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Holomorphic Curves into Projective Varieties Intersecting Closed Subschemes in Subgeneral Position*

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Abstract In this paper, the authors introduce the index of subgeneral position for closed subschemes and obtain a second main theorems based on this notion. They also give the corresponding Schmidt's subspace type theorem via the analogue between Nevanlinna theory and Diophantine approximation.

Keywords Nevanlinna theory, Second main theorem, Holomorphic curve, Subgeneral position, Closed subschemes, Schmidt's subspace theorem
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1 Introduction

In higher dimensional Nevanlinna theory, the second main theorem of holomorphic curves into complex varieties intersecting subvarieties is a main research content. Recently, there are many developments in extending the second main theorem for divisors to arbitrary closed subschemes. To state some of the results, we recall the following notions.

Definition 1.1 *Let $D_1, \cdots, D_q$ be effective (Weil or Cartier) divisors on a complex projective variety X.*

(i) *$D_1, \cdots, D_q$ are said to be in general position if for any subset $I \subset \{1, \cdots, q\}$ with $|I| \leq \dim X + 1$,*

$$\operatorname{codim} \bigcap_{i \in I} \operatorname{Supp} D_i \geq |I|.$$

(ii) *$D_1, \cdots, D_q$ are said to be in m-subgeneral position if for any subset $I \subset \{1, \cdots, q\}$ with $|I| \leq m + 1$,*

$$\dim \bigcap_{i \in I} \operatorname{Supp} D_i \leq m - |I|$$

(*Here we set* $\dim \emptyset = -1$).

Thus the divisors are in general position if they are in $(\dim X)$-subgeneral position.

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Let $\mathcal{L}$ be a line sheaf (invertible sheaf) on the projective variety X, and let D be an effective Cartier divisor on X. Ru and Vojta [4] introduced the number

$$\gamma(\mathcal{L}, D) = \limsup_{N\to\infty} \frac{Nh^0(\mathcal{L}^N)}{\sum\limits_{\alpha=1}^{\infty} h^0(\mathcal{L}^N(-\alpha D))},$$

where N passes over all positive integers such that $h^0(\mathcal{L}^N(-D)) \neq 0$. In [4], Ru and Vojta proved the following second main theorem in terms of this number.

Theorem 1.1 (see [4]) *Let X be a complex projective variety, and let $D_1, \cdots, D_q$ be effective Cartier divisors on X. Assume that $D_1, \cdots, D_q$ intersect properly, i.e., for any subset $I \subset \{1, \cdots, q\}$ and any $\mathbf{x} \in \bigcap_{i\in I} \operatorname{Supp} D_i$, the sequence $(\phi_i)_{i\in I}$ is a regular sequence in the local ring $\mathcal{O}_{X,\mathbf{x}}$, where ϕ_i is the local defining function of D_i. (We remark that this assumption is automatically true if X is smooth and $D_1, \cdots, D_q$ are in general position on X.) Let $D = D_1 + \cdots + D_q$ and let $\mathcal{L}$ be a line sheaf on X with $h^0(\mathcal{L}^N) > 1$ for N big enough. Let $f : \mathbb{C} \to X$ be an algebraically non-degenerate holomorphic curve. Then, for every $\varepsilon > 0$,*

$$\| \quad m_f(r, D) \le \Big(\max_{1\le j\le q} \gamma(\mathcal{L}, D_j) + \varepsilon\Big) T_{f,\mathcal{L}}(r),$$

where "$\|$" means the inequality holds for all r outside a set of finite Lebesgue measure.

Ji, Yan and Yu introduced the notion of index of subgeneral position as follows.

Definition 1.2 (see [2]) *Let $D_1, \cdots, D_q$ be effective (Weil or Cartier) divisors on a projective variety X of dimension n. Let $m \ge n$ and $\kappa \le n$ be two positive integers. We say $D_1, \cdots, D_q$ are in m-subgeneral position with index κ if $D_1, \cdots, D_q$ are in m-subgeneral position and for any subset $J \subset \{1, \cdots, q\}$ with $|J| \le \kappa$,*

$$\operatorname{codim} \bigcap_{j\in J} \operatorname{Supp} D_j \ge |J|.$$

Obviously, the index κ is at least one for any divisors in subgeneral position. Ji-Yan-Yu extended Ru-Vojta's result to arbitrary effective divisors in subgeneral position.

Theorem 1.2 (see [2]) *Let X be a smooth complex projective variety, and let $D_1, \cdots, D_q$ be effective divisors in m-subgeneral position with index κ on X. Let $\mathcal{L}$ be a line sheaf on X with $h^0(\mathcal{L}^N) > 1$ for N big enough. Let $f : \mathbb{C} \to X$ be an algebraically non-degenerate holomorphic curve. Then, for every $\varepsilon > 0$,*

$$\| \quad m_f(r, D) \le \Big(\frac{m}{\kappa} \max_{1\le j\le q} \gamma(\mathcal{L}, D_j) + \varepsilon\Big) T_{f,\mathcal{L}}(r), \tag{1.1}$$

where $D = D_1 + \cdots + D_q$.

The general and subgeneral position condition can be extended to the case of subschemes as follows.

Definition 1.3 *Let $Y_1, \cdots, Y_q$ be closed subschemes on a projective variety X of dimension n. Let $m \ge n$ be a positive integer. We say $Y_1, \cdots, Y_q$ are in m-subgeneral position if for any subset $I \subset \{1, \cdots, q\}$ with $|I| \le m + 1$,*

$$\dim \bigcap_{i\in I} \operatorname{Supp} Y_i \le m - \sum_{i\in I} \operatorname{codim} Y_i.$$

If $m = n$, then $Y_1, \cdots, Y_q$ are called in general position.

When the closed subschemes $Y_1, \cdots, Y_q$ are divisors, this definition is exactly the same with Definition 1.1.

Definition 1.4 (see [5–6, 10]) *Let $\mathcal{L}$ be a big line sheaf and let Y be a closed subscheme on a complete variety X. We define*

$$\beta(\mathcal{L}, Y) = \liminf_{N\to\infty} \frac{\sum\limits_{\alpha\geq 1} h^0(\mathcal{L}^N \otimes \mathscr{I}_Y^{\alpha})}{N h^0(\mathcal{L}^N)},$$

where $\mathscr{I}_Y$ is the ideal sheaf defining Y.

Definition 1.5 (see [6]) *Let $Y_1, \cdots, Y_q$ be closed subschemes on a projective variety X. We say that $Y_1, \cdots, Y_q$ intersect properly on X if for any subset $I \subset \{1, \cdots, q\}$ such that $\bigcap\limits_{i\in I} \operatorname{Supp} Y_i \neq \emptyset$, the sequence $\{\phi_{i,1} \cdots \phi_{i,\epsilon_i}, i \in I\}$ is a regular sequence in the local ring $\mathcal{O}_{X,\mathbf{x}}$ for $\mathbf{x} \in \bigcap\limits_{i\in I} \operatorname{Supp} Y_i$, where, for each $1 \leq i \leq q$, $\phi_{i,1} \cdots \phi_{i,\epsilon_i}$ are the local functions in $\mathcal{O}_{X,\mathbf{x}}$ defining Y_i. Therefore, $\epsilon_i \geq \operatorname{codim} Y_i$, and $\sum\limits_{i\in I} \epsilon_i \leq \dim X$.*

Ru and Wang proved the following result for subvarieties.

Theorem 1.3 (see [6]) *Let X be a smooth complex projective variety, and let $Y_1, \cdots, Y_q$ be closed subschemes intersecting properly. Let $\mathcal{L}$ be a big line sheaf on X. Let $f : \mathbb{C} \to X$ be an algebraically non-degenerate holomorphic curve. Then, for every $\varepsilon > 0$,*

$$\Big\| \quad \sum_{i=1}^{q} \beta(\mathcal{L}, Y_i) m_f(r, Y_i) \leq (1 + \varepsilon) T_{f,\mathcal{L}}(r). \tag{1.2}$$

We can also extend the index condition for subgeneral position to the case of closed subschemes.

Definition 1.6 *Let $Y_1, \cdots, Y_q$ be closed subschemes on a projective variety X of dimension n. Let $m \geq n$ and $\kappa \leq n$ be two positive integers. We say $Y_1, \cdots, Y_q$ are in m-subgeneral position with index κ if $Y_1, \cdots, Y_q$ are in m-subgeneral position and for any subset $J \subset \{1, \cdots, q\}$ with $|J| \leq \kappa$,*

$$\operatorname{codim} \bigcap_{j\in J} \operatorname{Supp} Y_j \geq \sum_{j\in J} \operatorname{codim} Y_j.$$

Remark 1.1 If $Y_1, \cdots, Y_q$ are of locally complete intersection closed subschemes intersecting properly on X, then they are in general position. The converse holds if X is Cohen-Macaulay (this is true if X is nonsingular) by [3, Theorem 17.4]. Thus, if X is a smooth projective variety, then the subschemes $Y_1, \cdots, Y_q$, which are of locally complete intersection, are in m-subgeneral position with index κ, if and only if, the subschemes $Y_1, \cdots, Y_q$ are in m-subgeneral position and for any set $J \subset \{1, \cdots, q\}$ with $|J| \leq \kappa$, $Y_j, j \in J$ intersect properly.

The purpose of this paper is to extend Theorem 1.2 to the case of closed subschemes. Here is our main result.

Theorem 1.4 (Main Theorem) *Let X be a smooth complex projective variety, and let $Y_1, \cdots, Y_q$ be closed subschemes, which are of locally complete intersection, in m-subgeneral* *position with index κ on X. Let $\mathcal{L}$ be a big line sheaf on X. Let $f:\mathbb{C}\to X$ be an algebraically non-degenerate holomorphic curve. Then, for every $\varepsilon>0$,*

$$\Big\| \quad \sum_{i=1}^{q}\beta(\mathcal{L},Y_i)m_f(r,Y_i)\le\Big(\frac{m}{\kappa}+\varepsilon\Big)T_{f,\mathcal{L}}(r). \tag{1.3}$$

Remark 1.2 We assume that X is smooth and the closed subschemes $Y_1,\cdots,Y_q$ are of locally complete intersection in Theorem 1.4, because we use the filtration which is valid for properly intersecting closed subschemes. If the closed subschemes $Y_1,\cdots,Y_q$ are in m-subgeneral position and for any set $J\subset\{1,\cdots,q\}$ with $|J|\le\kappa$, $Y_j, j\in J$ intersect properly, then we do not need these assumptions. When the closed subschemes intersect proplerly, we can take $m=\kappa=\dim X$, thus our main theorem recovers Ru-Wang's result in [6].

2 Preliminaries on Nevanlinna Theory

In this section, we briefly recall some definitions and facts in Nevanlinna theory.

2.1 Weil functions

We briefly recall the basic definition of Weil functions, one can refer to [7] for more details. Let Y be a closed subscheme of a projective variety X. One can associate a Weil function $\lambda_Y: X\setminus\operatorname{Supp}Y\to\mathbb{R}$, well-defined up to $O(1)$, which satisfies the following properties: If Y and Z are two closed subschemes of X, and $\phi: X'\to X$ is a morphism of projective varieties,

(i) $\lambda_{Y\cap Z}=\min\{\lambda_Y,\lambda_Z\}$;

(ii) $\lambda_{Y+Z}=\lambda_Y+\lambda_Z$;

(iii) $\lambda_Y\le\lambda_Z$, if $Y\subset Z$;

(iv) $\lambda_Y(\phi(\mathbf{x}))=\lambda_{\phi^*Y}(\mathbf{x})$.

In particular, let D be a Cartier divisor on a complex projective variety X. A Weil function with respect to D is a function $\lambda_D:(X\setminus\operatorname{Supp}D)\to\mathbb{R}$ such that for all $\mathbf{x}\in X$ there is an open neighborhood U of $\mathbf{x}$ in X, a nonzero rational function f on X with $D|_U=(f)$, and a continuous function $\alpha: U\to\mathbb{R}$ such that

$$\lambda_D(\mathbf{x})=-\log|f(\mathbf{x})|+\alpha(\mathbf{x})$$

for all $\mathbf{x}\in(U\setminus\operatorname{Supp}D)$. Note that a continuous fiber metric $\|\cdot\|$ on the line sheaf $\mathcal{O}_X(D)$ determines a Weil function for D given by $\lambda_D(\mathbf{x})=-\log\|s(\mathbf{x})\|$ where s is the rational section of $\mathcal{O}_X(D)$ such that $D=(s)$.

The Weil functions with respect to divisors satisfy the following properties:

(a) Functoriality: If λ is a Weil function for a Cartier divisor D on X, and if $\phi: X'\to X$ is a morphism such that $\phi(X')\not\subset\operatorname{Supp}D$, then $\mathbf{x}\mapsto\lambda(\phi(\mathbf{x}))$ is a Weil function for the Cartier divisor ϕ^*D on X'.

(b) Additivity: If λ_1 and λ_2 are Weil functions for Cartier divisors D_1 and D_2 on X, respectively, then $\lambda_1+\lambda_2$ is a Weil function for D_1+D_2.

(c) Uniqueness: If both λ_1 and λ_2 are Weil functions for a Cartier divisor on X, then $\lambda_1=\lambda_2+O(1)$.

(d) Boundedness from below: If D is an effective divisor and λ is a Weil function for D, then λ is bounded from below.

Let X be a projective variety, and let $Y \subset X$ be a closed subscheme.

Lemma 2.1 (see [7, Lemma 2.2]) *There exist effective Cartier divisors $D_1, \cdots, D_\ell$ such that*

$$Y = \bigcap_{i=1}^{\ell} D_i.$$

By Lemma 2.1, we can assume that $Y = D_1 \cap \cdots \cap D_\ell$, where $D_1, \cdots, D_\ell$ are effective Cartier divisors. This means that $\mathscr{I}_Y = \mathscr{I}_{D_1} + \cdots + \mathscr{I}_{D_\ell}$, where $\mathscr{I}_Y, \mathscr{I}_{D_1}, \cdots, \mathscr{I}_{D_\ell}$ are the defining ideal sheaves in $\mathcal{O}_X$. We set

$$\lambda_Y = \min\{\lambda_{D_1}, \cdots, \lambda_{D_\ell}\} + O(1). \tag{2.1}$$

Then we have $\lambda_Y : X \setminus \operatorname{Supp} Y \to \mathbb{R}$, which does not depend on the choice of Cartier divisors.

2.2 Characteristic function

Let X be a complex projective variety and $f : \mathbb{C} \to X$ be a holomorphic map. Let $\mathcal{L} \to X$ be an ample line sheaf and ω be its Chern form. We define the characteristic function of f with respect to $\mathcal{L}$ by

$$T_{f,\mathcal{L}}(r) = \int_1^r \frac{\mathrm{d}t}{t} \int_{|z|<t} f^*\omega.$$

Since any line sheaf $\mathcal{L}$ can be written as $\mathcal{L} = \mathcal{L}_1 \otimes \mathcal{L}_2^{-1}$ with $\mathcal{L}_1$, $\mathcal{L}_2$ being both ample, we define $T_{f,\mathcal{L}}(r) = T_{f,\mathcal{L}_1}(r) - T_{f,\mathcal{L}_2}(r)$. A divisor D on X defines a line bundle $\mathcal{O}(D)$, we denote by $T_{f,D}(r) = T_{f,\mathcal{O}(D)}(r)$. If $X = \mathbb{P}^n(\mathbb{C})$ and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(1)$, then we simply write $T_{f,\mathcal{O}_{\mathbb{P}^n(\mathbb{C})}(1)}(r)$ as $T_f(r)$.

The characteristic function satisfies the following properties:

(a) Functoriality: If $\phi : X \to X'$ is a morphism and if $\mathcal{L}$ is a line sheaf on X', then

$$T_{f,\phi^*\mathcal{L}}(r) = T_{\phi\circ f,\mathcal{L}}(r) + O(1).$$

(b) Additivity: If $\mathcal{L}_1$ and $\mathcal{L}_2$ are line sheaves on X, then

$$T_{f,\mathcal{L}_1\otimes\mathcal{L}_2}(r) = T_{f,\mathcal{L}_1}(r) + T_{f,\mathcal{L}_2}(r) + O(1).$$

(c) Positivity. If $\mathcal{L}$ is ample and $f : \mathbb{C} \to X$ is non-constant, then

$$T_{f,\mathcal{L}}(r) \to +\infty \quad \text{as } r \to +\infty.$$

(d) Base locus: If the image of f is not contained in the base locus of $|D|$, then $T_{f,D}(r)$ is bounded from below.

(e) Globally generated line sheaves: If $\mathcal{L}$ is a line sheaf on X, and is generated by its global sections, then $T_{f,\mathcal{L}}(r)$ is bounded from blow.

2.3 Proximity functions

Let X be a projective variety and let $Y \subset X$ be a closed subscheme. For a holomorphic curve $f : \mathbb{C} \to X$ with $f(\mathbb{C}) \not\subset \operatorname{Supp} Y$, the proximity function of f with respect to Y is defined by

$$m_f(r, Y) = \int_0^{2\pi} \lambda_Y(f(re^{i\theta}))\frac{d\theta}{2\pi}.$$

The proximity function satisfies the following properties:

(a) Functoriality: If $\phi : X \to X'$ is a morphism and Y' is a closed subscheme on X' with $\phi \circ f(\mathbb{C}) \not\subset \operatorname{Supp} Y'$, then

$$m_f(r, \phi^* Y') = m_{\phi\circ f}(r, Y') + O(1).$$

(b) Additivity: If Y_1 and Y_2 are two closed subschemes on X, then

$$m_f(r, Y_1 + Y_2) = m_f(r, Y_1) + m_f(r, Y_2) + O(1).$$

(c) Boundedness from below: If D is an effective divisor, then $m_f(r, D)$ is bounded from below.

2.4 A general form of Cartan's second main theorem

To prove our main theorem, we need the following general form of Cartan's second main theorem given by Ru and Vojta [4].

Theorem 2.1 (see [4, Theorem 2.8] *Let X be a complex projective variety, and let $\mathcal{L}$ be a line sheaf on X. Let V be a linear subspace of $H^0(X, \mathcal{L})$ with $\dim V > 1$, and let $s_1, \cdots, s_q$ be nonzero elements of V. For each $j = 1, \cdots, q$, let D_j be the Cartier divisor (s_j). Let $f : \mathbb{C} \to X$ be an algebraically non-degenerate holomorphic curve. Then, for every $\varepsilon > 0$,*

$$\Big\| \quad \int_0^{2\pi} \max_{\mathcal{K}} \sum_{j\in\mathcal{K}} \lambda_{D_j}(f(re^{i\theta}))\frac{d\theta}{2\pi} \le (\dim V + \varepsilon)T_{f,\mathcal{L}}(r), \tag{2.2}$$

where $\max_{\mathcal{K}}$ is taken over all subsets $\mathcal{K}$ of $\{1, \cdots, q\}$ such that the sections $\{s_j\}_{j\in\mathcal{K}}$ are linearly independent.

3 Proof of Main Theorem

Given $z \in \mathbb{C}$, we arrange so that

$$\begin{aligned}\beta(\mathcal{L}, Y_{1,z})\lambda_{Y_{1,z}}(f(z)) &\ge \beta(\mathcal{L}, Y_{2,z})\lambda_{Y_{2,z}}(f(z)) \ge \cdots \ge \beta(\mathcal{L}, Y_{\kappa,z})\lambda_{Y_{\kappa,z}}(f(z)) \\ &\ge \cdots \ge \beta(\mathcal{L}, Y_{m,z})\lambda_{Y_{m,z}}(f(z)) \ge \cdots \ge \beta(\mathcal{L}, Y_{q,z})\lambda_{Y_{q,z}}(f(z)).\end{aligned} \tag{3.1}$$

Since $Y_1, \cdots, Y_q$ are in m-subgeneral position, we have

$$\begin{aligned}\sum_{j=1}^{q} \beta(\mathcal{L}, Y_{j,z})\lambda_{Y_{j,z}}(f(z)) &\le \sum_{j=1}^{m} \beta(\mathcal{L}, Y_{j,z})\lambda_{Y_{j,z}}(f(z)) + O(1) \\ &\le \frac{m}{\kappa}\sum_{j=1}^{\kappa} \beta(\mathcal{L}, Y_{j,z})\lambda_{Y_{j,z}}(f(z)) + O(1).\end{aligned} \tag{3.2}$$

Note that $Y_{1,z}, \cdots, Y_{\kappa,z}$ are located in general position and X is smooth, then we know that $Y_{1,z}, \cdots, Y_{\kappa,z}$ intersect properly on X. Thus, we can use the filtration constructed in [1, 4, 6]. Now we consider the following filtration of $H^0(X, \mathcal{L}^N)$ with respect to $\{Y_{1,z}, \cdots, Y_{\kappa,z}\}$, where $N > 0$ is an integer to be specified.

Denote $\beta_i := \beta(\mathcal{L}, Y_i)$.

Let $\Delta_z = \big\{\mathbf{a}_z = (a_{i,z}) \in \prod_{i=1}^{\kappa} \beta_{i,z}^{-1}\mathbb{N} \big| \sum_{i=1}^{\kappa} \beta_{i,z} a_{i,z} = b\big\}$, where $b > 0$ is an integer to be specified.

For $\mathbf{a}_z \in \Delta_z$ and $x \in \mathbb{R}_{\geq 0}$, let

$$N(\mathbf{a}_z, x) = \left\{\mathbf{b}_z = (b_{i,z}) \in \mathbb{N}^\kappa \mid \sum_{i=1}^{\kappa} a_{i,z} b_{i,z} \geq x\right\}$$

and

$$\mathcal{I}(\mathbf{a}_z, x) = \sum_{\mathbf{b}_z \in N(\mathbf{a_z}, bx)} (\otimes_{i=1}^{\kappa} \mathscr{I}_{Y_{i,z}}^{b_{i,z}}) \tag{3.3}$$

be an ideal of $\mathcal{O}_X$. Set

$$\mathcal{F}(\mathbf{a}_z)_x = H^0(X, \mathcal{L}^N \otimes \mathcal{I}(\mathbf{a}_z, x))$$

and

$$F(\mathbf{a}_z) = \frac{1}{h^0(\mathcal{L}^N)} \int_0^{+\infty} (\dim \mathcal{F}(\mathbf{a}_z)_x) \mathrm{d}x.$$

Then $(\mathcal{F}(\mathbf{a}_z)_x)_{x \in \mathbb{R}_{\geq 0}}$ is a filtration of $H^0(X, \mathcal{L}^N)$ and for any basis $\mathcal{B}_{z,\mathbf{a}_z}$ of $H^0(X, \mathcal{L}^N)$ with respect to the above filtration $(\mathcal{F}(\mathbf{a}_z)_x)_{x \in \mathbb{R}_{\geq 0}}$, we have

$$F(\mathbf{a}_z) = \frac{1}{h^0(\mathcal{L}^N)} \sum_{s \in \mathcal{B}_{z,\mathbf{a}_z}} \mu_{\mathbf{a}_z}(s),$$

where $\mu_{\mathbf{a}_z}(s) = \sup\{\mu \in \mathbb{R}_{\geq 0} \mid s \in \mathcal{F}(\mathbf{a}_z)_\mu\}$.

We note that it suffices to use only the leading terms in (3.3). The union of the sets of leading terms as x ranges over the interval $[0, \mu_{\mathbf{a}_z}(s)]$ is finite, and each such $\mathbf{b}_z$ occurs in the sum (3.3) for a closed set of x. Therefore the supremum in the definition of $\mu_{\mathbf{a}_z}(s)$ is actually a maximum, i.e.,

$$\mu_{\mathbf{a}_z}(s) = \max\{\mu \in \mathbb{R}_{\geq 0} \mid s \in \mathcal{F}(\mathbf{a}_z)_\mu\}. \tag{3.4}$$

Then we have

$$s \in \mathcal{F}(\mathbf{a}_z)_{\mu_{\mathbf{a}_z}(s)} \tag{3.5}$$

and

$$\mathcal{L}^N \otimes \mathcal{I}(\mathbf{a}_z, \mu_{\mathbf{a}_z}(s)) = \sum_{\mathbf{b}_z \in K_{\mathbf{a}_z,s}} (\otimes_{i=1}^{\kappa} \mathscr{I}_{Y_{i,z}}^{b_{i,z}}), \tag{3.6}$$

where $K_{\mathbf{a}_z,s}$ is the set of minimal elements of $N(\mathbf{a}_z, \mu_{\mathbf{a}_z}(s))$ relative to the product partial ordering on $\mathbb{N}^\kappa$. This set is finite. Hence, using the properties of Weil functions, we get

$$\lambda_{(s)}(f(z)) \geq \min_{\mathbf{b}_z \in K_{\mathbf{a}_z,s}} \sum_{i=1}^{\kappa} b_{i,z} \lambda_{Y_{i,z}}(f(z)) + O(1). \tag{3.7}$$

By [6, Proposition 3.8], we know

$$F(\mathbf{a}_z) \geq \min_{1\leq j\leq q}\Big(\frac{b}{\beta_j h^0(\mathcal{L}^N)}\sum_{\alpha=1}^{\infty} h^0(\mathcal{L}^N\otimes \mathscr{I}_{Y_j}^{\alpha})\Big), \tag{3.8}$$

which implies

$$\sum_{s\in\mathcal{B}_{z,\mathbf{a}_z}}\mu_{\mathbf{a}_z}(s) \geq \min_{1\leq j\leq q}\frac{b}{\beta_j}\sum_{\alpha=1}^{\infty} h^0(\mathcal{L}^N\otimes \mathscr{I}_{Y_j}^{\alpha}). \tag{3.9}$$

Note that there are only finitely many choice of $\{Y_{1,z},\cdots,Y_{\kappa,z}\}\subset\{Y_1,\cdots,Y_q\}$ and $\mathbf{a}_z\in\Delta_z$, the number of basis $\mathcal{B}_{z,\mathbf{a}_z}$ is finite, write

$$\{\mathcal{B}_{z,\mathbf{a}_z}\mid z\in\mathbb{C},\mathbf{a}_z\in\Delta_z\}=\{\mathcal{B}_1,\cdots,\mathcal{B}_{T_1}\}$$

and

$$\mathcal{B}_1\cup\cdots\cup\mathcal{B}_{T_1}=\{s_1,\cdots,s_{T_2}\}.$$

For each $i=1,\cdots,T_1$, let $J_i\subseteq\{1,\cdots,T_2\}$ be the subset such that $\mathcal{B}_i=\{s_j : j\in J_i\}$.

Claim 3.1 (Key inequality)

$$\begin{aligned}&\sum_{j=1}^{\kappa}\beta_{j,z}\lambda_{Y_{j,z}}(f(z))\\ &\leq \frac{b+n}{b}\max_{1\leq i\leq q}\Big\{\frac{\beta_i}{\sum_{\alpha\geq1}h^0(\mathcal{L}^N\otimes\mathscr{I}_{Y_i}^{\alpha})}\Big\}\cdot\max_{1\leq i\leq T_1}\sum_{j\in J_i}\lambda_{(s_j)}(f(z))+O(1).\end{aligned} \tag{3.10}$$

For $j=1,\cdots,\kappa$, let

$$t_{j,z}:=\frac{\lambda_{Y_{j,z}}(f(z))}{\sum_{j=1}^{\kappa}\beta_{j,z}\lambda_{Y_{j,z}}(f(z))}, \tag{3.11}$$

then $\sum_{j=1}^{\kappa}\beta_{j,z}t_{j,z}=1$. Choose $\mathbf{a}_z=(a_{j,z})\in\Delta_z$ such that

$$a_{j,z}\leq(b+n)t_{j,z},\quad j=1,\cdots,\kappa. \tag{3.12}$$

Then (3.7), (3.11)–(3.12) and the definition of $K_{\mathbf{a}_z,s}$ imply

$$\begin{aligned}\sum_{j=1}^{\kappa}\beta_{j,z}\lambda_{Y_{j,z}}(f(z)) &\leq \sum_{j=1}^{\kappa}\beta_{j,z}\lambda_{Y_{j,z}}(f(z))\cdot\frac{\min\limits_{\mathbf{b}_z\in K_{\mathbf{a}_z,s}}\sum_{j=1}^{\kappa}a_{j,z}b_{j,z}}{\mu_{\mathbf{a}_z}(s)}+O(1)\\ &\leq \sum_{j=1}^{\kappa}\beta_{j,z}\lambda_{Y_{j,z}}(f(z))\cdot\frac{\min\limits_{\mathbf{b}_z\in K_{\mathbf{a}_z,s}}\sum_{j=1}^{\kappa}(b+n)t_{j,z}b_{j,z}}{\mu_{\mathbf{a}_z}(s)}+O(1)\\ &=(b+n)\frac{\min\limits_{\mathbf{b}_z\in K_{\mathbf{a}_z,s}}\sum_{j=1}^{\kappa}b_{j,z}\lambda_{Y_{j,z}}(f(z))}{\mu_{\mathbf{a}_z}(s)}+O(1)\\ &\leq(b+n)\frac{\lambda_{(s)}(f(z))}{\mu_{\mathbf{a}_z}(s)}+O(1).\end{aligned}$$

Combining the above inequality with (3.9), we have

$$\sum_{j=1}^{\kappa}\beta_{j,z}\lambda_{Y_{j,z}}(f(z)) \le (b+n)\cdot\frac{\sum\limits_{s\in\mathcal{B}_{z,\mathbf{a}_z}}\lambda_{(s)}(f(z))}{\sum\limits_{s\in\mathcal{B}_{z,\mathbf{a}_z}}\mu_{\mathbf{a}_z}(s)}+O(1)$$

$$\le \frac{b+n}{b}\max_{1\le i\le q}\Big\{\frac{\beta_i}{\sum\limits_{\alpha\ge 1}h^0(\mathcal{L}^N\otimes\mathscr{I}_{Y_i}^{\alpha})}\Big\}\cdot\max_{1\le i\le T_1}\sum_{j\in J_i}\lambda_{(s_j)}(f(z))+O(1).$$

Thus the claim is proved.

Combining (3.2) and (3.10), we obtain that

$$\sum_{j=1}^{q}\beta_j\lambda_{Y_j}(f(z))$$
$$\le \frac{m}{\kappa}\cdot\frac{b+n}{b}\max_{1\le i\le q}c\Big\{\frac{\beta_i}{\sum\limits_{\alpha\ge 1}h^0(\mathcal{L}^N\otimes\mathscr{I}_{Y_i}^{\alpha})}\Big\}\cdot\max_{1\le i\le T_1}\sum_{j\in J_i}\lambda_{(s_j)}(f(z))+O(1). \tag{3.13}$$

By using Theorem 2.1 with $V=H^0(X,\mathcal{L}^N)$, we have, for every $\varepsilon_1>0$,

$$\Big\|\quad\int_0^{2\pi}\max_{\mathcal{K}}\sum_{j\in\mathcal{K}}\lambda_{(s_j)}(f(re^{i\theta}))\frac{d\theta}{2\pi}\le (h^0(\mathcal{L}^N)+\varepsilon_1)T_{f,\mathcal{L}^N}(r), \tag{3.14}$$

where $\max\limits_{\mathcal{K}}$ is taken over all subsets $\mathcal{K}$ of $\{1,\cdots,T_2\}$ such that the sections $\{s_j\}_{j\in\mathcal{K}}$ are linearly independent. From the property of characteristic function, $T_{f,\mathcal{L}^N}(r)=NT_{f,\mathcal{L}}(r)$.

Choose $\varepsilon_1>0$, and positive integers N and b such that

$$\frac{b+n}{b}\max_{1\le i\le q}\Big\{\frac{\beta_i N(h^0(\mathcal{L}^N)+\varepsilon_1)}{\sum\limits_{\alpha\ge 1}h^0(\mathcal{L}^N\otimes\mathscr{I}_{Y_i}^{\alpha})}\Big\}<1+\frac{\kappa\epsilon}{2m}. \tag{3.15}$$

Then, it follows from (3.13)–(3.15) that

$$\Big\|\quad\sum_{i=1}^{q}\beta(\mathcal{L},Y_i)m_f(r,Y_i)\le\Big(\frac{m}{\kappa}+\epsilon\Big)T_{f,\mathcal{L}}(r).$$

Thus the main theorem is proved.

4 Schmidt's Subspace Theorems

In this section, we introduce the counterpart in number theory of our main theorem according to Vojta's dictionary which gives an analogue between Nevanlinna theory and Diophantine approximation (see [8–9]).

Let k be a number field. Denote by M_k the set of places (i.e., equivalence classes of absolute values) of k and write M_k^{∞} for the set of archimedean places of k.

Let X be a projective variety defined over k, let $\mathcal{L}$ be a line sheaf on X and let Y be a closed subscheme. For every place $v\in M_k$, we can associate the Weil functions (or local heights) $\lambda_{\mathcal{L},v}$ and $\lambda_{Y,v}$ with respect to v, which have similar properties as the Weil function introduced in Section 2 (see [7]). Define

$$h_{\mathcal{L}}(\mathbf{x})=\sum_{v\in M_k}\lambda_{\mathcal{L},v}(\mathbf{x})\quad\text{for }\mathbf{x}\in X$$

and

$$m_S(\mathbf{x}, Y) = \sum_{v\in S} \lambda_{Y,v}(\mathbf{x}) \quad \text{for } \mathbf{x} \in X \setminus \operatorname{Supp} Y,$$

where S is a finite subset of M_k containing M_k^∞.

Instead of Theorem 2.1, we shall use the following general form of Schmidt's subspace theorem given by Ru and Vojta [4].

Theorem 4.1 (see [4, Theorem 2.7] *Let X be a projective variety defined over k, and let $\mathcal{L}$ be a line sheaf on X. Let V be a linear subspace of $H^0(X, \mathcal{L})$ with $\dim V > 1$, and let $s_1, \cdots, s_q$ be nonzero elements of V. For each $j = 1, \cdots, q$, let D_j be the Cartier divisor (s_j). Let S be a finite subset of M_k containing M_k^∞, let $\varepsilon > 0$ and $c \in \mathbb{R}$. Then there is a proper Zariski-closed subset Z of X such that*

$$\sum_{v\in S} \max_{\mathcal{K}} \sum_{j\in\mathcal{K}} \lambda_{D_j,v}(\mathbf{x}) \le (\dim V + \varepsilon) h_{\mathcal{L}}(\mathbf{x}) + O(1)$$

holds for all $\mathbf{x} \in (X \setminus Z)(k)$. Here $\max_{\mathcal{K}}$ is taken over all subsets $\mathcal{K}$ of $\{1, \cdots, q\}$ such that the sections $\{s_j\}_{j\in\mathcal{K}}$ are linearly independent.

Now, we state the counterpart of Theorem 1.4, whose proof is similar and is therefore omitted here.

Theorem 4.2 *Let X be a smooth projective variety defined over k, and let $Y_1, \cdots, Y_q$ be closed subschemes, which are of locally complete intersection, in m-subgeneral position with index κ on X. Let $\mathcal{L}$ be a big line sheaf on X. Let S be a finite subset of M_k containing M_k^∞. Then, for every $\varepsilon > 0$,*

$$\sum_{i=1}^{q} \beta(\mathcal{L}, Y_i) m_S(\mathbf{x}, Y_i) \le \Big(\frac{m}{\kappa} + \varepsilon\Big) h_{\mathcal{L}}(\mathbf{x})$$

holds for all k-rational points outside a proper Zariski-closed subset of X.

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