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Holomorphic Retractions of Bounded Symmetric Domains onto Totally Geodesic Complex Submanifolds

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Abstract Given a bounded symmetric domain Ω the author considers the geometry of its totally geodesic complex submanifolds $S \subset \Omega$. In terms of the Harish-Chandra realization $\Omega \Subset \mathbb{C}^n$ and taking S to pass through the origin $0 \in \Omega$, so that $S = E \cap \Omega$ for some complex vector subspace of $\mathbb{C}^n$, the author shows that the orthogonal projection $\rho : \Omega \to E$ maps Ω onto S, and deduces that $S \subset \Omega$ is a holomorphic isometry with respect to the Carathéodory metric. His first theorem gives a new derivation of a result of Yeung's deduced from the classification theory by Satake and Ihara in the special case of totally geodesic complex submanifolds of rank 1 and of complex dimension ≥ 2 in the Siegel upper half plane $\mathcal{H}_g$, a result which was crucial for proving the nonexistence of totally geodesic complex suborbifolds of dimension ≥ 2 on the open Torelli locus of the Siegel modular variety $\mathcal{A}_g$ by the same author. The proof relies on the characterization of totally geodesic submanifolds of Riemannian symmetric spaces in terms of Lie triple systems and a variant of the Hermann Convexity Theorem giving a new characterization of the Harish-Chandra realization in terms of bisectional curvatures.

Keywords Bounded symmetric domain, Harish-Chandra embedding, Holomorphic retraction, Totally geodesy
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1 Introduction

Let $\Omega \Subset \mathbb{C}^n$ be a bounded symmetric domain in its Harish-Chandra realization (cf. Theorem 2.3). Denote by ds^2_Ω the G_0-invariant Kähler metric on Ω such that minimal disks on each irreducible factor of Ω are of constant Gaussian curvature -2. When Ω is irreducible, ds^2_Ω is a complete Kähler-Einstein metric. In general, the choice of a G_0-invariant Kähler metric on Ω depends on normalizing scalar constants, one for each irreducible factor.

In this article we consider complex linear slices S of Ω which are totally geodesic with respect to ds^2_Ω and prove the following theorem yielding a holomorphic retraction of Ω onto S.

Theorem 1.1 *Let $\Omega \Subset \mathbb{C}^n$ be a bounded symmetric domain in its Harish-Chandra realization. Let $E \subset \mathbb{C}^n$ be a complex vector subspace such that $S := E \cap \Omega \subset \Omega$ is a totally geodesic complex submanifold with respect to ds^2_Ω. Let $\rho : \mathbb{C}^n \to E$ be the orthogonal projection. Then, $\rho(\Omega) = S$.*

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We observe that any totally geodesic complex submanifold of (Ω, ds^2_Ω) passing through the origin $0 \in \Omega$ is necessarily of the form $S = E \cap \Omega$ for a complex vector subspace $E \subset \mathbb{C}^n$ satisfying additional conditions (cf. Proposition 2.1), hence Theorem 1.1 yields a holomorphic retraction of Ω onto any given totally geodesic complex submanifold of Ω with respect to one and hence all G_0-invariant Kähler metrics.

Theorem 1.1 in the special case where Ω is the type-III domain $D^{\rm III}_g := \{Z \in M(g,g;\mathbb{C}) : Z^t = Z \text{ and } I - \overline{Z}Z > 0\}$, $g \geq 2$, and S is biholomorphic to the complex unit ball $\mathbb{B}^m$ of dimension m, where $M(a,b;\mathbb{C})$ stands for the complex vector space of a-by-b matrices with complex coefficients and Z^t denotes the transposed matrix of Z, was established in Yeung [12, Theorem 1] by explicitly checking according to the classification of such embeddings due to Satake [10] and Ihara [2]. Theorem 1.1 in these special cases are crucial for the establishment of the following theorem in [12] concerning the open Torelli locus. For the understanding of the statement, note first of all that the type-III domain $D^{\rm III}_g$ is biholomorphic via the inverse Cayley transform $\tau = \lambda(Z) := -\imath(Z + \imath I_g)(Z - \imath I_g)^{-1}$, where I_g stands for the g-by-g identity matrix, to the Siegel upper half plane $\mathcal{H}_g := \{\tau : \mathrm{Im}(\tau) : \tau^t = \tau, \mathrm{Im}(\tau) > 0\}$ defined by the Riemann bilinear relations, so that $\mathcal{A}_g := \mathcal{H}_g/\mathbb{P}\mathrm{Sp}(g;\mathbb{Z})$ is the Siegel modular variety, the classification space of principally polarized abelian varieties.

The Torelli map $t_g : \mathcal{M}_g \to \mathcal{A}_g$, where $\mathcal{M}_g$ is the Teichmüller modular variety, i.e., the moduli space of compact Riemann surfaces C of genus $g \geq 2$, is the holomorphic map defined for a compact Riemann surface C of genus $g \geq 2$ by $t_g([C]) = [\mathrm{Jac}(C)]$, where $\mathrm{Jac}(C)$ stands for the Jacobian variety of C in its natural principal polarization, and $[\cdots]$ is here and henceforth a notation for the class of an object in some classification space. Denote by $\mathcal{M}_g \subset \overline{\mathcal{M}_g}$ the Deligne-Mumford compactification, and by $\mathcal{A}_g \subset \overline{\mathcal{A}_g}$ the Satake-Baily-Borel compactification, then it is known that $t_g : \mathcal{M}_g \to \mathcal{A}_g$ extends holomorphically to $\tau_g : \overline{\mathcal{M}_g} \to \overline{\mathcal{A}_g}$. The set $T^0_g := t_g(\mathcal{M}_g)$ is called the open Torelli locus, which is a Zariski open subset of the Zariski closed subset $\tau_g(\overline{\mathcal{M}_g}) \subset \overline{\mathcal{A}_g}$. Denoting by $H_g \subset \mathcal{M}_g$ the locus of hyperelliptic curves, then $H_g \subset \mathcal{M}_g$ is Zariski closed. It is well-known that the Torelli map $t_g : \mathcal{M}_g \to \mathcal{A}_g$ is injective and that $t_g|_{\mathcal{M}_g - H_g} : \mathcal{M}_g - H_g \to \mathcal{A}_g$ is immersive. The principal result of [12] is the following theorem.

Theorem 1.2 (cf. [12, Theorem 2]) *The set $T^0_g - t_g(H_g) \subset \mathcal{A}_g$ for $g > 2$ does not contain any complex hyperbolic complex ball quotient, compact or non-compact with finite volume, of complex dimension at least 2 as a totally geodesic complex suborbifold of $\mathcal{A}_g$.*

The above result of Yeung, in conjunction with known rigidity results in the higher rank case and existence results of Shimura curves on the open Torelli locus, yielded Yeung [12, Theorem 3], related to Oort's Conjecture, which described all Shimura varieties (necessarily of dimension 1) contained in the open Torelli locus $T^0_g - t_g(H_g)$. (For the statement of [12, Theorem 3] and related background and references cf. [12, §1].)

We give in this article a proof of Theorem 1.1 in the general situation, where the target bounded symmetric domain Ω may contain direct factors which are exceptional domains, and where the complex submanifold $S = E \cap \Omega$ is of arbitrary rank as a Hermitian symmetric manifold of the semisimple and noncompact type. Our proof is free from classification theory. It exploits the Harish-Chandra realization and a variant of the Hermann Convexity Theorem defining Ω in terms of inequalities involving bisectional curvatures. In Section 2 we collect basic materials on Riemannian symmetric spaces and bounded symmetric domains. In Section 3 we give in Theorem 3.1 a new description of independent interest of the Harish-Chandra realization of a bounded symmetric domain in terms of bisectional curvatures. In Section 4 we give the proof of Theorem 1.1 together with the immediate implication (Theorem 4.1) that totally geodesic complex submanifolds of Ω are holomorphic deformation retracts of Ω. In Section 5 we give in Theorem 5.1 an application of Theorem 1.1 to the geometry of the complex submanifold $S \subset \Omega$ in terms of the Carathéodory metric and equivalently the Kobayashi metric, which are equal on weakly convex bounded domains according to the celebrated work of Lempert [3] and a theorem of Royden-Wong (cf. Section 5 for remarks and references). In the Appendix we give a self-contained proof that the (infinitesimal) Carathéodory metric and the (infinitesimal) Kobayashi metrics agree with each other on a bounded symmetric domain Ω by means of Theorem 1.1. The article has been written in a somewhat expository style, supplied sometimes with more details than those are absolutely necessary, in order to make it more accessible to non-experts.

2 Background Materials

2.1 Basic materials in Lie theory and on Riemannian symmetric spaces

On a Riemannian symmetric space (M, ds_M^2) denote by G the identity component of the isometry group of (M, ds_M^2), and by $e \in G$ its identity element. We have $T_e(G) := \mathfrak{g}$. Here and in what follows, for real Lie groups in Roman letters we denote by the corresponding Gothic letters their associated Lie algebras, and vice versa. Let $K \subset G$ be the isotropy subgroup at a reference point $0 \in M$, so that $M = G/K$ as a homogeneous space, and $0 = eK$. Let s be the involution of (M, ds_M^2) as a Riemannian symmetric space at 0, $s = s^{-1}$, and $\sigma : G \to G$ be defined by $\sigma(g) = sgs = s^{-1}gs$, so that $d\sigma(e) : \mathfrak{g} \to \mathfrak{g}$, and we have the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ where $\mathfrak{k}$ (resp. $\mathfrak{m}$) is the eigenspace of $d\sigma(e)$ associated to the eigenvalue $+1$ (resp. -1), from which we have an identification $T_0(M) \cong \mathfrak{m}$. We have the following characterization of totally geodesic submanifolds of Riemannian symmetric spaces (cf. Helgason [1, Chapter IV, Theorem 7.2]).

Theorem 2.1 *On the Riemannian symmetric space (M, ds_M^2) and in the notation above, let $\mathfrak{m}_1 \subset \mathfrak{m} \cong T_0(M)$ be a vector subspace. Then, denoting by $\mathrm{Exp}_0 : T_0(M) \to M$ the exponential map in the sense of Riemannian geometry, $S := \mathrm{Exp}_0(\mathfrak{m}_1) \subset M$ is a totally geodesic submanifold if and only if $\mathfrak{m}_1 \subset \mathfrak{g}$ is a Lie triple system, i.e., if and only if $[\mathfrak{m}_1, [\mathfrak{m}_1, \mathfrak{m}_1]] \subset \mathfrak{m}_1$ for the Lie bracket $[\cdot, \cdot]$ on $\mathfrak{g}$. Moreover, in the notation above, writing $\mathfrak{k}_1 = [\mathfrak{m}_1, \mathfrak{m}_1] \subset [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$ and defining $\mathfrak{g}_1 := \mathfrak{m}_1 \oplus \mathfrak{k}_1 \subset \mathfrak{m} \oplus \mathfrak{k} = \mathfrak{g}$, $\mathfrak{g}_1 \subset \mathfrak{g}$ is a Lie subalgebra, and, denoting by $G_1 \subset G$ the Lie subgroup corresponding to the Lie subalgebra $\mathfrak{g}_1 \subset \mathfrak{g}$, $K_1 \subset K$ the Lie subgroup corresponding to the Lie subalgebra $\mathfrak{k}_1 \subset \mathfrak{k}$, $(S, ds_M^2|_S)$ is a Riemannian symmetric space on which G_1 acts transitively, and $S = G_1/K_1$ as a homogeneous space.*

For a Cartesian product of Riemannian manifolds $(N_1, ds_{N_1}^2) \times \cdots \times (N_p, ds_{N_p}^2) =: (N, ds_N^2)$, the Riemannian connection ∇ is unchanged if the background metric $ds_{N_k}^2$ of each Cartesian factor is replaced by $\lambda_k ds_{N_k}^2$ for some $\lambda_k > 0$. Since a smooth submanifold $Z \subset N$ is totally geodesic if and only if its tangent bundle $T(Z)$ is parallel along Z, which depends only on ∇, the latter occurs if and only if Z is totally geodesic with respect to any of the Riemannian metric h

thus obtained by scaling. It is therefore not surprising that the necessary and sufficient condition in Theorem 2.1 in terms of Lie triple systems is a purely Lie-theoretic condition independent of the choice of the background metric ds^2_M rendering (M, ds^2_M) Riemannian symmetric, noting also that the subset $S = \mathrm{Exp}_0(\mathfrak{m}_1)$, which is the union of geodesics emanating from 0, also remains unchanged by introducing the scaling constants.

2.2 Basic materials on bounded symmetric domains

Consider now the case where $(M, ds^2_M) = (X_0, g)$ is a Hermitian symmetric space of the noncompact type, so that X_0 is biholomorphic to a bounded symmetric domain. Here and henceforth by a Hermitian symmetric space of the noncompact (resp. compact) type we will mean one with negative (resp. positive) Ricci curvature, i.e., it is implicitly assumed that the Hermitian symmetric space is of the semisimple type.

Write G_0 for the identity component of the isometry group of X_0, which is equivalently the identity component of the group $\mathrm{Aut}(X_0)$ of biholomorphic automorphisms of X_0. Write $K \subset G_0$ for the isotropy subgroup at $0 \in X_0$, $0 = eK$. Denote by $X_c := G_c/K$ the Hermitian symmetric space of the compact type dual to X_0. Denote by $G^{\mathbb{C}}$ the identity component of $\mathrm{Aut}(X_c)$ and by $P \subset G^{\mathbb{C}}$ the isotropy (parabolic) subgroup at 0, so that $X_c = G^{\mathbb{C}}/P$ as a complex homogeneous space. Write $X_0 = G_0/K \hookrightarrow G^{\mathbb{C}}/P = X_c$ for the Borel embedding identifying X_0 as an open subset of its compact dual X_c. Whenever appropriate, we write $V^{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ for the complexification of a real vector space V. Write $\mathfrak{g}_0 = \mathfrak{k} + \mathfrak{m}$ for the Cartan decomposition of $\mathfrak{g}_0$ with respect to the involution at 0. Then, $\mathfrak{g}_c = \mathfrak{k} + \sqrt{-1}\mathfrak{m}$ stands for the corresponding Cartan decomposition of $\mathfrak{g}_c$.

For $u, v \in \mathfrak{g}_c$ we write $\mathrm{ad}(u)(w) := [u, w]$, for $\mathrm{ad}(u) \in \mathrm{End}(\mathfrak{g}_c)$, etc. and denote by $B(u, v) := \mathrm{Tr}\,\mathrm{ad}(u)\mathrm{ad}(v)$ the Killing form $B(\cdot, \cdot)$ on $\mathfrak{g}_c$. Since $\mathfrak{g}_c$ is a compact real form of a semisimple complex Lie algebra, $B(\cdot, \cdot)$ is negative definite. Extend $B(\cdot, \cdot)$ to the complexification $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{g}_c$ by complex bilinearity so that $B(\cdot, \cdot)$ is a nondegenerate complex bilinear form on $\mathfrak{g}^{\mathbb{C}}$. Denote by $(\cdot, \cdot)$ the Hermitian bilinear pairing defined by $(u, v) = B(u, -\tau_c(v))$, where τ_c stands for the conjugation on $\mathfrak{g}^{\mathbb{C}}$ with respect to the real form $\mathfrak{g}_c \subset \mathfrak{g}^{\mathbb{C}}$, and write $\|u\| = \sqrt{(u, u)}$. The isotropy subgroup $K \subset G_c$ is reductive, and the complex structure on X_c is induced by the adjoint action of some element z belonging to the center $\mathfrak{z}$ of $\mathfrak{k}$. We have the Harish-Chandra decomposition $\mathfrak{g}^{\mathbb{C}} = \mathfrak{m}^+ \oplus \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{m}^-$ which is eigenspace decomposition of $\mathrm{ad}(z) \in \mathrm{End}(\mathfrak{g}^{\mathbb{C}})$ corresponding to the eigenvalues $\sqrt{-1}$, 0 and $-\sqrt{-1}$, respectively. By considering the action of $\mathrm{ad}(z)$ it follows readily that $[\mathfrak{m}^+, \mathfrak{m}^+] = [\mathfrak{m}^-, \mathfrak{m}^-] = 0$, $[\mathfrak{k}^{\mathbb{C}}, \mathfrak{k}^{\mathbb{C}}] \subset \mathfrak{k}^{\mathbb{C}}$, $[\mathfrak{k}^{\mathbb{C}}, \mathfrak{m}^+] \subset \mathfrak{m}^+$, $[\mathfrak{k}^{\mathbb{C}}, \mathfrak{m}^-] \subset \mathfrak{m}^-$ and $[\mathfrak{m}^+, \mathfrak{m}^-] \subset \mathfrak{k}^{\mathbb{C}}$. In particular, the complex vector subspaces $\mathfrak{m}^+, \mathfrak{m}^- \subset \mathfrak{g}^{\mathbb{C}}$ are abelian subalgebras. We have $\mathfrak{m}^+ \oplus \mathfrak{m}^- = \mathfrak{m}^{\mathbb{C}}$ and $\overline{\mathfrak{m}^+} = \mathfrak{m}^-$. Here and in what follows, for $u \in \mathfrak{g}^{\mathbb{C}}$, $\overline{u}$ will be taken with respect to the conjugation τ_0 on the noncompact real form $\mathfrak{g}_0 \subset \mathfrak{g}^{\mathbb{C}}$. We have $\tau_0|_{\mathfrak{k}^{\mathbb{C}}} = \tau_c|_{\mathfrak{k}^{\mathbb{C}}}$ and $\tau_0|_{\mathfrak{m}^{\mathbb{C}}} = -\tau_c|_{\mathfrak{m}^{\mathbb{C}}}$.

The Hermitian symmetric space X_0 of the noncompact type can be identified as a bounded symmetric domain by means of the Harish-Chandra embedding, as follows (cf. Wolf [11]). For its formulation given the Harish-Chandra decomposition $\mathfrak{g}^{\mathbb{C}} = \mathfrak{m}^+ \oplus \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{m}^-$, we have correspondingly the abelian subgroups $M^+, M^- \subset G^{\mathbb{C}}$, and the reductive subgroup $K^{\mathbb{C}} \subset G^{\mathbb{C}}$.

Theorem 2.2 (Harish-Chandra Embedding Theorem) *The holomorphic map $F : M^+ \times K^{\mathbb{C}} \times M^- \to G^{\mathbb{C}}$ defined by $F(m^+, k, m^-) = m^+km^-$ is a biholomorphism of $M^+ \times K^{\mathbb{C}} \times M^-$*

onto a dense open subset of the complex Lie group $G^{\mathbb{C}}$ containing G_0. In particular, the map $\eta : \mathfrak{m}^+ \to G^{\mathbb{C}}/P = X_c$ defined by $\eta(m^+) = \exp(m^+)P$ is a biholomorphism onto a dense open subset of X_c containing $G_0/K = X_0$. Furthermore, $\eta^{-1}(X_0) =: \Omega$ is a bounded domain on $\mathfrak{m}^+ \cong \mathbb{C}^n$, $n = \dim_{\mathbb{C}} X_0$.

The bounded symmetric domain $\Omega \Subset \mathbb{C}^n$ in its Harish-Chandra realization can be precisely described in Lie-theoretic terms as the unit ball in $\mathfrak{m}^+ \cong \mathbb{C}^n$ with respect to a Banach norm, implying in particular its convexity, by the Hermann Convexity Theorem (cf. Wolf [11]), as follows.

Theorem 2.3 (Hermann Convexity Theorem) *Identify $\mathbb{C}^n$ with the holomorphic tangent space $T_0(\Omega)$. Then, $\Omega \Subset \mathbb{C}^n$ is the unit ball in $\mathbb{C}^n$ corresponding to the Banach norm $\|\cdot\|_H$ on $T_0(\Omega)$ defined by $\|\xi\|_H := \sup\{\|\mathrm{ad}(\mathrm{Re}\,\xi)(v)\| : v \in \mathfrak{g}^{\mathbb{C}}, (v, v) = 1\}$ for $\xi \in T_0(\Omega)$.*

Remark 2.1 Note that in the statement of the Hermann Convexity Theorem, the operator norm of $\mathrm{ad}(\mathrm{Re}\,\xi) \in \mathrm{End}(\mathfrak{g}^{\mathbb{C}})$ is unchanged when the Hermitian inner product $(\cdot, \cdot)$ is rescaled, i.e., when the Hermitian inner product on each of the simple factors of $\mathfrak{g}^{\mathbb{C}}$ is replaced by a scalar multiple.

2.3 Characterization of totally geodesic complex submanifolds of bounded symmetric domains in Harish-Chandra coordinates

In this subsection, we give a characterization of totally geodesic complex submanifolds $S \subset \Omega$ of bounded symmetric domains $\Omega \Subset \mathbb{C}^n$ in terms of Harish-Chandra coordinates. From the homogeneity of Ω under G_0 it suffices to characterize those $S \subset \Omega$ passing through 0.

Since the focus is now on complex manifolds, here and henceforth we adopt a convention common in complex geometry on the notation for tangent spaces. Given an n-dimensional complex manifold Z and a point $x \in Z$, we denote by $T_x^{\mathbb{R}}(Z)$ the real $(2n)$-dimensional tangent space at x of the real $(2n)$-dimensional smooth manifold underlying Z, while the notation $T_x(Z)$ is reserved for the complex n-dimensional holomorphic tangent space at x, as opposed to the meaning of the same notation in (2.1). Writing $T_x^{\mathbb{C}}(Z) = T_x^{\mathbb{R}}(Z) \otimes_{\mathbb{R}} \mathbb{C}$ and decomposing the $(2n)$-dimensional complex vector space $T_x^{\mathbb{C}}(Z) = T_x^{1,0}(Z) \oplus T_x^{0,1}(Z)$ as a direct sum of eigenspaces of the J-operator underlying the integrable almost complex structure of Z, the holomorphic tangent space $T_x(Z)$ is canonically identified with the complex vector subspace $T_x^{1,0}(Z) \subset T_x^{\mathbb{C}}(Z)$ of complexified tangent vectors of type (1,0). We have the following proposition.

Proposition 2.1 *Let $\Omega \Subset \mathbb{C}^n$ be a bounded symmetric domain in its Harish-Chandra realization, $\Omega = G_0/K$ as a homogeneous space in the notation above, and $S \subset \Omega$ be a complex submanifold passing through the origin $0 \in \Omega$. Then, identifying $T_0(S) \subset T_0(\Omega) \cong \mathfrak{m}^+$ as an abelian complex Lie subalgebra $\mathfrak{m}_1^+ \subset \mathfrak{m}^+ \subset \mathfrak{g}^{\mathbb{C}}$, $S \subset \Omega$ is totally geodesic with respect to a given invariant Kähler metric g on Ω if and only if $[\mathfrak{m}_1^+, [\mathfrak{m}_1^+, \mathfrak{m}_1^-]] \subset \mathfrak{m}_1^+$. Furthermore, $S = E \cap \Omega$ for the complex vector subspace $E \subset \mathbb{C}^n$ corresponding to $\mathfrak{m}_1^+ \subset \mathfrak{m}^+$ whenever $S \subset \Omega$ is totally geodesic.*

Proof Let $S \subset \Omega$ be a totally geodesic complex submanifold passing through $0 \in \Omega$. Under the identification $T_0(\Omega) \cong \mathfrak{m}^+$, we identify $T_0(S)$ with a complex vector subspace $\mathfrak{m}_1^+ \subset \mathfrak{m}^+$. The real tangent space $T_0^{\mathbb{R}}(S)$ is given by $\mathrm{Re}(\mathfrak{m}_1^+) =: \mathfrak{m}_1 \subset \mathfrak{m} = T_0^{\mathbb{R}}(\Omega)$. By Theorem 2.1, $\mathfrak{m}_1 \subset \mathfrak{g}_0$

is a Lie triple system, i.e., (†) $[\mathfrak{m}_1,[\mathfrak{m}_1,\mathfrak{m}_1]]\subset\mathfrak{m}_1$ holds. We claim that (†) is equivalent to (††) $[\mathfrak{m}_1^+,[\mathfrak{m}_1^+,\mathfrak{m}_1^-]]\subset\mathfrak{m}_1^+$.

Starting with (†) and complexifying, we have $[\mathfrak{m}_1^{\mathbb{C}},[\mathfrak{m}_1^{\mathbb{C}},\mathfrak{m}_1^{\mathbb{C}}]]\subset\mathfrak{m}_1^{\mathbb{C}}$. Since $\mathfrak{m}_1^{\mathbb{C}}=\mathfrak{m}_1^+\oplus\mathfrak{m}_1^-$, where $\mathfrak{m}_1^-=\overline{\mathfrak{m}_1^+}$, and $[\mathfrak{m}_1^+,\mathfrak{m}_1^+]=[\mathfrak{m}_1^-,\mathfrak{m}_1^-]=0$, (†) is equivalent to $[\mathfrak{m}_1^+,[\mathfrak{m}^+,\mathfrak{m}_1^-]]+[\mathfrak{m}_1^-,[\mathfrak{m}_1^-,\mathfrak{m}_1^+]]\subset\mathfrak{m}_1^+\oplus\mathfrak{m}_1^-$. Noting that $[\mathfrak{m}_1^-,[\mathfrak{m}_1^-,\mathfrak{m}_1^+]]=\overline{[\mathfrak{m}_1^+,[\mathfrak{m}_1^+,\mathfrak{m}_1^-]]}$ and that $[\mathfrak{m}_1^+,[\mathfrak{m}_1^+,\mathfrak{m}_1^-]]\subset[\mathfrak{m}^+,[\mathfrak{m}^+,\mathfrak{m}^-]]\subset[\mathfrak{m}^+,\mathfrak{k}^{\mathbb{C}}]\subset\mathfrak{m}^+$, we conclude that (†) is equivalent to (††) $[\mathfrak{m}_1^+,[\mathfrak{m}_1^+,\mathfrak{m}_1^-]]\subset\mathfrak{m}_1^+$, as claimed.

We have deduced from Theorem 2.1 that $S=\mathrm{Exp}_0(\mathfrak{m}_1^+)\subset\Omega$ is a totally geodesic complex submanifold if and only if (††) holds. To complete the proof of Proposition 2.1 it remains to show that $S\subset\Omega$ must be given by $S=E\cap\Omega$ for the complex vector subspace $E\subset\mathbb{C}^n$ corresponding to $\mathfrak{m}_1^+\subset\mathfrak{m}$, whenever the complex submanifold $S\subset\Omega$ is totally geodesic with respect to $(\Omega.g)$ and it passes through the origin $0\in\Omega$.

For $\theta\in\mathbb{R}$ define $\mu_\theta\in\mathrm{GL}(n;\mathbb{C})$ by $\mu_\theta(z)=e^{i\theta}z$. Consider now the circle group $S^1=\{\mu_\theta:\theta\in\mathbb{R}\}\subset K$, which acts on Ω by scalar multiplication, so that Ω is a circular domain. Write $S_\theta:=\mu_\theta(S)$. We have $0\in S_\theta$ and $T_0(S_\theta)=e^{i\theta}\cdot S=S$. Since there is exactly one totally geodesic submanifold $(Z,g|_Z)$ of (Ω,g) passing through 0 such that $T_0^{\mathbb{R}}(Z)=T_0^{\mathbb{R}}(S)$, we have $S_\theta=S$. Thus, for any point $x\in S$, $e^{i\theta}x\in S$ for any $\theta\in\mathbb{R}$, hence for $x\neq 0$, the complex analytic subset $\mathbb{C}x\cap S$ of the open disk $\mathbb{C}x\cap\Omega$ must be the whole disk as it contains the real analytic curve $S^1\cdot x$, so that $S\subset\Omega$ is a union of open disks centered at 0 on complex lines ℓ passing through 0. Thus, writing $\lambda:\mathbb{C}^n-\{0\}\to\mathbb{P}^{n-1}$ for the canonical projection, there is a complex analytic subvariety $A\subset\mathbb{P}^{n-1}$ such that $S=(\lambda^{-1}(A)\cup\{0\})\cap\Omega$. Finally, since $S\subset\Omega$ is smooth at 0, as is well-known the subvariety $A\subset\mathbb{P}^{n-1}$ must necessarily be a projective linear subspace, i.e., S must be of the form $E\cap\Omega$ for some complex linear subspace $E\subset\Omega$ (noting that $E\cap\Omega$ is connected as Ω is convex). Clearly $E\subset\mathbb{C}^n$ corresponds to $\mathfrak{m}_1^+$, as desired. The proof of Proposition 2.1 is complete.

3 Characterization of Harish-Chandra Realizations of Bounded Symmetric Domains in Terms of Bisectional Curvatures

Recall that the G_0-invariant Kähler metric ds^2_Ω on the bounded symmetric domain Ω has been chosen so that the minimal disks of each irreducible Cartesian factor Ω are of constant Gaussian curvature -2. For an irreducible bounded symmetric domain $\Omega=G_0/K$, we give a brief description of the root space decomposition of the complex simple Lie group $\mathfrak{g}^{\mathbb{C}}$ relevant to the study of bisectional curvatures, and refer the reader to [4, 11] and references therein for details. Here and in what follows we use the notation of the first two paragraphs in (2.2).

Writing $\mathfrak{k}_s=[\mathfrak{k},\mathfrak{k}]$ for the semisimple part of $\mathfrak{k}$, we have $\mathfrak{k}=\mathfrak{k}_s\oplus\mathfrak{z}$, where $\mathfrak{z}\subset\mathfrak{k}$ is the 1-dimensional center, containing an element z such that $\mathrm{ad}(z)$ defines the underlying integrable almost complex structure on Ω. Writing $h_s\subset\mathfrak{k}_s$ for a Cartan subalgebra and defining $\mathfrak{h}:=\mathfrak{h}_s\oplus\mathfrak{z}$, $\mathfrak{h}^{\mathbb{C}}\subset\mathfrak{g}^{\mathbb{C}}$ is a Cartan subalgebra. Denoting by Φ the space of $\mathfrak{h}^{\mathbb{C}}$-roots of $\mathfrak{g}^C$, we have $\Phi\subset\imath\mathfrak{h}^*$, where $\mathfrak{h}^*:=\mathrm{Hom}(\mathfrak{h},\mathbb{R})$. Recall the Harish-Chandra decomposition $\mathfrak{g}^{\mathbb{C}}=\mathfrak{m}^+\oplus\mathfrak{k}^{\mathbb{C}}\oplus\mathfrak{m}^-$. For the (complex 1-dimensional) root space $\mathfrak{g}_\varphi$ associated to $\varphi\in\Phi$, $\mathfrak{g}_\varphi$ is also an eigenspace of $\mathrm{ad}(z)$, and it follows that $\mathfrak{g}_\varphi\subset\mathfrak{m}^+,\mathfrak{k}^{\mathbb{C}}$ or $\mathfrak{m}^-$. We denote by $\Phi_c\subset\Phi$ the set of compact roots φ, i.e., those for which $\mathfrak{g}_\varphi\subset\mathfrak{k}^{\mathbb{C}}$, and the set Φ_0 of noncompact roots, i.e., those for which $\mathfrak{g}_\varphi\subset\mathfrak{m}^{\mathbb{C}}=\mathfrak{m}^+\oplus\mathfrak{m}^-$. With respect to a choice of the positive Weyl chamber in $\imath\mathfrak{h}^*$ determined by $\mathrm{ad}(z)$, we decompose Φ into the disjoint union of the set Φ^+ of positive roots and the set Φ^- of negative roots, so that, writing $\Phi_0^+ = \Phi_0 \cap \Phi^+$ for the set of positive noncompact roots, we have $\mathfrak{m}^+ = \mathrm{Span}_{\mathbb{C}}\{\mathfrak{g}_\varphi : \varphi \in \Phi_0^+\}$, and, $\Phi_0^- = \Phi_0 \cap \Phi^-$ for the set of negative noncompact roots, $\Phi_0^- = -\Phi_0^+$. We have $\mathfrak{m}^- = \mathrm{Span}_{\mathbb{C}}\{\mathfrak{g}_{-\varphi} : \varphi \in \Phi_0^+\}$, hence the direct sum decomposition

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{m}^+ \oplus \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{m}^- = \bigoplus_{\varphi \in \Phi_0^+} \mathfrak{g}_\varphi \;\oplus\; \Big(\mathfrak{h}^{\mathbb{C}} \oplus \bigoplus_{\rho \in \Phi_c} \mathfrak{g}_\rho\Big) \;\oplus\; \bigoplus_{\varphi \in \Phi_0^+} \mathfrak{g}_{-\varphi}.$$

In what follows, we will fix a lexicographic ordering of the roots compatible with the choice of positive Weyl chamber so that there is a unique highest root μ of $\mathfrak{g}$, and $\mu \in \Phi^+0$ is always a long root. When $\mathfrak{g}^{\mathbb{C}}$ is of type A, D or E, all roots in Φ are of equal length.

From now on, for Ω irreducible, we will replace the Killing form $B(\cdot,\cdot)$ on $\mathfrak{g}^{\mathbb{C}}$ in the definition of the Hermitian inner product $(\cdot;\cdot)$ and the Hermitian norm $\|\cdot\|$ by $B'(\cdot,\cdot) = cB(\cdot,\cdot)$ for some constant $c = c_{\mathfrak{g}} > 0$ such that the induced Hermitian inner product $(u;v) := B'(u, -\tau_c(v))$ is the standard Euclidean Hermitian inner product for $u, v \in T_0(\Omega)$, i.e., for the G_0-invariant Kähler metric g we have $g_{i\bar{j}}(0) = \delta_{ij}$, and g is precisely our choice of ds^2_Ω when Ω is irreducible. The restriction $B'|_{\imath\mathfrak{h}}$ is positive definite, and it defines a real linear isomorphism form $\mathfrak{h}^*_{\mathbb{R}} := \imath\mathfrak{h}^*$ to $\imath\mathfrak{h} =: \mathfrak{h}_{\mathbb{R}}$, and we identify $\varphi \in \Phi$ in this way with an element $H_\varphi \in \mathfrak{h}_{\mathbb{R}}$.

In the general case for $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}_1^{\mathbb{C}} \oplus \cdots \oplus \mathfrak{g}_s^{\mathbb{C}}$ we rescale the Killing form on each simple direct factor $\mathfrak{g}_i^{\mathbb{C}}$, $1 \le i \le s$, accordingly. Observe that the operator Banach norm $\|\mathrm{ad}(u)\|_H$ such as that appearing in the statement of the Hermann Convexity Theorem for $u \in \mathfrak{g}^{\mathbb{C}}$ remains unchanged by such a replacement of B by B' (cf. Remark 2.1).

For a positive noncompact root φ we write $\mathfrak{g}[\varphi] := \mathfrak{g}_\varphi \oplus \mathfrak{g}_{-\varphi} \oplus [\mathfrak{g}_\varphi, \mathfrak{g}_{-\varphi}]$, where $[\mathfrak{g}_\varphi, \mathfrak{g}_{-\varphi}] = \mathbb{C}H_\varphi$. Writing $e_\varphi \in \mathfrak{g}_\varphi$ for a unit root vector, $e_{-\varphi} = \overline{e_\varphi} \in \mathfrak{g}_{-\varphi}$, we have $[H_\varphi, e_\varphi] = 2e_\varphi$, $[H_\varphi, e_{-\varphi}] = -2e_{-\varphi}$ and $[e_\varphi, e_{-\varphi}] = H_\varphi$, so that $\mathfrak{g}[\varphi] \cong \mathfrak{sl}(2,\mathbb{C})$.

The orbit of $0 \in \Omega$ under the Lie group $G[\varphi] \subset G^{\mathbb{C}}$ corresponding to $\mathfrak{g}[\varphi] \subset \mathfrak{g}^{\mathbb{C}}$ is a rational curve $\ell_\varphi := G[\varphi] \cdot 0$ on the compact dual X_c of $X_0 \cong \Omega$, which is totally geodesic with respect to the G_c-invariant Kähler metric g_c on Ω dual to ds^2_Ω, $\Omega \subset X_c$ being the Borel embedding. When $\varphi \in \Phi_0^+$ is a long root, $\ell_\varphi \subset X_c$ is a minimal rational curve. Defining $\mathfrak{g}_0[\varphi] := \mathfrak{g}[\varphi] \cap \mathfrak{g}_0$, we have $\mathfrak{g}_0[\varphi] \cong \mathfrak{su}(1,1)$. The orbit of $0 \in \Omega$ under the corresponding Lie subgroup $G_0[\varphi] \subset G_0$ is a totally geodesic holomorphic disk $D_\varphi := G_0[\varphi] \cdot 0$ on Ω. When $\varphi \in \Phi_0^+$ is a long root, $D_\alpha = \ell_\varphi \cap \Omega$ is a minimal disk on Ω.

We say that two roots $\varphi_1, \varphi_2 \in \Phi$ are strongly orthogonal if and only if neither $\varphi_1 + \varphi_2$ nor $\varphi_1 - \varphi_2$ is a root. Let $\Psi = \{\psi_1, \cdots, \psi_s\} \subset \Phi_0^+$ be a maximal strongly orthogonal subset, i.e., a subset of maximal cardinality of mutually strongly orthogonal positive noncompact roots. Then, $s = r := \mathrm{rank}(\Omega)$. Note that $\psi_1, \psi_2 \in \Phi_0^+$ are strongly orthogonal to each other if and only if $\psi_1 - \psi_2 \notin \Phi$, since $\psi_1 + \psi_2$ is never a root, observing that $[\mathfrak{m}^+, \mathfrak{m}^+] = 0$.

For a strongly orthogonal set of positive noncompact roots $\Theta \subset \Phi_0^+$ we write

$$\mathfrak{g}[\Theta] := \bigoplus_{\theta \in \Theta} (\mathfrak{g}_\theta \oplus \mathfrak{g}_{-\theta} \oplus [\mathfrak{g}_\theta, \mathfrak{g}_{-\theta}]) = \bigoplus_{\theta \in \Theta} \mathfrak{g}[\theta].$$

Then, $\mathfrak{g}[\Theta]$ is a semisimple complex Lie algebra, $\mathfrak{g}[\Theta] \cong \mathfrak{sl}(2,\mathbb{C})^{|\Theta|}$. Writing $\mathfrak{g}_0[\Theta] = \mathfrak{g}[\Theta] \cap \mathfrak{g}_0$, then $\mathfrak{g}_0[\Theta] \subset \mathfrak{g}[\Theta]$ is a semisimple Lie algebra which is a noncompact real form of $\mathfrak{g}[\Theta]$ without compact factors, $\mathfrak{g}_0[\Theta] \cong \mathfrak{su}(1,1)^{|\Theta|}$, and the $G_0[\Theta]$-orbit of $0 \in \Omega$ is a Euclidean polydisk.

To extract a maximal strongly orthogonal subset $\Psi \subset \Phi_0^+$, we may start with choosing $\psi_1 = \mu \in \Phi_0^+$ being the highest root and consider the subset $\Sigma \subset \Phi_0^+$ consisting of roots φ strongly orthogonal to ψ_1. Then there exists a simple Lie subalgebra $\mathfrak{g}_0' \subset \mathfrak{g}_0$ such that, putting $\mathfrak{k}' = \mathfrak{k} \cap \mathfrak{g}_0'$, $G_0'/K_0' \subset G_0/K = X_0 \cong \Omega$ is an irreducible Hermitian symmetric space of the noncompact type embedded as a totally geodesic complex submanifold of $\Omega' \subset \Omega$, and such that $T_0(\Omega')$ is spanned by $\{\mathfrak{g}_\sigma : \sigma \in \Sigma\}$. Writing $\mathfrak{h}' := \mathfrak{h} \cap \mathfrak{k}$, $\mathfrak{h}'^{\mathbb{C}} \subset \mathfrak{g}'^{\mathbb{C}}$ is a Cartan subalgebra, and the restriction $\sigma' := \sigma|_{\mathfrak{h}'}$ for all $\sigma \in \Sigma$ gives the set of positive noncompact roots of $\mathfrak{g}'$. We have a totally geodesic complex submanifold $\Delta \times \Omega' \hookrightarrow \Omega$ such that $T_0(\Delta) = \mathbb{C}\mathfrak{g}_\mu$ and $T_0(\Omega') = \text{Span}\{\mathfrak{g}'_{\sigma'} : \sigma \in \Sigma\}$, $\mathfrak{g}'_{\sigma'} = \mathfrak{g}_\sigma$. Repeating the same procedure with Ω' in place of Ω, we obtain inductively a maximal strongly orthogonal subset $\Psi \subset \Phi_0^+$ of positive noncompact roots, $\Psi = \{\psi_1, \cdots, \psi_r\}$.

Since all roots $\psi \in \Psi$ are long roots, each direct factor $D_\psi := G_0[\psi] \cdot 0$ is the unit disk on $\mathbb{C}e_\varphi \cong T_0(D_\Psi)$, and $\Pi_0 = G_0[\Psi] \cdot 0$ is a maximal polydisk of polyradius $(1, \cdots, 1)$, i.e., $\Pi_0 = \Delta^r \times \{0\} \subset \Omega$ in terms of Harish-Chandra coordinates corresponding to an orthonormal basis consisting of unit root vectors e_φ arranged in a suitable order. In fact, there is more symmetry among the disks $D_\psi : \psi \in \Psi$. From the Restricted Root Theorem one can deduce that the full automorphism group $\text{Aut}(\Pi_0)$ (generated by $\text{Aut}_0(\Pi_0) \cong \text{Aut}(\Delta)^r$, and the permutation group on the r Cartesian factors) embeds into $G_0 := \text{Aut}_0(\Omega)$ (cf. [11]), so that for each pair (ψ_1, ψ_2) of distinct elements of Ψ, there exists $k \in K$ such that $D_{\psi_2} = k(D_\psi)$, and k stabilizes Π_0. The polydisk $\Pi_0 := G_0[\Psi] \cdot 0 \subset \Omega$ is a maximal polydisk on Ω passing through 0, where by a maximal polydisk in Ω we mean a totally geodesic complex submanifold of (Ω, ds^2_Ω) biholomorphic to Δ^r. All maximal polydisks in Ω passing through 0 are equivalent to each other (hence to Π_0) under conjugation by K. By the Polydisk Theorem (cf. [11]) we have $\Omega = \bigcup_{k \in K} k(\Pi_0)$, i.e., every $\nu \in T_0(\Omega)$ is tangent to some maximal polydisk $\Pi := k(\Pi_0)$.

Given any $\xi \in T_0(\Omega)$, there exists $k \in K$ such that $\eta := k(\xi)$ is tangent to the reference maximal polydisk $\Pi_0 \subset \Omega$. Since $\text{Aut}(\Pi_0)$ embeds into G_0, composing with the action of $(S^1)^r$ for the circle group S^1, acting according to $(e^{i\theta_1}, \cdots, e^{i\theta_r}) \cdot (z_1, \cdots, z_r) \mapsto (e^{i\theta_1} z_1, \cdots, e^{i\theta_r} z_r)$ and with permutations of the r Cartesian factors, we obtain some element $k' \in K$ such that $k'(\xi) = (a_1, \cdots, a_r; 0, \cdots, 0)$ such that all a_i, $1 \le i \le r$ are real and nonnegative, and such that $a_1 \ge \cdots \ge a_r \ge 0$. We call $(a_1, \cdots, a_r; 0, \cdots, 0)$, or simply $(a_1, \cdots, a_r)$, the normal form of ξ under the action of K.

Lemma 3.1 *For any irreducible bounded symmetric domain $\Omega_0 \Subset \mathbb{C}^{n_0}$, there exists an irreducible bounded symmetric domain $\Omega \Subset \mathbb{C}^n$ such that $\Omega_0 \subset \Omega$ as a totally geodesic complex submanifold passing through 0, and such that, writing $\Omega = G_0/K$, $\mathfrak{g}^{\mathbb{C}}$ is of type A, D or E. Hence, writing Φ for the set of all roots of $\mathfrak{g}^{\mathbb{C}}$ with respect to a Cartan subalgebra $\mathfrak{h}^{\mathbb{C}} \subset \mathfrak{g}^{\mathbb{C}}$, all roots $\varphi \in \Phi$ are of equal length.*

Proof Up to biholomorphisms, the only irreducible bounded symmetric domains Ω not of these types are those of types B or C. These include type II domains D_n^{II} where $n \ge 5$ is odd, type III domains D_n^{III} of rank n, $n \ge 3$, and type IV domains of odd dimension $n \ge 3$. For $\Omega_0 = D^{\text{II}}_{2m+1}$, $m \ge 2$, it suffices to take $\Omega = D^{\text{II}}_{2m+2}$. For $\Omega_0 = D_n^{\text{III}}$, it suffices to take $\Omega = D^{I}_{n,n}$. For $\Omega_0 = D^{\text{IV}}_{2m+1}, m \ge 1$, it suffices to take $\Omega = D^{\text{IV}}_{2m+2}$. All notations for bounded symmetric domains are standard ones and the embedding $\Omega_0 \subset \Omega$ are also standard embeddings.

We will need the following lemma on the combinatorics relating the space of positive noncompact roots Φ_0^+ with a maximal strongly orthogonal set Ψ of positive noncompact roots.

Lemma 3.2 *Let $\Omega \Subset \mathbb{C}^n$ be an irreducible bounded symmetric domain of type A, D or E. Let $\Psi = \{\psi_1, \cdots, \psi_r\}$ be a maximal strongly orthogonal set of positive noncompact roots and pick $\varphi \in \Phi_0^+ - \Psi$. Then either*

(a) *there exist exactly two distinct roots $\psi_{i_1}, \psi_{i_2} \in \Psi$ such that $\varphi - \psi_j \in \Phi$ if and only if $j = i_1$ or $j = i_2$; or*

(b) *there is exactly one root $\psi \in \Psi$ such that $\varphi - \psi \in \Phi$.*

Proof We use standard notation for irreducible bounded symmetric domains. For $\Omega = D^{\mathrm{I}}_{p,q}$, $1 \le p \le q$, and for D^{II}_{2m}, $m \ge 2$, the lemma is obvious by using the standard representation of $T_0(\Omega)$ as a complex vector space of matrices and the root vectors of $\varphi \in \Phi_0^+$ as a standard basis of such a vector space. For example, take in the case of a type I domain $D^{\mathrm{I}}_{p,q}$, $1 \le p \le q$ the standard choice of the Cartan subalgebra $\mathfrak{h}^{\mathbb{C}} \subset \mathfrak{g}^{\mathbb{C}}$ so that each of the 1-dimensional root spaces $\mathfrak{g}_\varphi$, $\varphi \in \Phi_0^+$, is spanned by an elementary matrix E_{ij}, with (i,j)-entry being 1, and all other entries being 0, where $1 \le i \le p$, $1 \le j \le q$, and choose the maximal polydisk Π to be such that $T_0(\Pi) = \mathrm{Span}_{\mathbb{C}}\{E_{kk} : 1 \le k \le p\}$. Then, case (a) occurs if and only if $\mathfrak{g}_\varphi = \mathbb{C}E_{ij}$, $1 \le i \le p, 1 \le j \le p$. and case (b) occurs if and only if $\mathfrak{g}_\varphi = \mathbb{C}E_{ij}$, $1 \le i \le p, p+1 \le j \le q$. The case of type II domains D^{II}_{2m} is very similar to the case of type I domains, except that only case (a) occurs. Since type-IV domains D^{IV}_{2m}, $m \ge 2$ are of rank 2, the lemma is vacuous in that case. The same is true for D^{V}, which is also of rank 2.

It remains to check the case of $\Omega = D^{\mathrm{VI}}$, which is of type E_7 and of rank 3. We will make use of the labeling of roots as in Zhong [13]. In the standard notation used in [13], a maximal set of strongly orthogonal positive noncompact roots $\Psi \subset \Phi_0^+$ is given by $\Psi = \{\psi_1, \psi_2, \psi_3\}$, in which $\psi_1 = x_1 - x_2, \psi_2 = x_1 + x_2 + x_3, \psi_3 = d - x_3$, where $d = x_1 + \cdots + x_7$. For each $\nu \in \Phi_0^+$ we define $\mathbf{H}_\nu := \{\varphi \in \Phi_0^+ : \nu - \varphi \in \Phi\}$. To complete the proof of the lemma, it suffices to show that $\mathbf{H}_{\psi_1} \cap \mathbf{H}_{\psi_2} \cap \mathbf{H}_{\psi_3} = \emptyset$. Any root $\nu \in \Phi_0^+$ is a long root, so that $[e_\nu] \in \mathscr{C}_0(X_c)$, the various of minimal rational tangents (VMRT for short) on the irreducible Hermitian symmetric space X_c of the compact type, and we have a decomposition of $T_0(\Omega)$ into a direct sum of eigenspaces of the Hermitian bilinear form $H_{e_\nu}(u,v) = \Theta_{e_\nu \overline{e_\nu} u \overline{v}}$, given by $T_0(\Omega) = \mathbb{C}e_\nu \oplus \mathcal{H}_{e_\nu} \oplus \mathcal{N}_{e_\nu}$, where $\mathbb{C}e_\nu$, $\mathcal{H}_{e_\nu}$ and $\mathcal{N}_{e_\nu}$ are the eigenspaces of H_{e_ν} corresponding to the eigenvalues 2, 1 and 0, respectively, and $\mathcal{H}_{e_\nu} = \mathrm{Span}\{e_\varphi : \varphi \in \mathbf{H}_\nu\}$. Now for each unit vector α such that $[\alpha] \in \mathscr{C}_0(X_c)$, $\mathcal{H}_\alpha$ can be identified with $T_{[\alpha]}(\mathscr{C}_0(X_c))$. In the case of X_c dual to $X_0 \cong D^{\mathrm{VI}}$, $\mathscr{C}_0(X_c)$ is dual to D^{V}, hence $\dim_{\mathbb{C}} \mathcal{H}_\alpha = 16$, $|\mathbf{H}_\nu| = 16$ for each $\nu \in \Phi_0^+$. Now $\mathbf{H}_{\psi_i} \subset \Phi_0^+ - \Psi$, $|\Phi_0^+ - \Psi| = 24$ and the maximal possible cardinality of $\mathbf{H}_{\psi_1} \cap \mathbf{H}_{\psi_2}$ is $16 + 16 - 24 = 8$. By direct checking we have $\mathbf{H}_{\psi_1} \cap \mathbf{H}_{\psi_2} = \{x_1 - x_4, x_1 - x_5, x_1 - x_6, x_1 - x_7, x_1 + x_3 + x_4, x_1 + x_3 + x_5, x_1 + x_3 + x_6, x_1 + x_3 + x_7\}$. Finally, $\psi_3 = d - x_3$, and none of the 8 elements of the set $\psi_3 - (\mathbf{H}_{\psi_1} \cap \mathbf{H}_{\psi_2}) = \{x_2 + 2x_4 + x_5 + x_6 + x_7, x_2 + x_4 + 2x_5 + x_6 + x_7, x_2 + x_4 + x_5 + 2x_6 + x_7, x_2 + x_4 + x_5 + x_6 + 2x_7, x_2 - x_3 + x_5 + x_6 + x_7, x_2 - x_3 + x_4 + x_6 + x_7, x_2 - x_3 + x_4 + x_5 + x_7, x_2 - x_3 + x_4 + x_5 + x_6\} \subset \mathfrak{h}^*_{\mathbb{R}}$ is a root, hence $\mathbf{H}_{\psi_1} \cap \mathbf{H}_{\psi_2} \cap \mathbf{H}_{\psi_3} = \emptyset$, as desired. The proof of Lemma 3.2 is complete.

The given proof of Lemma 3.2 relies on some direct checking on roots. While that has the advantage of being straightforward, it is also desirable to give a more conceptual proof of the lemma. We give here such a proof which relies on some knowledge of the VMRT $\mathscr{C}_0(X_c)$ of an irreducible Hermitian symmetric space X_c of the compact type, and on a curvature formula for $\mathscr{C}_0(X_c) \subset \mathbb{P}T_0(\Omega)$ as a Kähler submanifold, $\mathbb{P}T_0(\Omega)$ being endowed with the Fubini-study metric induced by $ds^2_\Omega(0)$ on $T_0(\Omega)$. Let g_c be the G_c-invariant Kähler metric on X_c such that g_c agrees with ds^2_Ω at 0. (X_c, g_c) is of nonnegative bisectional curvature, and, denoting by R^0 (resp. R^c) the curvature tensor of (Ω, ds^2_Ω) (resp. (X_c, g_c)), we have $R^c_{\alpha\overline{\beta}\gamma\overline{\delta}}(0) = -R^0_{\alpha\overline{\beta}\gamma\overline{\delta}}(0)$ for $\alpha, \beta, \gamma, \delta \in T_0(\Omega)$. For convenience we write $\Theta_{\alpha\overline{\beta}\gamma\overline{\delta}} = R^c_{\alpha\overline{\beta}\gamma\overline{\delta}}(0)$. We recall the following result from [4, Appendix (III.2)]. Here we will write (X, g) for (X_c, g_c).

Proposition 3.1 *Let (X, g) be an irreducible Hermitian symmetric space of the compact type, and denote by h the Fubini-study metric on $\mathbb{P}T_0(\Omega)$ induced by g. Then $(\mathscr{C}_0(X), h|_{\mathscr{C}_0(X)}) \hookrightarrow (\mathbb{P}T_0(\Omega, h))$ is a Hermitian symmetric space of the compact type of rank ≤ 2. Moreover, denoting by S the curvature tensor of $(\mathscr{C}_0(X), h|_{\mathscr{C}_0(X)})$, and identifying at each $[\alpha] \in \mathscr{C}_0(X)$, $T_{[\alpha]}(\mathbb{P}T_0(\Omega))$ with the orthogonal complement of $\mathbb{C}\alpha$ with respect to $ds^2_\Omega(0)$ for a unit characteristic vector α, S is the restriction of the curvature tensor Θ of (X, g) at 0 to $T_{[\alpha]}(\mathscr{C}_0(X))$, which corresponds under the aforementioned identification with $\mathcal{H}_\alpha$, the eigenspace belonging to the eigenvalue 1 of the Hermitian bilinear form $H_\alpha(u, v) = \Theta_{\alpha\overline{\alpha}u\overline{v}}$. In particular, for bisectional curvatures we have*

$$S_{\xi\overline{\xi}\eta\overline{\eta}} = \Theta_{\xi\overline{\xi}\eta\overline{\eta}}$$

for all $\xi, \eta \in \mathcal{H}_\alpha$.

Using Proposition 3.1 we prove the following statement, without requiring $\mathfrak{g}$ to be of type A, D or E, which implies Lemma 3.2.

Proposition 3.2 *Let X be an irreducible Hermitian symmetric space of the compact type, $\mathfrak{h} \subset \mathfrak{k}$ be a Cartan subalgebra of $\mathfrak{k} \subset \mathfrak{g}_0$, and Φ be the set of all $\mathfrak{h}^{\mathbb{C}}$-roots of $\mathfrak{g}^{\mathbb{C}}$. Let $\Psi = \{\psi_1, \cdots, \psi_r\} \subset \Phi_0^+$ be a maximal set of strongly orthogonal positive noncompact roots, and $\rho \in \Phi_0^+$ be a long root. Then, there are at most two distinct elements ψ of Ψ such that $\rho - \psi \in \Phi$.*

Proof Since $\rho \in \Phi_0^+$ is a long root, the unit root vector e_ρ is a minimal rational tangent, i.e., $\alpha := [e_\rho] \in \mathscr{C}_0(X)$. Suppose there exist distinct positive integers i, j and k such that $\rho - \psi_i$, $\rho - \psi_j$ and $\rho - \psi_k$ are roots. $\xi_\psi := e_\psi \mod \mathbb{C}\alpha \in T_0(X)/\mathbb{C}\alpha$ are unit tangent vectors of type $(1, 0)$ at $[\alpha]$ for $\psi = \psi_i, \psi_j, \psi_k$. For brevity we write also ξ_ℓ for ξ_{ψ_ℓ}, $1 \leq \ell \leq r$. By the definition of Ψ, we have $\Theta_{\xi_i\overline{\xi_i}\xi_j\overline{\xi_j}} = \Theta_{\xi_j\overline{\xi_j}\xi_k\overline{\xi_k}} = \Theta_{\xi_k\overline{\xi_k}\xi_i\overline{\xi_i}} = 0$. By Proposition 3.1, for the curvature tensor of $\mathscr{C}_0(X) \subset \mathbb{P}T_0(X)$, we have $S_{\xi_i\overline{\xi_i}\xi_j\overline{\xi_j}} = S_{\xi_j\overline{\xi_j}\xi_k\overline{\xi_k}} = S_{\xi_k\overline{\xi_k}\xi_i\overline{\xi_i}} = 0$. It follows that $\mathrm{Span}_{\mathbb{R}}\{\mathrm{Re}\,\xi_i, \mathrm{Re}\,\xi_j, \mathrm{Re}\,\xi_k\}$ is a real 3-dimensional abelian subalgebra in $T_0^{\mathbb{R}}(\mathscr{C}_0(X))$. Exponentiating, we get a real 3-dimensional totally geodesic flat submanifold $\Sigma \subset \mathscr{C}_0(X)$, so that the latter must be of rank ≥ 3 as a Riemannian symmetric space, which contradicts with the fact that $\mathrm{rank}(\mathscr{C}_0(X)) \leq 2$ as given in [4, Appendix III.2], proving the proposition.

On a complex affine line $\Lambda \subset \mathbb{C}^n$ and $x \in \Lambda, T_x(\Lambda) = \mathbb{C}\alpha$, where α is a unit vector, for $r > 0$ we denote by $\Delta_\alpha(x; r)$ the open disk on Λ centered at x of radius r, i.e., $\Delta_\alpha(x; r) := \mathbb{B}^n(x; r) \cap \Lambda$.

Theorem 3.1 *Let $\Omega \Subset \mathbb{C}^n$ be a bounded symmetric domain in its Harish-Chandra realization. Then, Ω is the union of open disks $\Delta_\alpha(0; r_\alpha) \subset \Lambda_\alpha$ on the complex lines $\Lambda_\alpha := \mathbb{C}\alpha$, as α*

ranges over unit vectors on $\mathbb{C}^n$, where

$$\frac{2}{r_\alpha^2} := \sup\{\Theta_{\alpha\overline{\alpha}\nu\overline{\nu}} : \nu \in T_0(\Omega), \|\nu\| = 1\}.$$

In other words, $\xi \in \Omega$ if and only if $\Theta_{\xi\overline{\xi}\nu\overline{\nu}} < 2$ for any unit vector $\nu \in T_0(\Omega)$.

Proof By Lemma 3.1 without loss of generality, we may assume that writing $\Omega = G_0/K$, each irreducible factor of the semisimple complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$ is of type A, D or E. Let $\alpha \in T_0(\Omega)$ be a unit vector. For simplicity in the ensuing arguments we will assume also that Ω is irreducible. The theorem for the general case where $\Omega = \Omega_1 \times \cdots \times \Omega_s$ and each irreducible factor Ω, $1 \le k \le s$ is of A, D or E type follows readily from the arguments in the special case where Ω is irreducible.

Writing $\Psi = \{\psi_1, \cdots, \psi_r\} \subset \Phi_0^+$ for a maximal set of strongly orthogonal positive noncompact roots, and writing e_φ for a unit root vector associated to a root $\varphi \in \Psi$, there is a reference maximal polydisk $\Pi_0 \subset \Omega$ passing through 0, such that $T_0(\Pi_0) = \mathbb{C}e_{\psi_1} \oplus \cdots \oplus \mathbb{C}e_{\psi_r}$, and, for any $\alpha \in T_0(\Omega)$ there exists $k \in K$ such that $k(\alpha) \in T_0(\Pi_0)$. An element $c_1 e_{\psi_1} + \cdots + e_{\psi_1}$ will be denoted as $(c_1, \cdots, c_r)$. Choosing k properly, we may take $k(\alpha)$ to be the normal form $(a_1, \cdots, a_r)$ such that each a_i is real and nonnegative, and furthermore $a_1 \ge \cdots \ge a_{r-1} \ge a_r$. For the proof of Theorem 3.1 without loss of generality we consider α to be the normal form $(a_1, \cdots, a_r)$ itself.

For the tangent space $T_0(\Omega)$, we write

$$T_0(\Omega) \cong \mathfrak{m}^+ = \bigoplus\{\mathfrak{g}_\varphi : \varphi \in \Phi_0^+\} = \Big(\bigoplus_{\psi\in\Psi} \mathbb{C}e_\psi\Big) \oplus \Big(\bigoplus_{\rho\in\Phi_0^+ - \Psi} \mathbb{C}e_\rho\Big).$$

Let now $\nu \in T_0(\Omega)$ be a unit vector and write

$$\nu = \sum_{\varphi\in\Phi_0^+} c_\varphi e_\varphi = \sum_{i=1}^r c_i e_{\psi_i} + \sum_{\rho\in\Phi_0^+ - \Psi} c_\rho e_\rho,$$

where for $\Psi = \{\psi_1, \cdots, \psi_r\}$ we write c_i for c_{ψ_i}, $1 \le i \le r$. We have

$$\Theta_{\alpha\overline{\alpha}\nu\overline{\nu}} = \sum_{i=1}^r a_i^2 |c_i|^2 \Theta_{i\bar{i}i\bar{i}} + \sum_{i=1}^r \sum_{\rho\in\Phi_0^+ - \Psi} a_i^2 |c_\rho|^2 \Theta_{i\bar{i}e_\rho\overline{e_\rho}},$$

where for brevity here and in what follows $\Theta_{i\bar{i}e_\rho\overline{e_\rho}}$ stands for $\Theta(e_{\psi_i}, e_{\psi_i}; e_\rho, \overline{e_\rho})$, etc. Here we have made use of the fact that $\Theta_{i\bar{j}u\overline{v}} = 0$ for $i \ne j$, $1 \le i, j \le r$, and for $u, v \in T_0(\Omega)$, which follows from $[e_{\psi_1}, \overline{e_{\psi_j}}] = 0$ as $\psi_i - \psi_j \notin \Phi$. Now we rewrite the curvature expression of $\Theta_{\alpha\overline{\alpha}\nu\overline{\nu}}$ in the above as

$$\begin{aligned}
\Theta_{\alpha\overline{\alpha}\nu\overline{\nu}} &= \sum_{i=1}^r a_i^2 |c_i|^2 \Theta_{i\bar{i}i\bar{i}} + \sum_{\rho\in\Phi_0^+ - \Psi} |c_\rho|^2 \Big(\sum_{\rho - \psi_i \in \Phi} a_i^2 \Theta_{i\bar{i}e_\rho\overline{e_\rho}}\Big) \\
&= 2\sum_{i=1}^r a_i^2 |c_i|^2 + \sum_{\rho\in\Phi_0^+ - \Psi} |c_\rho|^2 \Big(\sum\{a_i^2 : \rho - \psi_i \in \Phi\}\Big).
\end{aligned}$$

Noting that for $1 \le i \le r$ we have $\Theta_{i\bar{i}i\bar{i}} = 2$, and that for $\rho \in \Phi_0^+$ satisfying $\rho - \psi_i \in \Phi$ we have $e_{\psi_i} \in \mathcal{H}_{e_\rho}$, so that $\Theta_{i\bar{i}e_\rho\overline{e_\rho}} = 1$.

By Lemma 3.2, given $\rho \in \Phi_0^+ - \Psi$, there exist at most 2 elements in Ψ such that $\rho - \psi \in \Phi$. Together with the ordering $a_1 \geq \cdots \geq a_r \geq 0$, we have the curvature estimate

$$\begin{aligned} 0 \leq \Theta_{\alpha\overline{\alpha}\nu\overline{\nu}} &\leq 2a_1^2(|c_1|^2 + \cdots + |c_r|^2) + 2a_1^2\Big(\sum_{\rho\in\Phi_0^+-\Psi}|c_\rho|^2\Big) \\ &= 2a_1^2\Big(\sum_{\varphi\in\Phi_0^+}|c_\varphi|^2\Big) = 2a_1^2, \end{aligned}$$

where the last equality holds since $\nu = \sum\{c_\varphi e_\varphi : \varphi \in \Phi_0^+\}$ is a unit vector. Note that equality holds in the first line when $\nu = \mathrm{e}^{\mathrm{i}\theta}e_{\psi_1}$, $\theta \in \mathbb{R}$, so that $c_1 = \mathrm{e}^{\mathrm{i}\theta}$ and $c_\varphi = 0$ any other root $\varphi \in \Phi_0^+$ (although equality may also hold for some other unit vectors ν).

To determine r_α in another way, note that $\Pi_0 \cap \mathbb{C}\Lambda_\alpha$ consists of all $u = t(a_1, \cdots, a_r)$ such that $t \in \mathbb{C}$ and $|ta_i| < 1$ for $1 \leq i \leq r$, and it follows that $u \in \Pi_0$ if and only if $|t| < \frac{1}{a_1}$, i.e., $r_\alpha = \frac{1}{a_1}$. Combining with the curvature estimate in the above, we have

$$2a_1^2 = \frac{2}{r_\alpha^2} := \sup\{\Theta_{\alpha\overline{\alpha}\nu\overline{\nu}} : \nu \in T_0(\Omega), \|\nu\| = 1\}$$

as desired. Finally, for ξ nonzero, $\xi \in \Omega$ if and only if $\|\xi\| < r_\alpha, \alpha = \frac{\xi}{\|\xi\|}$ being a unit vector. Thus, $\xi \in \Omega$ if and only if $\Theta_{\xi\overline{\xi}\nu\overline{\nu}} < 2$ for any unit vector $\nu \in T_0(\Omega)$. The proof of Theorem 3.1 is complete.

Corollary 3.1 *The bounded domain $\Omega \Subset \mathbb{C}^n$ in its Harish-Chandra realization is a convex domain.*

Proof Suppose $\alpha, \beta \in \Omega$ and write $\gamma = t\alpha + (1-t)\beta$, $0 \leq t \leq 1$. To prove the convexity of Ω it suffices to show that $\gamma \in \Omega$, i.e, $\Theta_{\gamma\overline{\gamma}\nu\overline{\nu}} < 2$ for any vector $\nu \in T_0(\Omega)$. Now the Hermitian form H_ν given by $H_\alpha(u,v) := \Theta_{u\overline{v}\nu\overline{\nu}}$ is positive semidefinite, thus defining $\|u\|_{H_\nu} := \sqrt{H_\nu(u,u)}$ for $u \in T_0(\Omega) \cong \mathbb{C}^n$, $\|\cdot\|_{H_\nu}$ is a semi-norm. We have $\|\alpha\|_{H_\nu}, \|\beta\|_{H_\nu} < \sqrt{2}$ and by the triangular inequality

$$\begin{aligned} \Theta_{\gamma\overline{\gamma}\nu\overline{\nu}} &= H_\nu(t\alpha + (1-t)\beta, t\alpha + (1-t)\beta) = \|t\alpha + (1-t)\beta\|_{H_\nu}^2 \\ &\leq (t\|\alpha\|_{H_\nu} + (1-t)\|\beta\|)_{H_\nu})^2 < \sqrt{2}^2 = 2 \end{aligned}$$

as desired.

Remark 3.1 (a) The identification of each $\mathscr{C}_0(X) \subset \mathbb{P}T_0(\Omega)$ as a Hermitian symmetric space of the compact type and of rank ≤ 2 can be read off from the Dynkin diagram $\mathscr{D}(\mathfrak{g})$, $\mathfrak{g} = \mathfrak{g}^{\mathbb{C}}$. If the Hermitian symmetric space (X, ds_X^2) is of type $(\mathfrak{g}, \alpha_k)$ in standard notation, and Σ is the set of simple positive roots adjacent to α_k, then each connected component of $\mathscr{D}(\mathfrak{g}) - \{\alpha_k\}$, with a marking at $\sigma \in \Sigma$, corresponds to an ireducible factor of the Hermitian symmetric space $\mathscr{C}_0(X)$.

(b) Theorem 3.1 can be reformulated by stating that $\xi \in \Omega$ if and only if $\|\mathrm{ad}(\xi)(\overline{\nu})\| < \sqrt{2}$ for any unit vector $\nu \in \mathfrak{m}^+$, from which the convexity of Ω follows immediately. As such, Theorem 3.1 may be regarded as a variant of the Hermann Convexity Theorem. Here we prefer to formulate Theorem 3.1 as a statement concerning bisectional curvatures $\Theta_{\xi\overline{\xi}\nu\overline{\nu}}$, with an essentially geometric and self-contained proof.

(c) One can prove Theorem 3.1 by geometric means free from classification results. As given here, Theorem 3.1 is deduced from the Polydisk Theorem and the combinatorial Lemma 3.2. The main feature of the Polydisk Theorem relevant to us is that each Cartesian factor is of radius 1, which can be derived without using any classification theory. The alternative proof of Lemma 3.2 given here relies on identifying the VMRT of X as a Hermitian symmetric space $(\mathscr{C}_0(X), s)$ of rank ≤ 2, which follows from the pinching condition $1 \leq S_{\eta\overline{\eta}\eta\overline{\eta}} \leq 2$ on holomorphic sectional curvatures for unit vectors $\eta \in T_{[\alpha]}(\mathscr{C}_0(X))$ established in [4, Appendix III.2] by elementary means, and which by Ros [7] implies the parallelism of the second fundamental form of $\mathscr{C}_0(X) \subset \mathbb{P}T_0(\Omega)$ and hence the Hermitian symmetry of $(\mathscr{C}_0(X), s)$, as its curvature tensor S must then necessarily be parallel. By considering maximal polyspheres, the fact that $(\mathscr{C}_0(X), s)$ must be of rank ≤ 2 follows by observing that the Segre embedding of $(\mathbb{P}^1)^3$ into $\mathbb{P}^7$ does not have parallel second fundamental form.

(d) The result of [7] was accompanied by a complete listing of Kähler submanifolds satisfying the aforementioned pinching condition of the projective space $(\mathbb{P}^m, ds^2_{FS})$ equipped with the Fubini-study metric of constant holomorphic sectional curvature $+2$, according to Nakagawa-Takagi [6, Theorem 7.4], which miraculously corresponds exactly to the listing of VMRTs of (X, g_c) for irreducible Hermitian symmetric spaces of the compact type (cf. [4]).

4 Proof of Theorem 1.1

We will continue to adopt notation with the meaning of symbols as defined in Section 3. For the proof of Theorem 1.1 using results of Section 4 we will need furthermore a couple of preliminary results, as follows.

Proposition 4.1 *Let $S \subset \Omega$ be a totally geodesic complex submanifold passing through $0 \in \Omega$, so that $S = E \cap \Omega$ for $E \subset \Omega$ corresponding to $T_0(S) \subset T_0(\Omega)$. Then,*

$$\frac{2}{r_\alpha^2} := \sup\{\Theta_{\alpha\overline{\alpha}\nu\overline{\nu}} : \nu \in T_0(S), \|\nu\| = 1\}.$$

Proof By Theorem 3.1 we have

$$\frac{2}{r_\alpha^2} := \sup\{\Theta_{\alpha\overline{\alpha}\nu\overline{\nu}} : \nu \in T_0(\Omega), \|\nu\| = 1\},$$

hence the key point of Proposition 4.1 is that we can compute r_α by restricting to unit vectors $\nu \in T_0(S)$, for an arbitrary totally geodesic complex submanifold $S \subset \Omega$ passing through 0 such that $\alpha \in T_0(S)$. Noting that the set of totally geodesic complex submanifolds passing through 0 such that $\alpha \in T_0(S)$ is closed under intersection (of an arbitrary family of such manifolds), there is a unique minimal totally geodesic complex submanifold S_0 passing through 0 such that $T_0(S_0)$. For $\alpha \in T_0(\Omega)$ with normal form $(a_1, \cdots, a_r)$, $a_1 \geq \cdots \geq a_r$, we say that α is a general tangent vector if and only if all a_i, $1 \leq i \leq r$, are distinct and positive. Replacing without loss of generality α by $(a_1, \cdots, a_r) \in T_0(\Pi_0)$, the smallest totally geodesic complex submanifold S passing through 0 such that $\alpha \in T_0(S)$ is precisely given by $S = \Pi_0$, the reference maximal polydisk as defined in Section 3. Thus, to prove Proposition 4.1 for the case of a general tangent vector $\alpha \in T_0(\Omega)$, it suffices to show the validity of the formula for r_α for the special case where $\alpha \in T_0(\Pi_0)$, $S = \Pi_0$. But this already follows from the proof of Theorem 3.1, in which we showed that, writing $\eta_k := e_{\psi_1}$ for $1 \le k \le r$, for any unit vector $\nu \in T_0(\Omega)$ we have $\Theta_{\alpha\overline{\alpha}\nu\overline{\nu}} \le \Theta_{\alpha\overline{\alpha}\eta_1\overline{\eta_1}} = 2a_1^2$, so that $r_\alpha = \frac{1}{a_1}$. Since $\eta_1 \in T_0(\Pi_0)$, Proposition 4.1 for the case of a general unit tangent vector α follows.

Consider now the case where the normal form $(a_1, \cdots, a_r)$ of the unit vector $\alpha \in T_0$ (Ω is arbitrary). Writing

$$\begin{aligned} a_1 = \cdots = a_{m_1} > a_{m_1+1} = \cdots = a_{m_1+m_2} > \cdots > \cdots \\ > a_{m_1+\cdots+m_\ell} = \cdots = a_{m_1+\cdots+m_\ell} \ge 0, \end{aligned}$$

where $m_1 + \cdots + m_\ell$, then the smallest totally geodesic complex submanifold S passing through 0 such that $\alpha \in T_0(S)$ is an ℓ-dimensional polydisk given by $S = \mathrm{diag}(\Delta^{m_1}) \times \cdots \times \mathrm{diag}(\Delta^{m_\ell}) \subset \Delta^r =: S_0$, where $\Delta^r \subset \mathbb{C}^r$ is identified with Π_0 by the isomorphism $(z_1, \ldots, z_r) \mapsto z_1 e_{\psi_1} + \cdots z_r e_{\psi_r}$, $T_0(\Omega)$ being equated with $\mathbb{C}^n$. In this case, take $\nu_0 = \frac{1}{\sqrt{m_1}}(e_{\psi_1} + \cdots + e_{\psi_{m_1}})$, we have $\nu_0 \in T_0(S_0)$ and

$$\begin{aligned} \Theta_{\alpha\overline{\alpha}\nu_0\overline{\nu_0}} &= \frac{1}{m_1} a_1^2 (R_{\eta_1\overline{\eta_1}\eta_1\overline{\eta_1}} + \cdots + R_{\eta_{m_1}\overline{\eta_{m_1}}\eta_{m_1}\overline{\eta_{m_1}}}) \\ &= \frac{1}{m_1}(2m_1 a_1^2) = 2a_1^2 = \sup\{\Theta_{\alpha\overline{\alpha}\nu\overline{\nu}} : \nu \in T_0(\Omega), \|\nu\| = 1\}, \end{aligned}$$

where the last equation follows from $r_\alpha = \frac{1}{a_1}$ as given in the proof of Theorem 3.1. Thus, Proposition 4.1 holds for any totally geodesic complex submanifold $S \subset \Omega$ passing through 0, and for any unit vector $\alpha \in T_0(S)$, as desired.

Lemma 4.1 *Let $\Omega \Subset \mathbb{C}^n$ be a bounded symmetric domain in its Harish-Chandra realization, and $S \subset \Omega$ be a totally geodesic complex submanifold with respect to ds^2_Ω passing through the origin $0 \in \Omega$, so that S is an open subset of a complex vector subspace $E \subset \mathbb{C}^n$ corresponding to $\mathfrak{m}_1^+ \subset \mathfrak{m}^+$, $S = E \cap \Omega$. Let $\mu \in T_0(S) \cong \mathfrak{m}_1^+$. Then, for any $\xi_1 \in \mathfrak{m}_1^+$ and $\eta \in \mathfrak{m}^+ \cong T_0(\Omega)$ orthogonal to $\mathfrak{m}_1^+$ with respect to the Euclidean metric $ds^2_\Omega(0)$, we have $\Theta_{\xi_1\overline{\eta}\mu\overline{\mu}} = 0$.*

Proof Recall that

$$\Theta_{\xi_1\overline{\eta}\mu\overline{\mu}} = \Theta_{\mu\overline{\mu}\xi_1\overline{\eta}} = ([[\mu, \overline{\mu}]; \xi_1]; \eta),$$

where $(\cdot;\cdot)$ stands for the Hermitian inner product corresponding to $ds^2_\Omega(0)$. By Theorem 2.1, $S = E \cap \Omega$ is a totally geodesic complex submanifold if and only if $[[\mathfrak{m}_1^+, \mathfrak{m}_1^-], \mathfrak{m}_1^+] \subset \mathfrak{m}_1^+$. It follows readily that $[[\mu, \overline{\mu}], \xi_1] =: \gamma \in \mathfrak{m}_1^+$. Since $\eta \perp \mathfrak{m}_1^+$, we have

$$\Theta_{\xi_1\overline{\eta}\mu\overline{\mu}} = ([[\mu, \overline{\mu}], \xi_1], \eta) = (\gamma; \eta) = 0$$

as desired.

We are now ready to give a proof of the main result Theorem 1.1.

Proof Recall that $S \subset \Omega \Subset \mathbb{C}^n$ is a totally geodesic complex submanifold passing through 0, $S = E \cap \Omega$, in which $E \subset \mathbb{C}^n$ is a complex vector subspace identified with $T_0(S) \subset T_0(\Omega) \cong \mathbb{C}^n$, and that $\rho : \Omega \to E$ is the orthogonal projection with respect to the Euclidean metric $ds^2_0(0)$ on $T_0(\Omega) \cong \mathbb{C}^n$. Let $\xi \in \Omega$. Write $\xi = \xi_1 + \eta$ according to the orthogonal decomposition $T_0(\Omega) = T_0(S) \oplus T_0(S)^\perp$, in which $\xi_1 \in T_0(S)$ and $\eta \in T_0(S)^\perp$, where $A^\perp$ for a complex vector subspace $A \subset T_0(\Omega)$ denotes its orthogonal complement with respect to $ds^2_\Omega(0)$. By Theorem

3.1, we have $\Theta_{\xi\overline{\xi}\nu\overline{\nu}} < 2$ for any unit vector $\nu \in T_0(\Omega)$. For $\mu \in T_0(S)$, by Lemma 4.1 we have $\Theta_{\xi_1\overline{\eta}\mu\overline{\mu}} = 0$. Together with the orthogonal decomposition $\xi = \xi_1 + \eta$ we deduce that for $\mu \in T_0(S)$ we have

$$2 > \Theta_{\xi\overline{\xi}\mu\overline{\mu}} = \Theta_{\xi_1\overline{\xi_1}\mu\overline{\mu}} + \Theta_{\eta\overline{\eta}\mu\overline{\mu}} \geq \Theta_{\xi_1\overline{\xi_1}\mu\overline{\mu}},$$

where the last inequality follows from the nonnegativity of bisectional curvatures of (X_c, g_c). On the other hand, by Proposition 4.1 we know that for any unit vector $\nu \in T_0(\Omega)$ we have

$$\Theta_{\xi_1\overline{\xi_1}\nu\overline{\nu}} \leq \sup\{\Theta_{\xi_1\overline{\xi_1}\mu\overline{\mu}} : \mu \in T_0(S), \|\mu\| = 1\},$$

so that

$$\Theta_{\xi_1\overline{\xi_1}\nu\overline{\nu}} < 2$$

for any unit vector $\nu \in T_0(\Omega)$, hence $\xi_1 \in \Omega$ by Theorem 3.1. In other words, $\rho(\xi) = \xi_1 \in \Omega \cap E = S$, so that $\rho(\Omega) = S$, as desired. The proof of Theorem 1.1 is complete.

Remark 4.1 With an aim towards a specific application, the special case of Theorem 1.1 where the bounded symmetric domain $\Omega \Subset \mathbb{C}^n$ is irreducible and $S \subset \Omega$ is a minimal (totally geodesic) disk was proved in Mok-Ng [5]. The proof there relied on the Hermann Convexity Theorem.

From Theorem 1.1, we readily have the following theorem.

Theorem 4.1 *Let (X, g) be a Hermitian symmetric manifold of the noncompact type, and $(Y, g|_Y) \hookrightarrow (X, g)$ be a totally geodesic complex submanifold. Then, $Y \subset X$ is a holomorphic retract of X, i.e., there exists a holomorphic mapping $r : X \to Y$ such that $r|_Y = \mathrm{id}_Y$. Moreover the identity map id_X on X is homotopic through a continuous family $\{F_t : t \in [0, 1]\}$ of holomorphic maps $F_t : X \to X$ such that $F_0(x) = x$ and $F_1 = r$ is a holomorphic retract of X on Y and such that the continuous map $F : X \times [0, 1] \to X$ defined by $F(x, t) := F_t(x)$ is real analytic on $X \times (0, 1)$.*

Proof Since the total geodesy of Y in X does not depend on the choice of the $\mathrm{Aut}_0(X)$-invariant Kähler metric g on X, without loss of generality we may take g to correspond to the Kähler metric ds^2_Ω on $\Omega \cong X$. Denoting by $\xi : X \to \Omega$ the Harish-Chandra realization, define $S := \xi(Y)$. Then, writing $r : X \to Y$ to correspond to the orthogonal projection $\rho : \Omega \to S$, we have $r : X \to Y$ and $r|_Y = \mathrm{id}_Y$. Finally, for $0 \leq t \leq 1$, define $f_t : \Omega \to S$ by $f_t = \rho(x) + t(x - \rho(x))$, we have $f_t(x) \in \Omega$. As t ranges over $[0, 1]$, for each point $x \in \Omega$, $f_t(x)$ describes the closed interval joining x to $\rho(x) \in S \subset \Omega$. Hence, writing $f(x, t) = f_t(x)$ we have defined a continuous map $f : \Omega \times [0, 1] \to \Omega$, which corresponds under the inverse of the Harish-Chandra realization $\eta : X \xrightarrow{\cong} \Omega$ to a continuous map $F : X \times [0, 1] \to Y$ yielding a deformation of the identity map to the holomorphic retract $r : X \to Y$, as desired.

5 Holomorphic Totally Geodesic Isometric Embeddings with Respect to Carathéodory and Koba-yashi Metrics

For a bounded domain D in a complex Euclidean space $\mathbb{C}^N$, we denote by $\|\cdot\|_{C_D}$ its infinitesimal Carathéodory metric and by $\|\cdot\|_{K_D}$ its infinitesimal Kobayashi metric. On the unit disk Δ, we denote by $\|\cdot\|_\Delta$ its Poincaré metric of constant Gaussian curvature -2. By convention we have $\|\cdot\|_\Delta = \|\cdot\|_{C_\Delta} = \|\cdot\|_{K_\Delta}$. Concerning the Carathédory metric and the Kobayashi metric on bounded symmetric domains and those of their totally geodesic complex submanifolds (which are themselves biholomorphic to bounded symmetric domains) we have the following theorem.

Theorem 5.1 *Let (X, g) be a Hermitian symmetric manifold of the noncompact type, and $(Y, g|_Y) \hookrightarrow (X, g)$ be a totally geodesic complex submanifold. Then, the inclusion map $\imath : Y \hookrightarrow X$ is a holomorphic isometric embedding with respect to the Carathéodory (and equivalently the Kobayashi) metric.*

Proof In this proof for a complex manifold M we denote by $\|\cdot\|_M$ the Carathéodory pseudonorm on M. Write n (resp. m) for the complex dimension of X (resp. Y). Let $\chi : X \xrightarrow{\cong} \Omega$ be the Harish-Chandra realization of X as a bounded domain $\Omega \Subset \mathbb{C}^n$. Since $(Y, g|_Y) \hookrightarrow (X, g)$ is a totally geodesic complex manifold, $S := \chi(Y) = E \cap \Omega$ for an m-dimensional complex vector subspace $E \subset \mathbb{C}^n$. By a theorem of Wong-Royden based on Lempert's theorem on extremal holomorphic Kobayashi disks on strictly convex bounded domains, we know that the infinitesimal Carathéodory and Kobayashi metrics on Ω. $S = E \cap \Omega$, being the intersection of a bounded domain with a complex linear subspace, is itself a weakly convex domain in the complex vector space E, thus the infinitesimal Carathéodory and Kobayashi metrics agree on S. To prove the theorem it remains to show that the inclusion $\imath : S \hookrightarrow \Omega$ is an isometric embedding with respect to the Carathéodory metrics.

Given any point $x \in S$ and any vector η of type (1,0) tangent to S at x, among all holomorphic maps $h : S \to \Delta$ of S into the unit disk Δ as a consequence of Montel's theorem and the homogeneity of Δ that there exists $f : S \to \Delta$ such that $\|\partial f(\eta)\|_\Delta$ realizes the supremum of all $\|\partial h(\eta)\|_\Delta$. Let now $\rho : \Omega \to S$ be the holomorphic retract defined as in Theorem 1.1 as the orthogonal projection with respect to the Euclidean metric on $\mathbb{C}^n$. Then, $F := f \circ \rho : \Omega \to \Delta$ and we have $\partial F(\eta) = \partial f(\eta)$ since $\rho|_S = \mathrm{id}_S$. From the inclusion $S \subset \Omega$ we have

$$\|\eta\|_{C_\Omega} \le \|\eta\|_{C_S}.$$

On the other hand, by the choice of f we have $\|\eta\|_{C_S} = \|\partial f(\eta)\|_\Delta$, while the extension $F : \Omega \to \Delta$ yields

$$\|\eta\|_{C_\Omega} \ge \|\partial F(\eta)\|_\Delta = \|\partial f(\eta)\|_\Delta = \|\eta\|_{C_S}.$$

Combining the two inequalities we have $\|\eta\|_{C_\Omega} = \|\eta\|_{C_S}$, as desired. The proof of Theorem 5.1 is complete.

Remark 5.1 Regarding the Theorem of Royden-Wong referred to in the second paragraph of the proof, the original unpublished manuscript was elaborated and further developed posthumously leading to the published work of Royden-Wong-Krantz [8], and there was also a different proof by Salinas [9] using operator theory.

6 Appendix

In the proof of Theorem 5.1, in place of quoting the Theorem of Royden-Wong, for the special case of a bounded symmetric domain in its Harish-Chandra realization $\Omega \Subset \mathbb{C}^n$, one can assert the equivalence of the infinitesimal Carathéodory metric $\|\cdot\|_{C_\Omega}$ and the infinitesimal

Kobayashi metric $\|\cdot\|_{K_\Omega}$ by means of Theorem 1.1 itself, thus yielding a self-contained proof of the equivalence of $\|\cdot\|_{C_\Omega}$ and $\|\cdot\|_{K_\Omega}$ independent of Lempert's theorem in [3], as follows.

Proposition 6.1 *For any* $\eta \in T_\Omega$ *we have* $\|\eta\|_{C_\Omega} = \|\eta\|_{K_\Omega}$.

Proof Write r for the rank of Ω as a bounded symmetric domain. Let $\Pi \subset \Omega$ be a maximal polydisk passing through $0 \in \Omega$, $\Pi \cong \Delta^r$. $\Pi \subset \Omega$ is totally geodesic with respect to ds^2_Ω, and we have $\Pi = V \cap \Omega$, where $V \subset \mathbb{C}^n$ is a complex vector subspace, $\dim_{\mathbb{C}} V = r$. By the Polydisk Theorem, any tangent vector $\nu \in T_\Omega$ is equivalent under the action of G_0 to a vector $\eta \in T_0(\Pi)$. Then, by Theorem 1.1, the image of the orthogonal projection $\rho : \Omega \to V$ is exactly Π. Note that on the unit disk we have $\|\cdot\|_{C_\Delta} = \|\cdot\|_{K_\Delta}$ from the definitions. On $\Pi \cong \Delta^r$, for any $x \in \Pi$ and $\eta \in \Pi$ written as $\eta = (\eta_1, \cdots, \eta_r)$ in terms of Euclidean coordinates, we have obviously

$$\begin{aligned} \|\eta|_{C_\Pi} &= \max(\|\eta_1\|_{C_\Delta}, \cdots, \|\eta_r\|_{C_\Delta}) \\ &= \max(\|\eta_1\|_{K_\Delta}, \cdots, \|\eta_r\|_{K_\Delta}) = \|\eta\|_{K_\Pi}. \end{aligned}$$

The proof of Theorem 5.1, without quoting the theorem of Royden-Wong, shows that $\Pi \subset \Omega$ is an isometric embedding with respect to the Carathéodory metric. Thus, $\|\eta\|_{C_\Omega} = \|\eta\|_{C_\Pi}$. On the other hand,

$$\|\eta\|_{K_\Omega} = \inf\Big\{\frac{1}{R} : R > 0, \exists f : \Delta \xrightarrow{\text{hol.}} \Omega : f(0) = 0, \mathrm{d}f(0)\Big(\frac{\partial}{\partial z}\Big) = R\eta\Big\}.$$

Write $R = R_f$. Since $\rho : \Omega \to \Pi$, for any holomorphic map $f : \Delta \to \Omega$ in the above, we have $F := \rho \circ f : \Delta \to \Pi$ such that $F(0) = f(0) = 0$, and $\mathrm{d}F(0)\big(\frac{\partial}{\partial z}\big) = \mathrm{d}f(0)\big(\frac{\partial}{\partial z}\big) = R_f\eta$ since $\rho|_\Pi = \mathrm{id}_\Pi$. Taking infimum of $\frac{1}{R_f}$ over all holomorphic maps $f : \Delta \xrightarrow{\text{hol.}} \Omega$ such that $f(0) = 0$ and $\mathrm{d}f(0)\big(\frac{\partial}{\partial z}\big)$ is a positive multiple of η, we have

$$\|\eta\|_{K_\Pi} \le \|\eta\|_{K_\Omega}.$$

On the other hand, since $\Pi \subset \Omega$, any holomorphic map $F : \Delta \to \Pi$ is a holomorphic map into Ω, and it follows from the definition of the infinitesimal Kobayashi metric that $\|\eta\|_{K_\Pi} \ge \|\eta\|_{K_\Omega}$, forcing therefore $\|\eta\|_{K_\Pi} = \|\eta\|_{K_\Omega}$, and hence

$$\|\eta\|_{K_\Omega} = \|\eta\|_{K_\Pi} = \|\eta\|_{C_\Pi} = \|\eta\|_{C_\Omega},$$

proving $\|\cdot\|_{C_\Omega} = \|\cdot\|_{K_\Omega}$ on Ω, as desired.

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