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Turnpike Properties for Stochastic Linear-Quadratic Optimal Control Problems*

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Abstract This paper analyzes the limiting behavior of stochastic linear-quadratic optimal control problems in finite time-horizon $[0, T]$ as $T \to \infty$. The so-called turnpike properties are established for such problems, under stabilizability condition which is weaker than the controllability, normally imposed in the similar problem for ordinary differential systems. In dealing with the turnpike problem, a crucial issue is to determine the corresponding static optimization problem. Intuitively mimicking the deterministic situations, it seems to be natural to include both the drift and the diffusion expressions of the state equation to be zero as constraints in the static optimization problem. However, this would lead us to a wrong direction. It is found that the correct static problem should contain the diffusion as a part of the objective function, which reveals a deep feature of the stochastic turnpike problem.

Keywords Turnpike property, Stochastic optimal control, Static optimization, Linear-quadratic, Stabilizability, Riccati equation
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1 Introduction

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space on which a standard one-dimensional Brownian motion $W = \{W(t) \mid t \geq 0\}$ is defined. Denote by $\mathbb{F} = \{\mathscr{F}_t\}_{t\geq 0}$ the usual augmentation of the natural filtration generated by W. For a random variable ξ, we write $\xi \in \mathscr{F}_t$ if ξ is $\mathscr{F}_t$-measurable; and for a stochastic process X, we write $X \in \mathbb{F}$ if it is progressively measurable with respect to the filtration $\mathbb{F}$.

Consider the following controlled linear stochastic differential equation (SDE for short)

$$\begin{cases} \mathrm{d}X(t) = [AX(t) + Bu(t) + b]\mathrm{d}t + [CX(t) + Du(t) + \sigma]\mathrm{d}W(t), \quad t \geq 0, \\ X(0) = x, \end{cases} \tag{1.1}$$

and the following general quadratic cost functional

$$J_T(x; u(\cdot)) = \frac{1}{2}\,\mathbb{E}\int_0^T [\langle QX(t), X(t)\rangle + 2\langle SX(t), u(t)\rangle + \langle Ru(t), u(t)\rangle$$

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$$+ 2\langle q, X(t)\rangle + 2\langle r, u(t)\rangle]\mathrm{d}t, \tag{1.2}$$

where $A, C \in \mathbb{R}^{n\times n}$, $B, D \in \mathbb{R}^{n\times m}$, $Q \in \mathbb{S}^n$, $S \in \mathbb{R}^{m\times n}$, $R \in \mathbb{S}^m$, $b, \sigma, q \in \mathbb{R}^n$, and $r \in \mathbb{R}^m$ are constant matrices or vectors with $\mathbb{S}^k$ being the set of all $(k \times k)$ symmetric real matrices. The classical stochastic linear-quadratic (LQ for short) optimal control problem over the finite time-horizon $[0, T]$ is to find a control $\overline{u}_T(\cdot)$ from the space

$$\mathscr{U}[0,T] = \Big\{u : [0,T] \times \Omega \to \mathbb{R}^m \Big| u \in \mathbb{F} \text{ and } \mathbb{E}\int_0^T |u(t)|^2 \mathrm{d}t < \infty\Big\} \tag{1.3}$$

such that the cost functional (1.2) is minimized over $\mathscr{U}[0, T]$, for a given initial state $x \in \mathbb{R}^n$. More precisely, it can be stated as follows.

Problem (SLQ)$_T$ For any given initial state $x \in \mathbb{R}^n$, find a control $\overline{u}_T(\cdot) \in \mathscr{U}[0, T]$ such that

$$J_T(x; \overline{u}_T(\cdot)) = \inf_{u(\cdot)\in\mathscr{U}[0,T]} J_T(x; u(\cdot)) \equiv V_T(x). \tag{1.4}$$

The process $\overline{u}_T(\cdot)$ in (1.4) (if exists) is called an open-loop optimal control of Problem (SLQ)$_T$ for the initial state x, the corresponding state process $\overline{X}_T(\cdot)$ is called an open-loop optimal state process, $(\overline{X}_T(\cdot), \overline{u}_T(\cdot))$ an open-loop optimal pair, and $V_T(\cdot)$ the value function of Problem (SLQ)$_T$.

In this paper, we are concerned with the limiting behavior of the optimal pair $(\overline{X}_T(\cdot), \overline{u}_T(\cdot))$ of Problem (SLQ)$_T$ as the time-horizon T tends to infinity. More precisely, we want to seek conditions under which there exist positive constants $K, \mu > 0$, independent of T, such that for some $(x^*, u^*) \in \mathbb{R}^n \times \mathbb{R}^m$, it holds

$$|\mathbb{E}[\overline{X}_T(t) - x^*]| + |\mathbb{E}[\overline{u}_T(t) - u^*]| \le K[\mathrm{e}^{-\mu t} + \mathrm{e}^{-\mu(T-t)}], \quad \forall t \in [0, T]. \tag{1.5}$$

This is referred to as the exponential turnpike property of Problem (SLQ)$_T$. Such a property implies that for any small $\delta \in \big(0, \frac{1}{2}\big)$, the following is true:

$$|\mathbb{E}[\overline{X}_T(t) - x^*]| + |\mathbb{E}[\overline{u}_T(t) - u^*]| \le 2K\mathrm{e}^{-\mu\delta T}, \quad \forall t \in [\delta T, (1-\delta)T]. \tag{1.6}$$

Namely, in a big portion $[\delta T, (1 - \delta)T]$ of time interval $[0, T]$, the optimal pair $(\overline{X}_T(\cdot), \overline{u}_T(\cdot))$ is exponentially close to the point (x^*, u^*). This will give us an essential picture of the optimal pair without solving the problem analytically, which is very useful in applications.

The study of turnpike phenomena for deterministic problems can be traced back to the work of von Neumann [15] on problems in economics. In 1958, Dorfman, Samuelson and Solow [5] coined the name "turnpike" which was used in the highway system of the United States. Since then the turnpike phenomena have attracted considerable attentions, not only in mathematical economy (see [14]), but also in many other fields such as mathematical biology (see [9]) and chemical processes (see [19]). It is well-known by now that the turnpike property is a general phenomenon which holds for a large class of variational and optimal control problems. Numerous relevant results have been established for finite and infinite dimensional problems in the context of deterministic discrete-time and continuous-time systems (see, e.g., [2–3, 7, 13, 23–24, 27–30] and the references cited therein). In particular, we mention the papers [4, 6] for discrete-time LQ problems and the papers [17–18] for continuous-time LQ problems of ordinary differential equations.

The study of turnpike phenomena for stochastic optimal control problems is quite lacking in literature. In this paper, we shall carry out a thorough investigation on the turnpike property for the stochastic LQ optimal control problem introduced earlier. Note that when $C = 0$, $D = 0$ and $\sigma = 0$, Problem $(\text{SLQ})_T$ reduces to a deterministic LQ problem, for which the exponential turnpike property has been established in [17] and [24] under controllability and observability assumptions. For the deterministic LQ problem (i.e., the case of $C = 0, D = 0, \sigma = 0$), the associated static optimization problem, which is used to determine the point (x^*, u^*), reads

$$\begin{cases} \text{Minimize} \quad F_0(x,u) \equiv \dfrac{1}{2}[\langle Qx, x\rangle + 2\langle Sx, u\rangle + \langle Ru, u\rangle + 2\langle q, x\rangle + 2\langle r, u\rangle], \\ \text{subject to} \quad Ax + Bu + b = 0. \end{cases} \tag{1.7}$$

To establish the turnpike property for the stochastic LQ problem, suggested by the deterministic situation, one might naively introduce the following static optimization problem:

$$\begin{cases} \text{Minimize} \quad F_0(x,u), \\ \text{subject to} \quad Ax + Bu + b = 0, \quad Cx + Du + \sigma = 0. \end{cases} \tag{1.8}$$

Assume the above admits an optimal solution (x^*, u^*). Then one tries to show that the optimal pair $(\overline{X}_T(\cdot), \overline{u}_T(\cdot))$ of Problem $(\text{SLQ})_T$ satisfies (1.5). However, by a little careful observation of the above, one immediately realizes that (1.8) seems to be not natural because the condition that ensuring such an optimization problem to be feasible is already very restrictive: The two equality constraints might be contradicting each other. It turns out that (1.8) is not the correct one, which will be shown later in this paper. As a main contribution of this paper, we find that the correct formulation of the static optimization problem is as follows:

$$\begin{cases} \text{Minimize} \quad F(x,u) \equiv F_0(x,u) + \dfrac{1}{2}[\langle P(Cx + Du + \sigma), Cx + Du + \sigma\rangle], \\ \text{subject to} \quad Ax + Bu + b = 0, \end{cases} \tag{1.9}$$

where P is a positive definite solution to the following algebraic Riccati equation (ARE for short):

$$\begin{aligned} &PA + A^{\mathrm{T}}P + C^{\mathrm{T}}PC + Q \\ &\quad - (PB + C^{\mathrm{T}}PD + S^{\mathrm{T}})(R + D^{\mathrm{T}}PD)^{-1}(B^{\mathrm{T}}P + D^{\mathrm{T}}PC + S) = 0. \end{aligned} \tag{1.10}$$

By assuming the controlled homogenous state equation (denoted by $[A, C; B, D]$) to be stabilizable (see the next section for a precise definition) and the following strong standard condition:

$$R > 0, \quad Q - S^{\mathrm{T}}R^{-1}S > 0, \tag{1.11}$$

one will have a unique positive definite solution P to the above ARE (1.10), and problem (1.9) is not only feasible, but also admits a unique solution (x^*, u^*). We will show that there exist positive constants $K, \mu > 0$, independent of T, such that (1.5) holds, and the adjoint process $\overline{Y}_T(\cdot)$ will also have the same turnpike property. Note that by a (classical) standard condition, we mean that $Q - S^{\mathrm{T}}R^{-1}S$ is merely positive semidefinite, which could even be 0. In such cases, P might not be positive definite, and $(x,u) \mapsto F(x,u)$ might not be coercive. Therefore, it is unclear if the optimal solution (x^*, u^*) exists, or it might not be unique. This might bring some additional issues into the study and we will try to address that in our future publications. Also, we note that for the state equation, stabilizability is strictly weaker than the (null) controllability which was assumed in [13, 17, 24] for deterministic problems. For the study of controllability of linear SDEs, see [25].

The rest of the paper is organized as follows. In Section 2, we give the preliminaries and collect some relevant results on stochastic LQ optimal control problems. In Section 3 we recall the notion of stabilizability and formulate the correct static optimization problem. The convergence of the solution to a related differential Riccati equation as the time-horizon tends to infinity will be presented in Section 4. In Section 5, we study the static optimization problem associated to Problem $(\mathrm{SLQ})_T$ and establish the turnpike property of Problem $(\mathrm{SLQ})_T$ as well as of the adjoint process. Some concluding remarks are collected in Section 6.

2 Preliminaries

We begin with some notation that will be frequently used in the sequel. Let $\mathbb{R}^{n\times m}$ be the space of $(n\times m)$ real matrices equipped with the Frobenius inner product

$$\langle M,N\rangle=\mathrm{tr}\,(M^{\mathrm{T}}N),\quad M,N\in\mathbb{R}^{n\times m},$$

where M^{T} denotes the transpose of M and $\mathrm{tr}\,(M^{\mathrm{T}}N)$ is the trace of $M^{\mathrm{T}}N$. The norm induced by the Frobenius inner product is denoted by $|\cdot|$. For a subset $\mathbb{H}$ of $\mathbb{R}^{n\times m}$, we denote by $C([0,T];\mathbb{H})$ the space of continuous functions from $[0,T]$ map into $\mathbb{H}$, and by $L^\infty(0,T;\mathbb{H})$ the space of Lebesgue measurable, essentially bounded functions from $[0,T]$ map into $\mathbb{H}$. Let $\mathbb{S}^n$ be the subspace of $\mathbb{R}^{n\times n}$ consisting of symmetric matrices and $\mathbb{S}^n_+$ the subset of $\mathbb{S}^n$ consisting of positive definite matrices. For $\mathbb{S}^n$-valued functions $M(\cdot)$ and $N(\cdot)$, we write $M(\cdot)\geq N(\cdot)$ (respectively, $M(\cdot)>N(\cdot)$) if $M(\cdot)-N(\cdot)$ is positive semidefinite (respectively, positive definite) almost everywhere with respect to the Lebesgue measure. The identity matrix of size n is denoted by I_n, and a vector always refers to a column vector if not specified. Also, recall that $W=\{W(t)\mid t\geq 0\}$ is a standard one-dimensional Brownian motion, $\mathbb{F}=\{\mathscr{F}_t\}_{t\geq 0}$ is the usual augmentation of the natural filtration generated by W, and that $\mathscr{U}[0,T]$ is the space of $\mathbb{R}^m$-valued, $\mathbb{F}$-progressively measurable, square-integrable processes over $[0,T]$.

For the purpose of later presentation, we recall some results of time-variant stochastic LQ problem in finite time-horizon. Consider the state equation

$$\begin{cases}\mathrm{d}\mathbf{X}(t)=[\mathbf{A}(t)\mathbf{X}(t)+\mathbf{B}(t)\mathbf{u}(t)+\mathbf{b}(t)]\mathrm{d}t+[\mathbf{C}(t)\mathbf{X}(t)+\mathbf{D}(t)\mathbf{u}(t)+\boldsymbol{\sigma}(t)]\mathrm{d}W(t),\\ \mathbf{X}(0)=x\end{cases}\tag{2.1}$$

with the cost functional

$$\begin{aligned}\mathbf{J}(x;\mathbf{u}(\cdot))=\frac{1}{2}\mathbb{E}\Big\{&\langle\mathbf{G}\mathbf{X}(T),\mathbf{X}(T)\rangle+2\langle\mathbf{g},\mathbf{X}(T)\rangle\\&+\int_0^T\Big[\Big\langle\begin{pmatrix}\mathbf{Q}(t)&\mathbf{S}(t)^{\mathrm{T}}\\ \mathbf{S}(t)&\mathbf{R}(t)\end{pmatrix}\begin{pmatrix}\mathbf{X}(t)\\ \mathbf{u}(t)\end{pmatrix},\begin{pmatrix}\mathbf{X}(t)\\ \mathbf{u}(t)\end{pmatrix}\Big\rangle+2\Big\langle\begin{pmatrix}\mathbf{q}(t)\\ \mathbf{r}(t)\end{pmatrix},\begin{pmatrix}\mathbf{X}(t)\\ \mathbf{u}(t)\end{pmatrix}\Big\rangle\Big]\mathrm{d}t\Big\},\end{aligned}\tag{2.2}$$

where in (2.1), the coefficients satisfy

$$\mathbf{A}(\cdot),\mathbf{C}(\cdot)\in L^\infty(0,T;\mathbb{R}^{n\times n}),\quad \mathbf{B}(\cdot),\mathbf{D}(\cdot)\in L^\infty(0,T;\mathbb{R}^{n\times m}),\quad \mathbf{b}(\cdot),\boldsymbol{\sigma}(\cdot)\in L^\infty(0,T;\mathbb{R}^{n}),$$

and in (2.2), the weighting coefficients satisfy

$$\begin{aligned}&\mathbf{G}\in\mathbb{S}^n,\quad \mathbf{Q}(\cdot)\in L^\infty(0,T;\mathbb{S}^n),\quad \mathbf{S}(\cdot)\in L^\infty(0,T;\mathbb{R}^{m\times n}),\quad \mathbf{R}(\cdot)\in L^\infty(0,T;\mathbb{S}^m),\\&\mathbf{g}\in\mathbb{R}^n,\quad \mathbf{q}(\cdot)\in L^\infty(0,T;\mathbb{R}^n),\quad \mathbf{r}(\cdot)\in L^\infty(0,T;\mathbb{R}^m).\end{aligned}$$

The standard stochastic LQ optimal control problem on $[0,T]$ can be stated as follows.

Problem (SLQ) For a given initial state $x\in\mathbb{R}^n$, find a control $\overline{\mathbf{u}}(\cdot)\in\mathscr{U}[0,T]$ such that

$$\mathbf{J}(x;\overline{\mathbf{u}}(\cdot))=\inf_{\mathbf{u}(\cdot)\in\mathscr{U}[0,T]}\mathbf{J}(x;\mathbf{u}(\cdot))\equiv\mathbf{V}(x). \tag{2.3}$$

The process $\overline{\mathbf{u}}(\cdot)$ (if exists) in (2.3) is called an (open-loop) optimal control for the initial state x, and $\mathbf{V}(x)$ is called the value of Problem (SLQ) at x.

The following lemma summarizes a few results for Problem (SLQ). For proofs, the reader is referred to the book [22] by Sun and Yong; see also [20].

Lemma 2.1 *Suppose that for some constant $\delta>0$,*

$$\mathbf{G}\geq 0,\quad \mathbf{R}(t)\geq\delta I_m,\quad \mathbf{Q}(t)-\mathbf{S}(t)^{\mathrm{T}}\mathbf{R}(t)^{-1}\mathbf{S}(t)\geq 0. \tag{2.4}$$

Then, the following hold:

(i) *For every initial state x, Problem (SLQ) has a unique open-loop optimal control.*

(ii) *A pair $(\overline{\mathbf{X}}(\cdot),\overline{\mathbf{u}}(\cdot))$ is an open-loop optimal pair of Problem (SLQ) for the initial state x if and only if there exists a pair $(\overline{\mathbf{Y}}(\cdot),\overline{\mathbf{Z}}(\cdot))$ of adapted processes such that*

$$\begin{cases}\mathrm{d}\overline{\mathbf{X}}(t)=(\mathbf{A}\overline{\mathbf{X}}+\mathbf{B}\overline{\mathbf{u}}+\mathbf{b})\mathrm{d}t+(\mathbf{C}\overline{\mathbf{X}}+\mathbf{D}\overline{\mathbf{u}}+\boldsymbol{\sigma})\mathrm{d}W,\\ \mathrm{d}\overline{\mathbf{Y}}(t)=-(\mathbf{A}^{\mathrm{T}}\overline{\mathbf{Y}}+\mathbf{C}^{\mathrm{T}}\overline{\mathbf{Z}}+\mathbf{Q}\overline{\mathbf{X}}+\mathbf{S}^{\mathrm{T}}\overline{\mathbf{u}}+\mathbf{q})\mathrm{d}t+\overline{\mathbf{Z}}\mathrm{d}W,\\ \overline{\mathbf{X}}(0)=x,\quad \overline{\mathbf{Y}}(T)=\mathbf{G}\overline{\mathbf{X}}(T)+\mathbf{g},\end{cases} \tag{2.5}$$

(t is suppressed on the right-hand of (2.5)) and the following condition holds:

$$\mathbf{B}(t)^{\mathrm{T}}\overline{\mathbf{Y}}(t)+\mathbf{D}(t)^{\mathrm{T}}\overline{\mathbf{Z}}(t)+\mathbf{S}(t)\overline{\mathbf{X}}(t)+\mathbf{R}(t)\overline{\mathbf{u}}(t)+\mathbf{r}(t)=0,\quad \textit{a.e. } t\in[0,T],\ \textit{a.s.} \tag{2.6}$$

(iii) *The differential Riccati equation (t is also suppressed below)*

$$\begin{cases}\dot{\mathbf{P}}+\mathbf{P}\mathbf{A}+\mathbf{A}^{\mathrm{T}}\mathbf{P}+\mathbf{C}^{\mathrm{T}}\mathbf{P}\mathbf{C}+\mathbf{Q}\\ -(\mathbf{P}\mathbf{B}+\mathbf{C}^{\mathrm{T}}\mathbf{P}\mathbf{D}+\mathbf{S}^{\mathrm{T}})(\mathbf{R}+\mathbf{D}^{\mathrm{T}}\mathbf{P}\mathbf{D})^{-1}(\mathbf{B}^{\mathrm{T}}\mathbf{P}+\mathbf{D}^{\mathrm{T}}\mathbf{P}\mathbf{C}+\mathbf{S})=0,\\ \mathbf{P}(T)=\mathbf{G}\end{cases} \tag{2.7}$$

admits a unique positive semidefinite solution $\mathbf{P}(\cdot)\in C([0,T];\mathbb{S}^n)$. In particular, if

$$\mathbf{Q}(t)-\mathbf{S}(t)^{\mathrm{T}}\mathbf{R}(t)^{-1}\mathbf{S}(t)>0,$$

then $\mathbf{P}(t)>0$ for all $t\in[0,T)$.

(iv) *The unique open-loop optimal control $\overline{\mathbf{u}}(\cdot)$ for the initial state x is given by*

$$\overline{\mathbf{u}}(t)=\boldsymbol{\Theta}(t)\overline{\mathbf{X}}(t)-[\mathbf{R}(t)+\mathbf{D}(t)^{\mathrm{T}}\mathbf{P}(t)\mathbf{D}(t)]^{-1}[\mathbf{B}(t)^{\mathrm{T}}\boldsymbol{\varphi}(t)+\mathbf{D}(t)^{\mathrm{T}}\mathbf{P}(t)\boldsymbol{\sigma}(t)+\mathbf{r}(t)],$$

where

$$\boldsymbol{\Theta}(t)=-[\mathbf{R}(t)+\mathbf{D}(t)^{\mathrm{T}}\mathbf{P}(t)\mathbf{D}(t)]^{-1}[\mathbf{B}(t)^{\mathrm{T}}\mathbf{P}(t)+\mathbf{D}(t)^{\mathrm{T}}\mathbf{P}(t)\mathbf{C}(t)+\mathbf{S}(t)],$$

and $\boldsymbol{\varphi}(\cdot)$ is the solution to the terminal value problem of the ordinary differential equation (ODE for short)

$$\begin{cases}\dot{\boldsymbol{\varphi}}(t)+[\mathbf{A}(t)+\mathbf{B}(t)\boldsymbol{\Theta}(t)]^{\mathrm{T}}\boldsymbol{\varphi}(t)+[\mathbf{C}(t)+\mathbf{D}(t)\boldsymbol{\Theta}(t)]^{\mathrm{T}}\mathbf{P}(t)\boldsymbol{\sigma}(t)\\ +\boldsymbol{\Theta}(t)^{\mathrm{T}}\mathbf{r}(t)+\mathbf{P}(t)\mathbf{b}(t)+\mathbf{q}(t)=0,\quad t\in[0,T],\\ \boldsymbol{\varphi}(T)=\mathbf{g}.\end{cases} \tag{2.8}$$

(v) *The value function is given by*

$$\mathbf{V}(x) = \frac{1}{2}\langle \mathbf{P}(0)x, x\rangle + \langle \boldsymbol{\varphi}(0), x\rangle + \frac{1}{2}\int_0^T \{\langle \mathbf{P}(t)\boldsymbol{\sigma}(t), \boldsymbol{\sigma}(t)\rangle + 2\langle \boldsymbol{\varphi}(t), \mathbf{b}(t)\rangle \\ - |[\mathbf{R}(t) + \mathbf{D}(t)^{\mathrm{T}}\mathbf{P}(t)\mathbf{D}(t)]^{-\frac{1}{2}}[\mathbf{B}(t)^{\mathrm{T}}\boldsymbol{\varphi}(t) + \mathbf{D}(t)^{\mathrm{T}}\mathbf{P}(t)\boldsymbol{\sigma}(t) + \mathbf{r}(t)]|^2\}\mathrm{d}t.$$

Now, we return to our Problem $(\mathrm{SLQ})_T$. Let us make a reduction under the strong standard condition (1.11). Set

$$v(t) = u(t) + R^{-1}[SX(t) + r], \quad t \in [0, T]. \tag{2.9}$$

We observe the following (t is suppressed):

$$\begin{aligned} &\langle QX, X\rangle + 2\langle SX, u\rangle + \langle Ru, u\rangle + 2\langle q, X\rangle + 2\langle r, u\rangle \\ &= \langle (Q - S^{\mathrm{T}}R^{-1}S)X, X\rangle + \langle R(u + R^{-1}SX), u + R^{-1}SX\rangle \\ &\quad + 2\langle q - S^{\mathrm{T}}R^{-1}r, X\rangle + 2\langle r, u + R^{-1}SX\rangle \\ &= \langle \widehat{Q}X, X\rangle + \langle Rv, v\rangle + 2\langle q - S^{\mathrm{T}}R^{-1}r, X\rangle - \langle R^{-1}r, r\rangle \\ &\equiv \langle \widehat{Q}X, X\rangle + \langle Rv, v\rangle + 2\langle \widehat{q}, X\rangle - \varphi_0, \end{aligned}$$

where

$$\widehat{Q} = Q - S^{\mathrm{T}}R^{-1}S, \quad \widehat{q} = q - S^{\mathrm{T}}R^{-1}r, \quad \varphi_0 = r^{\mathrm{T}}R^{-1}r.$$

Also, under (2.9), one has

$$\begin{aligned} AX + Bu + b &= (A - BR^{-1}S)X + Bv + b - BR^{-1}r \equiv \widehat{A}X + Bv + \widehat{b}, \\ CX + Du + \sigma &= (C - DR^{-1}S)X + Dv + \sigma - DR^{-1}r \equiv \widehat{C}X + Dv + \widehat{\sigma}, \end{aligned}$$

where

$$\begin{aligned} \widehat{A} &= A - BR^{-1}S, \quad \widehat{b} = b - BR^{-1}r, \\ \widehat{C} &= C - DR^{-1}S, \quad \widehat{\sigma} = \sigma - DR^{-1}r. \end{aligned}$$

From the above reduction, we end up with the following state equation:

$$\begin{cases} \mathrm{d}X(t) = [\widehat{A}X(t) + Bv(t) + \widehat{b}]\mathrm{d}t + [\widehat{C}X(t) + Dv(t) + \widehat{\sigma}]\mathrm{d}W(t), \\ X(0) = x \end{cases} \tag{2.10}$$

with the cost functional

$$J_T(x; v(\cdot)) = \frac{1}{2}\mathbb{E}\int_0^T [\langle \widehat{Q}X(t), X(t)\rangle + \langle Rv(t), v(t)\rangle + 2\langle \widehat{q}, X(t)\rangle - \varphi_0]\mathrm{d}t. \tag{2.11}$$

It is clear that the (open-loop) optimal pair of the LQ problem associated with (2.10)–(2.11) is the same as that of the LQ problem associated with the state equation (2.10) and the cost functional

$$\widehat{J}_T(x; v(\cdot)) = \frac{1}{2}\mathbb{E}\int_0^T [\langle \widehat{Q}X(t), X(t)\rangle + \langle Rv(t), v(t)\rangle + 2\langle \widehat{q}, X(t)\rangle]\mathrm{d}t.$$

Because of the above reduction, one sees that it suffices to consider Problem $(\mathrm{SLQ})_T$ for the state equation (1.1) with the following cost functional:

$$J_T(x; u(\cdot)) = \frac{1}{2}\mathbb{E}\int_0^T [\langle QX(t), X(t)\rangle + \langle Ru(t), u(t)\rangle + 2\langle q, X(t)\rangle]\mathrm{d}t. \tag{2.12}$$

In the case that $b = \sigma = q = 0$, we denote the corresponding optimal control problem by Problem $(\mathrm{SLQ})_T^0$ and call it a homogenous LQ problem on $[0,T]$. The value function of Problem $(\mathrm{SLQ})_T^0$ is denoted by $V_T^0(x)$.

3 Stabilizability and the Static Optimization Problem

In this section, we are going to make some further preparations.

Let us denote by $[A,C]$ the following linear homogeneous uncontrolled SDE:

$$\mathrm{d}X(t) = AX(t)\mathrm{d}t + CX(t)\mathrm{d}W(t), \quad t \geq 0.$$

For every $x \in \mathbb{R}^n$, there exists a unique solution $X(\cdot) \equiv X(\cdot\,; x)$ of the above satisfying $X(0;x) = x$. We recall the following classical notion.

Definition 3.1 *System $[A,C]$ is said to be*

(i) *L^2-stable if*

$$\mathbb{E}\int_0^\infty |X(t;x)|^2 \mathrm{d}t < \infty, \quad \forall x \in \mathbb{R}^n;$$

(ii) *mean-square exponentially stable if there exists a $\beta > 0$ such that*

$$\sup_{t\in[0,\infty)} \mathrm{e}^{\beta t}\mathbb{E}|X(t;x)|^2 < \infty, \quad \forall x \in \mathbb{R}^n.$$

To characterize the above notions, we let $\Phi(\cdot)$ be the solution to the matrix SDE

$$\begin{cases} \mathrm{d}\Phi(t) = A\Phi(t)\mathrm{d}t + C\Phi(t)\mathrm{d}W(t), \quad t \geq 0, \\ \Phi(0) = I_n. \end{cases} \tag{3.1}$$

We have the following result.

Lemma 3.1 *The following are equivalent*:

(i) *The system $[A,C]$ is mean-square exponentially stable.*

(ii) *There exist constants $\alpha, \beta > 0$ such that*

$$\mathbb{E}|\Phi(t)|^2 \leq \alpha \mathrm{e}^{-\beta t}, \quad \forall t \geq 0. \tag{3.2}$$

(iii) *It holds that*

$$\mathbb{E}\int_0^\infty |\Phi(t)|^2 \mathrm{d}t < \infty. \tag{3.3}$$

(iv) *The system $[A,C]$ is L^2-stable.*

(v) *There exists a $P \in \mathbb{S}_+^n$ such that*

$$PA + A^{\mathrm{T}}P + C^{\mathrm{T}}PC < 0. \tag{3.4}$$

The proof is straightforward (see [1, 8, 21], or the book [22]).

Corollary 3.1 *If the system $[A,C]$ is L^2-stable, then*

$$|\mathrm{e}^{At}| \leq \sqrt{\alpha}\mathrm{e}^{-(\frac{\beta t}{2})}, \quad \forall t \geq 0,$$

where α and β are as in (3.2).

Proof It is easy to prove by noting that $e^{At} = \mathbb{E}[\Phi(t)]$.

Now we denote by $[A, C; B, D]$ the following controlled (homogeneous) linear system:

$$dX(t) = [AX(t) + Bu(t)]dt + [CX(t) + Du(t)]dW(t), \quad t \geq 0, \tag{3.5}$$

where $u(\cdot)$ is taken from the following set of admissible controls

$$\mathscr{U}[0, \infty) = \Big\{u : [0, \infty) \times \Omega \to \mathbb{R}^m \;\Big|\; u \in \mathbb{F} \text{ and } \mathbb{E}\int_0^\infty |u(t)|^2 dt < \infty\Big\}.$$

We recall the following notion (see [8, 21–22]).

Definition 3.2 *The system $[A, C; B, D]$ is said to be L^2-stabilizable if there exists a matrix $\Theta \in \mathbb{R}^{m\times n}$ such that the (closed-loop) system $[A + B\Theta, C + D\Theta]$ is L^2-stable. In this case, Θ is called a stabilizer of $[A, C; B, D]$.*

We point out that the L^2-stabilizability is a weaker condition than the null controllability, meaning that for every initial state x there exists a control to steer the system state from x to 0 in a finite time interval. In fact, according to [21, Theorem 3.2], the system $[A, C; B, D]$ is L^2-stabilizable if and only if for any initial state x, there exists a process $u(\cdot) \in \mathscr{U}[0, \infty)$ such that the solution $X(\cdot\,; x, u(\cdot))$ of (3.5) is also square-integrable over $[0, \infty)$. If the system $[A, C; B, D]$ is null controllable, then for any $x \in \mathbb{R}^n$, we can find a $v(\cdot) \in \mathscr{U}[0, T]$ for some $T > 0$, such that $X(T; x, v(\cdot)) = 0$. Thus, with

$$u(t) \triangleq \begin{cases} v(t), & t \in [0, T], \\ 0, & t > T, \end{cases}$$

the corresponding solution $X(\cdot) \equiv X(\cdot\,; x, u(\cdot))$ satisfies $\mathbb{E}\int_0^\infty |X(t)|^2 dt < \infty$. This, together with Example 3.1 below, shows that the null controllability is strictly stronger than the L^2-stabilizability.

Example 3.1 Consider the following one-dimensional linear system:

$$dX(t) = [X(t) + u(t)]dt + [X(t) + u(t)]dW(t), \quad t \geq 0. \tag{3.6}$$

Clearly, $\Theta = -2$ is a stabilizer of this system since

$$2(1 + \Theta) + (1 + \Theta)^2 = -1 < 0.$$

However, for any nonzero initial state x, the state cannot be transferred to 0 by any control. Indeed, the solution of (3.6) with initial state x has the following explicit representation:

$$X(t) = \Phi(t)\Big[x + \int_0^t \Phi(s)^{-1} u(s) dW(s)\Big],$$

where $\Phi(t) = e^{W(t)+\frac{t}{2}}$. If $X(t) = 0$ for some $t > 0$, then

$$x = -\int_0^t \Phi(s)^{-1} u(s) dW(s).$$

Taking expectations gives $x = 0$, a contradiction.

Controllability for linear ODEs is very standard in control theory, see [10]. For linear SDEs, the situation is much more complicated, see [12, 16, 25] for some known results.

We now introduce the following hypotheses.

(H1) The system $[A, C; B, D]$ is L^2-stabilizable.

(H2) The weighting matrices Q and R are positive definite, i.e., $Q \in \mathbb{S}^n_+$ and $R \in \mathbb{S}^m_+$.

Under the above (H1)–(H2), we may consider the state equation (3.5) with the cost functional

$$J^0_\infty(x; u(\cdot)) = \frac{1}{2}\mathbb{E}\int_0^\infty [\langle QX(t), X(t)\rangle + \langle Ru(t), u(t)\rangle]\mathrm{d}t.$$

We could formulate the following homogeneous stochastic LQ problem in the infinite time-horizon $[0, \infty)$.

Problem (SLQ)$^0_\infty$ For each $x \in \mathbb{R}^n$, find $\overline{u}(\cdot) \in \mathscr{U}[0, \infty)$ such that

$$J^0_\infty(x; \overline{u}(\cdot)) = \inf_{u(\cdot)\in\mathscr{U}[0,\infty)} J^0_\infty(x; u(\cdot)) \equiv V^0_\infty(x).$$

The following collects the relevant results of Problem (SLQ)$^0_\infty$. See [22] for a proof.

Proposition 3.1 *Let* (H1)–(H2) *hold. Then for each $x \in \mathbb{R}^n$, Problem $(SLQ)^0_\infty$ admits a unique open-loop optimal control $\overline{u}(\cdot)$. Moreover, the following ARE*

$$PA + A^{\mathrm{T}}P + C^{\mathrm{T}}PC + Q - (PB + C^{\mathrm{T}}PD)(R + D^{\mathrm{T}}PD)^{-1}(B^{\mathrm{T}}P + D^{\mathrm{T}}PC) = 0 \qquad (3.7)$$

admits a unique solution $P \in \mathbb{S}^n_+$ such that the open-loop optimal control $\overline{u}(\cdot)$ admits the following closed-loop representation:

$$\overline{u}(t) = -\Theta\overline{X}(t), \quad t \in [0, \infty),$$

where

$$\Theta = -(R + D^{\mathrm{T}}PD)^{-1}(B^{\mathrm{T}}P + D^{\mathrm{T}}PC)$$

is a stabilizer of $[A, C; B, D]$, and the value function has the following quadratic form:

$$V^0_\infty(x) = \frac{1}{2}\langle Px, x\rangle, \quad \forall x \in \mathbb{R}^n. \qquad (3.8)$$

In the above case, P is referred to as a stabilizing solution of the ARE with respect to the system $[A, C; B, D]$. Also, by a direct comparison, making use of (H2), we see that

$$V^0_T(x) \le V^0_\infty(x), \quad \forall x \in \mathbb{R}^n. \qquad (3.9)$$

Now, we define

$$\mathscr{V} = \{(x, u) \in \mathbb{R}^n \times \mathbb{R}^m \mid Ax + Bu + b = 0\},$$
$$F(x, u) = \frac{1}{2}[\langle Qx, x\rangle + \langle Ru, u\rangle + 2\langle q, x\rangle + \langle P(Cx + Du + \sigma), Cx + Du + \sigma\rangle],$$

and introduce the following static optimization problem.

Problem (O) Find $(x^*, u^*) \in \mathscr{V}$ such that

$$F(x^*, u^*) = \min_{(x,u)\in\mathscr{V}} F(x, u).$$

For the above static optimization problem, we have the following result.

Proposition 3.2 *Let* (H1)–(H2) *hold. Then Problem* (O) *admits a unique solution* $(x^*, u^*) \in \mathscr{V}$ *which, together with a Lagrange multiplier* $\lambda^* \in \mathbb{R}^n$, *is characterized by the following system of linear equations*:

$$\begin{cases} Qx^* + A^{\mathrm{T}}\lambda^* + C^{\mathrm{T}}P(Cx^* + Du^* + \sigma) + q = 0, \\ Ru^* + B^{\mathrm{T}}\lambda^* + D^{\mathrm{T}}P(Cx^* + Du^* + \sigma) = 0. \end{cases} \tag{3.10}$$

Proof Since $[A, C; B, D]$ is stabilizable, so must be $[A; B] \equiv [A, 0; B, 0]$. In fact, if Θ is a stabilizer of $[A, C; B, D]$, then there exists a $P \in \mathbb{S}^n_+$ such that

$$P(A + B\Theta) + (A + B\Theta)^{\mathrm{T}}P + (C + D\Theta)^{\mathrm{T}}P(C + D\Theta) < 0,$$

which leads to

$$P(A + B\Theta) + (A + B\Theta)^{\mathrm{T}}P < 0.$$

Thus, $[A; B]$ is stabilizable, which is equivalent to that the matrix $(A - \lambda I, B)$ is of full rank for any $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda \geq 0$ (see [11]). Consequently, by taking $\lambda = 0$, one sees that the matrix (A, B) has rank n. Then the feasible set $\mathscr{V}$ of Problem (O) is a non-empty closed convex set. Further, since $P > 0$,

$$F(x, u) \geq \frac{1}{2}[\langle Qx, x\rangle + \langle Ru, u\rangle + 2\langle q, x\rangle]$$

is coercive on $\mathbb{R}^n \times \mathbb{R}^m$ due to (H2). Therefore, Problem (O) admits a unique solution. Next, let

$$G(x, u) = Ax + Bu + b = (A, B)\begin{pmatrix} x \\ u \end{pmatrix} + b.$$

Then $G_{(x,u)}(x, u) = (A, B)$ is of full rank. Hence, the equality constraint is regular. Consequently, the optimal solution (x^*, u^*) can be obtained by the Lagrange multiplier method (see [26]). Now, we form Lagrange function:

$$L(x, u, \lambda) = F(x, u) + \lambda^{\mathrm{T}}(Ax + Bu + b).$$

Suppose that (x^*, u^*) is the unique optimal solution of Problem (O). Then

$$\begin{aligned} 0 &= L_x(x^*, u^*, \lambda^*)^{\mathrm{T}} = (Q + C^{\mathrm{T}}PC)x^* + C^{\mathrm{T}}PDu^* + q + C^{\mathrm{T}}P\sigma + A^{\mathrm{T}}\lambda^*, \\ 0 &= L_u(x^*, u^*, \lambda^*)^{\mathrm{T}} = (R + D^{\mathrm{T}}PD)u^* + D^{\mathrm{T}}PCx^* + D^{\mathrm{T}}P\sigma + B^{\mathrm{T}}\lambda^*. \end{aligned}$$

This leads to (3.10). Further, we may write the above as follows:

$$-\begin{pmatrix} Q + C^{\mathrm{T}}PC & C^{\mathrm{T}}PD \\ D^{\mathrm{T}}PC & R + D^{\mathrm{T}}PD \end{pmatrix}\begin{pmatrix} x^* \\ u^* \end{pmatrix} = \begin{pmatrix} C^{\mathrm{T}}P\sigma + q \\ D^{\mathrm{T}}P\sigma \end{pmatrix} + \begin{pmatrix} A^{\mathrm{T}} \\ B^{\mathrm{T}} \end{pmatrix}\lambda^*.$$

Since the coefficient matrix is invertible, there exists a unique solution (x^*, u^*) and

$$\begin{pmatrix} x^* \\ u^* \end{pmatrix} = -\begin{pmatrix} Q + C^{\mathrm{T}}PC & C^{\mathrm{T}}PD \\ D^{\mathrm{T}}PC & R + D^{\mathrm{T}}PD \end{pmatrix}^{-1}\Big[\begin{pmatrix} C^{\mathrm{T}}P\sigma + q \\ D^{\mathrm{T}}P\sigma \end{pmatrix} + \begin{pmatrix} A^{\mathrm{T}} \\ B^{\mathrm{T}} \end{pmatrix}\lambda^*\Big].$$

By the equality constraint, one has

$$b = -(Ax^* + Bu^*) = (A, B)\begin{pmatrix} Q + C^{\mathrm{T}}PC & C^{\mathrm{T}}PD \\ D^{\mathrm{T}}PC & R + D^{\mathrm{T}}PD \end{pmatrix}^{-1}\Big[\begin{pmatrix} C^{\mathrm{T}}P\sigma + q \\ D^{\mathrm{T}}P\sigma \end{pmatrix} + \begin{pmatrix} A^{\mathrm{T}} \\ B^{\mathrm{T}} \end{pmatrix}\lambda^*\Big].$$

Since $(A,B)\in\mathbb{R}^{n\times(n+m)}$ has rank n, we see that λ^* is uniquely determined by the following:

$$\lambda^* = \Big[(A,B)\begin{pmatrix} Q+C^{\rm T}PC & C^{\rm T}PD \\ D^{\rm T}PC & R+D^{\rm T}PD\end{pmatrix}^{-1}\begin{pmatrix}A^{\rm T}\\ B^{\rm T}\end{pmatrix}\Big]^{-1}$$
$$\cdot\Big[b-(A,B)\begin{pmatrix} Q+C^{\rm T}PC & C^{\rm T}PD \\ D^{\rm T}PC & R+D^{\rm T}PD\end{pmatrix}^{-1}\begin{pmatrix}C^{\rm T}P\sigma+q\\ D^{\rm T}P\sigma\end{pmatrix}\Big].$$

Hence, (x^*,u^*) is uniquely determined by (3.10).

Now we make some simple observation on problem (1.8) and Problem (O) (or (1.9)). Let

$$\mathscr{V}_0=\{(x,u)\in\mathbb{R}^n\times\mathbb{R}^m \mid Ax+Bu+b=0,\ Cx+Du+\sigma=0\}.$$

It is clear that $\mathscr{V}_0$ is a subset of $\mathscr{V}$, and $\mathscr{V}_0$ could even be empty. Moreover, if $(x^*,u^*)\in\mathscr{V}$ is an optimal solution of Problem (O) and $(x^*,u^*)\in\mathscr{V}_0$, then it is an optimal solution to the problem (1.8). However, if Problem (O) has a unique solution $(x^*,u^*)\notin\mathscr{V}_0$ (in particular, if $\mathscr{V}_0=\varnothing$), then problem (1.8) and Problem (O) are totally different. We present two illustrative examples below.

Example 3.2 Let $n=m=1$ and consider the following state equation:

$$\begin{cases} {\rm d}X(t)=u(t){\rm d}t+X(t){\rm d}W(t), \quad t\geq 0,\\ X(0)=x\end{cases}$$

with cost functional

$$J_T(x;u(\cdot))=\frac{1}{2}\mathbb{E}\int_0^T[2|X(t)|^2+|u(t)|^2+4X(t)]{\rm d}t.$$

We see that in this case,

$$A=D=0,\quad B=C=1,\quad b=\sigma=0,\quad Q=2,\quad R=1,\quad q=2.$$

Then the ARE reads:

$$0=P+2-P^2=-(P-2)(P+1).$$

Hence the positive definite solution is $P=2$. Thus, Problem (O) is equivalent to

$$\begin{cases}\text{Minimize} \quad 2x^2+2x+\frac{1}{2}u^2,\\ \text{Subject to} \quad u=0.\end{cases}$$

It is straightforward that the solution is given by

$$x^*=-\frac{1}{2},\quad u^*=0.$$

Whereas, (1.8) reads

$$\begin{cases}\text{Minimize} \quad x^2+2x+\frac{1}{2}u^2,\\ \text{Subject to} \quad x=0,\ u=0,\end{cases}$$

whose solution is trivially given by

$$\overline{x}^*=0,\quad \overline{u}^*=0.$$

Hence, the solutions to these two problems are different.

Example 3.3 Let $n=m=1$ and consider the following state equation:

$$\begin{cases}\mathrm{d}X(t)=[X(t)+u(t)+1]\mathrm{d}t+[X(t)+u(t)]\mathrm{d}W(t), \quad t\geq 0,\\ X(0)=x,\end{cases}$$

with cost functional

$$J_T(x;u(\cdot))=\frac{1}{2}\mathbb{E}\int_0^T[|X(t)|^2+|u(t)|^2]\mathrm{d}t.$$

We see that in this case,

$$A=B=C=D=Q=R=b=1, \quad \sigma=q=0.$$

Then

$$\mathscr{V}_0=\{(x,u)\mid x+u+1=0, x+u=0\}=\varnothing.$$

On the other hand, the ARE reads:

$$3P+1-\frac{4P^2}{1+P}=0,$$

which is equivalent to

$$P^2-4P-1=0.$$

Thus, the positive solution is $P=2+\sqrt{5}$. Hence,

$$F(x,u)=\frac{1}{2}[x^2+u^2+(2+\sqrt{5})(x+u)^2].$$

Consequently, Problem (O) is well-formulated and admits a unique optimal solution.

From the above examples, together with our main result on the turnpike property of Problem $(\mathrm{SLQ})_T$ which will be presented a little later, we see that (1.8) is not a suitable problem to be considered and the correct one is Problem (O).

4 Convergence of the Riccati Equation

For notational simplicity, we define, for each $P\in\mathbb{S}^n$:

$$\begin{cases}\mathcal{Q}(P)=PA+A^{\mathrm{T}}P+C^{\mathrm{T}}PC+Q,\\ \mathcal{S}(P)=B^{\mathrm{T}}P+D^{\mathrm{T}}PC,\\ \mathcal{R}(P)=R+D^{\mathrm{T}}PD,\\ \mathcal{K}(P)=-\mathcal{R}(P)^{-1}\mathcal{S}(P), \quad \text{provided } \mathcal{R}(P) \text{ is invertible.}\end{cases} \tag{4.1}$$

Then ARE (3.7) can be written as follows:

$$\mathcal{Q}(P)-\mathcal{S}(P)^{\mathrm{T}}\mathcal{R}(P)^{-1}\mathcal{S}(P)=0. \tag{4.2}$$

By Proposition 3.1, under (H1)–(H2), the above (4.2) admits a stabilizing solution $P\in\mathbb{S}^n_+$ (with respect to $[A,C;B,D]$), and (3.8) holds.

According to Lemma 2.1, we know that under (H1)–(H2), for each $T>0$, Problem $(\mathrm{SLQ})_T$ admits a unique optimal control for every initial state x. Moreover, the following conclusions hold:

(i) The optimal pair $(\overline{X}_T(\cdot), \overline{u}_T(\cdot))$, together with a pair of adapted processes $(\overline{Y}_T(\cdot), \overline{Z}_T(\cdot))$, satisfies the following optimality system:

$$\begin{cases} \mathrm{d}\overline{X}_T(t) = [A\overline{X}_T(t) + B\overline{u}_T(t) + b]\mathrm{d}t + [C\overline{X}_T(t) + D\overline{u}_T(t) + \sigma]\mathrm{d}W(t), \\ \mathrm{d}\overline{Y}_T(t) = -[A^{\mathrm{T}}\overline{Y}_T(t) + C^{\mathrm{T}}\overline{Z}_T(t) + Q\overline{X}_T(t) + q]\mathrm{d}t + \overline{Z}_T(t)\mathrm{d}W(t), \\ \overline{X}_T(0) = x, \quad \overline{Y}_T(T) = 0, \\ B^{\mathrm{T}}\overline{Y}_T(t) + D^{\mathrm{T}}\overline{Z}_T(t) + R\overline{u}_T(t) = 0, \quad \text{a.e. } t \in [0,T], \text{ a.s.} \end{cases} \tag{4.3}$$

(ii) The differential Riccati equation

$$\begin{cases} \dot{P}_T(t) + \mathcal{Q}(P_T(t)) - \mathcal{S}(P_T(t))^{\mathrm{T}}\mathcal{R}(P_T(t))^{-1}\mathcal{S}(P_T(t)) = 0, \quad t \in [0,T], \\ P_T(T) = 0 \end{cases} \tag{4.4}$$

admits a unique solution $P_T(\cdot) \in C([0,T];\mathbb{S}^n)$ satisfying $P_T(t) > 0$ for all $0 \le t < T$, and the (open-loop) optimal control has the following state feedback representation:

$$\overline{u}_T(t) = \mathcal{K}(P_T(t))\overline{X}_T(t) - \mathcal{R}(P_T(t))^{-1}[B^{\mathrm{T}}\varphi_T(t) + D^{\mathrm{T}}P_T(t)\sigma],$$

where $\varphi_T(\cdot)$ is the solution to the following ODE:

$$\begin{cases} \dot{\varphi}_T(t) + [A + B\mathcal{K}(P_T(t))]^{\mathrm{T}}\varphi_T(t) + [C + D\mathcal{K}(P_T(t))]^{\mathrm{T}}P_T(t)\sigma(t) \\ +P_T(t)b + q = 0, \quad t \in [0,T], \\ \varphi_T(T) = 0. \end{cases} \tag{4.5}$$

(iii) The value function is given by

$$\begin{aligned} V_T(x) = \frac{1}{2}\langle P_T(0)x, x\rangle + \langle \varphi_T(0), x\rangle + \frac{1}{2}\int_0^T \{\langle P_T(t)\sigma, \sigma\rangle + 2\langle \varphi_T(t), b\rangle \\ - |\mathcal{R}(P_T(t))^{-\frac{1}{2}}[B^{\mathrm{T}}\varphi_T(t) + D^{\mathrm{T}}P_T(t)\sigma]|^2\}\mathrm{d}t. \end{aligned}$$

Note that by setting $b = \sigma = q = 0$, we have the corresponding result for the (homogeneous) Problem $(\mathrm{SLQ})^0_T$. For such a case, $\varphi_T(\cdot) = 0$, and in particular, also taking into account of (3.8)–(3.9),

$$V^0_T(x) = \frac{1}{2}\langle P_T(0)x, x\rangle \le \frac{1}{2}\langle Px, x\rangle = V^0_\infty(x), \quad \forall x \in \mathbb{R}^n. \tag{4.6}$$

We now look at the convergence property of $P_T(\cdot)$ as $T \to \infty$, which plays an essential role in the turnpike property of Problem $(\mathrm{SLQ})_T$. For this, let us present the following result first.

Proposition 4.1 *Let* (H1)–(H2) *hold. Then the equation*

$$\begin{cases} \dot{\Sigma}(t) - \mathcal{Q}(\Sigma(t)) + \mathcal{S}(\Sigma(t))^{\mathrm{T}}\mathcal{R}(\Sigma(t))^{-1}\mathcal{S}(\Sigma(t)) = 0, \quad t \in [0,\infty), \\ \Sigma(0) = 0 \end{cases} \tag{4.7}$$

admits a unique solution $\Sigma(\cdot) \in C([0,\infty);\mathbb{S}^n)$ satisfying

$$0 < \Sigma(s) \le \Sigma(t) \le P, \quad \forall\, 0 < s < t < \infty \tag{4.8}$$

with $P \in \mathbb{S}^n_+$ being the stabilizing solution of (4.2) *with respect to* $[A,C;B,D]$. *Moreover, $\Sigma(T-t) = P_T(t)$ for every $0 \le t \le T$.*

Proof For fixed but arbitrary $0 < T_1 < T_2 < \infty$, we define

$$\Sigma_1(t) \triangleq P_{T_1}(T_1 - t), \quad 0 \le t \le T_1,$$
$$\Sigma_2(t) \triangleq P_{T_2}(T_2 - t), \quad 0 \le t \le T_2.$$

On the interval $[0, T_1]$, both Σ_1 and Σ_2 solve the same equation

$$\begin{cases} \dot{\Sigma}(t) - \mathcal{Q}(\Sigma(t)) + \mathcal{S}(\Sigma(t))^{\mathrm{T}}\mathcal{R}(\Sigma(t))^{-1}\mathcal{S}(\Sigma(t)) = 0, \\ \Sigma(0) = 0. \end{cases}$$

By the uniqueness, we must have

$$\Sigma_1(t) = \Sigma_2(t), \quad \forall t \in [0, T_1].$$

Then the function $\Sigma : [0, \infty) \to \mathbb{S}^n$ defined by

$$\Sigma(t) \triangleq P_T(T - t)$$

is independent of the choice of $T \ge t$ and is a solution of (4.7). We now claim that

$$0 < \Sigma(s) \le \Sigma(t) \le P, \quad \forall\, 0 < s < t < \infty.$$

To see this, we note that for any $0 \le T_1 < T_2 < \infty$, by (H2), we have

$$\langle P_{T_1}(0)x, x\rangle = 2V^0_{T_1}(x) \le 2V^0_{T_2}(x) = \langle P_{T_2}(0)x, x\rangle, \quad \forall x \in \mathbb{R}^n.$$

Thus,

$$\Sigma(T_1) = P_{T_1}(0) \le P_{T_2}(0) = \Sigma(T_2).$$

Finally, by (4.6), we obtain our conclusion.

The above result leads to the following convergence.

Proposition 4.2 *Let* (H1)–(H2) *hold and* $\Sigma(\cdot)$ *be the solution to the ODE* (4.7). *Then the limit* $P_\infty \triangleq \lim_{t\to\infty} \Sigma(t)$ *exists, which is the stabilizing solution of the ARE* (4.2) *with respect to* $[A, C; B, D]$, *and is the one appearing in the representation* (3.8) *of the value function* $V^0_\infty(\cdot)$ *of Problem (SLQ)*$^0_\infty$.

Proof From (4.8), we see that

$$P_\infty \triangleq \lim_{t\to\infty} \Sigma(t) \le P$$

exists and is positive definite. To see that P_∞ satisfies the ARE (4.2), we observe that (by (4.7))

$$\Sigma(T+1) - \Sigma(T) = \int_T^{T+1} [\mathcal{Q}(\Sigma(t)) - \mathcal{S}(\Sigma(t))^{\mathrm{T}}\mathcal{R}(\Sigma(t))^{-1}\mathcal{S}(\Sigma(t))]\mathrm{d}t.$$

Letting $T \to \infty$ yields (4.2). Finally, we observe that (4.2) can be written as

$$P[A+B\mathcal{K}(P)] + [A+B\mathcal{K}(P)]^{\mathrm{T}}P + [C+D\mathcal{K}(P)]^{\mathrm{T}}P[C+D\mathcal{K}(P)] + Q + \mathcal{K}(P)^{\mathrm{T}}R\mathcal{K}(P) = 0.$$

Since $Q, R > 0$ and P_∞ satisfies (4.2), the above implies

$$P_\infty[A + B\mathcal{K}(P_\infty)] + [A + B\mathcal{K}(P_\infty)]^{\mathrm{T}}P_\infty + [C + D\mathcal{K}(P_\infty)]^{\mathrm{T}}P_\infty[C + D\mathcal{K}(P_\infty)] < 0.$$

Since $P_\infty > 0$, we conclude from Lemma 3.1 that $\mathcal{K}(P_\infty)$ is a stabilizer of $[A, C; B, D]$. Because the stabilizing solution of (4.2) is unique, we have $P_\infty = P$.

An interesting further issue is how fast $\Sigma(t)$ converges to the solution P of (4.2) as $t \to \infty$. To address this issue, we need the following lemma.

Lemma 4.1 *Suppose that the system $[A, C]$ is L^2-stable and let β be the constant in* (3.2). *Let $f : [0, \infty) \times \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ be a continuous function satisfying $f(t, 0) = 0$ and*

$$|f(t, M) - f(t, N)| \le \rho|M - N|(|M| + |N|), \quad \forall t \ge 0,\ \forall M, N \in \mathbb{R}^{n\times n}$$

for some constant $\rho > 0$. Then for small initial state $\varPi_0 \in \mathbb{R}^{n\times n}$, the ODE

$$\begin{cases} \dot{\varPi}(t) = \varPi(t)A + A^{\mathrm{T}}\varPi(t) + C^{\mathrm{T}}\varPi(t)C + f(t, \varPi(t)), \quad t \in [0, \infty), \\ \varPi(0) = \varPi_0 \end{cases}$$

has a unique exponentially stable solution with decay rate β, i.e., for some constant $\delta > 0$,

$$|\varPi(t)| \le \delta \mathrm{e}^{-\beta t}, \quad \forall t \ge 0.$$

Proof Let $\delta > 0$ be an undetermined constant, and consider the complete metric space (with respect to the uniform metric)

$$\mathscr{M} = \big\{M(\cdot) \in C([0, \infty); \mathbb{R}^{n\times n}); |M(t)| \le \delta \mathrm{e}^{-\beta t},\ \forall t \ge 0\big\}.$$

Clearly, for each $M(\cdot) \in \mathscr{M}$ and each initial state $\varPi_0 \in \mathbb{R}^{n\times n}$, the ODE

$$\begin{cases} \dot{\varPi}(t) = \varPi(t)A + A^{\mathrm{T}}\varPi(t) + C^{\mathrm{T}}\varPi(t)C + f(t, M(t)), \quad t \in [0, \infty), \\ \varPi(0) = \varPi_0 \end{cases}$$

has a unique solution $\varPi(\cdot) \equiv \mathscr{T}[M(\cdot)]$. Let $\varPhi(\cdot)$ be the solution of (3.1). By Itô's rule, we have for $0 \le s \le t < \infty$,

$$\begin{aligned} \mathrm{d}\big[\varPhi(s)^{\mathrm{T}}\varPi(t-s)\varPhi(s)\big] = &-\varPhi(s)^{\mathrm{T}} f(t-s, M(t-s))\varPhi(s)\mathrm{d}s \\ &+ \varPhi(s)^{\mathrm{T}}\big[C^{\mathrm{T}}\varPi(t-s) + \varPi(t-s)C\big]\varPhi(s)\mathrm{d}W(s). \end{aligned}$$

Consequently,

$$\begin{aligned} \varPi(t-s) = \mathbb{E}\Big\{&[\varPhi(t)\varPhi(s)^{-1}]^{\mathrm{T}}\varPi(0)[\varPhi(t)\varPhi(s)^{-1}] \\ &+ \int_s^t [\varPhi(r)\varPhi(s)^{-1}]^{\mathrm{T}} f(t-r, M(t-r))[\varPhi(r)\varPhi(s)^{-1}]\mathrm{d}r\Big\}. \end{aligned}$$

Taking $s = 0$ yields

$$\begin{aligned} \varPi(t) &= \mathbb{E}[\varPhi(t)^{\mathrm{T}}\varPi_0\varPhi(t)] + \mathbb{E}\int_0^t \varPhi(r)^{\mathrm{T}} f(t-r, M(t-r))\varPhi(r)\mathrm{d}r \\ &= \mathbb{E}[\varPhi(t)^{\mathrm{T}}\varPi_0\varPhi(t)] + \mathbb{E}\int_0^t \varPhi(t-s)^{\mathrm{T}} f(s, M(s))\varPhi(t-s)\mathrm{d}s. \end{aligned} \tag{4.9}$$

It follows from (3.2) and the assumption on f that

$$|\varPi(t)| \le |\varPi_0|\alpha \mathrm{e}^{-\beta t} + \int_0^t \rho\alpha \mathrm{e}^{-\beta(t-s)}|M(s)|^2\mathrm{d}s$$

$$\leq |\Pi_0|\alpha \mathrm{e}^{-\beta t} + \int_0^t \rho\alpha\delta^2 \mathrm{e}^{-\beta(t+s)}\mathrm{d}s \leq \alpha\Big(|\Pi_0| + \frac{\rho\delta^2}{\beta}\Big)\mathrm{e}^{-\beta t}.$$

Take $\delta \in (0, \frac{\beta}{2\alpha\rho}]$. Then

$$\alpha\Big(\frac{\delta}{2\alpha} + \frac{\rho\delta^2}{\beta}\Big) \leq \delta.$$

Thus, for $|\Pi_0| \leq \frac{\delta}{2\alpha}$, $\mathscr{T}$ maps $\mathscr{M}$ into itself. Further, for any $M(\cdot), N(\cdot) \in \mathscr{M}$, we have from (4.9) that

$$\begin{aligned}
|\mathscr{T}[M(\cdot)](t) - \mathscr{T}[N(\cdot)](t)| &\leq \alpha\int_0^t \mathrm{e}^{-\beta(t-s)}|f(s,M(s)) - f(s,N(s))|\mathrm{d}s \\
&\leq \alpha\rho\int_0^t \mathrm{e}^{-\beta(t-s)}|M(s)-N(s)|(|M(s)|+|N(s)|)\mathrm{d}s \\
&\leq 2\alpha\rho\delta\int_0^t \mathrm{e}^{-\beta t}|M(s)-N(s)|\mathrm{d}s \\
&\leq (2\alpha\rho\delta t\mathrm{e}^{-\beta t})\sup_{s\geq 0}|M(s)-N(s)|.
\end{aligned}$$

If we take $\delta > 0$ small enough so that also $2\alpha\rho\delta t\mathrm{e}^{-\beta t} \leq \frac{1}{2}$ for all $t \geq 0$, then $\mathscr{T}$ is a contraction mapping on $\mathscr{M}$. The desired result therefore follows.

The following result shows that the rate of convergence of $\Sigma(t)$ is exponential.

Theorem 4.1 *Let* (H1)–(H2) *hold. There exist positive constants $K, \lambda > 0$ such that*

$$|P - \Sigma(t)| \leq K\mathrm{e}^{-2\lambda t}, \quad \forall t \geq 0. \tag{4.10}$$

Proof Let $\Pi(t) = P - \Sigma(t)$. Then

$$\begin{aligned}
\dot{\Pi} &= \Pi A + A^{\mathrm{T}}\Pi + C^{\mathrm{T}}\Pi C - \mathcal{S}(P)^{\mathrm{T}}\mathcal{R}(P)^{-1}\mathcal{S}(P) + \mathcal{S}(\Sigma)^{\mathrm{T}}\mathcal{R}(\Sigma)^{-1}\mathcal{S}(\Sigma) \\
&= \Pi[A + B\mathcal{K}(P)] + [A + B\mathcal{K}(P)]^{\mathrm{T}}\Pi + [C + D\mathcal{K}(P)]^{\mathrm{T}}\Pi[C + D\mathcal{K}(P)] \\
&\quad - \mathcal{S}(\Pi)^{\mathrm{T}}\mathcal{K}(P) - \mathcal{K}(P)^{\mathrm{T}}\mathcal{S}(\Pi) - \mathcal{K}(P)^{\mathrm{T}}D^{\mathrm{T}}\Pi D\mathcal{K}(P) \\
&\quad + \mathcal{K}(P)^{\mathrm{T}}\mathcal{S}(P) + \mathcal{S}(\Sigma)^{\mathrm{T}}\mathcal{R}(\Sigma)^{-1}\mathcal{S}(\Sigma) \\
&= \Pi[A + B\mathcal{K}(P)] + [A + B\mathcal{K}(P)]^{\mathrm{T}}\Pi + [C + D\mathcal{K}(P)]^{\mathrm{T}}\Pi[C + D\mathcal{K}(P)] \\
&\quad - [\mathcal{S}(\Pi) + D^{\mathrm{T}}\Pi D\mathcal{K}(P)]^{\mathrm{T}}\mathcal{K}(P) + [\mathcal{K}(P)^{\mathrm{T}} + \mathcal{S}(\Sigma)^{\mathrm{T}}\mathcal{R}(\Sigma)^{-1}]\mathcal{S}(\Sigma).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&[\mathcal{K}(P)^{\mathrm{T}} + \mathcal{S}(\Sigma)^{\mathrm{T}}\mathcal{R}(\Sigma)^{-1}]\mathcal{S}(\Sigma) - [\mathcal{S}(\Pi) + D^{\mathrm{T}}\Pi D\mathcal{K}(P)]^{\mathrm{T}}\mathcal{K}(P) \\
&= [\mathcal{K}(P)^{\mathrm{T}} + \mathcal{S}(\Sigma)^{\mathrm{T}}\mathcal{R}(\Sigma)^{-1}]\mathcal{S}(\Sigma) + [\mathcal{S}(\Pi) + D^{\mathrm{T}}\Pi D\mathcal{K}(P)]^{\mathrm{T}}\mathcal{R}(P)^{-1}[\mathcal{S}(\Pi) + \mathcal{S}(\Sigma)] \\
&= [-\mathcal{S}(P)^{\mathrm{T}} + \mathcal{S}(\Sigma)^{\mathrm{T}}\mathcal{R}(\Sigma)^{-1}\mathcal{R}(P) + \mathcal{S}(\Pi)^{\mathrm{T}} + \mathcal{K}(P)^{\mathrm{T}}D^{\mathrm{T}}\Pi D]\mathcal{R}(P)^{-1}\mathcal{S}(\Sigma) \\
&\quad + [\mathcal{S}(\Pi) + D^{\mathrm{T}}\Pi D\mathcal{K}(P)]^{\mathrm{T}}\mathcal{R}(P)^{-1}\mathcal{S}(\Pi) \\
&= \big[\mathcal{S}(\Sigma)^{\mathrm{T}}\mathcal{R}(\Sigma)^{-1}D^{\mathrm{T}}\Pi D + \mathcal{K}(P)^{\mathrm{T}}D^{\mathrm{T}}\Pi D\big]\mathcal{R}(P)^{-1}\mathcal{S}(\Sigma) \\
&\quad + [\mathcal{S}(\Pi) + D^{\mathrm{T}}\Pi D\mathcal{K}(P)]^{\mathrm{T}}\mathcal{R}(P)^{-1}\mathcal{S}(\Pi) \\
&= \big[-\mathcal{K}(\Sigma) + \mathcal{K}(P)\big]^{\mathrm{T}}D^{\mathrm{T}}\Pi D\mathcal{R}(P)^{-1}\mathcal{S}(\Sigma) + [\mathcal{S}(\Pi) + D^{\mathrm{T}}\Pi D\mathcal{K}(P)]^{\mathrm{T}}\mathcal{R}(P)^{-1}\mathcal{S}(\Pi) \\
&= -\big[\mathcal{S}(\Pi) + D^{\mathrm{T}}\Pi D\mathcal{K}(\Sigma)\big]^{\mathrm{T}}\mathcal{R}(P)^{-1}D^{\mathrm{T}}\Pi D\mathcal{R}(P)^{-1}\mathcal{S}(\Sigma) \\
&\quad + [\mathcal{S}(\Pi) + D^{\mathrm{T}}\Pi D\mathcal{K}(P)]^{\mathrm{T}}\mathcal{R}(P)^{-1}\mathcal{S}(\Pi).
\end{aligned}$$

Set $\mathcal{A} \triangleq A + B\mathcal{K}(P)$, $\mathcal{C} \triangleq C + D\mathcal{K}(P)$, and

$$\begin{aligned} f(t,\Pi) \triangleq\ & [\mathcal{S}(\Pi) + D^{\mathrm{T}}\Pi D\mathcal{K}(P)]^{\mathrm{T}}\mathcal{R}(P)^{-1}\mathcal{S}(\Pi) \\ & - [\mathcal{S}(\Pi) + D^{\mathrm{T}}\Pi D\mathcal{K}(\Sigma(t))]^{\mathrm{T}}\mathcal{R}(P)^{-1}D^{\mathrm{T}}\Pi D\mathcal{R}(P)^{-1}\mathcal{S}(\Sigma(t)). \end{aligned} \tag{4.11}$$

Then we can rewrite the equation for $\Pi(\cdot)$ as follows:

$$\dot{\Pi}(t) = \Pi(t)\mathcal{A} + \mathcal{A}^{\mathrm{T}}\Pi(t) + \mathcal{C}^{\mathrm{T}}\Pi(t)\mathcal{C} + f(t,\Pi(t)).$$

From Proposition 4.2 we know that the system $[\mathcal{A},\mathcal{C}]$ is L^2-stable. Thus, by Lemma 3.1, there exist constants $K,\lambda > 0$ such that the solution $\Psi(\cdot)$ to

$$\begin{cases} \mathrm{d}\Psi(t) = \mathcal{A}\Psi(t)\mathrm{d}t + \mathcal{C}\Psi(t)\mathrm{d}W(t), \quad t \geq 0, \\ \Psi(0) = I_n \end{cases}$$

satisfies

$$\mathbb{E}|\Psi(t)|^2 \leq K\mathrm{e}^{-2\lambda t}, \quad \forall t \geq 0. \tag{4.12}$$

Also, it is easy to see that the function defined by (4.11) satisfies the properties in Lemma 4.1. Since $\lim_{t\to\infty}\Pi(t) = 0$, we conclude from Lemma 4.1 that (4.10) holds for large t and hence all $t \geq 0$ (with a possibly different constant $K > 0$).

5 The Turnpike Property

Let $(\overline{X}_T(\cdot), \overline{u}_T(\cdot))$ be the optimal pair of Problem $(\mathrm{SLQ})_T$ for the given initial state x and $(\overline{Y}_T(\cdot), \overline{Z}_T(\cdot))$ the adapted solution to the corresponding adjoint equation in (4.3). Let (x^*, u^*) be the unique solution of Problem (O) and $\lambda^* \in \mathbb{R}^n$ the corresponding Lagrange multiplier. Define

$$\widehat{X}_T(\cdot) = \overline{X}_T(\cdot) - x^*, \quad \widehat{u}_T(\cdot) = \overline{u}_T(\cdot) - u^*, \quad \widehat{Y}_T(\cdot) = \overline{Y}_T(\cdot) - \lambda^*. \tag{5.1}$$

We are now ready to state the main result of this paper, which establishes the exponential turnpike property of Problem $(\mathrm{SLQ})_T$ as well as of the adjoint process.

Theorem 5.1 *Let* (H1)–(H2) *hold. Then there exist positive constants $K, \mu > 0$, independent of T, such that*

$$|\mathbb{E}[\widehat{X}_T(t)]| + |\mathbb{E}[\widehat{u}_T(t)]| + |\mathbb{E}[\widehat{Y}_T(t)]| \leq K[\mathrm{e}^{-\mu t} + \mathrm{e}^{-\mu(T-t)}], \quad \forall t \in [0,T]. \tag{5.2}$$

As an immediate consequence of Theorem 5.1, we have the following corollary, which shows that the integral and the mean-square turnpike properties also hold for Problem $(\mathrm{SLQ})_T$.

Corollary 5.1 *Let* (H1)–(H2) *hold. Then as $T \to \infty$,*

$$\begin{aligned} &\frac{1}{T}\int_0^T \mathbb{E}[\overline{X}_T(t)]\mathrm{d}t \to x^*, \quad \frac{1}{T}\int_0^T |\mathbb{E}[\overline{X}_T(t)] - x^*|^2\mathrm{d}t \to 0, \\ &\frac{1}{T}\int_0^T \mathbb{E}[\overline{u}_T(t)]\mathrm{d}t \to u^*, \quad \frac{1}{T}\int_0^T |\mathbb{E}[\overline{u}_T(t)] - u^*|^2\mathrm{d}t \to 0. \end{aligned}$$

In order to prove Theorem 5.1, let us first make some observations. For convenience, we rewrite the optimality system (4.3) of Problem $(\mathrm{SLQ})_T$ and the characterization (3.10) of the optimal solution to Problem (O) together here

$$\begin{cases}\mathrm{d}\overline{X}_T(t)=[A\overline{X}_T(t)+B\overline{u}_T(t)+b]\mathrm{d}t+[C\overline{X}_T(t)+D\overline{u}_T(t)+\sigma]\mathrm{d}W(t),\\ \mathrm{d}\overline{Y}_T(t)=-[A^{\mathrm{T}}\overline{Y}_T(t)+C^{\mathrm{T}}\overline{Z}_T(t)+Q\overline{X}_T(t)+q]\mathrm{d}t+\overline{Z}_T(t)\mathrm{d}W(t),\\ \overline{X}_T(0)=x,\quad \overline{Y}_T(T)=0,\\ B^{\mathrm{T}}\overline{Y}_T(t)+D^{\mathrm{T}}\overline{Z}_T(t)+R\overline{u}_T(t)=0,\quad \text{a.e. } t\in[0,T],\ \text{a.s.}\end{cases}\tag{5.3}$$

and

$$\begin{cases}Qx^*+A^{\mathrm{T}}\lambda^*+C^{\mathrm{T}}P(Cx^*+Du^*+\sigma)+q=0,\\ Ru^*+B^{\mathrm{T}}\lambda^*+D^{\mathrm{T}}P(Cx^*+Du^*+\sigma)=0.\end{cases}\tag{5.4}$$

Noting that $(x^*,u^*)\in\mathscr{V}$, we have

$$Ax^*+Bu^*+b=0.$$

Also, we denote

$$\sigma^*=Cx^*+Du^*+\sigma.$$

Then, a direction calculation yields the following:

$$\begin{cases}\mathrm{d}\widehat{X}_T(t)=[A\widehat{X}_T(t)+B\widehat{u}_T(t)]\mathrm{d}t+[C\widehat{X}_T(t)+D\widehat{u}_T(t)+\sigma^*]\mathrm{d}W(t),\\ \mathrm{d}\widehat{Y}_T(t)=-[A^{\mathrm{T}}\widehat{Y}_T(t)+C^{\mathrm{T}}\overline{Z}_T(t)+Q\widehat{X}_T(t)-C^{\mathrm{T}}P\sigma^*]\mathrm{d}t+\overline{Z}_T(t)\mathrm{d}W(t),\\ \widehat{X}_T(0)=x-x^*,\quad \widehat{Y}_T(T)=-\lambda^*,\\ B^{\mathrm{T}}\widehat{Y}_T(t)+D^{\mathrm{T}}\overline{Z}_T(t)+R\widehat{u}_T(t)-D^{\mathrm{T}}P\sigma^*=0,\quad \text{a.e. } t\in[0,T],\ \text{a.s}\end{cases}\tag{5.5}$$

Comparing the above with (2.5)–(2.6) in Lemma 2.1(ii), we see that $(\widehat{X}_T(\cdot),\widehat{u}_T(\cdot))$ is an optimal pair of the stochastic LQ problem with state equation

$$\begin{cases}\mathrm{d}X(t)=[AX(t)+Bu(t)]\mathrm{d}t+[CX(t)+Du(t)+\sigma^*]\mathrm{d}W(t),\\ X(0)=x-x^*\end{cases}$$

and cost functional

$$\begin{aligned}J(x;u)=\frac{1}{2}\mathbb{E}\Big\{-2\langle\lambda^*,X(T)\rangle+\int_0^T[\langle QX(t),X(t)\rangle+\langle Ru(t),u(t)\rangle\\ -2\langle C^{\mathrm{T}}P\sigma^*,X(t)\rangle-2\langle D^{\mathrm{T}}P\sigma^*,u(t)\rangle]\mathrm{d}t\Big\}.\end{aligned}$$

Now applying Lemma 2.1(iv), we obtain the following result immediately.

Proposition 5.1 *Let* (H1)–(H2) *hold. Let* $P_T(\cdot)$ *be the solution to* (4.4) *and*

$$\Theta_T(t)\triangleq\mathcal{K}(P_T(t))=-\mathcal{R}(P_T(t))^{-1}\mathcal{S}(P_T(t)),\tag{5.6}$$

and let $\varphi_T(\cdot)$ *be the solution to the ODE*

$$\begin{cases}\dot{\varphi}_T(t)+[A+B\Theta_T(t)]^{\mathrm{T}}\varphi_T(t)+[C+D\Theta_T(t)]^{\mathrm{T}}(P_T(t)-P)\sigma^*,\\ \varphi_T(T)=-\lambda^*.\end{cases}\tag{5.7}$$

Then the process $\widehat{u}_T(\cdot)$ *defined in* (5.1) *is given by*

$$\widehat{u}_T(t)=\Theta_T(t)\widehat{X}_T(t)-\mathcal{R}(P_T(t))^{-1}[B^{\mathrm{T}}\varphi_T(t)+D^{\mathrm{T}}(P_T(t)-P)\sigma^*].\tag{5.8}$$

To prove Theorem 5.1, we also need the following lemma.

Lemma 5.1 *Let* (H1)–(H2) *hold. The solution* $\varphi_T(\cdot)$ *to the ODE* (5.7) *satisfies*

$$|\varphi_T(t)| \le K\mathrm{e}^{-\lambda(T-t)}, \quad \forall\, 0 \le t \le T \tag{5.9}$$

for some constants $K, \lambda > 0$ *independent of* T.

Proof For notational simplicity, we let

$$\Theta \triangleq \mathcal{K}(P) = -(R + D^{\mathrm{T}}PD)^{-1}(B^{\mathrm{T}}P + D^{\mathrm{T}}PC), \quad \mathcal{A} \triangleq A + B\Theta, \quad \mathcal{C} \triangleq C + D\Theta \tag{5.10}$$

and write (5.7) as

$$\begin{cases} \dot{\varphi}_T(t) = -\mathcal{A}^{\mathrm{T}}\varphi_T(t) - [B(\Theta_T(t) - \Theta)]^{\mathrm{T}}\varphi_T(t) - [C + D\Theta_T(t)]^{\mathrm{T}}(P_T(t) - P)\sigma^*, \\ \varphi_T(T) = -\lambda^*. \end{cases} \tag{5.11}$$

By the variation of constants formula,

$$\varphi_T(t) = \mathrm{e}^{\mathcal{A}^{\mathrm{T}}(T-t)}\Big[-\lambda^* + \int_t^T \mathrm{e}^{\mathcal{A}^{\mathrm{T}}(s-T)}\rho(s)\mathrm{d}s\Big], \tag{5.12}$$

where

$$\rho(s) = [B(\Theta_T(s) - \Theta)]^{\mathrm{T}}\varphi_T(s) + [C + D\Theta_T(s)]^{\mathrm{T}}(P_T(s) - P)\sigma^*.$$

Recall from the proof of Theorem 4.1 that there exist constants $K, \lambda > 0$, independent of T, such that (4.10) and (4.12) hold, and note that $\Sigma(T-t) = P_T(t)$ (Proposition 4.1). We have

$$|\mathrm{e}^{\mathcal{A}^{\mathrm{T}}t}| \le K\mathrm{e}^{-\lambda t}, \quad |P_T(t) - P| \le K\mathrm{e}^{-2\lambda(T-t)}, \quad \forall\, 0 \le t \le T < \infty, \tag{5.13}$$

where and hereafter, K represents a generic constant (independent of T) which can be different from line to line, but λ is the fixed constant in (4.10) and (4.12). Observe that

$$\begin{aligned} \Theta_T(s) - \Theta &= \mathcal{R}(P)^{-1}\mathcal{S}(P) - \mathcal{R}(P_T(s))^{-1}\mathcal{S}(P_T(s)) \\ &= \mathcal{R}(P)^{-1}\mathcal{S}(P - P_T(s)) + [\mathcal{R}(P)^{-1} - \mathcal{R}(P_T(s))^{-1}]\mathcal{S}(P_T(s)) \\ &= \mathcal{R}(P)^{-1}\mathcal{S}(P - P_T(s)) + \mathcal{R}(P)^{-1}D^{\mathrm{T}}[P_T(s) - P]D\mathcal{R}(P_T(s))^{-1}\mathcal{S}(P_T(s)). \end{aligned}$$

Since $P_T(\cdot)$, and hence $\Theta_T(\cdot)$, is bounded uniformly in T, we have

$$|\Theta_T(s) - \Theta| \le K\mathrm{e}^{-2\lambda(T-s)}, \quad \forall\, 0 \le s \le T < \infty. \tag{5.14}$$

It follows that

$$|\rho(s)| \le K\mathrm{e}^{-2\lambda(T-s)}[|\varphi_T(s)| + 1].$$

If we let $h(t) = \mathrm{e}^{\lambda(T-t)}|\varphi_T(t)|$, then by (5.12),

$$\begin{aligned} h(t) &\le K\Big[|\lambda^*| + \int_t^T K\mathrm{e}^{-\lambda(s-T)}|\rho(s)|\mathrm{d}s\Big] \\ &\le K + K\int_t^T [\mathrm{e}^{-\lambda(T-s)}|\varphi_T(s)| + \mathrm{e}^{-\lambda(T-s)}]\mathrm{d}s \\ &= K + K\int_t^T [\mathrm{e}^{-2\lambda(T-s)}h(s) + \mathrm{e}^{-\lambda(T-s)}]\mathrm{d}s. \end{aligned}$$

Applying Gronwall's inequality we obtain that for some $K > 0$ independent of T,

$$h(t) \leq K, \quad \forall t \in [0, T].$$

The desired result then follows.

Proof of Theorem 5.1 For notational simplicity, we let

$$\widehat{v}_{T}(t) \triangleq -\mathcal{R}(P_{T}(t))^{-1}[B^{\mathrm{T}}\varphi_{T}(t) + D^{\mathrm{T}}(P_{T}(t) - P)\sigma^{*}].$$

Substituting (5.8) into the state equation for $\widehat{X}_{T}(\cdot)$ yields

$$\begin{cases} \mathrm{d}\widehat{X}_{T}(t) = \{[A + B\Theta_{T}(t)]\widehat{X}_{T}(t) + B\widehat{v}_{T}(t)\}\mathrm{d}t \\ \qquad\qquad + \{[C + D\Theta_{T}(t)]\widehat{X}_{T}(t) + D\widehat{v}_{T}(t) + \sigma^{*}\}\mathrm{d}W(t), \quad t \in [0, T], \\ \widehat{X}_{T}(0) = x - x^{*}. \end{cases}$$

Using the notation (5.10), we can rewrite the above as

$$\begin{cases} \mathrm{d}\widehat{X}_{T}(t) = [\mathcal{A}\widehat{X}_{T}(t) + \xi(t)]\mathrm{d}t + [\mathcal{C}\widehat{X}_{T}(t) + \eta(t)]\mathrm{d}W(t), \quad t \in [0, T], \\ \widehat{X}_{T}(0) = x - x^{*} \equiv \widehat{x}, \end{cases} \tag{5.15}$$

where

$$\xi(t) = B[\Theta_{T}(t) - \Theta]\widehat{X}_{T}(t) + B\widehat{v}_{T}(t), \quad \eta(t) = D[\Theta_{T}(t) - \Theta]\widehat{X}_{T}(t) + D\widehat{v}_{T}(t) + \sigma^{*}.$$

Taking expectations in (5.15), we get

$$\begin{cases} \mathrm{d}\mathbb{E}[\widehat{X}_{T}(t)] = \{\mathcal{A}\mathbb{E}[\widehat{X}_{T}(ft)] + B[\Theta_{T}(t) - \Theta]\mathbb{E}[\widehat{X}_{T}(t)] + B\widehat{v}_{T}(t)\}\mathrm{d}t, \quad t \in [0, T], \\ \mathbb{E}[\widehat{X}_{T}(0)] = \widehat{x}. \end{cases} \tag{5.16}$$

Noting

$$P\mathcal{A} + \mathcal{A}^{\mathrm{T}}P + \mathcal{C}^{\mathrm{T}}P\mathcal{C} + \Theta^{\mathrm{T}}R\Theta + Q = 0,$$

we obtain

$$\begin{aligned} &\langle P\mathbb{E}[\widehat{X}_{T}(t)], \mathbb{E}[\widehat{X}_{T}(t)]\rangle - \langle P\widehat{x}, \widehat{x}\rangle \\ &= \int_{0}^{t} \{-\langle(\mathcal{C}^{\mathrm{T}}P\mathcal{C} + \Theta^{\mathrm{T}}R\Theta + Q)\mathbb{E}[\widehat{X}_{T}(s)], \mathbb{E}[\widehat{X}_{T}(s)]\rangle \\ &\quad + 2\langle P\mathbb{E}[\widehat{X}_{T}(s)], B[\Theta_{T}(s) - \Theta]\mathbb{E}[\widehat{X}_{T}(s)] + B\widehat{v}_{T}(s)\rangle\}\mathrm{d}s. \end{aligned}$$

Further, since $P > 0$ and $\mathcal{C}^{\mathrm{T}}P\mathcal{C} + \Theta^{\mathrm{T}}R\Theta + Q > 0$, we have from the above that

$$\begin{aligned} |\mathbb{E}[\widehat{X}_{T}(t)]|^{2} \leq \alpha_{1} + \mathbb{E}\int_{0}^{t} &[-\alpha_{2}|\mathbb{E}[\widehat{X}_{T}(s)]|^{2} + \alpha_{1}|\Theta_{T}(s) - \Theta| \cdot |\mathbb{E}[\widehat{X}_{T}(s)]|^{2} \\ &+ 2\alpha_{1}|\mathbb{E}[\widehat{X}_{T}(s)]| \cdot |\widehat{v}_{T}(s)|]\mathrm{d}s \end{aligned}$$

for some constants $\alpha_{1}, \alpha_{2} > 0$ independent of T. Using the Cauchy-Schwarz inequality we can obtain that with two possibly different constants $\alpha_{1}, \alpha_{2} > 0$,

$$|\mathbb{E}[\widehat{X}_{T}(t)]|^{2} \leq \alpha_{1} + \mathbb{E}\int_{0}^{t} [-\alpha_{2}|\mathbb{E}[\widehat{X}_{T}(s)]|^{2} + \alpha_{1}|\Theta_{T}(s) - \Theta| \cdot |\mathbb{E}[\widehat{X}_{T}(s)]|^{2} + \alpha_{1}|\widehat{v}_{T}(s)|^{2}]\mathrm{d}s.$$

Recalling (5.9) and (5.14), we see that with another two possibly different constants $\alpha_1, \alpha_2 > 0$ independent of T,

$$|\mathbb{E}[\widehat{X}_T(t)]|^2 \leq \alpha_1 + \mathbb{E}\int_0^t [(\alpha_1 e^{-2\lambda(T-s)} - \alpha_2)|\mathbb{E}[\widehat{X}_T(s)]|^2 + \alpha_1 e^{-2\lambda(T-s)}]ds.$$

For any $0 \leq s \leq t \leq T$, we have

$$\begin{aligned}\phi(s,t) &\triangleq \exp\Big\{\int_s^t (\alpha_1 e^{-2\lambda(T-r)} - \alpha_2)dr\Big\} \\ &= \exp\Big\{\frac{\alpha_1}{2\lambda}[e^{-2\lambda(T-t)} - e^{-2\lambda(T-s)}] - \alpha_2(t-s)\Big\} \\ &\leq e^{\frac{\alpha_1}{2\lambda}} e^{-\alpha_2(t-s)},\end{aligned}$$

and thus

$$\int_0^t \phi(s,t)e^{-2\lambda(T-s)}ds \leq e^{\frac{\alpha_1}{2\lambda}}\int_0^t e^{-2\lambda(T-s)}ds \leq \frac{1}{2\lambda}e^{\frac{\alpha_1}{2\lambda}}e^{-2\lambda(T-t)}.$$

Hence, by Gronwall's inequality,

$$\begin{aligned}|\mathbb{E}[\widehat{X}_T(t)]|^2 &\leq \alpha_1\phi(0,t) + \alpha_1\int_0^t \phi(s,t)e^{-2\lambda(T-s)}ds \\ &\leq \alpha_1 e^{\frac{\alpha_1}{2\lambda}}e^{-\alpha_2 t} + \frac{\alpha_1}{2\lambda}e^{\frac{\alpha_1}{2\lambda}}e^{-2\lambda(T-t)} \\ &\leq L[e^{-\mu t} + e^{-\mu(T-t)}], \quad \forall t \in [0,T], \end{aligned} \tag{5.17}$$

where $L = \big(\alpha_1 + \frac{\alpha_1}{2\lambda}\big)e^{\frac{\alpha_1}{2\lambda}}$ and $\mu = \alpha_2 \wedge 2\lambda$. Now using the relation (5.8) and noting that

$$|\widehat{v}_T(t)| \leq Ke^{-\lambda(T-t)},$$

we can show that

$$|\mathbb{E}[\widehat{u}_T(t)]|^2 \leq 2|\Theta_T(t)|^2\,|\mathbb{E}[\widehat{X}_T(t)]|^2 + 2|\widehat{v}_T(t)|^2 \leq L[e^{-\mu t} + e^{-\mu(T-t)}], \quad \forall t \in [0,T] \tag{5.18}$$

for a possibly different constant L that is independent of T. Finally, we can verify that the following relation holds:

$$\widehat{Y}_T(t) = P_T(t)\widehat{X}_T(t) + \varphi_T(t), \quad \overline{Z}_T(t) = P_T(t)[C\widehat{X}_T(t) + D\widehat{u}_T(t) + \sigma^*],$$

from which it follows that (with a possibly different constant $L > 0$)

$$|\mathbb{E}[\widehat{Y}_T(t)]|^2 \leq L[e^{-\mu t} + e^{-\mu(T-t)}], \quad \forall t \in [0,T]. \tag{5.19}$$

Combining (5.17)–(5.19), we get the desired (5.2).

We conclude this section by showing that the value function $V_T(x)$ of Problem (SLQ)$_T$ converges in the sense of time-average to the optimal value $V \triangleq F(x^*, u^*)$ of the associated static optimization problem.

Theorem 5.2 *Let* (H1)–(H2) *hold. Then*

$$\frac{1}{T}V_T(x) \to V \quad \textit{as } T \to \infty. \tag{5.20}$$

Proof We observe first that

$$V_T(x) = J_T(x; \overline{u}_T(\cdot))$$

can be written as follows:

$$\begin{aligned} J_T(x; \overline{u}_T(\cdot)) = & \frac{1}{2} \int_0^T [f(\mathbb{E}\overline{X}_T(t)) + g(\mathbb{E}\overline{u}_T(t))]\mathrm{d}t \\ & + \frac{1}{2} \mathbb{E} \int_0^T [\langle Q\check{X}_T(t), \check{X}_T(t)\rangle + \langle R\check{u}_T(t), \check{u}_T(t)\rangle]\mathrm{d}t, \end{aligned} \tag{5.21}$$

where $f(x) \triangleq \langle Qx, x\rangle + 2\langle q, x\rangle$, $g(u) \triangleq \langle Ru, u\rangle$, and

$$\check{X}_T(\cdot) \triangleq \overline{X}_T(\cdot) - \mathbb{E}\overline{X}_T(\cdot), \quad \check{u}_T(\cdot) \triangleq \overline{u}_T(\cdot) - \mathbb{E}\overline{u}_T(\cdot).$$

By Corollary 5.1,

$$\frac{1}{T} \int_0^T [f(\mathbb{E}\overline{X}_T(t)) + g(\mathbb{E}\overline{u}_T(t))]\mathrm{d}t \to f(x^*) + g(u^*). \tag{5.22}$$

On the other hand, $\check{X}_T(\cdot)$ evolves according to the following SDE:

$$\begin{cases} \mathrm{d}\check{X}_T(t) = [A\check{X}_T(t) + B\check{u}_T(t)]\mathrm{d}t + [C\check{X}_T(t) + D\check{u}_T(t) + \overline{\sigma}(t)]\mathrm{d}W, \\ \check{X}_T(0) = 0, \end{cases}$$

where

$$\overline{\sigma}(t) \triangleq C\mathbb{E}\overline{X}_T(t) + D\mathbb{E}\overline{u}_T(t) + \sigma.$$

Let $P_T(\cdot)$ be the solution to the Riccati equation (4.4). Then

$$\begin{aligned} 0 &= \mathbb{E}\langle P_T(T)\check{X}_T(T), \check{X}_T(T)\rangle - \mathbb{E}\langle P_T(0)\check{X}_T(0), \check{X}_T(0)\rangle \\ &= \mathbb{E} \int_0^T \{\langle \dot{P}_T(t)\check{X}_T(t), \check{X}_T(t)\rangle + 2\langle P_T(t)\check{X}_T(t), A\check{X}_T(t) + B\check{u}_T(t)\rangle \\ &\quad + \langle P_T(t)[C\check{X}_T(t) + D\check{u}_T(t) + \overline{\sigma}(t)], C\check{X}_T(t) + D\check{u}_T(t) + \overline{\sigma}(t)\rangle\}\mathrm{d}t. \end{aligned} \tag{5.23}$$

Noting the fact

$$\mathbb{E}\check{X}_T(t) = 0, \quad \mathbb{E}\check{u}_T(t) = 0, \quad \forall t \in [0, T],$$

and using (4.4), we derive from (5.23) that

$$\begin{aligned} 0 = \mathbb{E} \int_0^T & \{\big\langle [\mathcal{S}(P_T(t))^{\mathrm{T}}\mathcal{R}(P_T(t))^{-1}\mathcal{S}(P_T(t)) - Q]\check{X}_T(t), \check{X}_T(t)\big\rangle \\ & + 2\langle \check{u}_T(t), \mathcal{S}(P_T(t))\check{X}_T(t)\rangle + \langle D^{\mathrm{T}}P_T(t)D\check{u}_T(t), \check{u}_T(t)\rangle + \langle P_T(t)\overline{\sigma}(t), \overline{\sigma}(t)\rangle\}\mathrm{d}t. \end{aligned} \tag{5.24}$$

It follows that

$$\begin{aligned} & \mathbb{E} \int_0^T [\langle Q\check{X}_T(t), \check{X}_T(t)\rangle + \langle R\check{u}_T(t), \check{u}_T(t)\rangle]\mathrm{d}t \\ &= \mathbb{E} \int_0^T \{\big\langle [\mathcal{S}(P_T)^{\mathrm{T}}\mathcal{R}(P_T)^{-1}\mathcal{S}(P_T)]\check{X}_T, \check{X}_T\big\rangle + 2\langle \check{u}_T, \mathcal{S}(P_T)\check{X}_T\rangle \\ &\quad + \langle \mathcal{R}(P_T)\check{u}_T, \check{u}_T\rangle + \langle P_T\overline{\sigma}, \overline{\sigma}\rangle\}\mathrm{d}t \\ &= \mathbb{E} \int_0^T \{\langle \mathcal{R}(P_T)[\check{u}_T - \Theta_T\check{X}_T], \check{u}_T - \Theta_T\check{X}_T\rangle + \langle P_T\overline{\sigma}, \overline{\sigma}\rangle\}\mathrm{d}t. \end{aligned}$$

By (5.8), we have

$$\begin{aligned}\check{u}_T(t)-\Theta_T(t)\check{X}_T(t)&=(\widehat{u}_T(t)-\mathbb{E}[\widehat{u}_T(t)])-\Theta_T(t)(\widehat{X}_T(t)-\mathbb{E}[\widehat{X}_T(t)])\\&=(\widehat{u}_T(t)-\Theta_T(t)\widehat{X}_T(t))-\mathbb{E}(\widehat{u}_T(t)-\Theta_T(t)\widehat{X}_T(t))=0,\end{aligned}$$

and by (4.10) and (5.2), we have

$$\frac{1}{T}\int_0^T\langle P_T(t)\overline{\sigma}(t),\overline{\sigma}(t)\rangle\mathrm{d}t\to\langle P\sigma^*,\sigma^*\rangle\quad\text{as } T\to\infty.$$

Therefore, as $T\to\infty$,

$$\frac{1}{T}\mathbb{E}\int_0^T[\langle Q\check{X}_T(t),\check{X}_T(t)\rangle+\langle R\check{u}_T(t),\check{u}_T(t)\rangle]\mathrm{d}t\to\langle P\sigma^*,\sigma^*\rangle.\tag{5.25}$$

Combining (5.22) and (5.25) yields (5.20).

We point out that for general situation, namely, the state equation (1.1) and the cost functional (1.2), we may carry out the procedure (with more complicated notation) to get the same results, or transform back from the results for the reduced problem. The general conditions ensuring the results are the stabilizability of the system $[A,C;B,D]$ and the stronger standard condition (1.11) for the weighting functions of the cost functional.

6 Concluding Remarks

For linear-quadratic stochastic optimal control problems in finite time-horizon, we have established the turnpike property under the natural condition of stabilizability of the controlled linear SDE and the strong standard condition of the quadratic cost functional. The crucial contribution of the current paper is to find the correct form of the corresponding static optimization problem in which the diffusion part of the state equation should be getting into the cost functional, rather than taking it to be as an additional equality constraint. Such an idea should have big impact on the study of turnpike type problems for general stochastic optimal control problems. We will report some further results along this line in our future publications.

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