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Deformations of Compact Complex Manifolds with Ample Canonical Bundles*

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Abstract In this paper, the author discusses the deformations of compact complex manifolds with ample canonical bundles. It is known that a complex manifold has unobstructed deformations when it has a trivial canonical bundle or an ample anti-canonical bundle. When the complex manifold has an ample canonical bundle, the author can prove that this manifold also has unobstructed deformations under an extra condition.

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1 Introduction

Let (X,ω) be a compact Kähler manifold with dimension $\dim_{\mathbb{C}} X = n$, and we denote its canonical bundle by K_X. In the last several decades, there have been a large amount of results about the deformations of complex structures on compact complex manifolds, for example [11, 16]. The most fundamental theorem established by Kodaira and Spencer states that on a compact complex manifold X, an element $\varphi \in \mathrm{A}^{0,1}(X, T^{1,0}X)$, which we usually call a Beltrami differential, determines a new complex structure once it solves the Maurer-Cartan equation

$$\begin{cases} \overline{\partial}\varphi = \dfrac{1}{2}[\varphi,\varphi], \\ \varphi(0) = 0. \end{cases} \tag{1.1}$$

They also showed that the obstruction of the deformations lies in the cohomology group $\mathrm{H}^2(X, T^{1,0}X)$. Consequently, when X is a Fano manifold, i.e., K_X^{-1} is ample, by the Kodaira vanishing theorem, we see that

$$\mathrm{H}^2(X, T^{1,0}X) \cong \mathrm{H}^{n-2}(X, \Omega^1(K_X)) = 0$$

because of the negativity of the line bundle K_X, which yields that all Fano manifolds have unobstructed deformations.

When the manifold X is Calabi-Yau, i.e., the canonical bundle K_X is trivial, the deformations are also unobstructed according to Bogomolov, Tian and Todorov, which is now widely known as the Bogomolov-Tian-Todorov theorem (see [2, 24–25]). Besides, there are also many

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noteworthy results concerning the deformations of logarithmic Calabi-Yau pairs, for example, [9, 13]. It is worth pointing out that the research concerning the deformations of other interesting structures in complex geometry also have a lot of breakthrough in recent years, for example [19–21]. Note that in [15] there is a more global method to deal with the deformation theory.

When K_X is ample, it is much more complicated. There are examples that the deformations may be obstructed. For example, Horikawa [8, Section 10] constructed an example as follows. First, by studying the deformations of holomorphic maps, he got that the monoidal transformation Y of the complex projective space $\mathbb{CP}^3$ has obstructed deformations, where the center C is a curve of degree 14 and of genus 24 in $\mathbb{CP}^3$ which was constructed by Mumford [17]. Horikawa then showed that if X is a general element of a sufficiently ample linear system on Y, then X is non-singular, irreducible, and has an ample canonical bundle, and then he showed that X has obstructed deformations by showing that its Kodaira-Spencer map is not surjective.

However, there are also examples that some certain compact complex manifolds with ample canonical bundles have unobstructed deformations, such as ample hypersurfaces in an Abelian variety (see [4]) and surfaces of type IIb, which are birational to the quintic hypersurface in $\mathbb{CP}^3$ (see [7]).

Thus, it is natural to ask what the obstruction of the deformations is and whether it has a Hodge theoretic characterization when the canonical bundle is ample.

In this paper, we use the Hodge theory and the iteration method to explore the obstruction. We will solve (1.1) and express the solution as a formal power series

$$\varphi(t) = \sum_{i=1}^{\infty} \varphi_i t^i,$$

when K_X is ample.

Explicitly speaking, we begin with an arbitrary harmonic initial value $\varphi_1 \in \mathbb{H}^{0,1}(X, T^{1,0}X)$ and solve the reduced equations (2.8) by induction with an extra condition that the essential obstruction vanishes:

$$\mathbb{H}(\nabla' \circ i_\varphi \circ i_\varphi \Omega_0) = 0. \tag{1.2}$$

The solution at step 2 (which means the coefficient of t^2 in $\varphi(t)$) is expressed as

$$\varphi_2 = \Omega_0^* \lrcorner \overline{\partial}^* \mathbb{G}\Big(-\frac{1}{2} \nabla' \circ i_{\varphi_1} \circ i_{\varphi_1} \Omega_0 \Big).$$

Here $\mathbb{H}$ is the orthogonal projection of differential forms to their harmonic parts, $\mathbb{G}$ is the Green operator of $\overline{\partial}$, ∇' is the (1,0)-component of the Chern connection on the anticanonical bundle and Ω_0 is a globally defined and nowhere vanishing element in $\mathrm{A}^{n,0}(X, K_X^{-1})$, which can be written as

$$\Omega_0 = \mathrm{d}z \otimes \frac{\partial}{\partial z} := \mathrm{d}z^1 \wedge \cdots \wedge \mathrm{d}z^n \otimes \frac{\partial}{\partial z^1} \wedge \cdots \wedge \frac{\partial}{\partial z^n} \tag{1.3}$$

under a local coordinate $(z^1, \cdots, z^n)$.

The notion $\bullet \lrcorner \Omega_0$ denotes the contraction between elements in $\mathrm{A}^{0,q}(X, T^{1,0}X)$ and Ω_0, which induces an isomorphism

$$\bullet \lrcorner \Omega_0 : \mathrm{A}^{0,q}(X, T^{1,0}X) \to \mathrm{A}^{n-1,q}(X, K_X^{-1}).$$

And we denote the inverse by $\Omega_0^* \lrcorner \bullet$.

By running induction, the solution we obtain at the N-th step can be expressed as

$$\varphi_N = \Omega_0^* \lrcorner \Big[\overline{\partial}^* \mathbb{G} \Big(-\frac{1}{2} \sum_{i+j=N} \nabla' \circ i_{\varphi_i} \circ i_{\varphi_j} \Omega_0 \Big) \Big]$$

for any positive integer N.

The solutions we have obtained till the N-th step can be put together and written as

$$\varphi^N = \Omega_0^* \lrcorner \Big\{ \overline{\partial}^* \mathbb{G} \Big(-\frac{1}{2} \sum_{2 \le K \le N} \sum_{i+j=K} \nabla' \circ i_{\varphi_i} \circ i_{\varphi_j} \Omega_0 \Big) t^i t^j + (i_{\varphi_1} \Omega_0) t^1 \Big\},$$

where φ_i is the solution at step i (which means the coefficient of t^i in $\varphi(t)$), $1 \le i \le N-1$. Here $\varphi^N = \varphi_1 t + \cdots + \varphi_N t^N$.

By doing so, the solution $\varphi(t)$ can eventually be expressed as

$$\begin{cases} \varphi = \Omega_0^* \lrcorner \Big\{ \overline{\partial}^* \mathbb{G} \Big(-\frac{1}{2} \nabla' \circ i_\varphi \circ i_\varphi \Omega_0 \Big) + i_{\varphi_1} \Omega_0 \Big\}, \\ \dfrac{\partial \varphi}{\partial t}\Big|_{t=0} = \varphi_1, \\ \varphi(0) = 0, \end{cases} \tag{1.4}$$

which is uniquely determined by the harmonic initial value φ_1.

Note that at each step the condition (1.2) means $\mathbb{H}(\nabla' \circ i_{\varphi_i} \circ i_{\varphi_j} \Omega_0) = 0$ for the corresponding i, j.

In conclusion, we obtain the following theorem.

Theorem 1.1 *Let X be a compact complex manifold with an ample canonical bundle K_X. If $\mathbb{H}(\nabla' \circ i_\varphi \circ i_\varphi \Omega_0) = 0$, where φ is defined by* (1.4), *then X has unobstructed deformations. Here Ω_0 is a nowhere vanishing element in $\mathrm{A}^{n,0}(X, K_X^{-1})$ defined in* (1.3).

Remark 1.1 There are examples satisfying our condition $\mathbb{H}(\nabla' \circ i_\varphi \circ i_\varphi \Omega_0) = 0$, e.g.

(1) Compact Riemann surfaces with genus at least 2.

(2) The manifolds like $X = X_1 \times \cdots \times X_m$ for any integer $m \ge 2$ where each X_i is a compact Riemann surface with genus at least 2, $i = 1, \cdots, m$.

Both of them have ample canonical bundles and thus by Theorem 1.1 they have unobstructed deformations.

In addition, we need to point out that our method also works when $c_1(X) = 0$, i.e., when K_X is a torsion line bundle.

Corollary 1.1 (see [24–25]) *If $c_1(X) = 0$, i.e., K_X is a torsion line bundle, then X has unobstructed deformations.*

This paper is organized as follows. In Section 2, we present some basic notions and reduce the Maurer-Cartan equation (1.1) into two equations (2.8). In Section 3, we solve the reduced equations when the canonical bundle is ample and discuss some examples about the obstruction. Besides, we also show that our method still works when K_X is a torsion line bundle.

2 Reduction of the Equation

Inspired by the work of Liu, Rao and Wan [13], we first reduce the Maurer-Cartan equation (1.1) into two equations.

Let (X, ω) be a compact Kähler manifold. In terms of a local coordinate,

$$\omega = \sqrt{-1} g_{i\overline{j}} \mathrm{d}z^i \wedge \mathrm{d}\overline{z}^j.$$

Selecting a nowhere vanishing section Ω of $\mathrm{A}^{n,0}(X, K_X^{-1})$, we have an isomorphism obtained by contraction:

$$\bullet \lrcorner \Omega : \mathrm{A}^{0,1}(X, T^{1,0}X) \to \mathrm{A}^{n-1,1}(X, K_X^{-1}).$$

And we denote the inverse by

$$\Omega^* \lrcorner \bullet : \mathrm{A}^{n-1,1}(X, K_X^{-1}) \to \mathrm{A}^{0,1}(X, T^{1,0}X).$$

Here the notion $\varphi \lrcorner (\bullet)$ denotes the contraction between tangent vectors and differential forms that dual to each other. Sometimes we also use the notion $i_\varphi(\bullet)$ to denote the same operation.

Throughout this paper, we need the following technical lemma.

Lemma 2.1 *For any $\varphi, \psi \in \mathrm{A}^{0,1}(X, T^{1,0}X)$ and $\Omega \in \mathrm{A}^{n,q}(X)$, we have*

$$[\varphi, \psi] \lrcorner \Omega = -\partial(\psi \lrcorner \varphi \lrcorner \Omega) + \psi \lrcorner \partial(\varphi \lrcorner \Omega) + \varphi \lrcorner \partial(\psi \lrcorner \Omega).$$

For the proof, the generalizations and further applications of this lemma, one can refer to [12, 14].

There is a unique Chern connection $\nabla = \nabla' + \overline{\partial}$ on the Hermitian line bundle $(K_X^{-1}, \det(g_{i\overline{j}}))$. Therefore, similar to Lemma 2.1, we have the following Tian-Todorov lemma (e.g. in [12, Theorem 3.4])

$$[\varphi, \psi] \lrcorner \Omega = -\nabla'(\psi \lrcorner \varphi \lrcorner \Omega) + \psi \lrcorner \nabla'(\varphi \lrcorner \Omega) + \varphi \lrcorner \nabla'(\psi \lrcorner \Omega). \tag{2.1}$$

Before reducing the Maurer-Cartan equation, we need some preparations.

Definition 2.1 *For an element $\varphi \in \mathrm{A}^{0,1}(X, T^{1,0}X)$, the divergence operator is defined by*

$$\mathrm{div} = \mathrm{tr} \circ \nabla : \mathrm{A}^{0,1}(X, T^{1,0}X) \to \mathrm{A}^{0,1}(X).$$

In terms of a local coordinate $(z^1, \cdots, z^n)$, we write $\varphi = \varphi^i_{\overline{j}} \mathrm{d}\overline{z}^j \otimes \frac{\partial}{\partial z^i}$. Thus

$$\mathrm{div}(\varphi) = (\partial_i \varphi^i_{\overline{j}} + \varphi^i_{\overline{j}} \partial_i \log \det(g)) \mathrm{d}\overline{z}^j.$$

Since $\mathrm{div}(\varphi)$ is a (0,1)-form, it is obvious that

$$\varphi \lrcorner (\mathrm{div}(\varphi) \wedge \Omega) = \mathrm{div}(\varphi) \wedge (\varphi \lrcorner \Omega). \tag{2.2}$$

Proposition 2.1 *Let φ be an element in $\mathrm{A}^{0,1}(X, T^{1,0}X)$ and Ω be a nowhere vanishing element in $\mathrm{A}^{n,0}(X, K_X^{-1})$.*

If

$$\begin{cases} \Big(\overline{\partial} + \dfrac{1}{2}\nabla' \circ i_\varphi + \operatorname{div}(\varphi)\Big) i_\varphi \Omega = 0, \\ (\overline{\partial} + \nabla' \circ i_\varphi + \operatorname{div}(\varphi))\Omega = 0, \end{cases} \tag{2.3}$$

then

$$\overline{\partial}\varphi = \frac{1}{2}[\varphi, \varphi].$$

Proof We assume that the equations in (2.3) hold. Note that

$$\overline{\partial}(\varphi \lrcorner \Omega) = \overline{\partial}\varphi \lrcorner \Omega + \varphi \lrcorner \overline{\partial}\Omega. \tag{2.4}$$

By the assumption, the left-hand side of (2.4) is

$$\begin{aligned} \text{LHS} &= -\frac{1}{2}\nabla' \circ i_\varphi \circ i_\varphi \Omega - \operatorname{div}(\varphi) \wedge i_\varphi \Omega \\ &= \frac{1}{2}[\varphi, \varphi] \lrcorner \Omega - i_\varphi \circ \nabla' \circ i_\varphi \Omega - \varphi \lrcorner (\operatorname{div}(\varphi) \wedge \Omega), \end{aligned}$$

and the right-hand side of (2.4) is

$$\begin{aligned} \text{RHS} &= \overline{\partial}\varphi \lrcorner \Omega - \varphi \lrcorner (\nabla' \circ i_\varphi \Omega + \operatorname{div}(\varphi) \wedge \Omega) \\ &= \overline{\partial}\varphi \lrcorner \Omega - i_\varphi \circ \nabla' \circ i_\varphi \Omega - \varphi \lrcorner (\operatorname{div}(\varphi) \wedge \Omega). \end{aligned}$$

Comparing the two sides of (2.4) we have

$$\Big(\overline{\partial}\varphi - \frac{1}{2}[\varphi, \varphi]\Big) \lrcorner \Omega = 0,$$

and we get

$$\overline{\partial}\varphi = \frac{1}{2}[\varphi, \varphi]$$

since the operation $\bullet \lrcorner \Omega$ is an isomorphism.

In order to simplify the subsequent calculations, we need the following lemma.

Lemma 2.2 *Denote*

$$\Phi = \Phi(z) \mathrm{d}z \otimes \frac{\partial}{\partial z} = -\nabla' \circ i_\varphi \Omega - \operatorname{div}(\varphi) \wedge \Omega,$$

where $\Phi(z) \in \mathrm{A}^{0,1}(X)$ *and*

$$\Omega = \Omega(z) \mathrm{d}z \otimes \frac{\partial}{\partial z},$$

where $\Omega(z)$ *is a smooth function on* X. *Then we have*

$$\Phi(z) = i_\varphi \partial \Omega(z).$$

Here the notions $\mathrm{d}z$ *and* $\frac{\partial}{\partial z}$ *can be locally written as*

$$\mathrm{d}z = \mathrm{d}z^1 \wedge \cdots \wedge \mathrm{d}z^n, \quad \frac{\partial}{\partial z} = \frac{\partial}{\partial z^1} \wedge \cdots \wedge \frac{\partial}{\partial z^n}.$$

Proof On one hand, we know that

$$\begin{aligned}\nabla'\circ i_\varphi\Omega&=\nabla'\circ i_\varphi\Big(\Omega(z)\mathrm{d}z\otimes\frac{\partial}{\partial z}\Big)\\&=\nabla'(\varphi^i_{\overline{j}})\mathrm{d}\overline{z}^j\wedge\Omega(z)(-1)^{i-1}\mathrm{d}z^1\wedge\cdots\wedge\widehat{\mathrm{d}z^i}\wedge\cdots\wedge\mathrm{d}z^n\otimes\frac{\partial}{\partial z}\\&=-\nabla_i(\varphi^i_{\overline{j}}\Omega(z))\mathrm{d}\overline{z}^j\wedge\mathrm{d}z\otimes\frac{\partial}{\partial z}.\end{aligned}$$

On the other hand,

$$\mathrm{div}(\varphi)\wedge\Omega=(\nabla_i\varphi^i_{\overline{j}})\Omega(z)\mathrm{d}\overline{z}^j\wedge\mathrm{d}z\otimes\frac{\partial}{\partial z}.$$

Hence

$$\begin{aligned}-\nabla'\circ i_\varphi\Omega-\mathrm{div}(\varphi)\wedge\Omega&=\varphi^i_{\overline{j}}\partial_i\Omega(z)\mathrm{d}\overline{z}^j\wedge\mathrm{d}z\otimes\frac{\partial}{\partial z}\\&=(\varphi\lrcorner\partial\Omega(z))\wedge\mathrm{d}z\otimes\frac{\partial}{\partial z},\end{aligned}$$

which implies the conclusion.

From now on, our aim is to solve equations (2.3) by using the Hodge theory and the iteration method. To do this, following the approach of Kodaira and Spencer [11, 16], we expand the terms φ and Ω into power series

$$\varphi(t)=\sum_{i=1}^{\infty}\varphi_i t^i,\quad \Omega(t)=\Omega_0+\sum_{i=1}^{\infty}\Omega_i t^i.$$

Thus the terms $\Phi(z)$ and $\Omega(z)$ defined in Lemma 2.2 can also be expanded into power series in t.

Throughout this paper, we usually choose a harmonic φ_1 as the initial value, i.e., $\overline{\partial}\varphi_1=0$ and $\overline{\partial}^*\varphi_1=0$.

The following proposition reveals the legality of the iteration method in the study of deformation theory.

Proposition 2.2 *If for any $k\le N-1$ we have*

$$\begin{cases}\Big[\Big(\overline{\partial}+\frac{1}{2}\nabla'\circ i_\varphi+\mathrm{div}(\varphi)\Big)i_\varphi\Omega\Big]_{k+1}=0,\\ [(\overline{\partial}+\nabla'\circ i_\varphi+\mathrm{div}(\varphi))\Omega]_k=0,\end{cases}\tag{2.5}$$

we then derive that

$$\begin{cases}\overline{\partial}[(\nabla'\circ i_\varphi+\mathrm{div}(\varphi))\Omega]_N=0,\\ \overline{\partial}\Big[\Big(\frac{1}{2}\nabla'\circ i_\varphi+\mathrm{div}(\varphi)\Big)i_\varphi\Omega\Big]_{N+1}=0.\end{cases}\tag{2.6}$$

Here the subscript $[\bullet]_k$ denotes the coefficient of t^k once we expand both equations in (2.3) *into power series of the variant t.*

Proof According to Proposition 2.1, the condition implies that

$$\Big[\overline{\partial}\varphi-\frac{1}{2}[\varphi,\varphi]\Big]_{\le k+1}=0$$

for any $k \leq N-1$.

Note that the first equation in (2.6) to be proved is equivalent to $\overline{\partial}\Phi(z)_N = 0$ while the second one in (2.5) that we assumed is equivalent to $(\overline{\partial}\Omega(z) - \Phi(z))_{N-1} = 0$. Then by explicit calculations we have

$$\begin{aligned}\overline{\partial}\Phi(z)_N &= \overline{\partial}[i_\varphi \partial\Omega(z)]_N \\ &= \Big[\frac{1}{2}[\varphi,\varphi]\lrcorner\partial\Omega(z) - \varphi\lrcorner\partial\overline{\partial}\Omega(z)\Big]_N \\ &= [\varphi\lrcorner\partial(\varphi\lrcorner\partial\Omega(z)) - \varphi\lrcorner\partial(\varphi\lrcorner\partial\Omega(z))]_N \\ &= 0,\end{aligned}$$

where in the third equality we used the Tian-Todorov lemma.

Meanwhile, we have

$$\Big[\Big(\frac{1}{2}\nabla'\circ i_\varphi + \operatorname{div}(\varphi)\Big)i_\varphi\Omega\Big]_{N+1} = [\varphi\lrcorner(\nabla'\circ i_\varphi\Omega + \operatorname{div}(\varphi)\wedge\Omega) - i_{\overline{\partial}\varphi}\Omega]_{N+1},$$

and then

$$\begin{aligned}\overline{\partial}\Big[\Big(\frac{1}{2}\nabla'\circ i_\varphi + \operatorname{div}(\varphi)\Big)i_\varphi\Omega\Big]_{N+1} &= [\overline{\partial}\varphi\lrcorner(\nabla'\circ i_\varphi\Omega + \operatorname{div}(\varphi)\wedge\Omega) + i_{\overline{\partial}\varphi}\overline{\partial}\Omega]_{N+1} \\ &= [\overline{\partial}\varphi\lrcorner(\nabla'\circ i_\varphi\Omega + \operatorname{div}(\varphi)\wedge\Omega + \overline{\partial}\Omega)]_{N+1} \\ &= 0,\end{aligned}$$

where in the first equality we used the Tian-Todorov lemma and in the third equality we used the assumption that the equations in (2.5) hold in lower degrees and the fact that the initial value φ_1 is harmonic so that $\overline{\partial}\varphi_1 = 0$.

Although Proposition 2.2 enables us to solve the equations (2.3) by induction and then solve the Maurer-Cartan equation

$$\overline{\partial}\varphi = \frac{1}{2}[\varphi,\varphi],$$

there is a straightforward way to deal with the problem. Indeed, as we pointed out in the proof of Proposition 2.2, the second equation in (2.3) is equivalent to

$$\overline{\partial}\Omega(z) = \Phi(z) = i_\varphi\partial\Omega(z),$$

which has a trivial solution. Then the original equation also has a trivial solution

$$\Omega(t) = \Omega_0 = \mathrm{d}z \otimes \frac{\partial}{\partial z},$$

where $\mathrm{d}z$ and $\frac{\partial}{\partial z}$ are defined in Lemma 2.2.

Then it suffices to solve the equation

$$\Big(\overline{\partial} + \frac{1}{2}\nabla'\circ i_\varphi + \operatorname{div}(\varphi)\Big)i_\varphi\Omega_0 = 0.$$

By direct calculations, we have

$$\nabla'(\varphi\lrcorner\Omega_0) = (-1)^i\nabla'\Big(\varphi^i_{\overline{j}}\mathrm{d}\overline{z}^j \otimes \mathrm{d}z^1\wedge\cdots\wedge\widehat{\mathrm{d}z^i}\wedge\cdots\wedge\mathrm{d}z^n\otimes\frac{\partial}{\partial z}\Big)$$

$$= -\nabla_i\varphi^i_{\bar{j}}\mathrm{d}\overline{z}^j \wedge \mathrm{d}z \otimes \frac{\partial}{\partial z}$$
$$= -\mathrm{div}(\varphi) \wedge \Omega_0. \tag{2.7}$$

Then by (2.2) we have

$$\mathrm{div}(\varphi) \wedge i_\varphi\Omega_0 = i_\varphi(\mathrm{div}(\varphi) \wedge \Omega_0) = -i_\varphi \circ \nabla'(i_\varphi\Omega_0).$$

In conclusion, the equations that we need to solve can be reduced to the following equations

$$\begin{cases} \Big(\overline{\partial} + \dfrac{1}{2}\nabla' \circ i_\varphi\Big) i_\varphi\Omega_0 = 0, \\ \nabla'(i_\varphi\Omega_0) = 0. \end{cases} \tag{2.8}$$

3 Solving the Equations

In this section, we solve the equations (2.8) on a compact Kähler manifold (X,ω) when the canonical bundle K_X is ample or a torsion line bundle separately.

First, we state a technical lemma about the divergence of the Beltrami differential $\mathrm{div}(\varphi)$ which is known to experts in this area (see [22–23, 28]). For the readers' convenience, we present the proof here.

Lemma 3.1 (see [22, 28]) *Let (X,ω) be a compact Kähler manifold. Let $\varphi \in \mathrm{A}^{0,1}(X, T^{1,0}X)$ and Δ'' be the Laplacian operator of $\overline{\partial}$. Then we have*

(1) *If $\overline{\partial}^*\varphi = 0$, then $\overline{\partial}^*(\varphi \lrcorner \omega) = \sqrt{-1}\mathrm{div}(\varphi)$.*

(2) *If $\overline{\partial}(\varphi \lrcorner \omega) = 0$ and $\overline{\partial}^*\varphi = 0$, then*

$$\Delta''(\varphi \lrcorner \omega) = \sqrt{-1}\mathrm{div}(\overline{\partial}\varphi) + \varphi \lrcorner \mathrm{Ric}(\omega).$$

Proof Locally we write $\varphi = \varphi^i_{\bar{j}}\mathrm{d}\overline{z}^j \otimes \frac{\partial}{\partial z^i}$ and $\omega = \sqrt{-1}g_{k\bar{l}}\mathrm{d}z^k \wedge \mathrm{d}\overline{z}^l$. The lemma can be proved by direct calculations.

(1) For the first term, we have

$$\begin{aligned}\overline{\partial}^*(\varphi \lrcorner \omega) &= \sqrt{-1}\partial_l[(\varphi^m_{\bar{p}}g_{m\bar{j}} - \varphi^m_{\bar{j}}g_{m\bar{p}})g^{k\bar{p}}]g^{l\bar{j}}g_{k\bar{i}}\mathrm{d}\overline{z}^i \\ &= \sqrt{-1}[\partial_l(\varphi^m_{\bar{i}}g_{m\bar{j}} - \varphi^m_{\bar{j}}g_{m\bar{i}})g^{l\bar{j}} + (\varphi^m_{\bar{p}}g_{m\bar{j}} - \varphi^m_{\bar{j}}g_{m\bar{p}})\partial_l g^{k\bar{p}}g^{l\bar{j}}g_{k\bar{i}}]\mathrm{d}\overline{z}^i \\ &= \sqrt{-1}[\partial_l(\varphi^m_{\bar{i}}g_{m\bar{j}})g^{l\bar{j}} - \partial_l(\varphi^m_{\bar{j}}g_{m\bar{i}})g^{l\bar{j}} + \varphi^m_{\bar{p}}\partial_m g^{k\bar{p}}g_{k\bar{i}} + \varphi^m_{\bar{j}}\partial_l g_{m\bar{p}}g^{k\bar{p}}g^{l\bar{j}}g_{k\bar{i}}]\mathrm{d}\overline{z}^i \\ &= \sqrt{-1}\mathrm{div}(\varphi),\end{aligned}$$

where in the last equality we used the condition $\overline{\partial}^*\varphi = 0$, i.e., $\partial_k(\varphi^i_{\bar{l}}g_{i\bar{j}})g^{k\bar{l}} = 0$.

(2) For the second term, we have

$$\begin{aligned}\Delta''(\varphi \lrcorner \omega) &= (\overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial})(\varphi \lrcorner \omega) \\ &= \overline{\partial}\,\overline{\partial}^*(\varphi \lrcorner \omega) \\ &= \overline{\partial}(\sqrt{-1}\mathrm{div}(\varphi)) \\ &= \sqrt{-1}\overline{\partial}[(\partial_i\varphi^i_{\bar{j}} + \varphi^i_{\bar{j}}\partial_i \log g)\mathrm{d}z^i] \\ &= \sqrt{-1}[\overline{\partial}_k(\partial_i\varphi^i_{\bar{j}}) + (\overline{\partial}\varphi^i_{j})\partial_i \log g + \varphi^i_{\bar{j}}\overline{\partial}_k\partial_i \log g]\mathrm{d}\overline{z}^k \wedge \mathrm{d}\overline{z}^j \\ &= \sqrt{-1}\mathrm{div}(\overline{\partial}\varphi) + \varphi \lrcorner \mathrm{Ric}(\omega).\end{aligned}$$

3.1 When K_X is ample

Let X be a compact Kähler manifold with an ample canonical bundle K_X. Since K_X is ample, there is a Hermitian metric h on K_X such that its curvature form gives rise to a Kähler metric

$$\omega = \sqrt{-1}\,\overline{\partial}\partial \log h$$

on X. For any harmonic initial value $\varphi_1 \in \mathbb{H}^{0,1}(X, T^{1,0}X)$, we try to construct a power series

$$\varphi(t) = \varphi_1 t + \varphi_2 t^2 + \cdots \in \mathrm{A}^{0,1}(X, T^{1,0}X)$$

satisfying the Maurer-Cartan equation

$$\overline{\partial}\varphi(t) = \frac{1}{2}[\varphi(t), \varphi(t)].$$

As we did in the last section, we denote

$$\Omega_0 := \mathrm{d}z \otimes \frac{\partial}{\partial z} \in \mathrm{A}^{n,0}(X, K_X^{-1}),$$

which gives rise to an isomorphism between $\mathrm{A}^{0,q}(X, T^{1,0}X)$ and $\mathrm{A}^{n-1,q}(X, K_X^{-1})$ through contraction $\bullet\lrcorner\Omega_0$. The inverse is denoted by $\Omega_0^*\lrcorner\bullet$. Clearly, for any elements α, $\beta \in \mathrm{A}^{0,q}(X, T^{1,0}X)$, we have the following equalities

$$\overline{\partial}(\alpha\lrcorner\Omega_0) = (\overline{\partial}\alpha)\lrcorner\Omega_0, \quad \overline{\partial}^*(\alpha\lrcorner\Omega_0) = (\overline{\partial}^*\alpha)\lrcorner\Omega_0, \quad \langle \alpha\lrcorner\Omega_0, \beta\lrcorner\Omega_0\rangle = \langle \alpha, \beta\rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on the space of (bundle-valued) differential forms. Then the operation $\bullet\lrcorner\Omega_0$ preserves the inner product and the Hodge decomposition

$$\mathbb{I} = \mathbb{H} + \Delta''\mathbb{G},$$

where $\mathbb{H}$ is the orthogonal projection of a (bundle valued) differential form to its harmonic part, Δ'' is the Laplacian operator of $\overline{\partial}$ and $\mathbb{G}$ is the Green operator of Δ''.

In other words, we have an isomorphism between two spaces of harmonic forms

$$\bullet\lrcorner\Omega_0 : \mathbb{H}^{0,q}(X, T^{1,0}X) \to \mathbb{H}^{n-1,q}(X, K_X^{-1}).$$

The following lemma wonderfully reflects the spirit of the iteration method and is of significant importance in the proof of the main theorem.

Lemma 3.2 *Assume that for $\varphi_\nu \in \mathrm{A}^{0,1}(X, T^{1,0}X)$, $\nu = 2, \cdots, K$,*

$$\overline{\partial}\varphi_\nu = \frac{1}{2}\sum_{\alpha+\beta=\nu}[\varphi_\alpha, \varphi_\beta], \quad \overline{\partial}\varphi_1 = 0.$$

Then one has

$$\overline{\partial}\Big(\sum_{\nu+\gamma=K+1}[\varphi_\nu, \varphi_\gamma]\Big) = 0.$$

The readers who are interested in the proof can refer to [14, Lemma 4.2].

Now we are ready to solve the reduced equations (2.8) when K_X is ample with an extra condition which is an essential obstruction in this case.

Theorem 3.1 *Let X be a compact complex manifold with an ample canonical bundle. If $\mathbb{H}(\nabla' \circ i_\varphi \circ i_\varphi \Omega_0) = 0$ for any $\varphi_1 \in \mathbb{H}^{0,1}(X, T^{1,0}X)$, where φ is defined by* (1.4)*, then there exists a power series solving* (2.8)*. Therefore, X has unobstructed deformations.*

Proof As we are going to solve the equations (2.8) upward from φ_1 with respect to the degree of the formal variant t, the condition $\mathbb{H}(\nabla' \circ i_\varphi \circ i_\varphi \Omega_0) = 0$ means

$$\mathbb{H}(\nabla' \circ i_{\varphi_i} \circ i_{\varphi_j} \Omega_0) = 0$$

for any positive integers i and j, where φ_i is what we get at the i-th step of the iteration process as the coefficient of t^i.

For any $\alpha \in \mathrm{A}^{p,q}(X, K_X^{-1})$, the Bochner-Kodaira identity states that

$$\begin{aligned} \Delta'' &= \Delta' + [\sqrt{-1} R^{K_X^{-1}}, \Lambda] \\ &= \Delta' - [L, \Lambda] \\ &= \Delta' - (p + q - n)\mathrm{Id}, \end{aligned} \tag{3.1}$$

where Δ' is the Laplacian operator of ∇'.

Since $\varphi_1 \in \mathbb{H}^{0,1}(X, T^{1,0}X)$, so $\varphi_1 \lrcorner \Omega_0 \in \mathbb{H}^{n-1,1}(X, K_X^{-1})$. By (3.1), we have

$$\varphi_1 \lrcorner \Omega_0 \in \operatorname{Ker} \Delta',$$

i.e., $\nabla'(\varphi_1 \lrcorner \Omega_0) = 0$ and $\nabla'^*(\varphi_1 \lrcorner \Omega_0) = 0$. Then by the Tian-Todorov lemma, we have

$$\begin{aligned} \frac{1}{2}\nabla'(\varphi_1 \lrcorner \varphi_1 \lrcorner \Omega_0) &= -\frac{1}{2}[\varphi_1, \varphi_1] \lrcorner \Omega_0 + \varphi_1 \lrcorner \nabla'(\varphi_1 \lrcorner \Omega_0) \\ &= -\frac{1}{2}[\varphi_1, \varphi_1] \lrcorner \Omega_0. \end{aligned}$$

Thus

$$\begin{aligned} \overline{\partial}\Big(\frac{1}{2}\nabla'(\varphi_1 \lrcorner \varphi_1 \lrcorner \Omega_0)\Big) &= \overline{\partial}\Big(-\frac{1}{2}[\varphi_1, \varphi_1] \lrcorner \Omega_0\Big) \\ &= -[\overline{\partial}\varphi_1, \varphi_1] \lrcorner \Omega_0 \\ &= 0. \end{aligned} \tag{3.2}$$

According to the Hodge theorem [6, p. 84], the condition $\mathbb{H}(\nabla' \circ i_{\varphi_1} \circ i_{\varphi_1} \Omega_0) = 0$ implies that we can take the solution φ_2 as

$$\begin{aligned} i_{\varphi_2}\Omega_0 &= \overline{\partial}^* \mathbb{G}\Big(-\frac{1}{2}\nabla' \circ i_{\varphi_1} \circ i_{\varphi_1} \Omega_0\Big) \\ &= \mathbb{G}\overline{\partial}^* \Big(-\frac{1}{2}\nabla' \circ i_{\varphi_1} \circ i_{\varphi_1} \Omega_0\Big). \end{aligned} \tag{3.3}$$

This is the solution of the first equation in (2.8) at the second step.

Note that $\overline{\partial}^*\big(-\frac{1}{2}\nabla' \circ i_{\varphi_1} \circ i_{\varphi_1}\Omega_0\big) \in \mathrm{A}^{n-1,1}(X, K_X^{-1})$, then the Bochner-Kodaira identity (3.1) implies that when acting on it, the two Laplacian operators Δ' and Δ'' coincide, so do the two Green operators, i.e., $\mathbb{G} = \mathbb{G}'$.

As a consequence, we have

$$\nabla'(i_{\varphi_2}\Omega_0) = \nabla' \mathbb{G}\overline{\partial}^* \Big(-\frac{1}{2}\nabla' \circ i_{\varphi_1} \circ i_{\varphi_1} \Omega_0\Big)$$

$$
\begin{aligned}
&= \nabla'\mathbb{G}'\overline{\partial}^*\Big(-\frac{1}{2}\nabla'\circ i_{\varphi_1}\circ i_{\varphi_1}\Omega_0\Big)\\
&= \mathbb{G}'\nabla'\overline{\partial}^*\Big(-\frac{1}{2}\nabla'\circ i_{\varphi_1}\circ i_{\varphi_1}\Omega_0\Big)\\
&= \mathbb{G}'\overline{\partial}^*\nabla'\Big(\frac{1}{2}\nabla'\circ i_{\varphi_1}\circ i_{\varphi_1}\Omega_0\Big)\\
&= 0, \qquad (3.4)
\end{aligned}
$$

where in the fourth equality we used the fact that $[\overline{\partial}^*, \nabla'] = 0$. This is the second equation of (2.8) at the second step.

By running induction, we assume that we have obtained the solutions up to the N-th step, i.e., we have already constructed φ_k, $1 \le k \le N$. The proof will be accomplished as soon as we construct the solution φ_{N+1}.

As the $\varphi_k' s$ are assumed to be constructed ($k \le N$), by the Tian-Todorov lemma again we have

$$
\begin{aligned}
\frac{1}{2}\nabla'(\varphi_i\lrcorner\varphi_j\lrcorner\Omega_0) &= -\frac{1}{2}[\varphi_i, \varphi_j]\lrcorner\Omega_0 + \varphi_i\lrcorner\nabla'(\varphi_j\lrcorner\Omega_0)\\
&= -\frac{1}{2}[\varphi_i, \varphi_j]\lrcorner\Omega_0 \qquad (3.5)
\end{aligned}
$$

for any positive integers i, j such that $i + j = N + 1$.

Then combining Lemma 3.2 with the calculations above, one has

$$
\overline{\partial}\Big(\sum_{i+j=N+1}\frac{1}{2}\nabla'(\varphi_i\lrcorner\varphi_j\lrcorner\Omega_0)\Big) = 0. \qquad (3.6)
$$

Since $\mathbb{H}(\nabla'\circ i_{\varphi_i}\circ i_{\varphi_j}\Omega_0) = 0$, we can take φ_{N+1} as

$$
i_{\varphi_{N+1}}\Omega_0 = \overline{\partial}^*\mathbb{G}\Big(-\frac{1}{2}\sum_{i+j=N+1}\nabla'\circ i_{\varphi_i}\circ i_{\varphi_j}\Omega_0\Big). \qquad (3.7)
$$

Then, similar to (3.4), it holds that

$$
\nabla'(i_{\varphi_{N+1}}\Omega_0) = 0. \qquad (3.8)
$$

Remark that in the view point of iteration one has

$$
i_{\varphi^{N+1}}\Omega_0 = \overline{\partial}^*\mathbb{G}\Big(-\frac{1}{2}\sum_{2\le K\le N+1}\sum_{i+j=K}\nabla'\circ i_{\varphi_i}\circ i_{\varphi_j}\Omega_0 t^i t^j\Big) + i_{\varphi_1}\Omega_0 t^1, \qquad (3.9)
$$

where $\varphi^N = \varphi_1 t^1 + \cdots + \varphi_N t^N$ can be treated as the truncation of $\varphi(t)$ at the N-th step.

Therefore, we eventually obtain a solution given by

$$
\begin{cases}
i_\varphi\Omega_0 = \overline{\partial}^*\mathbb{G}\Big(-\frac{1}{2}\nabla'\circ i_\varphi\circ i_\varphi\Omega_0\Big) + i_{\varphi_1}\Omega_0,\\
\dfrac{\partial\varphi}{\partial t}\Big|_{t=0} = \varphi_1,\\
\varphi(0) = 0,
\end{cases} \qquad (3.10)
$$

which is uniquely determined by the chosen harmonic initial value φ_1.

Remark 3.1 Due to (3.3) and (3.7), we have $\varphi_k \in \mathrm{Im}\overline{\partial}^*$ for $k \geq 2$.

Example 3.1 It is clear that on a compact Riemann surface, the condition $\mathbb{H}(\nabla' \circ i_\varphi \circ i_\varphi \Omega_0) = 0$ holds due to the dimension.

Let $X = X_1 \times X_2$, where each X_i is a compact Riemann surface of genus $g_i \geq 2$, $i = 1, 2$. Then K_X is clearly ample.

We take a local coordinate $\{z^1, z^2\}$ on X such that each z^i is the local coordinate of X_i, $i = 1, 2$. Then we have

$$\Omega = \Omega_1 \wedge \Omega_2 \in \mathrm{A}^{2,0}(X, K_X^{-1}),$$

where $\Omega_i = \mathrm{d}z^i \otimes \frac{\partial}{\partial z^i} \in \mathrm{A}^{1,0}(X, K_{X_i}^{-1})$, $i = 1, 2$.

From the last example, we know that there is a Beltrami differential $\widetilde{\varphi}_i \in \mathrm{A}^{0,1}(X_i, T^{1,0}X_i)$ on each X_i determining the unobstructed deformations of X_i, $i = 1, 2$. Under the local coordinate we can write them as $\widetilde{\varphi}_i = \widetilde{\varphi}^i_{\bar{i}} \mathrm{d}\overline{z}^i \otimes \frac{\partial}{\partial z^i}$ where $\widetilde{\varphi}^i_{\bar{i}}$ is a smooth function only in z^i $(i = 1, 2)$. Then we have $i_{\widetilde{\varphi_1}} \Omega_2 = i_{\widetilde{\varphi_2}} \Omega_1 = 0$.

Repeating the calculations in Section 2 we have

$$\begin{cases} \Big(\overline{\partial} + \dfrac{1}{2}\nabla' \circ i_{\widetilde{\varphi}_i}\Big) i_{\widetilde{\varphi}_i} \Omega_i = 0, \\ \nabla'(i_{\widetilde{\varphi}_i} \Omega_i) = 0 \end{cases} \tag{3.11}$$

on each X_i, $i = 1, 2$.

We take $\varphi = \widetilde{\varphi}_1 + \widetilde{\varphi}_2$ which lies in $\mathrm{A}^{0,1}(X, T^{1,0}X)$ and by elementary calculations, we have

$$\begin{aligned} i_\varphi \circ i_\varphi \Omega &= (i_{\widetilde{\varphi}_1} + i_{\widetilde{\varphi}_2}) \circ (i_{\widetilde{\varphi}_1} + i_{\widetilde{\varphi}_2}) \Omega \\ &= (i_{\widetilde{\varphi}_1} + i_{\widetilde{\varphi}_2})[(i_{\widetilde{\varphi}_1} \Omega_1) \wedge \Omega_2 + \Omega_1 \wedge (i_{\widetilde{\varphi}_2} \Omega_2)] \\ &= (i_{\widetilde{\varphi}_1} \circ i_{\widetilde{\varphi}_1} \Omega_1) \wedge \Omega_2 + 2(i_{\widetilde{\varphi}_1} \Omega_1) \wedge (i_{\widetilde{\varphi}_2} \Omega_2) \\ &\quad + \Omega_1 \wedge (i_{\widetilde{\varphi}_2} \circ i_{\widetilde{\varphi}_2} \Omega_2). \end{aligned} \tag{3.12}$$

Then

$$\begin{aligned} \nabla'(i_\varphi \circ i_\varphi \Omega) &= \nabla'_1(i_{\widetilde{\varphi}_1} \circ i_{\widetilde{\varphi}_1} \Omega_1) \wedge \Omega_2 + 2\nabla'_1(i_{\widetilde{\varphi}_1} \Omega_1) \wedge (i_{\widetilde{\varphi}_2} \Omega_2) \\ &\quad + 2(i_{\widetilde{\varphi}_1} \Omega_1) \wedge \nabla'_2(i_{\widetilde{\varphi}_2} \wedge \Omega_2) + \Omega_1 \wedge \nabla'_1(i_{\widetilde{\varphi}_2} \circ i_{\widetilde{\varphi}_2} \Omega_2) \\ &= \nabla'_1(i_{\widetilde{\varphi}_1} \circ i_{\widetilde{\varphi}_1} \Omega_1) \wedge \Omega_2 + \Omega_1 \wedge \nabla'_1(i_{\widetilde{\varphi}_2} \circ i_{\widetilde{\varphi}_2} \Omega_2) \\ &= -2\overline{\partial}(i_{\widetilde{\varphi}_1} \Omega_1) \wedge \Omega_2 - 2\Omega_1 \wedge 2\overline{\partial}(i_{\widetilde{\varphi}_2} \Omega_2) \\ &= -2\overline{\partial}[(i_{\widetilde{\varphi}_1} \Omega_1) \wedge \Omega_2 + \Omega_1 \wedge (i_{\widetilde{\varphi}_2} \Omega_2)] \\ &\in \mathrm{Im}(\overline{\partial}), \end{aligned} \tag{3.13}$$

which implies that $\mathbb{H}(\nabla' \circ i_\varphi \circ i_\varphi \Omega_0) = 0$. Then $X = X_1 \times X_2$ has unobstructed deformations. Throughout the calculations above, the notion ∇'_i denotes the covariant derivative in z^i, $i = 1, 2$.

By the same arguments, one easily knows that the manifolds of the form $X = X_1 \times \cdots \times X_m$ also have ample canonical bundles and satisfy the condition $\mathbb{H}(\nabla' \circ i_\varphi \circ i_\varphi \Omega_0) = 0$, where each X_i is a compact Riemann surface with genus $g_i \geq 2$ $(i = 1, \cdots, m)$. Therefore they have unobstructed deformations.

Remark 3.2 The condition $\mathbb{H}(\nabla' \circ i_\varphi \circ i_\varphi \Omega_0) = 0$ is essential in the proof of our main theorem. It may look a little complicated at first, but it can be improved into a somewhat more geometric form.

First, we claim that

$$\overline{\partial}^*(i_\varphi \Omega_0) = (\overline{\partial}^* \varphi) \lrcorner \Omega_0. \tag{3.14}$$

Indeed, for any $\alpha \in \mathrm{A}^{n-1,0}(X, K_X^{-1})$, one has

$$\begin{aligned}
\langle \overline{\partial}^*(i_\varphi \Omega_0), \alpha \rangle &= \langle \overline{\partial}^*(i_\varphi \Omega_0), V \lrcorner \Omega_0 \rangle \\
&= \langle i_\varphi \Omega_0, \overline{\partial}(V \lrcorner \Omega_0) \rangle \\
&= \langle i_\varphi \Omega_0, \overline{\partial} V \lrcorner \Omega_0 \rangle \\
&= \langle \varphi, \overline{\partial} V \rangle = \langle \overline{\partial}^* \varphi, V \rangle \\
&= \langle \overline{\partial}^* \varphi \lrcorner \Omega_0, V \lrcorner \Omega_0 \rangle \\
&= \langle \overline{\partial}^* \varphi \lrcorner \Omega_0, \alpha \rangle,
\end{aligned}$$

which implies the claim. Here V is some vector field of $(1,0)$-type.

Proposition 3.1 *If K_X is ample, and X satisfies $\mathbb{H}^{n-1,2}(X, K_X^{-1}) \subset \mathrm{Ker}(\nabla'^*)$, then X has unobstructed deformations.*

Proof If $\mathbb{H}^{n-1,2}(X, K_X^{-1}) \subset \mathrm{Ker}(\nabla'^*)$, for any harmonic element $\gamma \in \mathbb{H}^{n-1,2}(X, K_X^{-1})$, we have

$$\langle \nabla' \circ i_\varphi \circ i_\varphi \Omega_0, \gamma \rangle = \langle i_\varphi \circ i_\varphi \Omega_0, \nabla'^* \gamma \rangle = 0,$$

which implies $\mathbb{H}(\nabla' \circ i_\varphi \circ i_\varphi \Omega_0) = 0$. By Theorem 3.1, the deformations are unobstructed.

Recall that the contraction $\bullet \lrcorner \Omega_0$ and its inverse $\Omega_0^* \lrcorner \bullet$ give rise to an isomorphism between harmonic spaces

$$\mathbb{H}^{n-1,2}(X, K_X^{-1}) \cong \mathbb{H}^{0,2}(X, T^{1,0}X). \tag{3.15}$$

Locally the operator Λ can be written as

$$\Lambda = -\sqrt{-1} g^{k\bar{l}} i_{\frac{\partial}{\partial \bar{z}^l}} \wedge i_{\frac{\partial}{\partial z^k}}. \tag{3.16}$$

Then we have

$$\Lambda : \mathrm{A}^{0,2}(X, T^{1,0}X) \to \mathrm{A}^{0,1}(X, \wedge^2 T^{1,0}X). \tag{3.17}$$

Thus the condition $\mathbb{H}^{n-1,2}(X, K_X^{-1}) \subset \mathrm{Ker}(\nabla'^*)$ is equivalent to

$$\mathbb{H}^{0,2}(X, T^{1,0}X) \subset \mathrm{Ker}(\overline{\partial}\Lambda). \tag{3.18}$$

The characterization (3.18) seems make more sense in geometry than the original one since the harmonic space $\mathbb{H}^{0,2}(X, T^{1,0}X)$ is isomorphic to the cohomology group $\mathrm{H}^2(X, T^{1,0}X)$, which contains the obstructions of the deformations (see [11, 16]).

Remark 3.3 Note that a projective variety X is said to satisfy the Bott vanishing theorem, if $\mathrm{H}^i(X,\Omega^j(L))=0$ for all the ample line bundles over X, where $i>0, j\geq 0$. Bott showed that it holds for projective spaces. A good reference about it is [10, Chapter 3.4]. Later this theorem was generalized to the toric case (the proof can be found in [1, 3, 5, 18]) and some certain Del Pezzo surfaces and $K3$ surfaces (see [26]). But they are all beyond our consideration. We remark that any smooth variety with ample canonical bundle has unobstructed deformations, once it satisfies the Bott vanishing theorem.

Remark 3.4 If X is a nonsingular irreducible hypersurface of $\mathbb{CP}^3$ of degree d. According to [11, (6.49)], we have the fact that $\dim \mathrm{H}^{n-1,2}(X,K_X^{-1})=\frac{1}{2}(d-2)(d-3)(d-5)$. When $d=5$, by the adjunction formula, we see that $K_X\cong\mathcal{O}_X(1)$, which is ample. In this case, the cohomology group containing the obstruction $\mathbb{H}(\nabla'\circ i_\varphi\circ i_\varphi\Omega_0)$ vanishes. So we see that the quintic surface in $\mathbb{CP}^3$ has unobstructed deformations.

3.2 When K_X is a torsion line bundle

In this subsection, we show that our method also works when the compact Kähler manifold X has a torsion canonical bundle K_X, i.e., there is an integer m such that $K_X^{\otimes m}\cong\mathcal{O}_X$, the trivial line bundle over X.

Corollary 3.1 *If $c_1(X)=0$, i.e., K_X is a torsion line bundle, then X has unobstructed deformations.*

Proof According to Yau's celebrated work [27], there exists a Kähler metric ω on X such that $\mathrm{Ric}(\omega)=0$. Similar to the ample case, we start with an arbitrary harmonic initial value $\varphi_1\in\mathbb{H}^{0,1}(X,T^{1,0}X)$ and try to construct a power series

$$\varphi(t)=\varphi_1 t+\varphi_2 t^2+\cdots\in \mathrm{A}^{0,1}(X,T^{1,0}X),$$

which satisfies the Maurer-Cartan equation

$$\overline{\partial}\varphi(t)=\frac{1}{2}[\varphi(t),\varphi(t)].$$

By the arguments in Section 2, it suffices for us to solve the following equations

$$\begin{cases}\Big(\overline{\partial}+\frac{1}{2}\nabla'\circ i_\varphi\Big)i_\varphi\Omega_0=0,\\ \nabla'(i_\varphi\Omega_0)=0,\end{cases}$$

where $\Omega_0=\mathrm{d}z\otimes\frac{\partial}{\partial z}\in\mathrm{A}^{n,0}(X,K_X^{-1})$. By Lemma 3.1 we have

$$\Delta''(\varphi_1\lrcorner\omega)=0,\quad \overline{\partial}^*(\varphi_1\lrcorner\omega)=\sqrt{-1}\mathrm{div}(\varphi_1)$$

since $\overline{\partial}\varphi_1=\overline{\partial}^*\varphi_1=0$ and $\mathrm{Ric}(\omega)=0$. Then we have $\mathrm{div}(\varphi_1)=0$ and it follows by (2.7) that $\nabla'(i_{\varphi_1}\Omega_0)=0$. Thus we have the solution at the second step

$$i_{\varphi_2}\Omega_0=\overline{\partial}^*\mathbb{G}\Big(-\frac{1}{2}\nabla'\circ i_{\varphi_1}\circ i_{\varphi_1}\Omega_0\Big).$$

Under the Ricci-flat setting, the Bochner-Kodaira identity states that

$$\Delta''=\Delta'+[\sqrt{-1}R^{K_X^{-1}},\Lambda]=\Delta'$$

for any K_X^{-1}- valued differential forms. Since the two Laplacian operators coincide, it follows that $\mathbb{G}\nabla' = \nabla'\mathbb{G}$, which, together with the fact that $\nabla'\overline{\partial}^* = -\overline{\partial}^*\nabla'$, implies that

$$i_{\varphi_2}\Omega_0 = -\nabla'\overline{\partial}^*\mathbb{G}\Big(\frac{1}{2}i_{\varphi_1} \circ i_{\varphi_1}\Omega_0\Big) \in \mathrm{Im}(\nabla').$$

Thus $\nabla'(i_{\varphi_2}\Omega_0) = 0$. By running induction, we assume that the solutions φ_k satisfying $\nabla'(i_{\varphi_k}\Omega_0) = 0$ have already been constructed for $k \leq N-1$. By the same operation in the last subsection, we obtain the solution φ_N given by

$$i_{\varphi_N}\Omega_0 = \overline{\partial}^*\mathbb{G}\Big(-\frac{1}{2}\sum_{i+j=N}\nabla' \circ i_{\varphi_i} \circ i_{\varphi_j}\Omega_0\Big) \tag{3.19}$$

such that $i_{\varphi_N}\Omega_0 \in \mathrm{Im}(\nabla')$. Hence the proof is completed.

Remark 3.5 For the convergence and the regularity of the solution $\varphi(t)$ in both the K_X ample case and the K_X torsion case, there are many works concerning this, for example, [11, 16] and more recently, [14, Theorem 4.3, Theorem 4.4] or [13, Proposition 4.10], etc. By repeating the calculations therein, one can obtain the convergence and the regularity of $\varphi(t)$ by standard analytic theory.

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