 - \sum\limits_{j=0}^{+\infty} \eta_j$. Also, for $l_1 \le l_2 \in \mathbb{Z}_+$,

$$\eta_{[l_1,l_2]} = \sum_{l=l_1}^{l_2} \eta_l \quad \text{and} \quad \eta_{\le l_2} = \sum_{l=0}^{l_2} \eta_1.$$

For any $k \in \mathbb{Z}_+$, we define the operators P_k, $P_{\le k}$, $P_{\ll k}$ with respective to the variable x by the formulas

$$\widehat{P_k u}(\xi) = \eta_k(|\xi|)\widehat{u}(\xi), \quad \widehat{P_{\le k} u}(\xi) = \eta_{\le k}(|\xi|)\widehat{u}(\xi), \quad \widehat{P_{\ll k} u}(\xi) = \eta_{\ll k}(|\xi|)\widehat{u}(\xi).$$

For $l \in \mathbb{Z}$, let $I_l = \{\xi \in \mathbb{R}^n : |\xi| \in [2^{l-1}, 2^{l+1}]\}$. For $l \in \mathbb{Z}_+$, let $\widetilde{I}_l = I_l$ if $k \ge 1$ and $\widetilde{I}_0 = [-2, 2]$. For $k \in \mathbb{Z}_+$ and $j \ge 0$, let

$$D_{k,j} = \{(\xi,\tau) \in \mathbb{R}^n \times \mathbb{R} : \xi \in \widetilde{I}_k, \tau - \phi(\xi) \in \widetilde{I}_j\}.$$

We define first the Banach spaces $X_k = X_k(\mathbb{R}^n \times \mathbb{R})$ for $k \in \mathbb{Z}_+$,

$$X_k(\mathbb{R}^n \times \mathbb{R}) = \Big\{ f(\xi,\tau) \in L^2(\mathbb{R}^n \times \mathbb{R}) : f \text{ is supported in } \mathbb{R} \times \widetilde{I}_k \text{ and}$$
$$\|f\|_{X_k} = \sum_{j=0}^{+\infty} 2^{\frac{j}{2}} \beta_{(j,k)} \|\eta_j(\tau - \phi(\xi)) f(\xi,\tau)\|_{L^2_{\xi,\tau}} < \infty \Big\}, \tag{2.4}$$

where $\beta_{(j,k)}$ satisfies

$$\beta_{(j,k)} := \begin{cases} 1+T^{-\frac{\widetilde{\theta}}{2}}, & \text{if } 2^j \geq T^{-2\widetilde{\theta}}2^{\widetilde{\theta}k}, \\ 1+\dfrac{2^{\frac{j}{2}}}{T^{-\widetilde{\theta}}2^{\frac{\widetilde{\theta}k}{2}}}, & \text{if } 2^j \geq 2^{2k} \gtrsim T^{-2\widetilde{\theta}}2^{\widetilde{\theta}k}, \\ 1, & \text{if } 2^j \leq T^{-2\widetilde{\theta}}2^{\widetilde{\theta}k} \quad \text{for some fixed } \widetilde{\theta}, T>0; \\ 1, & \text{if } k \leq 1. \end{cases} \tag{2.5}$$

Here $\widetilde{\theta}, T$ will be chosen in the later.

For $k \in \mathbb{Z}_+$ define the frequency localized initial data spaces

$$E_k = \{\phi : \mathbb{R} \to \mathbb{R} : \mathcal{F}(\phi) = \eta_k(\xi)\mathcal{F}(\phi) \text{ and } \|\phi\|_{E_k} = \|\widehat{\phi}\|_{L^2_\xi} < \infty\}. \tag{2.6}$$

The corresponding frequency localized energy spaces for the solutions are

$$C_0(\mathbb{R} : E_k) = \{u_k \in C(\mathbb{R} : E_k) : u_k \text{ is supported in } \mathbb{R} \times [-4,4]\}.$$

At frequency $T^{-\widetilde{\theta}}2^{\widetilde{\theta}k}$ we will use the $X^{s,b}$ structure given by the X_k, on the $T^{\widetilde{\theta}}2^{-\widetilde{\theta}k}$ time scale. For $k \in \mathbb{Z}_+$ we define the normed spaces

$$F_k = \Big\{u_k \in C_0(\mathbb{R} : E_k) : \|u_k\|_{F_k} = \sup_{t_k \in \mathbb{R}} \|\mathcal{F}[u_k \cdot \eta_0(T^{-\widetilde{\theta}}2^{\widetilde{\theta}k}(t-t_k))]\|_{X_k} < \infty\Big\}. \tag{2.7}$$

For $k \in \mathbb{Z}_+$ we define the normed spaces $N_k = C_0(\mathbb{R} : E_k)$, which are used to measure the frequency $T^{\widetilde{\theta}}2^{-\widetilde{\theta}k}$ part of the nonlinear term, with norms

$$\|f_k\|_{N_k} = \sup_{t_k \in \mathbb{R}} \|(\tau - \phi(\xi) + \mathrm{i}T^{-\widetilde{\theta}}2^{\widetilde{\theta}k})^{-1} \cdot \mathcal{F}(f_k \cdot \eta_0(T^{-\widetilde{\theta}}2^{\widetilde{\theta}k}(t-t_k)))\|_{X_k}. \tag{2.8}$$

In the following proof, we will use the localized spaces defined above, for any time $T \in (0,1]$ we define the normed spaces

$$\begin{cases} F_k(T) = \Big\{u_k \in C([-T,T] : E_k) : \|u_k\|_{F_k(T)} = \inf\limits_{\widetilde{u}_k = u_k \text{ in } \mathbb{R}\times[-T,T]} \|\widetilde{u}_k\|_{F_k} < \infty\Big\}, \\ N_k(T) = \Big\{f_k \in C([-T,T] : E_k) : \|f_k\|_{N_k(T)} = \inf\limits_{\widetilde{f}_k = f_k \text{ in } \mathbb{R}\times[-T,T]} \|\widetilde{f}_k\|_{N_k} < \infty\Big\}. \end{cases} \tag{2.9}$$

Next, we will assemble these dyadic function spaces above using a Littlewood-Paley decomposition to obtain the global function spaces. For $u \in C([-T,T] : H^s)$ and $s \geq 0$, we define

$$\|u\|^2_{\mathbf{E}^s(T)} = \|P_{\leq 0}(u(0))\|^2_{H^s} + \sum_{k\geq 1} \sup_{t_k \in [-T,T]} 2^{2sk}\|P_k(u(t_k))\|^2_{E_k}. \tag{2.10}$$

Finally, the $X_{s,b}$- type control of the solutions, respectively, the nonlinearity is achieved using the normed spaces

$$\begin{aligned}
&\mathbf{F}^s(T)=\Big\{u\in C([-T,T]:H^s):\|u\|^2_{\mathbf{F}^s(T)}=\sum_{k=0}^{\infty}2^{2sk}\|P_k(u)\|^2_{F_k(T)}<\infty\Big\},\\
&\mathbf{N}^s(T)=\Big\{f\in C([-T,T]:H^s):\|f\|^2_{\mathbf{N}^s(T)}=\sum_{k=0}^{\infty}2^{2sk}\|P_k(f)\|^2_{N_k(T)}<\infty\Big\},\\
&\mathbf{F}^{s-r}(T)=\Big\{u\in C([-T,T]:H^{s-r}):\|u\|^2_{\mathbf{F}^{s-r}(T)}=\sum_{k=0}^{\infty}2^{2(s-r)k}\|P_k(u)\|^2_{F_k(T)}<\infty\Big\},\\
&\mathbf{N}^{s-r}(T)=\Big\{f\in C([-T,T]:H^{s-r}):\|f\|^2_{\mathbf{N}^{s-r}(T)}=\sum_{k=0}^{\infty}2^{2(s-r)k}\|P_k(f)\|^2_{N_k(T)}<\infty\Big\}.
\end{aligned}\tag{2.11}$$

For any $k\in\mathbb{Z}_+$, we define the set S_k of k-acceptable time multiplication factors

$$S_k=\Big\{m_k:\mathbb{R}\to\mathbb{R}:\|m_k\|_{S_k}=\sum_{j=0}^{10}2^{-jk}\|\partial^j m_k\|_{L^\infty}<\infty\Big\}.\tag{2.12}$$

Direct estimates using the definitions and Lemma 4.1 show that for $T\in(0,1]$,

$$\begin{cases}\Big\|\sum_{k\in\mathbb{Z}}m_k(t)\cdot P_k(u)\Big\|_{\mathbf{F}^s(T)}\lesssim\Big(\sup_{k\in\mathbb{Z}}\|m_k\|_{S_k}\Big)\cdot\|u\|_{\mathbf{F}^s(T)},\\
\Big\|\sum_{k\in\mathbb{Z}}m_k(t)\cdot P_k(u)\Big\|_{\mathbf{N}^s(T)}\lesssim\Big(\sup_{k\in\mathbb{Z}}\|m_k\|_{S_k}\Big)\cdot\|u\|_{\mathbf{N}^s(T)},\\
\Big\|\sum_{k\in\mathbb{Z}}m_k(t)\cdot P_k(u)\Big\|_{\mathbf{E}^s(T)}\lesssim\Big(\sup_{k\in\mathbb{Z}}\|m_k\|_{S_k}\Big)\cdot\|u\|_{\mathbf{E}^s(T)}.\end{cases}\tag{2.13}$$

Denote $A\sim B$ by the statement: $A\le C_1B$ and $B\le C_1A$ for some constant $C_1>0$, and $A\ll B$ by the statement: $A\le\frac{1}{C_2}B$ for some large enough constant $C_2>0$, and $A\lesssim B$ by the statement: $A\le C_3B$ for some constant $C_3>0$. We use $a+$ and $a-$ to denote expressions of the form $a+\varepsilon$ and $a-\varepsilon$, where $0<\varepsilon\ll1$.

Denote the convolution integral $\int_\star \mathrm{d}\delta$ by the form

$$\int_{\xi=\xi_1+\xi_2+\xi_3;\tau=\tau_1+\tau_2+\tau_3}\mathrm{d}\tau_1\mathrm{d}\tau_2\mathrm{d}\tau_3\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3.$$

First, we introduce some variables for convenience

$$\widetilde{\sigma}=\tau\pm|\xi|^2,\quad \widetilde{\sigma}_l=\tau_l\pm|\xi_l|^2,\quad l=1,2,3,\tag{2.14}$$

$$\overline{\sigma}=\tau+|\xi|^2,\quad \sigma=\tau-|\xi|^2,\quad \overline{\sigma}_l=\tau_l+|\xi_l|^2,\quad \sigma_l=\tau_l-|\xi_l|^2,\quad l=1,2,3.\tag{2.15}$$

Define $N_j:=|\xi_j|$ and $L_j:=|\widetilde{\sigma}_j|$, we adopt the notation that

$$1\le\text{soprano},\text{alto},\text{tenor},\text{baritone}\le4$$

are the distinct indices such that

$$N_{\text{soprano}}\ge N_{\text{alto}}\ge N_{\text{tenor}}\ge N_{\text{baritone}}$$

are the highest, second highest, third highest, and fourth highest values of the frequencies N_1,N_2,N_3,N_4, respectively. Since $\xi_1+\xi_2+\xi_3+\xi_4=0$, we must have $N_{\text{soprano}}\sim N_{\text{alto}}$. Similarly define $L_{\text{soprano}}\ge L_{\text{alto}}\ge L_{\text{tenor}}\ge L_{\text{baritone}}$ whenever $L_1,L_2,L_3,L_4>0$.

Let $\psi\in C_0^\infty(\mathbb{R})$ with $\psi=1$ on $[-\frac12,\frac12]$ and supp $\psi\subset[-1,1]$. Denote $\psi_\delta(\cdot)=\psi(\delta^{-1}(\cdot))$ for some $\delta\in\mathbb{R}\setminus\{0\}$.

3 Multilinear Estimates

In Sections 5 and 7, the trilinear and multilinear estimates are obtained by using Tao's $[k;Z]$-multiplier method (see [15]). We firstly list some useful notations and properties for multi-linear expressions. Let Z be any Abelian additive group with an invariant measure $\mathrm{d}\xi$. For any integer $k \geq 2$, we denote $\Gamma_k(Z)$ by the "hyperplane"

$$\Gamma_k(Z) = \{(\xi_1, \cdots, \xi_k) \in Z^k : \xi_1 + \cdots + \xi_k = 0\},$$

which is endowed with the measure

$$\int_{\Gamma_k(Z)} f = \int_{Z^{k-1}} f(\xi_1, \cdots, \xi_{k-1}, -\xi_1 - \cdots - \xi_{k-1}) \mathrm{d}\xi_1 \cdots \mathrm{d}\xi_{k-1},$$

and define a $[k;Z]$-multiplier to be any function m: $\Gamma_k(Z) \to \mathbb{C}$. If m is a $[k;Z]$-multiplier, we define $\|m\|_{[k;Z]}$ to be the best constant, such that the inequality

$$\Big| \int_{\Gamma_k(Z)} m(\xi) \prod_{j=1}^{k} f_j(\xi_j) \Big| \leq \|m\|_{[k;Z]} \prod_{j=1}^{k} \|f_j\|_{L_2(Z)}$$

holds for all test functions f_j defined on Z. It is clear that $\|m\|_{[k;Z]}$ determines a norm on m, for test functions at least. We are interested in obtaining the good boundedness on the norm. We will also define $\|m\|_{[k;Z]}$ in situations when m is defined on all of Z^k by restricting to $\Gamma_k(Z)$.

We give some properties of $\|m\|_{[k;Z]}$, especially for the case $k = 3$. This corresponds to the bilinear $X_{s,b}$ estimates of Schrödinger equation since multilinear estimates can be reduced to some bilinear estimates (we can find it later).

Let

$$\xi_1 + \xi_2 + \xi_3 = 0, \quad \tau_1 + \tau_2 + \tau_3 = 0, \tag{3.1}$$

$$\widetilde{\sigma}_j = \tau_j + h_j(\xi_j), \quad h_j(\xi_j) = \pm|\xi_j|^2, \quad j = 1, 2, 3. \tag{3.2}$$

Then we will study the problem of obtaining

$$\|m((\xi_1, \tau_1), (\xi_2, \tau_2), (\xi_3, \tau_3))\|_{[3, \mathbb{R}^n \times \mathbb{R}]} \lesssim 1, \tag{3.3}$$

where $m((\xi_1, \tau_1), (\xi_2, \tau_2), (\xi_3, \tau_3))$ is some $[k;Z]$-multiplier in $\Gamma_3(\mathbb{R}^n \times \mathbb{R})$.

From (3.1) and (3.2), it follows that

$$\widetilde{\sigma}_1 + \widetilde{\sigma}_2 + \widetilde{\sigma}_3 = h(\xi_1, \xi_2, \xi_3). \tag{3.4}$$

By symmetry, there are only two possibilities for the h_j: The $(+++)$ case

$$h_1(\xi) = h_2(\xi) = h_3(\xi) = |\xi|^2; \tag{3.5}$$

and the $(++-)$ case

$$h_1(\xi) = h_2(\xi) = |\xi|^2; \quad h_3(\xi) = -|\xi|^2. \tag{3.6}$$

Of the two cases, the $(+++)$ case is substantially easier, because the resonance function

$$h(\xi_1, \xi_2, \xi_3) := |\xi_1|^2 + |\xi_2|^2 + |\xi_3|^2 \tag{3.7}$$

does not vanish except at the origin. The $(++-)$ case is more delicate, because the resonance function,

$$h(\xi_1,\xi_2,\xi_3) := |\xi_1|^2+|\xi_2|^2-|\xi_3|^2 \tag{3.8}$$

vanishes when ξ_1 and ξ_2 are orthogonal. Notice that for ξ_1, ξ_2, $\xi_3\in\mathbb{R}^2$, the resonance identity is given by

$$|h(\xi_1,\xi_2,\xi_3)| = ||\xi_1|^2+|\xi_2|^2-|\xi_3|^2| = 2|\xi_1\cdot\xi_2| \sim |\xi_1||\xi_2|\Big|\frac{\pi}{2}-\angle(\xi_1,\xi_2)\Big|.$$

In particular, we may assume

$$|h(\xi_1,\xi_2,\xi_3)| \lesssim |\xi_1||\xi_2|, \tag{3.9}$$

and that

$$\angle(\xi_1,\xi_2) = \frac{\pi}{2}+O\Big(\frac{|h(\xi_1,\xi_2,\xi_3)|}{|\xi_1||\xi_2|}\Big). \tag{3.10}$$

We assume that $|\xi_j|\sim N_j$, $|\widetilde{\sigma}_j|\sim L_j$ and $|h(\xi_1,\xi_2,\xi_3)|\sim H$. Where N_j, L_j and H are presumed to be dyadic, i.e., these variables range over numbers of form $2^k(k\in\mathbb{Z})$. It is convenient to define $N_{\max}\ge N_{\text{med}}\ge N_{\min}$ to be the maximum, median and minimum of N_1, N_2, N_3. Similarly define $L_{\max}\ge L_{\text{med}}\ge L_{\min}$ whenever $L_1,L_2,L_3>0$.

Then we estimate the following expression to replace (3.3),

$$\|m((N_1,L_1),(N_2,L_2),(N_3,L_3))X_{N_1,N_2,N_3;H;L_1,L_2,L_3}\|_{[3,\mathbb{R}^n\times\mathbb{R}]}\lesssim 1, \tag{3.11}$$

where $X_{N_1,N_2,N_3;H;L_1,L_2,L_3}$ is the multiplier

$$X_{N_1,N_2,N_3;H;L_1,L_2,L_3}(\xi,\tau) := \chi_{|h(\xi)|\sim H}\prod_{j=1}^{3}\chi_{|\xi_j|\sim N_j}\chi_{|\widetilde{\sigma}_j|\sim L_j}. \tag{3.12}$$

From the identities (3.1) and (3.4), $X_{N_1,N_2,N_3;H;L_1,L_2,L_3}$ vanishes unless

$$N_{\max}\sim N_{\text{med}} \tag{3.13}$$

and

$$L_{\max}\sim\max(H,L_{\text{med}}). \tag{3.14}$$

Therefore, we only need to estimate

$$\|X_{N_1,N_2,N_3;H;L_1,L_2,L_3}\|_{[3,\mathbb{R}^n\times\mathbb{R}]}. \tag{3.15}$$

Then we have the following lemma about the boundedness of (3.15).

Lemma 3.1 (see [15]) *Let* $H,N_1,N_2,N_3,L_1,L_2,L_3>0$ *obey* (3.13)–(3.14).

• *For the* $(+++)$ *case, let the dispersion relations be given by* (3.5), *then* $H\sim N_{\max}^2$. *It follows that*

$$(3.15)\lesssim L_{\min}^{\frac12}N_{\max}^{-\frac12}N_{\min}^{\frac{n-1}{2}}\min(N_{\max}N_{\min},L_{\text{med}})^{\frac12}. \tag{3.16}$$

• *For the* $(++-)$ *case, let the dispersion relations be given by* (3.6)*, then* $H \lesssim N_1N_2$*. It follows that*

– *The* $((++)$ *case). If* $N_1 \sim N_2 \gg N_3$*, then* (3.15) *vanishes unless* $H \sim N_1^2$*, in which case one has*

$$(3.15) \lesssim L_{\min}^{\frac{1}{2}} N_{\max}^{-\frac{1}{2}} N_{\min}^{\frac{n-1}{2}} \min(N_{\max}N_{\min}, L_{\mathrm{med}})^{\frac{1}{2}}. \tag{3.17}$$

– *The* $((+-)$ *coherence). If we have*

$$N_1 \sim N_3 \gg N_2; \quad H \sim L_2 \gg L_1, L_3, N_2^2, \tag{3.18}$$

then we have

$$(3.15) \lesssim L_{\min}^{\frac{1}{2}} N_{\max}^{-\frac{1}{2}} N_{\min}^{\frac{n-1}{2}} \min\Big(H, \frac{H}{N_{\min}^2} L_{\mathrm{med}}\Big)^{\frac{1}{2}}. \tag{3.19}$$

Similarly with the roles of 1 *and* 2 *reversed.*

– *In other cases, we have*

$$(3.15) \lesssim L_{\min}^{\frac{1}{2}} N_{\max}^{-\frac{1}{2}} N_{\min}^{\frac{n-1}{2}} \min(H, L_{\mathrm{med}})^{\frac{1}{2}} \min\Big(1, \frac{H}{N_{\min}^2}\Big)^{\frac{1}{2}}. \tag{3.20}$$

Lemma 3.2 (Comparison principle) (see [15]) *If* m *and* M *are* $[k;Z]$ *multipliers and satisfy* $|m(\xi)| \le |M(\xi)|$ *for all* $\xi \in \Gamma_k(Z)$*. Then* $\|m\|_{[k;Z]} \le \|M\|_{[k;Z]}$*. Also, if* m *is a* $[k,Z]$ *multiplier, and* $a_1, \cdots, a_k$ *are functions from* Z *to* $\mathbb{R}$*, then*

$$\Big\| m(\xi) \prod_{j=1}^{k} a_j(\xi_j) \Big\|_{[k;Z]} \le \|m\|_{[k;Z]} \prod_{j=1}^{k} \|a_j\|_{L_\infty}. \tag{3.21}$$

Lemma 3.3 (Composition and TT^*) (see [15]) *If* $k_1, k_2 \ge 1$ *and* m_1*,* m_2 *are functions on* Z^{k_1} *and* Z^{k_2}*, respectively, then*

$$\begin{aligned}&\|m_1(\xi_1, \cdots, \xi_{k_1}) m_2(\xi_{k_1+1}, \cdots, \xi_{k_1+k_2})\|_{[k_1+k_2;Z]} \\ &\le \|m_1(\xi_1, \cdots, \xi_{k_1})\|_{[k_1+1;Z]} \|m_2(\xi_1, \cdots, \xi_{k_2})\|_{[k_2+1;Z]}.\end{aligned} \tag{3.22}$$

As a special case, for all functions $m : Z^k \to \mathbb{R}$*, we have the* TT^* *identity*

$$\|m(\xi_1, \cdots, \xi_k)\overline{m(-\xi_{k+1}, \cdots, -\xi_{2k})}\|_{[2k;Z]} = \|m(\xi_1, \cdots, \xi_k)\|_{[k+1;Z]}^2. \tag{3.23}$$

4 Linear Estimates, Trilinear Estimates, Energy Estimates, Well-posedness of DNLS Equation

In this section, we give the proof of Theorem 1.1. In Theorem 1.1, we show that the Cauchy problem (1.1) is locally well-posed in space H^s for any large initial data.

Lemma 4.1 *If* $\mathcal{F}u_k \in X_k$ *with* $k \ge 1$*, then*

$$\sum_{2^j \ge T^{-\widetilde{\theta}} 2^{\widetilde{\theta} k}} \beta_{(j,k)} 2^{\frac{j}{2}} \|\eta_j(\tau - \phi(\xi)) \mathcal{F}(\eta_0(T^{-\widetilde{\theta}} 2^{\widetilde{\theta} k} t) u_k)(\xi, \tau)\|_{L_\tau^2 L_\xi^2}$$

$$+\sum_{2^j\le T^{-\widetilde{\theta}}2^{\widetilde{\theta}k}} T^{-\frac{\widetilde{\theta}}{2}}2^{-\frac{\widetilde{\theta}k}{2}}\|\eta_j(\tau-\phi(\xi))\mathcal{F}(T^{-\widetilde{\theta}}\eta_0(2^{\widetilde{\theta}k}t)u_k)(\xi,\tau)\|_{L^2_\tau L^2_\xi}$$
$$\lesssim \sum_{j=0}^{+\infty} 2^{\frac{j}{2}}\beta_{(j,k)}\|\eta_j(\tau-\phi(\xi))\widehat{u}_k(\xi,\tau)\|_{L^2_{\xi,\tau}}. \tag{4.1}$$

Proof In fact, using the fact

$$\|f(\lambda t)\|_{B^s_{2,1}} \lesssim (\lambda^{-\frac{1}{2}}+\lambda^{s-\frac{1}{2}})\|f(t)\|_{B^s_{2,1}}, \quad \lambda>0, \tag{4.2}$$

we can have the results. This completes the proof of Lemma 4.1.

Lemma 4.2 (Global linear estimate) (see [5]) *Assume $T\in(0,1]$ and $s\in\mathbb{R}$, $u,f\in C([-T,T):H^\infty)$,*

$$u_t-\mathrm{i}\Delta u=f \quad \text{on } \mathbb{R}^n\times(-T,T). \tag{4.3}$$

Then

$$\|u\|_{\mathbf{F}^s(T)}\lesssim \|u\|_{\mathbf{E}^s(T)}+\|f\|_{\mathbf{N}^s(T)}. \tag{4.4}$$

Proof Using (4.2) and the arguments in [5], we can obtain the results.
This completes the proof of Lemma 4.2.

Lemma 4.3 *Let $T\in(0,1]$ and $s\in\mathbb{R}$, and $u\in\mathbf{F}^s(T)$, then*

$$\sup_{t\in[-T,T]}\|u(t)\|_{H^s}\lesssim\|u\|_{\mathbf{F}^s(T)}. \tag{4.5}$$

Proof From the definition of $\mathbf{F}^s(T)$, we can easily obtain the results.

Theorem 4.1 (Trilinear estimates) *For $T\in(0,1)$, we have for $s>\frac{n}{2}$ and some $\theta>0$,*

$$\|u_1\overline{u_2}(\overrightarrow{\gamma}\cdot\nabla u_3)\|_{\mathbf{N}^s(T)}\le CT^\theta\|u_1\|_{\mathbf{F}^s(T)}\|u_2\|_{\mathbf{F}^s(T)}\|u_3\|_{\mathbf{F}^s(T)}, \tag{4.6}$$
$$\|u_1(\overrightarrow{\lambda}\cdot\nabla\overline{u}_2)u_1\|_{\mathbf{N}^s(T)}\le CT^\theta\|u_1\|_{\mathbf{F}^s(T)}\|u_2\|_{\mathbf{F}^s(T)}\|u_3\|_{\mathbf{F}^s(T)}, \tag{4.7}$$
$$\begin{aligned}\|u_1\overline{u}_2(\overrightarrow{\gamma}\cdot\nabla u_3)\|_{\mathbf{N}^{\beta_n}(T)}\le{}& CT^\theta\|u_1\|_{\mathbf{F}^{\beta_n}(T)}\|u_2\|_{\mathbf{F}^s(T)}\|u_3\|_{\mathbf{F}^s(T)}\\&+CT^\theta\|u_1\|_{\mathbf{F}^s(T)}\|u_2\|_{\mathbf{F}^{\beta_n}(T)}\|u_3\|_{\mathbf{F}^s(T)}\\&+CT^\theta\|u_1\|_{\mathbf{F}^s(T)}\|u_2\|_{\mathbf{F}^s(T)}\|u_3\|_{\mathbf{F}^{\beta_n}(T)},\end{aligned} \tag{4.8}$$
$$\begin{aligned}\|u_1(\overrightarrow{\lambda}\cdot\nabla\overline{u}_2)u_1\|_{\mathbf{N}^{\beta_n}(T)}\le{}& CT^\theta\|u_1\|_{\mathbf{F}^{\beta_n}(T)}\|u_2\|_{\mathbf{F}^s(T)}\|u_3\|_{\mathbf{F}^s(T)}\\&+CT^\theta\|u_1\|_{\mathbf{F}^s(T)}\|u_2\|_{\mathbf{F}^{\beta_n}(T)}\|u_3\|_{\mathbf{F}^s(T)}\\&+CT^\theta\|u_1\|_{\mathbf{F}^s(T)}\|u_2\|_{\mathbf{F}^s(T)}\|u_3\|_{\mathbf{F}^{\beta_n}(T)},\end{aligned} \tag{4.9}$$

where the constant C is independent of $\|u_0\|_{H^s}$ and T depends on $\|u_0\|_{H^s}$; $\beta_n>\frac{n}{2}+1$ is large enough number obtained in [4], here the Cauchy problem (1.1) is locally well-posed in H^{β_n} with $\beta_n>\frac{n}{2}+1$ for any large initial data.

Theorem 4.2 (Energy estimates) *Assume $s>\frac{n}{2}$ and β_n is defined in Theorem 4.1. Let $T\in(0,1)$, and u be a solution of the Cauchy problem (1.1) in $C([0,T];H^\infty)$. Then for some $\theta>0$,*

$$\|u(t)\|^2_{\mathbf{E}^s(T)}\le C\|u_0\|^2_{H^s}+CT^\theta\|u(t)\|^4_{\mathbf{F}^s(T)}, \tag{4.10}$$

$$\|u(t)\|^2_{\mathbf{E}^{s-\widetilde{\theta}}(T)} \le C\|u_0\|^2_{H^{s-\widetilde{\theta}}} + CT^{\theta}\|u(t)\|^2_{\mathbf{F}^{s-\widetilde{\theta}}(T)}\|u(t)\|^2_{\mathbf{F}^s(T)}, \tag{4.11}$$

$$\|u(t)\|^2_{\mathbf{E}^{\beta_n}(T)} \le C\|u_0\|^2_{H^{\beta_n}} + CT^{\theta}\|u(t)\|^2_{\mathbf{F}^{\beta_n}(T)}\|u(t)\|^2_{\mathbf{F}^s(T)}, \tag{4.12}$$

where the constant C is independent of $\|u_0\|_{H^s}$ and T depends on $\|u_0\|_{H^s}$.

Lemma 4.4 (see [4, 10]) *Let $s > \frac{n}{2}$. Assume $u_0 \in H^\infty$. Then there is $T = T(\|u_0\|_{H^s}) \in (0,1]$ and a unique solution $u = \mathcal{A}^\infty(u_0) \in C([-T,T]; H^\infty)$ of the initial value problem* (1.1). *In addition, for any $s > \frac{n}{2}$,*

$$\sup_{t\in[-T,T]} \|u(t)\|_{H^s} \le C(s, \|u_0\|_{H^s}, T). \tag{4.13}$$

Proof By Lemma 4.2 and Theorems 4.1–4.2, similarly with the proof (a) in Theorem 1.1 in the following, we can obtain a priori estimate for some $T = T(\|u_0\|_{H^s}) \in \mathbb{R}$,

$$\sup_{t\in(0,T)} \|u(t)\|_{H^s} \le C\|u_0\|_{H^s}, \tag{4.14}$$

where the constant C is independent of $\|u_0\|_{H^s}$ and T depends on $\|u_0\|_{H^s}$ when $u(t)$ is a solution of the initial value problem (1.1). Moreover, from [4], it follows that the Cauchy problem (1.1) is locally well-posed in H^{β_n} with $\beta_n > \frac{n}{2} + 1$ for any large initial data. Thus, we can obtain Lemma 4.4.

In Sections 6–7, we will give the proofs of Theorems 4.1–4.2, respectively. Now we turn to the proof of Theorem 1.1. First, we prove that if $T \in (0,1]$ and $u \in C([-T,T] : H^\infty)$ is a solution of (1.1) with $u_0 \in H^{\beta_n}$ with $\beta_n > \frac{n}{2} + 1$, then

$$\sup_{t\in[-T,T]} \|u(t)\|_{H^{\beta_n}} \lesssim \|u\|_{\mathbf{F}^{\beta_n}(T)} \lesssim \|u_0\|_{H^{\beta_n}}. \tag{4.15}$$

We first use a continuity argument to establish an $\mathbf{F}^s$ bound on u in the interval $[-T,T]$. It follows from Lemma 4.2 and Theorems 4.1–4.2 that for any $T' \in [0,T]$, we have for $s > \frac{n}{2}$,

$$\begin{cases} \|u\|_{\mathbf{F}^s(T')} \lesssim \|u\|_{\mathbf{E}^s(T')} + \||u|^2(\overrightarrow{\gamma}\cdot\nabla u) + u^2(\overrightarrow{\lambda}\cdot\nabla\overline{u})\|_{\mathbf{N}^s(T')}; \\ \||u|^2(\overrightarrow{\gamma}\cdot\nabla u) + u^2(\overrightarrow{\lambda}\cdot\nabla\overline{u})\|_{\mathbf{N}^s(T')} \le CT^\theta\|u\|^3_{\mathbf{F}^s(T')}; \\ \|u\|^2_{\mathbf{E}^s(T')} \le C\|u_0\|^2_{H^s} + CT^\theta\|u\|^4_{\mathbf{F}^s(T')}, \end{cases} \tag{4.16}$$

where the constant C is independent of $\|u_0\|_{H^s}$ and T' depends on $\|u_0\|_{H^s}$.

We denote $X(T') = \|u\|_{\mathbf{E}^s(T')} + \||u|^2(\overrightarrow{\gamma}\cdot\nabla u) + u^2(\overrightarrow{\lambda}\cdot\nabla\overline{u})\|_{\mathbf{N}^s(T')}$ and eliminate $\|u\|_{\mathbf{F}^s(T')}$ to obtain

$$X(T')^2 \lesssim \|u_0\|^2_{H^s} + X(T')^4 + X(T')^6. \tag{4.17}$$

Assuming that $X(T')$ is continuous and satisfies

$$\lim_{T'\to 0} X(T') \lesssim \|u_0\|_{H^s}. \tag{4.18}$$

Using (4.16), we have

$$\|u\|_{\mathbf{F}^s(T)} \lesssim \|u_0\|_{H^s}. \tag{4.19}$$

To obtain (4.18), we first show that for $u \in C(-T,T;H^s)$ the mapping $T' \to \|u\|_{\mathbf{E}^s(T')}$ is increasing and continuous on the interval $[-T,T]$ and

$$\lim_{T'\to 0} \|u\|_{\mathbf{E}^s(T')} \le \|u_0\|_{H^s}. \tag{4.20}$$

The continuous property of $\||u|^2(\overrightarrow{\gamma}\cdot\nabla u)+u^2(\overrightarrow{\lambda}\cdot\nabla\overline{u})\|_{\mathbf{N}^s(T')}$ is obtained by applying the following lemma to $f=(|u|^2(\overrightarrow{\gamma}\cdot\nabla u)+u^2(\overrightarrow{\lambda}\cdot\nabla\overline{u}))$.

Lemma 4.5 (see [5]) *Assume $T\in(0,1)$ and $f\in C([0,T];H^\infty)$. Then the mapping $T'\to \|f\|_{\mathbf{N}^s(T')}$ is increasing, depends on $\|u_0\|_{H^s}$ and is continuous on the interval $[0,T]$ and*

$$\lim_{T'\to 0}\|f\|_{\mathbf{N}^s(T')}=0. \tag{4.21}$$

To prove (4.15), using Lemma 4.2 and Theorems 4.1–4.2, we have

$$\begin{cases} \|u\|_{\mathbf{F}^{\beta_n}(T')} \lesssim \|u\|_{\mathbf{E}^{\beta_n}(T')} + \||u|^2(\overrightarrow{\gamma}\cdot\nabla u)+u^2(\overrightarrow{\lambda}\cdot\nabla\overline{u})\|_{\mathbf{N}^{\beta_n}(T')};\\ \||u|^2(\overrightarrow{\gamma}\cdot\nabla u)+u^2(\overrightarrow{\lambda}\cdot\nabla\overline{u})\|_{\mathbf{N}^{\beta_n}(T')} \le CT^\theta\|u\|_{\mathbf{F}^{\beta_n}(T')}\|u\|^2_{\mathbf{F}^s(T')};\\ \|u\|^2_{\mathbf{E}^{\beta_n}(T')} \le C\|u_0\|^2_{H^{\beta_n}} + CT^\theta\|u\|^2_{\mathbf{F}^s(T')}\|u\|^2_{\mathbf{F}^{\beta_n}(T')}. \end{cases} \tag{4.22}$$

Using (4.19) and (4.22), we have

$$\|u\|_{\mathbf{F}^{\beta_n}(T)} \lesssim \|u_0\|_{H^{\beta_n}}. \tag{4.23}$$

This completes the proof of Theorem 1.1(a). Next we continuously prove Theorem 1.1(b). Assume $u_0\in H^s$ with $s>\frac{n}{2}$,

$$\{\phi_m : m\in\mathbb{Z}_+\}\subseteq H^\infty \quad\text{and}\quad \lim_{m\to\infty}\phi_m = u_0 \text{ in } H^s.$$

Let u_m be the solutions of the Cauchy problem (1.1) with initial data ϕ_m. It suffices to prove that the sequence $u_m\in C([-T,T]:H^\infty)$ is a Cauchy sequence in $C([-T,T]:H^s)$.

It suffices to prove that for any $\delta>0$, there is M_δ such that

$$\sup_{t\in[-1,1]}\|u_m(t)-u_l(t)\|_{H^s}\le\delta \quad \text{for any } m,l\ge M_\delta. \tag{4.24}$$

For $N\in\mathbb{Z}_+$, let $\phi_m^N = P_{\le N}\phi_m$. We show first that for any $N\in\mathbb{Z}_+$, there is $M_{\delta,N}$ such that

$$\sup_{t\in[-1,1]}\|(\mathcal{A}^\infty\phi_m^N)(t)-(\mathcal{A}^\infty\phi_l^N)(t)\|_{H^s}\le\delta \quad\text{for any } m,l\ge M_{\delta,N}. \tag{4.25}$$

Using $\|(\mathcal{A}^\infty\phi_m^N)(t)\|_{H^{\beta_n}}\le C(N)$, we can easily obtain (4.25) by Lemma 4.6. Moreover, by Lemma 4.6 below, we have

$$\begin{aligned} &\sup_{t\in[-1,1]}\|(\mathcal{A}^\infty\phi_m)(t)-(\mathcal{A}^\infty\phi_m^N)(t)\|_{H^s} \lesssim \|(\mathcal{A}^\infty\phi_m)(t)-(\mathcal{A}^\infty\phi_m^N)(t)\|_{\mathbf{F}^s(1)}\\ &\lesssim C_{(\|\phi_m\|_{H^s},\|\phi_m^N\|_{H^s})}\|\phi_m-\phi_m^N\|_{H^s} + C_{(\|\phi_m\|_{H^s},\|\phi_m^N\|_{H^s})}\|\phi_m^N\|_{H^{s+\tilde\theta}}\|\phi_m-\phi_m^N\|_{H^{s-\tilde\theta}}. \end{aligned} \tag{4.26}$$

Using above, we have for enough large n,m,

$$\sup_{t\in[-1,1]}\|(\mathcal{A}^\infty\phi_n)(t)-(\mathcal{A}^\infty\phi_m)(t)\|_{H^s}\lesssim\|(\mathcal{A}^\infty\phi_n)(t)-(\mathcal{A}^\infty\phi_m)(t)\|_{\mathbf{F}^s(T)}$$

$$
\begin{aligned}
&\lesssim \|(\mathcal{A}^\infty\phi_n)(t)-(\mathcal{A}^\infty\phi_n^N)(t)\|_{\mathbf{F}^s(T)}+\|(\mathcal{A}^\infty\phi_m)(t)-(\mathcal{A}^\infty\phi_m^N)(t)\|_{\mathbf{F}^s(T)}\\
&\quad+\|(\mathcal{A}^\infty\phi_n^N)(t)-(\mathcal{A}^\infty\phi_m^N)(t)\|_{\mathbf{F}^s(T)}\\
&\lesssim C_{\|u_0\|_{H^s}}\|\phi_n-\phi_n^N\|_{H^s}. \qquad (4.27)
\end{aligned}
$$

This completes the proof Theorem 1.1(b).

Lemma 4.6 *Assume* $s>\frac{n}{2}$. *Let* $u_1,u_2\in\mathbf{F}^s(1)$ *be solutions to* (1.1) *with initial data* $\phi_1,\phi_2\in H^\infty$ *satisfying*

$$\|\phi_1\|_{H^s}+\|\phi_2\|_{H^s}\ll 1.$$

Then

$$\|u_1-u_2\|_{\widetilde{\mathbf{F}}^{s-\widetilde{\theta}}}\lesssim C_{\|\phi_1\|_{H^s},\|\phi_2\|_{H^s}}\|\phi_1-\phi_2\|_{H^{s-\widetilde{\theta}}}. \qquad (4.28)$$

$$
\begin{aligned}
\|u_1-u_2\|_{\widetilde{\mathbf{F}}^s}&\lesssim C_{\|\phi_1\|_{H^s},\|\phi_2\|_{H^s}}\|\varphi_2\|_{H^s}\|\phi_1-\phi_2\|_{H^s}\\
&\quad+C_{\|\phi_1\|_{H^s},\|\phi_2\|_{H^s}}\|\varphi_2\|_{H^{s+\widetilde{\theta}}}\|\phi_1-\phi_2\|_{H^{s-\widetilde{\theta}}}. \qquad (4.29)
\end{aligned}
$$

Proof From the arguments in Section 7, we can obtain Lemma 4.6.

5 Dyadic Trilinear Estimates

In this section, by Lemma 4.1, we prove several dyadic bounds which are used in the proof of Theorems 4.1. For the nonlinear terms $(|u|^2(\overrightarrow{\gamma}\cdot\nabla u)+u^2(\overrightarrow{\lambda}\cdot\nabla\overline{u}))$, by symmetry, we only consider $(|u|^2(\overrightarrow{\gamma}\cdot\nabla u))$ and the estimate of $(u^2(\overrightarrow{\lambda}\cdot\nabla\overline{u}))$.

Define

$$
\begin{aligned}
2^{k_{\text{soprano}}}&=N_{\text{soprano}}=\text{Soprano}\{N_1,N_2,N_3,N_4\},\\
2^{k_{\text{alto}}}&=N_{\text{alto}}=\text{Alto}\{N_1,N_2,N_3,N_4\},\\
2^{k_{\text{tenor}}}&=N_{\text{tenor}}=\text{Tenor}\{N_1,N_2,N_3,N_4\},\\
2^{k_{\text{baritone}}}&=N_{\text{baritone}}=\text{Baritone}\{N_1,N_2,N_3,N_4\}.
\end{aligned}
$$

Lemma 5.1 (Trilinear estimate-I) *Let* $s>\frac{n}{2}$. *Assume that* $|\xi_l|\sim N_l=2^{k_l}$, $L_l=|\widetilde{\sigma}_l|=|\tau_l-\phi(\xi_l)|=2^{j_l}$. *Let* $\mathcal{L}_l=\max\{L_l,T^{-\widetilde{\theta}}N_l^{\widetilde{\theta}}\}$, $l=1,2,3,4$. *Let* $N_{\text{soprano}}\gtrsim T^{-\frac14}$. *By symmetry we assume* $N_3\ge N_2\ge N_1$. *Then for small enough* $\varepsilon,\theta>0$,

$$
\begin{aligned}
&\frac{\beta_{(j_4,k_4)}}{\beta_{(j_1,k_1)}\beta_{(j_2,k_2)}\beta_{(j_3,k_3)}}\int_\star\frac{N_{\text{soprano}}^{\widetilde{\theta}}}{\langle N_4\rangle^{\widetilde{\theta}}}\frac{N_{\text{soprano}}\langle\xi_4\rangle^s}{\langle\xi_1\rangle^s\langle\xi_2\rangle^s\langle\xi_3\rangle^s}\frac{f_1(\tau_1,\xi_1)f_2(\tau_2,\xi_2)f_3(\tau_3,\xi_3)f_4(\tau_4,\xi_4)}{\langle\mathcal{L}_1\rangle^{\frac12}\langle\mathcal{L}_2\rangle^{\frac12}\langle\mathcal{L}_3\rangle^{\frac12}\langle\mathcal{L}_4\rangle^{\frac12}}\mathrm{d}\delta\\
&\le\langle N_{\text{tenor}}\rangle^{-\varepsilon}T^\theta\prod_{j=1}^4\|f_j\|_{L_2}. \qquad (5.1)
\end{aligned}
$$

Remark 5.1 In this lemma, we use (5.1) to prove the following trilinear estimates

$$\|u_1\overline{u}_2(\overrightarrow{\gamma}\cdot\nabla u_3)\|_{\mathbf{N}^s(T)}\le CT^\theta\|u_1\|_{\mathbf{F}^s(T)}\|u_2\|_{\mathbf{F}^s(T)}\|u_3\|_{\mathbf{F}^s(T)} \qquad (5.2)$$

and

$$\|u_1\overline{u}_2(\overrightarrow{\gamma}\cdot\nabla u_3)\|_{\mathbf{N}^{\beta_n}(T)}\le CT^\theta\|u_1\|_{\mathbf{F}^{\beta_n}(T)}\|u_2\|_{\mathbf{F}^s(T)}\|u_3\|_{\mathbf{F}^s(T)}$$

$$+ CT^{\theta}\|u_1\|_{\mathbf{F}^s(T)}\|u_2\|_{\mathbf{F}^{\beta_n}(T)}\|u_3\|_{\mathbf{F}^s(T)}$$
$$+ CT^{\theta}\|u_1\|_{\mathbf{F}^s(T)}\|u_2\|_{\mathbf{F}^s(T)}\|u_3\|_{\mathbf{F}^{\beta_n}(T)}. \quad (5.3)$$

Remark 5.2 In this lemma, let $f_i(\xi_i,\tau_i)=\langle\sigma_i\rangle^{\frac12}\mathcal{F}(\eta_0(T^{-\widetilde{\theta}}2^{\widetilde{\theta}k_i}(t-t_{k_i}))P_{k_i}(u))$, $i=1,2,3,4$. Here notice that $\theta\le\widetilde{\theta}$. If $N_{\rm soprano}\lesssim T^{-\frac14}$, then we can easily obtain Theorem 4.1.

Proof By duality and the Plancheral identity, it suffices to show

$$\begin{aligned}
&\|m((\xi_1,\tau_1),(\xi_2,\tau_2),(\xi_3,\tau_3),(\xi_4,\tau_4))\|_{[4,\mathbb{R}^n\times\mathbb{R}]}\\
&:=\Big\|\frac{\frac{N_{\rm soprano}^{\widetilde{\theta}}}{\langle N_4\rangle^{\widetilde{\theta}}}\frac{N_4\langle\xi_4\rangle^s}{\langle\xi_1\rangle^s\langle\xi_2\rangle^s\langle\xi_3\rangle^s}}{\langle\mathcal{L}_1\rangle^{\frac12}\langle\mathcal{L}_2\rangle^{\frac12}\langle\mathcal{L}_3\rangle^{\frac12}\langle L_4\rangle^{\frac12}}\Big\|_{[4,\mathbb{R}^n\times\mathbb{R}]}\\
&\lesssim\langle N_{\rm tenor}\rangle^{-\varepsilon}T^{\theta}.
\end{aligned}\quad (5.4)$$

We often use the inequality

$$\Big\|\frac{f}{\langle\tau-\phi(\xi)\rangle^b}\Big\|_{L^2_{\xi,\tau}}\lesssim\|f\|_{L^2_{\xi,\tau}}T^{b\widetilde{\theta}}2^{-b\widetilde{\theta}k},\quad \operatorname{supp}\check{f}\subset\{t:|t|\lesssim T^{\widetilde{\theta}}2^{-\widetilde{\theta}k}\},\quad 0<b<\frac12. \quad (5.5)$$

Case 1 Assume $N_4\sim N_{\rm soprano}$ and $L_4\lesssim N_{\rm soprano}^{\widetilde{\theta}}T^{-\widetilde{\theta}}$. By symmetry, we assume that $N_3\sim N_4\sim N_{\rm soprano}\ge N_2\ge N_1$.

Notice that

$$\operatorname{supp}\check{f}_i\subset\{t:|t|\lesssim T^{\widetilde{\theta}}2^{-\widetilde{\theta}k_{\rm soprano}}\},\quad 0<b<\frac12,\quad i=1,2,3,4.$$

Then

$$\frac{N_{\rm soprano}^{\widetilde{\theta}}}{\langle N_4\rangle^{\widetilde{\theta}}}\frac{N_4\langle\xi_4\rangle^s}{\langle\xi_1\rangle^s\langle\xi_2\rangle^s\langle\xi_3\rangle^s}\lesssim N_4\langle N_1\rangle^{-s}\langle N_2\rangle^{-s}.$$

Subcase 1-1 Assume $N_2\gtrsim T^{-\frac14}$.

Sub-subcase 1-1-1 If $L_2\gtrsim N_2^2\gtrsim N_2^{\widetilde{\theta}}T^{-2\widetilde{\theta}}$, then $\frac{\beta_{(j_4,k_4)}}{\beta_{(j_1,k_1)}\beta_{(j_2,k_2)}\beta_{(j_3,k_3)}}\lesssim 2^{-\frac{j_2}{2}}T^{-\widetilde{\theta}}2^{\frac{\widetilde{\theta}k}{2}}$. Using (3.19) in Lemmas 3.1(We only consider the bad case.), 3.2–3.3, we have for $s\ge\frac n2+10\varepsilon$,

$$\begin{aligned}
&\|m((\xi_1,\tau_1),(\xi_2,\tau_2),(\xi_3,\tau_3),(\xi_4,\tau_4))\|_{[4,\mathbb{R}^n\times\mathbb{R}]}\\
&\lesssim\langle N_1\rangle^{-s}\langle N_2\rangle^{-\frac12+\varepsilon}T^{\theta}N_{\rm soprano}2^{-\frac{j_2}{2}}T^{-\widetilde{\theta}}2^{\frac{\widetilde{\theta}k}{2}}\Big\|\frac{1}{\langle\mathcal{L}_1\rangle^{\frac12}\langle\mathcal{L}_2\rangle^{\frac12}\langle\mathcal{L}_3\rangle^{\frac12}\langle L_4\rangle^{\frac12}}\Big\|_{[4,\mathbb{R}^n\times\mathbb{R}]}\\
&\lesssim\langle N_1\rangle^{-s}\langle N_2\rangle^{-s}T^{\widetilde{\theta}}N_{\rm soprano}2^{-\frac{j_2}{2}}T^{-\widetilde{\theta}}2^{\frac{\widetilde{\theta}k}{2}}\Big\|\frac{1}{\langle\mathcal{L}_1\rangle^{\frac12}\langle\mathcal{L}_3\rangle^{\frac12}}\Big\|_{[3,\mathbb{R}^n\times\mathbb{R}]}\Big\|\frac{1}{\langle\mathcal{L}_2\rangle^{\frac12}\langle L_4\rangle^{\frac12}}\Big\|_{[3,\mathbb{R}^n\times\mathbb{R}]}\\
&\lesssim\langle N_1\rangle^{-s\frac n2+\varepsilon}\langle N_2\rangle^{-s+\frac n2+\varepsilon}T^{\theta}2^{-\frac{j_2}{2}}T^{-\widetilde{\theta}}2^{\frac{\widetilde{\theta}k}{2}}L_2^{\frac12}N_2^{-1}\lesssim N_{\rm tenor}^{-\varepsilon}T^{\theta}.
\end{aligned}\quad (5.6)$$

Sub-subcase 1-1-2 If $L_2\lesssim N_2^2$, then we can assume $\frac{\beta_{(j_4,k_4)}}{\beta_{(j_1,k_1)}\beta_{(j_2,k_2)}\beta_{(j_3,k_3)}}\lesssim 1$. Using (3.19) in Lemmas 3.1–3.3, we have for $s\ge\frac n2+10\varepsilon$,

$$\begin{aligned}
&\|m((\xi_1,\tau_1),(\xi_2,\tau_2),(\xi_3,\tau_3),(\xi_4,\tau_4))\|_{[4,\mathbb{R}^n\times\mathbb{R}]}\\
&\lesssim\langle N_1\rangle^{-s}\langle N_2\rangle^{-\frac12+\varepsilon}T^{\theta}N_{\rm soprano}\Big\|\frac{1}{\langle\mathcal{L}_1\rangle^{\frac12}\langle\mathcal{L}_2\rangle^{\frac12}\langle\mathcal{L}_3\rangle^{\frac12}\langle L_4\rangle^{\frac12}}\Big\|_{[4,\mathbb{R}^n\times\mathbb{R}]}\\
&\lesssim\langle N_1\rangle^{-s}\langle N_2\rangle^{-s}T^{\widetilde{\theta}}N_{\rm soprano}\Big\|\frac{1}{\langle\mathcal{L}_1\rangle^{\frac12}\langle\mathcal{L}_3\rangle^{\frac12}}\Big\|_{[3,\mathbb{R}^n\times\mathbb{R}]}\Big\|\frac{1}{\langle\mathcal{L}_2\rangle^{\frac12}\langle L_4\rangle^{\frac12}}\Big\|_{[3,\mathbb{R}^n\times\mathbb{R}]}
\end{aligned}$$

$$\lesssim \langle N_1\rangle^{-s\frac{n}{2}+\varepsilon}\langle N_2\rangle^{-s\frac{n}{2}+\varepsilon}T^{\theta} \lesssim N_{\rm tenor}^{-\varepsilon}T^{\theta}. \tag{5.7}$$

Subcase 1-2 If $L_2 \lesssim N_2^{\widetilde{\theta}}T^{-2\widetilde{\theta}}$ and $N_2 \lesssim T^{-\frac{1}{4}}$, then $\frac{\beta_{(j_4,k_4)}}{\beta_{(j_1,k_1)}\beta_{(j_2,k_2)}\beta_{(j_3,k_3)}} \lesssim 1$. Using Lemmas 3.1–3.3 and (5.5), we have for $s \geq \frac{n}{2}+10\varepsilon$,

$$\begin{aligned}
&\|m((\xi_1,\tau_1),(\xi_2,\tau_2),(\xi_3,\tau_3),(\xi_4,\tau_4))\|_{[4,\mathbb{R}^n\times\mathbb{R}]}\\
&\lesssim \langle N_1\rangle^{-s}\langle N_2\rangle^{-s}N_{\rm soprano}N_2^{b\widetilde{\theta}}T^{-2b\widetilde{\theta}}\Big\|\frac{1}{\langle \mathcal{L}_1\rangle^{\frac12}\langle \mathcal{L}_2\rangle^{\frac12+b}\langle \mathcal{L}_3\rangle^{\frac12}\langle L_4\rangle^{\frac12}}\Big\|_{[4,\mathbb{R}^n\times\mathbb{R}]}\\
&\lesssim \langle N_1\rangle^{-s}\langle N_2\rangle^{-s}N_{\rm soprano}N_2^{b\widetilde{\theta}}T^{-2b\widetilde{\theta}}\Big\|\frac{1}{\langle \mathcal{L}_1\rangle^{\frac12}\langle \mathcal{L}_3\rangle^{\frac12}}\Big\|_{[3,\mathbb{R}^n\times\mathbb{R}]}\Big\|\frac{1}{\langle \mathcal{L}_2\rangle^{\frac12+b}\langle L_4\rangle^{\frac12}}\Big\|_{[3,\mathbb{R}^n\times\mathbb{R}]}\\
&\lesssim \langle N_1\rangle^{-s\frac{n}{2}+\varepsilon}\langle N_2\rangle^{-s+\frac{2}{n}+\varepsilon}N_2^{b\widetilde{\theta}}T^{-b2\widetilde{\theta}}T^{\widetilde{\theta}}N_{\rm soprano}^{-b\widetilde{\theta}} \lesssim N_{\rm tenor}^{-\varepsilon}T^{\theta}.
\end{aligned}\tag{5.8}$$

Subcase 1-3 If $L_2 \gtrsim N_2^{\widetilde{\theta}}T^{-2\widetilde{\theta}}$ and $N_2 \lesssim T^{-\frac{1}{4}}$, then $\frac{\beta_{(j_4,k_4)}}{\beta_{(j_1,k_1)}\beta_{(j_2,k_2)}\beta_{(j_3,k_3)}} \lesssim T^{\frac{\widetilde{\theta}}{4}}$. Using Lemmas 3.1–3.3, we have for $s \geq \frac{n}{2}+10\varepsilon$,

$$\begin{aligned}
&\|m((\xi_1,\tau_1),(\xi_2,\tau_2),(\xi_3,\tau_3),(\xi_4,\tau_4))\|_{[4,\mathbb{R}^n\times\mathbb{R}]}\\
&\lesssim \langle N_1\rangle^{-s}\langle N_2\rangle^{-\frac12+\varepsilon}T^{\frac{\widetilde{\theta}}{4}}N_{\rm soprano}\Big\|\frac{1}{\langle \mathcal{L}_1\rangle^{\frac12}\langle \mathcal{L}_2\rangle^{\frac12}\langle \mathcal{L}_3\rangle^{0}\langle L_4\rangle^{\frac12}}\Big\|_{[4,\mathbb{R}^n\times\mathbb{R}]}\\
&\lesssim \langle N_1\rangle^{-s}\langle N_2\rangle^{-s}T^{\widetilde{\theta}}N_{\rm soprano}\Big\|\frac{1}{\langle \mathcal{L}_1\rangle^{\frac12}\langle \mathcal{L}_3\rangle^{\frac12}}\Big\|_{[3,\mathbb{R}^n\times\mathbb{R}]}\Big\|\frac{1}{\langle \mathcal{L}_2\rangle^{\frac12+b}\langle L_4\rangle^{\frac12}}\Big\|_{[3,\mathbb{R}^n\times\mathbb{R}]}\\
&\lesssim \langle N_1\rangle^{-s\frac{n}{2}+\varepsilon}\langle N_2\rangle^{-s+\frac{2}{n}+\varepsilon}T^{\theta} \lesssim N_{\rm tenor}^{-\varepsilon}.
\end{aligned}\tag{5.9}$$

Case 2 Assume $N_4 \sim N_{\rm soprano}$ and $L_4 \gtrsim N_{\rm soprano}^{\widetilde{\theta}}T^{-\widetilde{\theta}}$. By symmetry, we assume that $N_3 \sim N_4 \sim N_{\rm soprano} \geq N_2 \geq N_1$. It implies that L_3 or L_2 or $L_1 \gtrsim N_{\rm soprano}^{\widetilde{\theta}}T^{-\widetilde{\theta}}$. Thus we have $\frac{\beta_{(j_4,k_4)}}{\beta_{(j_1,k_1)}\beta_{(j_2,k_2)}\beta_{(j_3,k_3)}} \lesssim 1$. Similarly to Case 1, we have the results.

Case 3 Assume $N_4 \ll N_{\rm soprano}$. By symmetry, we assume that $N_3 \sim N_2 \sim N_{\rm soprano} \geq N_4 \geq N_1$. Then for $s > \frac{n}{2}$, we have

$$\frac{N_{\rm soprano}^{\widetilde{\theta}}}{\langle N_4\rangle^{\widetilde{\theta}}}\frac{N_4\langle \xi_4\rangle^s}{\langle \xi_1\rangle^s\langle \xi_2\rangle^s\langle \xi_3\rangle^s} \lesssim N_{\rm soprano}^{1-s}\langle N_1\rangle^{-s}.$$

Similarly to Case 1, we have the results.

This completes the proof of Lemma 5.1.

Lemma 5.2 (Trilinear estimate-II) *Let $s > \frac{n}{2}$. Assume that $|\xi_l| \sim N_l = 2^{k_l}$, $L_l = |\widetilde{\sigma}_l| = |\tau_l - \phi(\xi_l)| = 2^{j_l}$. Let $\mathcal{L}_l = \max\{L_l, T^{-\widetilde{\theta}}N_l^{\widetilde{\theta}}\}$, $l = 1,2,3,4$. Let $N_{\rm soprano} \gtrsim T^{-\frac14}$. By symmetry we assume $N_3 \geq N_2 \geq N_1$. Then for small enough $\varepsilon, \theta > 0$,*

$$\begin{aligned}
&\frac{\beta_{(j_4,k_4)}}{\beta_{(j_1,k_1)}\beta_{(j_2,k_2)}\beta_{(j_3,k_3)}}\int_{\star}\frac{N_{\rm soprano}}{\langle N_4\rangle}\frac{N_4\langle \xi_4\rangle^{s-\widetilde{\theta}}}{\langle \xi_1\rangle^s\langle \xi_2\rangle^s\langle \xi_3\rangle^{s-\widetilde{\theta}}}\\
&\cdot\frac{f_1(\tau_1,\xi_1)f_2(\tau_2,\xi_2)f_3(\tau_3,\xi_3)f_4(\tau_4,\xi_4)}{\langle \mathcal{L}_1\rangle^{\frac12}\langle \mathcal{L}_2\rangle^{\frac12}\langle \mathcal{L}_3\rangle^{\frac12}\langle L_4\rangle^{\frac12}}{\rm d}\delta\\
&\leq C\langle N_{\rm tenor}\rangle^{-\varepsilon}T^{\theta}\prod_{i=1}^{4}\|f_i\|_{L_2}.
\end{aligned}\tag{5.10}$$

Remark 5.3 In this lemma, we use (5.10) to prove the following trilinear estimates

$$\|u_1\overline{u}_2(\overrightarrow{\gamma}\cdot\nabla u_3)\|_{\mathbf{N}^{s-\widetilde{\theta}}(T)} \le CT^\theta\|u_1\|_{\mathbf{F}^s(T)}\|u_2\|_{\mathbf{F}^s(T)}\|u_3\|_{\mathbf{F}^{s-\widetilde{\theta}}(T)}. \tag{5.11}$$

Proof Similarly to the proof of Lemma 5.1, we have the results.

6 Trilinear Estimates

In this section, we give the proof of Theorem 4.1. For the nonlinear terms $|u|^2(\overrightarrow{\gamma}\cdot\nabla u)+u^2(\overrightarrow{\lambda}\cdot\nabla\overline{u})$, by symmetry, we only consider $(|u|^2(\overrightarrow{\gamma}\cdot\nabla u))$, and the estimate of $(u^2(\overrightarrow{\lambda}\cdot\nabla\overline{u}))$ is similar with that of $(|u|^2(\overrightarrow{\gamma}\cdot\nabla u))$.

Theorem 6.1 *Assume $k,k_1,k_2\in\mathbb{Z}_+$, $k=\max\{k,k_1,k_2,k_3\}\ge 100$ and $u_{k_j}\in F_{k_j}$, $j=1,2,3$. Then for $s>\frac{n}{2}$ and small enough $\varepsilon,\theta>0$,*

$$2^{ks}\|u_{k_1}\overline{u_{k_2}}\overrightarrow{\gamma}\cdot\nabla u_{k_3}\|_{N_k} \le C2^{-k_{\rm tenor}\varepsilon}2^{(k_1+k_2+k_3)s}T^\theta\|u_{k_1}\|_{F_{k_1}}\|u_{k_2}\|_{F_{k_2}}\|u_{k_3}\|_{F_{k_3}}, \tag{6.1}$$

$$\begin{aligned}2^{k\beta_n}\|u_{k_1}\overline{u_{k_2}}\overrightarrow{\gamma}\cdot\nabla u_{k_3}\|_{N_k} &\le C2^{-k_{\rm tenor}\varepsilon}2^{\beta_n k_1}2^{(k_2+k_3)s}T^\theta\|u_{k_1}\|_{F_{k_1}}\|u_{k_2}\|_{F_{k_2}}\|u_{k_3}\|_{F_{k_3}}\\ &\quad + C2^{-k_{\rm tenor}\varepsilon}2^{(k_1+k_3)s}2^{\beta_n k_2}T^\theta\|u_{k_1}\|_{F_{k_1}}\|u_{k_2}\|_{F_{k_2}}\|u_{k_3}\|_{F_{k_3}}\\ &\quad + C2^{-k_{\rm tenor}\varepsilon}2^{(k_1+k_2)s}2^{\beta_n k_3}T^\theta\|u_{k_1}\|_{F_{k_1}}\|u_{k_2}\|_{F_{k_2}}\|u_{k_3}\|_{F_{k_3}},\end{aligned} \tag{6.2}$$

$$2^{k(s-\widetilde{\theta})}\|u_{k_1}\overline{u_{k_2}}\overrightarrow{\gamma}\cdot\nabla u_{k_3}\|_{N_k} \le C2^{-k_{\rm tenor}\varepsilon}2^{(k_1+k_2)s}2^{k_3(s-\widetilde{\theta})}T^\theta\|u_{k_1}\|_{F_{k_1}}\|u_{k_2}\|_{F_{k_2}}\|u_{k_3}\|_{F_{k_3}}, \tag{6.3}$$

where the constant C is independent of $\|u_0\|_{H^s}$; $\beta_n>\frac{n}{2}+1$, here the Cauchy problem (1.1) *is locally well-posed in H^{β_n} with $\beta_n>\frac{n}{2}+1$ for any large initial data.*

Remark 6.1 In this lemma, we use (6.1) to prove the following trilinear estimates

$$\|u_1\overline{u}_2(\overrightarrow{\gamma}\cdot\nabla u_3)\|_{\mathbf{N}^s(T)} \le CT^\theta\|u_1\|_{\mathbf{F}^s(T)}\|u_2\|_{\mathbf{F}^s(T)}\|u_3\|_{\mathbf{F}^s(T)}; \tag{6.4}$$

we use (6.2) to prove the following trilinear estimates

$$\begin{aligned}\|u_1\overline{u}_2(\overrightarrow{\gamma}\cdot\nabla u_3)\|_{\mathbf{N}^{\beta_n}(T)} &\le CT^\theta\|u_1\|_{\mathbf{F}^{\beta_n}(T)}\|u_2\|_{\mathbf{F}^s(T)}\|u_3\|_{\mathbf{F}^s(T)}\\ &\quad + CT^\theta\|u_1\|_{\mathbf{F}^s(T)}\|u_2\|_{\mathbf{F}^{\beta_n}(T)}\|u_3\|_{\mathbf{F}^s(T)}\\ &\quad + CT^\theta\|u_1\|_{\mathbf{F}^s(T)}\|u_2\|_{\mathbf{F}^s(T)}\|u_3\|_{\mathbf{F}^{\beta_n}(T)}.\end{aligned} \tag{6.5}$$

Proof We only prove (6.1). The proofs of (6.2) and (6.3) are similar with that of (6.1). From the definition of N_k, it follows that

$$\begin{aligned}2^{sk}\|u_{k_1}\overline{u_{k_2}}\overrightarrow{\gamma}\cdot\nabla u_{k_3}\|_{N_k} &= 2^{sk}\sup_{t_k\in\mathbb{R}}\|(\tau-\phi(\xi)+{\rm i}T^{-\widetilde{\theta}}2^{\widetilde{\theta}k})^{-1}\\ &\qquad\cdot\mathcal{F}(u_{k_1}\overline{u_{k_2}}\overrightarrow{\gamma}\cdot\nabla u_{k_3}\cdot\eta_0(T^{-\widetilde{\theta}}2^{\widetilde{\theta}k}(t-t_k)))\|_{X_k}.\end{aligned} \tag{6.6}$$

Using the definitions and (2.13), we have

$$\begin{aligned}(6.6) \lesssim{}& 2^{k_3}2^{sk}\sup_{t_k\in\mathbb{R}}\|(\tau-\phi(\xi)+{\rm i}T^{-\widetilde{\theta}}2^{\widetilde{\theta}k})^{-1}\mathbf{1}_{I_k}(\xi)\cdot\eta_0(T^{-\widetilde{\theta}}2^{\widetilde{\theta}k}(t-t_k))\\ &\mathcal{F}[u_{k_1}\cdot\eta_0(T^{-\widetilde{\theta}}2^{\widetilde{\theta}k_{\rm soprano}}(t-t_{k_1}))]*\mathcal{F}[\overline{u_{k_2}}\cdot\eta_0(T^{-\widetilde{\theta}}2^{\widetilde{\theta}k_{\rm soprano}}(t-t_{k_2}))]\end{aligned}$$

$$
\begin{aligned}
&* \mathcal{F}[u_{k_3} \cdot \eta_0(T^{-\widetilde{\theta}} 2^{\widetilde{\theta} k_{\text{soprano}}}(t - t_{k_3}))]\|_{X_k} \\
\lesssim\ & 2^{k_3} 2^{sk} \sup_{t_k \in \mathbb{R}} \|(\tau - \phi(\xi) + \mathrm{i} T^{-\widetilde{\theta}} 2^{\widetilde{\theta} k})^{-1} \mathbf{1}_{I_k}(\xi) \cdot \eta_0(T^{-\widetilde{\theta}} 2^{\widetilde{\theta} k}(t - t_k)) \\
& \mathcal{F}[u_{k_1} \cdot \eta_0(T^{-\widetilde{\theta}} 2^{\widetilde{\theta} k_{\text{soprano}}}(t - t_{k_1}))] * \mathcal{F}[(\overline{u_{k_2}}) \cdot \eta_0(T^{-\widetilde{\theta}} 2^{\widetilde{\theta} k_{\text{soprano}}}(t - t_{k_2}))] \\
& * \mathcal{F}[(u_{k_3}) \cdot \eta_0(T^{-\widetilde{\theta}} 2^{\widetilde{\theta} k_{\text{soprano}}}(t - t_{k_3}))]\|_{X_k}.
\end{aligned}
$$

Let

$$
\begin{aligned}
f_{k_m} &= \mathcal{F} g(u_{k_m} \cdot \eta_0 T^{-\widetilde{\theta}} 2^{\widetilde{\theta} k_m}(t - t_{k_m}))), \quad m = 1, 3, \\
f_{k_2} &= \mathcal{F}(\overline{u_{k_2}} \cdot \eta_0(T^{-\widetilde{\theta}} 2^{\widetilde{\theta} k_2}(t - t_{k_2})))
\end{aligned}
$$

and

$$
\begin{aligned}
f_{k_m, j_m} &= \eta_{j_m}(\tau - \phi(\xi)) \mathcal{F}(u_{k_m} \cdot \eta_0(T^{-\widetilde{\theta}} 2^{\widetilde{\theta} k_m}(t - t_{k_m}))), \quad m = 1, 3, \\
f_{k_2, j_2} &= \eta_{j_2}(\tau + \phi(\xi)) \mathcal{F}(\overline{u_{k_2}} \cdot \eta_0(T^{-\widetilde{\theta}} 2^{\widetilde{\theta} k_2}(t - t_{k_2}))).
\end{aligned}
$$

Using Lemma 4.1 and (2.13), it suffices to prove that if $f_{k_l, j_l} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}_+$ are supported in D_{k_l, j_l}, $2^{j_l} \geq T^{-\widetilde{\theta}} 2^{\widetilde{\theta} k_l}$, $l = 1, 2, 3$, then for $s > \frac{n}{2}$,

$$
\begin{aligned}
(6.6) \lesssim\ & 2^{k_3} 2^{sk} \sum_{2^j \geq T^{-\widetilde{\theta}} 2^{\widetilde{\theta} k}} \beta_{(j,k)} 2^{\frac{j}{2}} (2^j + \mathrm{i} T^{-\widetilde{\theta}} 2^{\widetilde{\theta} k})^{-1} \|\mathbf{1}_{D_{k,j}} \cdot f_{k_1, j_1} * f_{k_2, j_2} * f_{k_3, j_3}\|_{L^2} \\
\lesssim\ & 2^{(k_1 + k_2 + k_3)s} 2^{-k_{\text{tenor}} \varepsilon} T^{\theta} \sum_{2^{j_l} \geq T^{-\widetilde{\theta}} 2^{k_l}, l = 1,2,3} \prod_{l=1}^{3} 2^{\frac{j_l}{2}} \beta_{(j_l, k_l)} \|f_{k_l, j_l}\|_{L^2}.
\end{aligned} \tag{6.7}
$$

In fact, using Lemma 5.1, we can obtain the above. Then performing the j summations for (6.7), we have (6.1). This completes the proof of Theorem 6.1.

Now we turn to the proof of (4.6) in Theorem 4.1. The proofs of (4.7)–(4.9) are similar with that of (4.6). From the definition of $\mathbf{N}^s$, it follows that for $s > \frac{n}{2}$,

$$
\|u_1 \overline{u_2} \overrightarrow{\gamma} \cdot \nabla u_3\|_{\mathbf{N}^s}^2 = \sum_{k=0}^{\infty} 2^{2ks} \|P_k(u_1 \overline{u_2} \overrightarrow{\gamma} \cdot \nabla u_3)\|_{N_k}^2. \tag{6.8}
$$

For $2^{2ks} \|P_k(u_1 \overline{u_2} \overrightarrow{\gamma} \cdot \nabla u_3)\|_{N_k}^2$, using Theorem 6.1, we have

$$
\begin{aligned}
& 2^{2ks} \|P_k(u_1 \overline{u_2} \overrightarrow{\gamma} \cdot \nabla u_3)\|_{N_k}^2 \\
\leq & \sum_{k_1, k_2, k_3 \geq 0} 2^{2k_3} 2^{2ks} \|P_k(P_{k_1}(u_1) P_{k_2}(\overline{u_2}) P_{k_3}(u_3))\|_{N_k}^2 \\
\lesssim & \sum_{k_1, k_2, k_3 \geq 0} 2^{2(k_1 + k_2 + k_3)s} 2^{-k_{\text{tenor}} \varepsilon} T^{\theta} \|P_{k_1}(u_1)\|_{F_{k_1}}^2 \|P_{k_2}(u_2)\|_{F_{k_2}}^2 \|P_{k_3}(u_3)\|_{F_{k_3}}^2.
\end{aligned} \tag{6.9}
$$

Then performing the k summations for (6.9), we have for $s > \frac{n}{2}$,

$$
\|u_1 \overline{u_2} \overrightarrow{\gamma} \cdot \nabla u_3\|_{\mathbf{N}^s}^2 = \sum_{k=0}^{\infty} 2^{2ks} \|P_k(u_1 \overline{u_2} \overrightarrow{\gamma} \cdot \nabla u_3)\|_{N_k}^2 \lesssim T^{\theta} \|u_1\|_{\mathbf{F}^s}^2 \|u_2\|_{\mathbf{F}^s}^2 \|u_3\|_{\mathbf{F}^s}^2. \tag{6.10}
$$

This completes the proof of (4.6) in Theorem 4.1.

7 Energy Estimates

In this section, we give the proofs of Theorem 4.2 and Lemma 4.6. For the DNLS equation (1.1), we have

$$\begin{cases} u_t - \mathrm{i}\Delta u = f, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}, \\ u(x,0) = u_0(x), \end{cases} \tag{7.1}$$

where $f = |u|^2(\overrightarrow{\gamma} \cdot \nabla u) + \overrightarrow{\gamma} \cdot u^2 \nabla \overline{u}$. We apply the operator P_k to the equation (7.1), then multiply by $P_k\overline{u}$, integrate and take the real parts to conclude that

$$2^{2ks}\|P_k u(t)\|_{L^2}^2 \le 2^{2ks}\|P_k u_0\|_{L^2}^2 + 2^{2ks} \sup_{|t|\le T} \Big|\mathrm{Re} \int_{\mathbb{R}\times[0,t]} P_k(f)P_k(\overline{u})\mathrm{d}x\mathrm{d}t\Big|. \tag{7.2}$$

Theorem 7.1 (Coifman-Meyers theorem see [3]) *Let $1 < p_1, \cdots, p_m \le +\infty$ and $1 \le p < +\infty$ satisfy $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m}$. Assume that $f_1, f_2, \cdots, f_m \in \mathcal{S}(\mathbb{R})$ are functions with Fourier variables supported in $|\xi_i| \sim N_i$ for some dyadic numbers N_i with $i = 1, 2, \cdots, m$. Assume also that $\gamma(\xi) = \gamma(\xi_1, \cdots, \xi_m) \in C^\infty(\mathbb{R}^m)$ satisfies the Marcinkiewicz type condition*

$$\beta = (\beta_1, \cdots, \beta_m) \in \mathbb{N}_+^m, \quad |\partial^\beta \gamma(\xi)| \lesssim \prod_i^m |\xi|^{-\beta} \tag{7.3}$$

on the support of $\prod_i^m \widehat{f}_i(\xi_i)$. Then

$$\Big\| \int_{\mathbb{R}^m} \gamma(\xi_1, \cdots, \xi_m) \prod_i^m \widehat{f}_i(\xi_i) \mathrm{e}^{\mathrm{i}t(\xi_1+\cdots+\xi_m)} \mathrm{d}\xi_1 \cdots \mathrm{d}\xi_m \Big\|_{L^p} \lesssim \prod_i^m \|f_i\|_{L^{p_i}} \tag{7.4}$$

with an implicit constant that does not depend on dyadic numbers N_i, $i = 1, 2, \cdots, m$.

Theorem 7.2 *Assume $s > \frac{n}{2}$. If $u(t)$ and $v(t)$ are solutions of the Cauchy problem* (1.1) *with initial datum $u_0 \in H^s$ and $v_0 \in H^s$, respectively ($u(t)$ and $v(t)$ only exist). Then the difference $w = u - v$ satisfies that*

$$\begin{aligned} \sup_{t\in(0,T)} \|w(t)\|_{H^s} \lesssim \|w\|_{\mathbf{E}^s(T)} \le \|w_0\|_{H^s} + T^\theta(\|u\|_{\mathbf{F}^s}^2 + \|w\|_{\mathbf{F}^s}^2 + \|u\|_{\mathbf{F}^s}\|w\|_{\mathbf{F}^s})\|w\|_{\mathbf{F}^s}\|w\|_{\mathbf{F}^s} \\ T^\theta(\|u\|_{\mathbf{F}^s}\|w\|_{\mathbf{F}^s} + \|w\|_{\mathbf{F}^s}^2)\|u\|_{\mathbf{F}^{s+\tilde{\theta}}}\|w\|_{\mathbf{F}^{s-\tilde{\theta}}}. \end{aligned} \tag{7.5}$$

Proof First, the difference $w = u - v$ satisfies that

$$\begin{cases} w_t - \mathrm{i}\Delta w = (|u|^2(\overrightarrow{\gamma} \cdot \nabla w) + (w\overline{u}\overrightarrow{\gamma} \cdot \nabla u - w\overline{u}\overrightarrow{\gamma} \cdot \nabla w + u\overline{w}\overrightarrow{\gamma} \cdot \nabla u - u\overline{w}\overrightarrow{\gamma} \cdot \nabla w \\ \qquad -|w|^2\overrightarrow{\gamma} \cdot \nabla u + |w|^2\overrightarrow{\gamma} \cdot \nabla w) + u^2(\overrightarrow{\lambda} \cdot \nabla \overline{w}) + (uw\overrightarrow{\lambda} \cdot \nabla \overline{u} - uw\overrightarrow{\lambda} \cdot \nabla \overline{w} \\ \qquad +uw\overrightarrow{\lambda} \cdot \nabla \overline{u} - uw\overrightarrow{\lambda} \cdot \nabla \overline{w} - w^2\overrightarrow{\lambda} \cdot \nabla \overline{u} + w^2\overrightarrow{\lambda} \cdot \nabla \overline{w})), \\ w(x,0) = w_0(x). \end{cases} \tag{7.6}$$

We apply the operator P_k to the equation (7.6). By symmetry, we only consider the operator P_k, multiply by $P_k\overline{w}$, then integrate and take the real parts, we have

$$\begin{aligned} 2^{2ks}\|P_k w(t)\|_{L^2} \lesssim 2^{2ks}\|P_k w_0\|_{L^2} + \mathrm{Re} \int_{\mathbb{R}^n\times[0,t]} P_k(|u|^2(\overrightarrow{\gamma} \cdot \nabla w) \\ + (w\overline{u}\overrightarrow{\gamma} \cdot \nabla u - w\overline{u}\overrightarrow{\gamma} \cdot \nabla w + u\overline{w}\overrightarrow{\gamma} \cdot \nabla u - u\overline{w}\overrightarrow{\gamma} \cdot \nabla w \end{aligned}$$

$$
\begin{aligned}
& - |w|^2\overrightarrow{\gamma}\cdot\nabla u + |w|^2\overrightarrow{\gamma}\cdot\nabla w) + u^2(\overrightarrow{\lambda}\cdot\nabla\overline{w}) \\
& + (uw\overrightarrow{\lambda}\cdot\nabla\overline{u} - uw\overrightarrow{\lambda}\cdot\nabla\overline{w} + uw\overrightarrow{\lambda}\cdot\nabla\overline{u} - uw\overrightarrow{\lambda}\cdot\nabla\overline{w} \\
& - w^2\overrightarrow{\lambda}\cdot\nabla\overline{u} + w^2\overrightarrow{\lambda}\cdot\nabla\overline{w}))_k(\overline{w})\mathrm{d}x\mathrm{d}t. \qquad (7.7)
\end{aligned}
$$

By symmetry, we only consider the three terms

$$(A_k):\ \Big|\mathrm{Re}\int_{\mathbb{R}^n\times[0,t]} P_k(u\overline{w}(\overrightarrow{\gamma}\cdot\nabla w))P_k(\overline{w})\mathrm{d}x\mathrm{d}t\Big|.$$

$$(B_k):\ \Big|\mathrm{Re}\int_{\mathbb{R}^n\times[0,t]} P_k(\overline{u}w(\overrightarrow{\gamma}\cdot\nabla w))P_k(\overline{w})\mathrm{d}x\mathrm{d}t\Big|.$$

$$(D_k):\ \Big|\mathrm{Re}\int_{\mathbb{R}^n\times[0,t]} P_k(w\overline{u}\overrightarrow{\gamma}\cdot\nabla u)P_k(\overline{w})\mathrm{d}x\mathrm{d}t\Big|.$$

The other cases in (7.7) can be considered similarly to those of A_k, B_k, D_k.

We first consider the term $(A_k):\ \big|\mathrm{Re}\int_{\mathbb{R}^n\times[0,T]} P_k(u\overline{w}(\overrightarrow{\gamma}\cdot\nabla w))P_k(\overline{w})\mathrm{d}x\mathrm{d}t\big|$. Then we have

$$
\begin{aligned}
& \mathrm{Re}\int_{\mathbb{R}^n\times[0,t]} P_k(u\overline{w}(\overrightarrow{\gamma}\cdot\nabla w))P_k(\overline{w})\mathrm{d}x\mathrm{d}t \\
&= \mathrm{Re}\int_{\mathbb{R}^n\times[0,t]} P_k(u\overline{w}(\overrightarrow{\gamma}\cdot\nabla P_{\sim k}w))P_k(\overline{w})\mathrm{d}x\mathrm{d}t + \mathrm{Re}\int_{\mathbb{R}^n\times[0,t]} P_k(u\overline{w}(\overrightarrow{\gamma}\cdot\nabla P_{\ll k}w))P_k(\overline{w})\mathrm{d}x\mathrm{d}t \\
&\quad + \mathrm{Re}\int_{\mathbb{R}^n\times[0,t]} P_k(u\overline{w}(\overrightarrow{\gamma}\cdot\nabla P_{\gg k}w))P_k(\overline{w})\mathrm{d}x\mathrm{d}t \\
&:= A_k^1 + A_k^2 + A_k^3. \qquad (7.8)
\end{aligned}
$$

For A_k^2 and A_k^3, we will use the following Lemma 7.1 to consider them. For A_k^1, we have

$$
\begin{aligned}
A_k^1 &= \mathrm{Re}\int_{\mathbb{R}^n\times[0,t]} P_k(u\overline{w}(\overrightarrow{\gamma}\cdot\nabla P_{\sim k}w))P_k(\overline{w})\mathrm{d}x\mathrm{d}t \\
&= \int_{\mathbb{R}^n\times[0,t]} P_k(u\overline{w}(\overrightarrow{\gamma}\cdot\nabla P_{\sim k}w))P_k(\overline{w})\mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}^n\times[0,t]} P_k(\overline{u}w(\overrightarrow{\gamma}\cdot\nabla P_{\sim k}\overline{w}))P_k(w)\mathrm{d}x\mathrm{d}t \\
&:= A_k^{11} + A_k^{12}. \qquad (7.9)
\end{aligned}
$$

Next we consider the term $(B_k):\ |\mathrm{Re}\int_{\mathbb{R}^n\times[0,t]} P_k(\overline{u}w(\overrightarrow{\gamma}\cdot\nabla w)P_k(\overline{w})\mathrm{d}x\mathrm{d}t|$. Then we have

$$
\begin{aligned}
B_k &= \mathrm{Re}\int_{\mathbb{R}^n\times[0,t]} P_k(\overline{u}w(\overrightarrow{\gamma}\cdot\nabla w))P_k(\overline{w})\mathrm{d}x\mathrm{d}t \\
&= \mathrm{Re}\int_{\mathbb{R}^n\times[0,t]} P_k(\overline{u}w(\overrightarrow{\gamma}\cdot\nabla P_{\sim k}w))P_k(\overline{w})\mathrm{d}x\mathrm{d}t + \mathrm{Re}\int_{\mathbb{R}^n\times[0,t]} P_k(\overline{u}w(\overrightarrow{\gamma}\cdot\nabla P_{\ll k}w))P_k(\overline{w})\mathrm{d}x\mathrm{d}t \\
&\quad + \mathrm{Re}\int_{\mathbb{R}^n\times[0,t]} P_k(\overline{u}w(\overrightarrow{\gamma}\cdot\nabla P_{\gg k}w))P_k(\overline{w})\mathrm{d}x\mathrm{d}t \\
&:= B_k^1 + B_k^2 + B_k^3. \qquad (7.10)
\end{aligned}
$$

For B_k^2 and B_k^3, we will use the following Lemma 7.1 to consider them. For B_k^1, we have

$$B_k^1 = \mathrm{Re}\int_{\mathbb{R}^n\times[0,t]} P_k(\overline{u}w(\overrightarrow{\gamma}\cdot\nabla P_{\sim k}w))P_k(\overline{w})\mathrm{d}x\mathrm{d}t$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^n\times[0,t]} P_k(u\overline{w}(\overrightarrow{\gamma}\cdot\nabla P_{\sim k}\overline{w}))P_k(w)\mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}^n\times[0,t]} P_k(\overline{u}w(\overrightarrow{\gamma}\cdot\nabla P_{\sim k}w))P_k(\overline{w})\mathrm{d}x\mathrm{d}t \\
&:= B_k^{11} + B_k^{12}. \qquad (7.11)
\end{aligned}
$$

Thus $A_k^1 + B_k^1$, we have for some $k_{12} \le k$,

$$
\begin{aligned}
|A_k^{11} + B_k^{11}| &= \Big| \int_{\mathbb{R}^n\times[0,t]} P_k(u\overline{w}(\overrightarrow{\gamma}\cdot\nabla P_{\sim k}w))P_k(\overline{w})\mathrm{d}x\mathrm{d}t \\
&\quad + \int_{\mathbb{R}^n\times[0,t]} P_k(u\overline{w}(\overrightarrow{\gamma}\cdot\nabla P_{\sim k}\overline{w}))P_k(w)\mathrm{d}x\mathrm{d}t \Big| \\
&\sim \Big| \int_{\mathbb{R}^n\times[0,t]} P_{k_{12}}(u\overline{w})\nabla(|P_kw|^2)\mathrm{d}x\mathrm{d}t \Big| \qquad (7.12)
\end{aligned}
$$

and

$$
\begin{aligned}
|A_k^{12} + B_k^{12}| &= \Big| \int_{\mathbb{R}^n\times[0,t]} P_k(\overline{u}w(\overrightarrow{\gamma}\cdot\nabla P_{\sim k}\overline{w}))P_k(w)\mathrm{d}x\mathrm{d}t \\
&\quad + \int_{\mathbb{R}^n\times[0,t]} P_k(\overline{u}w(\overrightarrow{\gamma}\cdot\nabla P_{\sim k}w))P_k(\overline{w})\mathrm{d}x\mathrm{d}t \Big| \\
&\sim \Big| \int_{\mathbb{R}^n\times[0,t]} P_{k_{12}}(\overline{u}w)\nabla(|P_kw|^2)\mathrm{d}x\mathrm{d}t \Big|. \qquad (7.13)
\end{aligned}
$$

For $A_k^{11} + B_k^{11}$ and $A_k^{12} + B_k^{12}$, we can use the following Lemma 7.1 to consider them.

Lemma 7.1 (Dyadic energy estimates-I) *Let $s > \frac{n}{2}$. Assume that $|\xi_l| \sim N_l = 2^{k_l}$, $L_l = |\sigma_l| = |\tau_l - \phi(\xi_l)| = 2^{j_l}$. Let $\mathcal{L}_l = \max\{L_l, T^{-\widetilde{\theta}}N_{\mathrm{soprano}}^{\widetilde{\theta}}\}$, $l = 1,2,3,4$. Assume $N_{\mathrm{soprano}} \gtrsim T^{-\frac{1}{4}}$. Then for some small $\varepsilon, \theta > 0$,*

$$
\begin{aligned}
&\frac{1}{\beta_{(j_1,k_1)}\beta_{(j_2,k_2)}\beta_{(j_3,k_3)}\beta_{(j_4,k_4)}} \\
&\quad\cdot \int_\star \frac{T^{-\widetilde{\theta}}N_{\mathrm{soprano}}^{\widetilde{\theta}}N_{\mathrm{tenor}}\langle\xi_4\rangle^s}{\langle\xi_1\rangle^s\langle\xi_2\rangle^s\langle\xi_3\rangle^s} \frac{f_1(\tau_1,\xi_1)f_2(\tau_2,\xi_2)f_3(\tau_3,\xi_3)f_4(\tau_4,\xi_4)}{\langle\mathcal{L}_1\rangle^{\frac{1}{2}}\langle\mathcal{L}_2\rangle^{\frac{1}{2}}\langle\mathcal{L}_3\rangle^{\frac{1}{2}}\langle\mathcal{L}_4\rangle^{\frac{1}{2}}}\mathrm{d}\delta \\
&\lesssim \langle N_{\mathrm{tenor}}\rangle^{-\varepsilon}T^\theta \prod_{j=1}^4 \|f_j\|_{L_2}. \qquad (7.14)
\end{aligned}
$$

Remark 7.1 In this lemma, let

$$
f_i(\xi_i,\tau_i) = \langle\sigma_i\rangle^{\frac{1}{2}}\mathcal{F}(\eta_0(T^{-\widetilde{\theta}}N_i^{\widetilde{\theta}}t - \widetilde{m})P_{k_i}(\widetilde{u})), \quad i = 1,2,3,4
$$

Here $\widetilde{u} = w$ or u or $\overline{w}$ or $\overline{u}$.

Proof By duality and the Plancheral identity, it suffices to show

$$
\begin{aligned}
&\|m((\xi_1,\tau_1),(\xi_2,\tau_2),(\xi_3,\tau_3),(\xi_4,\tau_4))\|_{[4,\mathbb{R}^n\times\mathbb{R}]} \\
&:= \Bigg\| \frac{\frac{T^{-\widetilde{\theta}}N_{\mathrm{soprano}}^{\widetilde{\theta}}N_{\mathrm{tenor}}\langle\xi_4\rangle^s}{\langle\xi_1\rangle^s\langle\xi_2\rangle^s\langle\xi_3\rangle^s}}{\langle\mathcal{L}_1\rangle^{\frac{1}{2}}\langle\mathcal{L}_2\rangle^{\frac{1}{2}}\langle\mathcal{L}_3\rangle^{\frac{1}{2}}\langle\mathcal{L}_4\rangle^{\frac{1}{2}}} \Bigg\|_{[4,\mathbb{R}^n\times\mathbb{R}]} \lesssim \langle N_{\mathrm{tenor}}\rangle^{-\varepsilon}T^\theta. \qquad (7.15)
\end{aligned}
$$

Similarly to the proof of Lemma 5.1. Then using Lemmas 3.1–3.3, we have for $s \geq \frac{n}{2} + \varepsilon$,

$$\begin{aligned}
&\|m((\xi_1,\tau_1),(\xi_2,\tau_2),(\xi_3,\tau_3),(\xi_4,\tau_4))\|_{[4,\mathbb{R}^n\times\mathbb{R}]} \\
&\lesssim \langle N_1\rangle^{-s} N_{\text{tenor}}^{-s} T^{-\widetilde{\theta}} N_{\text{soprano}}^{\widetilde{\theta}} N_{\text{tenor}} \Big\| \frac{1}{\langle \mathcal{L}_1\rangle^{\frac12}\langle \mathcal{L}_2\rangle^{\frac12}\langle \mathcal{L}_3\rangle^{\frac12}\langle \mathcal{L}_4\rangle^{\frac12}} \Big\|_{[4,\mathbb{R}^n\times\mathbb{R}]} \\
&\lesssim \langle N_1\rangle^{-s} N_{\text{tenor}}^{-s} T^{-\widetilde{\theta}} N_{\text{soprano}}^{\widetilde{\theta}} N_{\text{tenor}} \Big\| \frac{1}{\langle \mathcal{L}_1\rangle^{\frac12}\langle \mathcal{L}_3\rangle^{\frac12}} \Big\|_{[3,\mathbb{R}^n\times\mathbb{R}]} \Big\| \frac{1}{\langle \mathcal{L}_2\rangle^{\frac12}\langle \mathcal{L}_4\rangle^{\frac12}} \Big\|_{[3,\mathbb{R}^n\times\mathbb{R}]} \\
&\lesssim \langle N_1\rangle^{-s+\frac{n}{2}} N_{\text{tenor}}^{-s+\frac{n}{2}} T^{-\widetilde{\theta}} N_{\text{soprano}}^{\widetilde{\theta}} N_{\text{tenor}} N_{\text{soprano}}^{-1} \lesssim \langle N_{\text{tenor}}\rangle^{-\varepsilon} T^{\theta}. \qquad (7.16)
\end{aligned}$$

This completes the proof of Lemma 7.1.

For D_k, we can use the following Lemma 7.2 to consider them.

Lemma 7.2 (Dyadic energy estimates-II) *Let $s > \frac{n}{2}$. Assume that $|\xi_l| \sim N_l = 2^{k_l}$, $L_l = |\sigma_l| = |\tau_l - \phi(\xi_l)| = 2^{j_l}$. Let $\mathcal{L}_l = \max\{L_l, T^{-\widetilde{\theta}} N_{\text{soprano}}^{\widetilde{\theta}}\}$, $l = 1,2,3,4$. Assume $N_{\text{soprano}} \gtrsim T^{-\frac14}$. Then for some small $\theta > 0$,*

$$\begin{aligned}
&\frac{1}{\beta_{(j_1,k_1)}\beta_{(j_2,k_2)}\beta_{(j_3,k_3)}\beta_{(j_4,k_4)}} \\
&\cdot \int_\star \frac{T^{-\widetilde{\theta}} N_{\text{soprano}}^{\widetilde{\theta}} N_{\text{soprano}} \langle \xi_4\rangle^s}{\langle \xi_1\rangle^s \langle \xi_2\rangle^s \langle \xi_3\rangle^s} \frac{f_1(\tau_1,\xi_1) f_2(\tau_2,\xi_2) f_3(\tau_3,\xi_3) f_4(\tau_4,\xi_4)}{\langle \mathcal{L}_1\rangle^{\frac12}\langle \mathcal{L}_2\rangle^{\frac12}\langle \mathcal{L}_3\rangle^{\frac12}\langle \mathcal{L}_4\rangle^{\frac12}} \mathrm{d}\delta \\
&\lesssim \langle N_{\text{tenor}}\rangle^{-\varepsilon} T^{\theta} N_{\text{tenor}}^{-\widetilde{\theta}} N_{\text{soprano}}^{\widetilde{\theta}} \prod_{j=1}^{4} \|f_j\|_{L_2}. \qquad (7.17)
\end{aligned}$$

Proof Similarly to the proof of Lemma 7.1, we have the results. This completes the proof of Lemma 7.2.

Thus using Lemmas 7.1–7.2, we have

$$\begin{aligned}
&\sum_k (A_k^2 + A_k^3 + B_k^2 + B_k^3 + [A_k^{11} + B_k^{11}] + [A_k^{12} + B_k^{12}] + D_k) \\
&\lesssim T^{\theta} \|u\|_{\mathbf{F}^s}^2 \|w\|_{\mathbf{F}^s}^2 + T^{\theta} \|u\|_{\mathbf{F}^s} \|w\|_{\mathbf{F}^s} \|w\|_{\mathbf{F}^{s-\widetilde{\theta}}} \|u\|_{\mathbf{F}^{s+\widetilde{\theta}}}. \qquad (7.18)
\end{aligned}$$

Thus using (7.7) and (7.18), we have

$$\begin{aligned}
2^{2sk} \|P_k w(t)\|_{L_x^2}^2 \lesssim{}& 2^{2sk} \|w_0\|_{L_x^2}^2 + T^{\theta}(\|u\|_{\mathbf{F}^s}^2 + \|w\|_{\mathbf{F}^s}^2) \|w\|_{\mathbf{F}^s} \|w\|_{\mathbf{F}^s} \\
&+ T^{\theta}(\|u\|_{\mathbf{F}^s} + \|w\|_{\mathbf{F}^s}) \|w\|_{\mathbf{F}^{s-\widetilde{\theta}}} \|u\|_{\mathbf{F}^{s+\widetilde{\theta}}}. \qquad (7.19)
\end{aligned}$$

This completes the proof Theorem 7.2.

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