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Exact Boundary Synchronization by Groups for a Kind of System of Wave Equations Coupled with Velocities*

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Abstract This paper deals with the exact boundary controllability and the exact boundary synchronization for a 1-D system of wave equations coupled with velocities. These problems can not be solved directly by the usual HUM method for wave equations, however, by transforming the system into a first order hyperbolic system, the HUM method for 1-D first order hyperbolic systems, established by Li-Lu (2022) and Lu-Li (2022), can be applied to get the corresponding results.

Keywords Exact boundary controllability, Exact boundary synchronization, Coupled system of wave equations, HUM method
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1 Introduction

The synchronization is a widespread natural phenomenon (see [5, 23]) which has been intensively studied in PDEs case in recent years (see [16] and the references therein, and [1–3]). The study of synchronization for the coupled system of wave equations

$$U_{tt} - \Delta U + AU = 0 \tag{1.1}$$

in a bounded smooth domain with various boundary conditions, in which $U = U(t, x) = (u^{(1)}, \cdots, u^{(n)})^{\mathrm{T}}$ is the state variable, Δ is the Laplacian operator, and A is a coupling matrix with constant elements, has been carried out in [4, 7–19, 22], etc.

However, for the system of wave equations coupled with velocities

$$U_{tt} - \Delta U + AU_t = 0, \tag{1.2}$$

the situation is quite different: Its exact boundary controllability can not be obtained by usual HUM method. In fact, since (1.2) does not possess the property of energy conservation, one can not establish the corresponding observability inequalities for the adjoint system by the energy estimate and the multiplier method directly from (1.2). On the other hand, if we regard AU_t as a perturbation term, unlike AU, which is not a compact one, the compact perturbation

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method given in [16] does not work. Thus, the exact boundary controllability and then the exact boundary synchronization for system (1.2) is still an open problem up to now.

In this paper, we will specially consider the corresponding problem in the 1-D case, namely, we will consider the following 1-D system of wave equations coupled with velocities

$$U_{tt} - U_{xx} + AU_t = 0, \quad t \in (0, +\infty),\ x \in (0, L), \tag{1.3}$$

where $U = U(t, x) = (u^{(1)}, \cdots, u^{(n)})^{\mathrm{T}}$ is the state variable, and A is a coupling matrix of order n.

We give the following Dirichlet boundary condition on $x = 0$:

$$x = 0:\ U = 0, \quad t \in (0, +\infty). \tag{1.4}$$

While, on $x = L$ we take any one of boundary conditions of Dirichlet type, Neumann type and coupled dissipative type:

$$x = L:\ U = DH(t), \quad t \in (0, +\infty), \tag{1.5a}$$

$$x = L:\ U_x = DH(t), \quad t \in (0, +\infty), \tag{1.5b}$$

$$x = L:\ U_x + BU_t = DH(t), \quad t \in (0, +\infty), \tag{1.5c}$$

where the boundary control matrix D is an $n \times M (M \le n)$ full column-rank matrix, and B is a boundary coupling matrix of order n. All A, B and D have real constant elements, and $H = (h^{(1)}, \cdots, h^{(M)})^{\mathrm{T}}$ denotes the boundary control.

The initial data is given by

$$t = 0:\ U = \widehat{U}_0, \quad U_t = \widehat{U}_1, \quad x \in (0, L). \tag{1.6}$$

The basic idea is to transform system (1.3)–(1.5) into a first order hyperbolic system, then the characteristic method can be applied to establish the corresponding observability inequalities for the corresponding adjoint system, so that the HUM method still works (see [21]), in other words, by means of the general result given in [6], we can get the desired exact boundary synchronization by groups for system (1.3)–(1.5).

For this purpose, we first transform system (1.3)–(1.5) into a first order hyperbolic system without zero eigenvalues. Let

$$V = (V^-, V^+)^{\mathrm{T}} \quad \text{with} \quad V^- = \frac{1}{2}(U_t + U_x), \quad V^+ = \frac{1}{2}(U_t - U_x). \tag{1.7}$$

It is easy to see that V satisfies

$$V_t + \Lambda V_x + \mathbb{A} V = 0, \quad t \in (0, +\infty),\ x \in (0, L), \tag{1.8}$$

where

$$\Lambda = \begin{pmatrix} -I_n & \\ & I_n \end{pmatrix} \quad \text{and} \quad \mathbb{A} = \frac{1}{2}\begin{pmatrix} A & A \\ A & A \end{pmatrix}, \tag{1.9}$$

in which I_n is the identity matrix of order n.

By (1.4)–(1.5), on $x = 0$ we have

$$x = 0:\ V^+(t, 0) = -V^-(t, 0), \quad t \in (0, +\infty), \tag{1.10}$$

and, assuming that -1 is not an eigenvalue of B in case (1.5c), on $x = L$ we have any one of the following boundary conditions:

$$x = L: \ V^-(t,L) = -V^+(t,L) + DH'(t), \quad t \in (0,+\infty), \tag{1.11a}$$

$$x = L: \ V^-(t,L) = V^+(t,L) + DH(t), \quad t \in (0,+\infty), \tag{1.11b}$$

$$x = L: \ V^-(t,L) = G_1 V^+(t,L) + (I_n + B)^{-1} DH(t), \quad t \in (0,+\infty), \tag{1.11c}$$

where

$$G_1 = (I_n + B)^{-1}(I_n - B). \tag{1.12}$$

Moreover, by (1.6)–(1.7), the initial data is given by

$$t = 0: \ V = V_0 = \frac{1}{2}(\widehat{U}_1 + \widehat{U}_0', \widehat{U}_1 - \widehat{U}_0'). \tag{1.13}$$

In 1-D case, instead of discussing separately system (1.3)–(1.4) with different boundary conditions (1.5a), (1.5b) and (1.5c), respectively, by transforming system (1.3)–(1.5) into a first order hyperbolic system (1.8) and (1.10)–(1.11) with different parameters on the boundary conditions on $x = L$, we can use the theory of controllability and synchronization for first order hyperbolic systems obtained in [6] to uniformly get the boundary controllability and the boundary synchronization for system (1.3)–(1.5).

We first present the well-posedness of system (1.3)–(1.5) in Section 2, then the exact boundary synchronization by groups, corresponding exactly synchronizable states by groups and some necessary conditions will be studied in Sections 3–6, respectively. In Section 7 we give remarks for a more general coupled system.

2 Well-Posedness

Let

$$\mathcal{H} = \begin{cases} (H^1(0,T))^M & \text{for (1.5a) and (1.11a)}, \\ (L^2(0,T))^M & \text{for (1.5b), (1.5c) and (1.11b), (1.11c)}. \end{cases} \tag{2.1}$$

In what follows, we always assume that the following conditions of C^0 compatibility at the points $(t,x) = (0,0)$ and $(0,L)$ are satisfied, respectively:

$$\begin{cases} \widehat{U}_0(0) = 0 \quad \text{and} \quad \widehat{U}_0(L) = DH(0) & \text{in case (1.5a)}, \\ \widehat{U}_0(0) = 0 & \text{in cases (1.5b) and (1.5c)}. \end{cases} \tag{2.2}$$

Applying [20, Theorem 3.1] to the first order hyperbolic system (1.8) and (1.10)–(1.11), for any given $(\widehat{U}_0, \widehat{U}_1) \in (H^1(0,L) \times L^2(0,L))^n$ we have the following lemma.

Lemma 2.1 *For any given $T > 0$, for any given initial data $V_0 \in (L^2(0,L))^{2n}$ and any given boundary function $H \in \mathcal{H}$, satisfying the conditions of C^0 compatibility* (2.2) *at the points $(t,x) = (0,0)$ and $(0,L)$, respectively, the mixed problem* (1.8), (1.10)–(1.11) *and* (1.13) *admits a unique weak solution $V = V(t,x) \in (L^2(0,T; L^2(0,L)))^{2n}$, satisfying*

$$\|V(T,\cdot)\|_{(L^2(0,L))^{2n}} \le c(\|V_0\|_{(L^2(0,L))^{2n}} + \|H\|_{\mathcal{H}}) \tag{2.3}$$

and

$$\|V(\cdot,L)\|_{(L^2(0,T))^{2n}} \le c(\|V_0\|_{(L^2(0,L))^{2n}} + \|H\|_{\mathcal{H}}), \tag{2.4}$$

here and hereafter, c denotes a positive constant.

By Lemma 2.1, noting (1.13), we have the following theorem.

Theorem 2.1 *Assume that -1 is not an eigenvalue of B in case* (1.5c)*. For any given $T>0$, for any given initial data $(\widehat{U}_0,\widehat{U}_1)\in(H^1(0,L)\times L^2(0,L))^n$ and any given boundary function $H\in\mathcal{H}$, satisfying the conditions of C^0 compatibility* (2.2) *at the points $(t,x)=(0,0)$ and $(0,L)$, respectively, problem* (1.3)–(1.6) *admits a unique weak solution $(U,U_t)\in(L^2(0,T;H^1(0,L)\times L^2(0,L)))^n$, satisfying*

$$\|(U,U_t)\|_{(L^2(0,T;H^1(0,L)\times L^2(0,L)))^n} \le c(\|(\widehat{U}_0,\widehat{U}_1)\|_{(H^1(0,L)\times L^2(0,L))^n} + \|H\|_{\mathcal{H}}), \tag{2.5}$$
$$\|(U_x,U_t)(t,L)\|_{(L^2(0,T)\times L^2(0,T))^n} \le c(\|(\widehat{U}_0,\widehat{U}_1)\|_{(H^1(0,L)\times L^2(0,L))^n} + \|H\|_{\mathcal{H}}). \tag{2.6}$$

3 Exact Boundary Synchronization by p-Groups

We now take a look to the exact boundary synchronization by p-groups for system (1.3)–(1.5). Let $p\ge 1$ be an integer, $n_i(\ge 2, i=1,\cdots,p)$ be any given positive integers, and let $n=\sum\limits_{i=1}^{p} n_i$. Let the state variable $U=U(t,x)$ in system (1.3)–(1.5) be divided into p groups, and for $i=1,\cdots,p$, the ith group consists of n_i components, $U_i=(u_i^{(1)},u_i^{(2)},\cdots,u_i^{(n_i)})^{\mathrm{T}}$. Assume that for any given initial data $(\widehat{U}_0,\widehat{U}_1)\in(H^1(0,L)\times L^2(0,L))^n$, there exists a $T>0$ such that by boundary control $H\in\mathcal{H}$ given by (2.1), the exact synchronization is realized in each group at the time $t=T(>0)$:

$$t\ge T:\ u_i^{(1)}(t,x)\equiv u_i^{(2)}(t,x)\equiv\cdots\equiv u_i^{(n_i)}(t,x)\overset{\text{def.}}{=}\widetilde{u}_i(t,x),\quad i=1,\cdots,p, \tag{3.1}$$

where $(\widetilde{u}_1(t,x),\cdots,\widetilde{u}_p(t,x))^{\mathrm{T}}$ is a priori unknown, then we say that system (1.3)–(1.5) possesses the exact boundary synchronization by p-groups, and $(\widetilde{u}_1(t,x),\cdots,\widetilde{u}_p(t,x))^{\mathrm{T}}$ is called the exactly synchronizable state by p-groups. In the special case $p=1$, system (1.3)–(1.5) is exactly synchronizable.

Correspondingly, let C_p be the following $(n-p)\times n$ full row-rank matrix of synchronization

$$C_p=\begin{pmatrix}\widetilde{C}_1 & & & \\ & \widetilde{C}_2 & & \\ & & \ddots & \\ & & & \widetilde{C}_p\end{pmatrix}\quad\text{with}\quad \widetilde{C}_i=\begin{pmatrix}1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1\end{pmatrix} \tag{3.2}$$

an $(n_i-1)\times n_i$ full row-rank matrix for $i=1,\cdots,p$. We have

$$\mathrm{Ker}(\widetilde{C}_i)=\mathrm{Span}\{\widetilde{e}_i\}\quad\text{with}\quad \widetilde{e}_i=(\underbrace{1,\cdots,1}_{n_i})^{\mathrm{T}},\ i=1,\cdots,p \tag{3.3}$$

and

$$\mathrm{Ker}(C_p)=\mathrm{Span}\{\epsilon_1,\epsilon_2,\cdots,\epsilon_p\}\quad\text{with}\quad \epsilon_i=(\underbrace{0,\cdots,0}_{\sum_{j=1}^{i-1}n_j},\widetilde{e}_i^{\mathrm{T}},\underbrace{0,\cdots,0}_{\sum_{j=i+1}^{p}n_j})^{\mathrm{T}},\ i=1,\cdots,p. \tag{3.4}$$

If system (1.3)–(1.5) is exactly synchronizable by p-groups at the time $t = T$, then

$$t \geq T: \ C_pU = 0 \quad \text{or} \quad U = \sum_{i=1}^{p} \widetilde{u}_i \epsilon_i, \tag{3.5}$$

where $\epsilon_i (i = 1, \cdots, p)$ are given by (3.4), or, equivalently,

$$t \geq T: \ \widetilde{C}_i U_i = 0 \quad \text{or} \quad U_i = \widetilde{u}_i \widetilde{e}_i, \quad i = 1, \cdots, p, \tag{3.6}$$

where $\widetilde{e}_i (i = 1, \cdots, p)$ are given by (3.3).

Let

$$\mathbf{D} = \begin{cases} D & \text{in cases (1.11a) and (1.11b)}, \\ (I_n + B)^{-1} D & \text{in case (1.11c)} \end{cases} \tag{3.7}$$

and let $\mathcal{H}$ be given by (2.1). Applying [6, Lemma 2.7] to system (1.8) and (1.10)–(1.11), for any given $(\widehat{U}_0, \widehat{U}_1) \in (H^1(0, L) \times L^2(0, L))^n$ we have the following lemma.

Lemma 3.1 *Assume that* -1 *is not an eigenvalue of* B *in case* (1.11c). *Let* $T \geq 2L$. *If* $M = \text{rank}(\mathbf{D}) = n$, *then for any given initial data* $V_0(x) \in (L^2(0, L))^{2n}$ *given by* (1.13), *there exists a boundary control* $H(t) \in \mathcal{H}$, *satisfying*

$$\|H\|_{\mathcal{H}} \leq c \|(\widehat{U}_0, \widehat{U}_1)\|_{(H^1(0,L) \times L^2(0,L))^n}, \tag{3.8}$$

such that system (1.8) *and* (1.10)–(1.11) *is exactly null controllable.*

Remark 3.1 In case (1.11a), applying [6, Lemma 2.7] to system (1.8)–(1.10) and (1.11a), we can find $H' \in (L^2(0, T))^M$, satisfying

$$\|H'\|_{(L^2(0,T))^M} \leq c \|V_0\|_{(L^2(0,L))^{2n}} \leq c \|(\widehat{U}_0, \widehat{U}_1)\|_{(H^1(0,L) \times L^2(0,L))^n},$$

such that the system is exactly null controllable. Then, noting the condition of C^0 compatibility at the point $(0, L)$ in (2.2), (3.8) still holds.

Noting (1.4) and (1.7), it is easy to get the following lemma.

Lemma 3.2 *The exact boundary* (*null*) *controllability for system* (1.8) *and* (1.10)–(1.11) *is equivalent to that for system* (1.3)–(1.5).

By Lemmas 3.1–3.2, we immediately get the following theorem.

Theorem 3.1 *Assume that* -1 *is not an eigenvalue of* B *in case* (1.5c). *Let* $T \geq 2L$. *If* $M = \text{rank}(D) = n$, *then system* (1.3)–(1.5) *is exactly null controllable for any given initial data* $(\widehat{U}_0, \widehat{U}_1) \in (H^1(0, L) \times L^2(0, L))^n$, *and the boundary control* $H(t) \in \mathcal{H}$ *satisfies* (3.8).

Remark 3.2 By [6, Lemma 2.7], system (1.8) and (1.10)–(1.11) is in fact exactly controllable at the time $t = T$ if $M = \text{rank}(D) = n$ for $T \geq 2L$. Thus, by Lemma 3.2, system (1.3)–(1.5) is also exactly controllable under the conditions of Theorem 3.1.

However, the exact boundary null controllability and the exact boundary controllability for system (1.3)–(1.5) is not always equivalent. By applying [20, Remark 3.4] to system (1.8) and (1.10)–(1.11), we have that system (1.3)–(1.4) with boundary condition of Dirichlet type (1.5a) or Neumann type (1.5b) are time reversible; moreover, assuming that 1 is not an eigenvalue of B, then G_1 given by (1.12) is invertible, and system (1.3)–(1.4) with dissipative boundary condition (1.5c) is also time reversible. Thus by [20, Theorem 4.1], we have the following corollary.

Corollary 3.1 *For system* (1.3)–(1.4) *with* (1.5a) *or* (1.5b)*, the exact boundary null controllability and the exact boundary controllability are equivalent. Moreover, if* 1 *is not an eigenvalue of* B*, then we also have the same result for system* (1.3)–(1.4) *and* (1.5c).

Remark 3.3 The exact boundary (null) controllability is important for getting the exact boundary synchronization by groups for system (1.3)–(1.5). It is usually complicated to establish the exact boundary (null) controllability especially in higher-dimensional space. In this paper we do it in the one-dimensional space with the aid of the controllability results on first order hyperbolic systems given in [6]. It is challenging to deal with higher-dimensional case, system with (1.5c) will be more difficult because of the coupling on the boundary.

Once the exact boundary (null) controllability is established, the exact boundary synchronization by p-groups and corresponding exactly synchronizable states by p-groups can be discussed under the following conditions of C_p-compatibility for the coupling matrices.

Definition 3.1 *Let* $\epsilon_i (i = 1, \cdots, p)$ *be given by* (3.4). *Matrix* A *satisfies the condition of* C_p*-compatibility if*

$$A\mathrm{Ker}(C_p) \subseteq \mathrm{Ker}(C_p), \quad \textit{namely,} \quad C_p A = \overline{A}_p C_p$$
$$\textit{or} \quad A\epsilon_i = \sum_{j=1}^{p} \alpha_{ij}\epsilon_j \quad \textit{for } i = 1, \cdots, p, \tag{3.9}$$

in which $\overline{A}_p$ *is a matrix of order* $(n-p)$*, and* $\alpha_{ij} (i, j = 1, \cdots, p)$ *are constants.*

Matrix B *satisfies the condition of* C_p*-compatibility if*

$$B\mathrm{Ker}(C_p) \subseteq \mathrm{Ker}(C_p), \quad \textit{namely,} \quad C_p B = \overline{B}_p C_p$$
$$\textit{or} \quad B\epsilon_i = \sum_{j=1}^{p} \beta_{ij}\epsilon_j \quad \textit{for } i = 1, \cdots, p, \tag{3.10}$$

in which $\overline{B}_p$ *is matrix of order* $(n-p)$*, and* $\beta_{ij} (i, j = 1, \cdots, p)$ *are constants.*

Theorem 3.2 *Assume that* A *satisfies the condition of* C_p*-compatibility* (3.9). *Assume furthermore that* -1 *is not an eigenvalue of* B*, and* B *satisfies the condition of* C_p*-compatibility* (3.10) *in case* (1.5c). *If* $\mathrm{rank}(C_p D) = n - p$*, then there exists a boundary control* $H(t) \in \mathcal{H}$*, satisfying*

$$\|H\|_{\mathcal{H}} \le c\|C_p(\widehat{U}_0, \widehat{U}_1)\|_{(H^1(0,L)\times L^2(0,L))^{n-p}} \le c\|(\widehat{U}_0, \widehat{U}_1)\|_{(H^1(0,L)\times L^2(0,L))^{n}}, \tag{3.11}$$

such that system (1.3)–(1.5) *is exactly synchronizable by* p*-groups, where* $\mathcal{H}$ *is given by* (2.1).

Proof Under the condition of C_p-compatibility (3.9) for A and (3.10) for B in case (1.5c), let $W = C_p U$, where U satisfies system (1.3)–(1.5) with (1.6). We have the following reduced system of W:

$$W_{tt} - W_{xx} + \overline{A}_p W_t = 0, \quad t \in (0, +\infty),\ x \in (0, L), \tag{3.12}$$

$$x = 0 :\ W = 0, \quad t \in (0, +\infty) \tag{3.13}$$

and any one of

$$x = L :\ W = C_p D H(t), \quad t \in (0, +\infty), \tag{3.14a}$$

$$x = L: \; W_x = C_p DH(t), \quad t \in (0, +\infty), \tag{3.14b}$$

$$x = L: \; W_x + \overline{B}_p W_t = C_p DH(t), \quad t \in (0, +\infty), \tag{3.14c}$$

where $\overline{A}_p$ and $\overline{B}_p$ are given by (3.9) and (3.10), respectively. By Theorem 3.1, the reduced system (3.12)–(3.14) is exactly null controllability when $\text{rank}(C_p D) = n - p$. Noting that the exact boundary null controllability of the reduced system (3.12)–(3.14) is equivalent to the exact boundary synchronization by p-groups of the original system (1.3)–(1.5), we immediately get Theorem 3.2.

Remark 3.4 Under the conditions of C_p-compatibility for the coupling matrices, Theorem 3.2 and the following results on exactly synchronizable states by p-groups are discussed directly from the viewpoint of wave equations as in [16].

On the other hand, these results can be also built as an application of the results for first order hyperbolic systems obtained in [6] by transforming the exact boundary synchronization by p-groups for system (1.3)–(1.5) into the exact boundary synchronization for system (1.8) and (1.10)–(1.11) with respect to the matrix of synchronization $\mathbf{C}_1 = \begin{pmatrix} C_p & \\ & C_p \end{pmatrix}$. The perspective of first order hyperbolic system is practical since no matter what p is for the exact boundary synchronization by p-groups for system (1.3)–(1.5), it is always exact boundary synchronization for system (1.8) and (1.10)–(1.11) but with different size of $\mathbf{C}_1$.

However, for a system of wave equations coupled with velocities, since there is a lack of compactness, the conditions of C_p-compatibility can not be directly derived from both viewpoints.

4 Exactly Synchronizable States by p-Groups

Under the conditions of C_p-compatibility for A and B, by inserting (3.5) into (1.3)–(1.5), it is easy to get the system satisfied by the exactly synchronizable state by p-groups, and similarly to [6, Theorem 4.2, Lemma 4.3], if the system of exactly synchronizable states by p-groups is time reversible, then the attainable set of exactly synchronizable states $(\widetilde{u}_i, \widetilde{u}_{it})(i = 1, \cdots, p)$ at the time $t = T$ is the whole space $(H^1(0, L) \times L^2(0, L))^p$.

Theorem 4.1 *Assume that A satisfies the condition of C_p-compatibility* (3.9). *Assume furthermore that -1 is not an eigenvalue of B, and B satisfies the condition of C_p-compatibility* (3.10) *in case* (1.5c). *If system* (1.3)–(1.5) *is exactly synchronizable by p-groups at the time $t = T$, then, as $t \geq T$, the exactly synchronizable state by p-groups $(\widetilde{u}_1, \cdots, \widetilde{u}_p)^{\mathrm{T}}$ satisfies*

$$\widetilde{u}_{itt} - \widetilde{u}_{ixx} + \sum_{j=1}^{p} \alpha_{ji} \widetilde{u}_{jt} = 0, \quad t \in (T, +\infty), \; x \in (0, L), \tag{4.1}$$

$$x = 0: \; \widetilde{u}_i = 0, \quad t \in (T, +\infty) \tag{4.2}$$

and any one of

$$x = L: \; \widetilde{u}_i = 0, \quad t \in (T, +\infty), \tag{4.3a}$$

$$x = L: \; \widetilde{u}_{ix} = 0, \quad t \in (T, +\infty), \tag{4.3b}$$

$$x = L: \; \widetilde{u}_{ix} + \sum_{j=1}^{p} \beta_{ji} \widetilde{u}_{jt} = 0, \quad t \in (T, +\infty) \tag{4.3c}$$

for $i = 1, \cdots, p$, where α_{ij} and $\beta_{ij} (i, j = 1, \cdots, p)$ are given by (3.9) *and* (3.10), *respectively.*

Moreover, the attainable set of exactly synchronizable states $(\widetilde{u}_i, \widetilde{u}_{it})(i = 1, \cdots, p)$ *at the time* $t = T$ *is the whole space* $(H^1(0, L) \times L^2(0, L))^p$ *for cases* (1.5a) *and* (1.5b). *Assume furthermore that* $\mathrm{Ker}(G_1) \cap \mathrm{Ker}(C_p) = \{0\}$, *then this result is also true for case* (1.5c), *where* G_1 *is given by* (1.12).

Remark 4.1 By [16, Proposition 2.21], in order to have $\mathrm{Ker}(G_1) \cap \mathrm{Ker}(C_p) = \{0\}$, we can assume that 1 is not an eigenvalue of B.

In order to further determine corresponding exactly synchronizable states, let

$$\varepsilon_i \in \mathbb{R}^n (i = 1, \cdots, p) \tag{4.4}$$

satisfy that $\mathrm{Span}\{\varepsilon_1, \cdots, \varepsilon_p\}$ and $\mathrm{Ker}(C_p) = \mathrm{Span}\{\epsilon_1, \cdots, \epsilon_p\}$ are bi-orthonormal, where $\epsilon_i(i = 1, \cdots, p)$ are given by (3.4).

Theorem 4.2 *Assume that* A *satisfies the condition of* C_p*-compatibility* (3.9). *Assume furthermore that* -1 *is not an eigenvalue of* B, *and* B *satisfies the condition of* C_p*-compatibility* (3.10) *in case* (1.5c). *Define* D *by* $\mathrm{Ker}(D^{\mathrm{T}}) = \mathrm{Span}\{\varepsilon_1, \cdots, \varepsilon_p\}$. *Then we have*

$$\mathrm{rank}(C_p D) = \mathrm{rank}(D) = n - p \tag{4.5}$$

and system (1.3)–(1.5) *is exactly synchronizable by p-groups.*

Moreover, if $\mathrm{Span}\{\varepsilon_1, \cdots, \varepsilon_p\}$ *is an invariant subspace of* A^{T}, *then the exactly synchronizable state by p-groups* $(\widetilde{u}_1, \cdots, \widetilde{u}_p)^{\mathrm{T}}$ *of system* (1.3)–(1.4) *with* (1.5a) (*resp.* (1.5b)) *is independent of applied boundary controls. If* $\mathrm{Span}\{\varepsilon_1, \cdots, \varepsilon_p\}$ *is a common invariant subspace of* A^{T} *and* B^{T}, *then the exactly synchronizable state by p-groups* $(\widetilde{u}_1, \cdots, \widetilde{u}_p)^{\mathrm{T}}$ *of system* (1.3)–(1.4) *with* (1.5c) *is also independent of applied boundary controls.*

Proof Since $\mathrm{Ker}(D^{\mathrm{T}}) = \mathrm{Span}\{\varepsilon_1, \cdots, \varepsilon_p\}$, noting $\mathrm{Span}\{\varepsilon_1, \cdots, \varepsilon_p\}$ and $\mathrm{Ker}(C_p)$ are bi-orthonormal, by [16, Propositions 2.5, 2.11], we immediately get (4.5), then, by Theorem 3.2, system (1.3)–(1.5) is exactly synchronizable by p-groups.

Assume that $\mathrm{Span}\{\varepsilon_1, \cdots, \varepsilon_p\}$ is an invariant subspace of A^{T} and B^{T}, respectively, noting (3.9)–(3.10) and that $\mathrm{Span}\{\varepsilon_1, \cdots, \varepsilon_p\}$ and $\mathrm{Ker}(C_p) = \mathrm{Span}\{\epsilon_1, \cdots, \epsilon_p\}$ are bi-orthonormal, it is easy to check that $A^{\mathrm{T}}\varepsilon_i = \sum\limits_{j=1}^{p} \alpha_{ji}\varepsilon_j$ and $B^{\mathrm{T}}\varepsilon_i = \sum\limits_{j=1}^{p} \beta_{ji}\varepsilon_j (i = 1, \cdots, p)$, where α_{ij} and $\beta_{ij}(i, j = 1, \cdots, p)$ are given by (3.9) and (3.10), respectively. Let $\phi_i = (U, \varepsilon_i)(i = 1, \cdots, p)$, where $U = U(t, x)$ is the solution to system (1.3)–(1.5), which realizes the exact boundary synchronization by p-groups at the time $t = T$. Then, multiplying $\varepsilon_i(i = 1, \cdots, p)$ on system (1.3)–(1.5), for $i = 1, \cdots, p$ we have

$$\phi_{itt} - \phi_{ixx} + \sum_{j=1}^{p} \alpha_{ji}\phi_{jt} = 0, \quad t \in (0, +\infty),\ x \in (0, L), \tag{4.6}$$

$$x = 0:\ \phi_i = 0, \quad t \in (0, +\infty) \tag{4.7}$$

and any one of

$$x = L:\ \phi_i = 0, \quad t \in (0, +\infty), \tag{4.8a}$$

$$x = L:\ \phi_{ix} = 0, \quad t \in (0, +\infty), \tag{4.8b}$$

$$x = L: \ \phi_{ix} + \sum_{j=1}^{p} \beta_{ji}\phi_{jt} = 0, \quad t \in (0, +\infty) \tag{4.8c}$$

with the initial data

$$t = 0: \ \phi_i = (\widehat{U}_0, \varepsilon_i), \quad \phi_{it} = (\widehat{U}_1, \varepsilon_i), \quad x \in (0, L). \tag{4.9}$$

Noting (3.5), since $\mathrm{Span}\{\varepsilon_1, \cdots, \varepsilon_p\}$ and $\mathrm{Ker}(C_p) = \mathrm{Span}\{\epsilon_1, \cdots, \epsilon_p\}$ are bi-orthonormal, we have

$$t \geq T: \ \phi_i = (U, \varepsilon_i) = \Big(\sum_{j=1}^{p} \widetilde{u}_j \epsilon_j, \varepsilon_i\Big) = \widetilde{u}_i, \quad x \in (0, L). \tag{4.10}$$

Thus the exactly synchronizable state by p-groups $(\widetilde{u}_1, \cdots, \widetilde{u}_p)^{\mathrm{T}}$ of system (1.3)–(1.5) is entirely determined by the solution to problem (4.6)–(4.9), which is independent of applied boundary controls.

Inversely to Theorem 4.2, we have the following theorem.

Theorem 4.3 *Assume that A satisfies the condition of C_p-compatibility* (3.9)*. Assume furthermore that -1 is not an eigenvalue of B, and B satisfies the condition of C_p-compatibility* (3.10) *in case* (1.5c)*. Assume finally that system* (1.3)*–*(1.5) *is exactly synchronizable by p-groups under the condition* $\mathrm{rank}(C_pD) = n - p$*. Let U be the solution to problem* (1.3)*–*(1.6)*, which realizes the exact boundary synchronization by p-groups at the time $t = T$. If (U, ε_i) with $\varepsilon_i (i = 1, \cdots, p)$ given by* (4.4) *are independent of applied boundary controls, then* $\mathrm{Span}\{\varepsilon_1, \cdots, \varepsilon_p\}$ *is an invariant subspace of A^{T} for system* (1.3)*–*(1.4) *with* (1.5a) (*resp.* (1.5b))*, while* $\mathrm{Span}\{\varepsilon_1, \cdots, \varepsilon_p\}$ *is a common invariant subspace of A^{T} and B^{T} for system* (1.3)*–*(1.4) *with* (1.5c)*. Moreover, we have $\varepsilon_i \in \mathrm{Ker}(D^{\mathrm{T}}) (i = 1, \cdots, p)$. In particular, if D satisfies* (4.5)*, then we have* $\mathrm{Ker}(D^T) = \mathrm{Span}\{\varepsilon_1, \cdots, \varepsilon_p\}$.

Proof We only give a sketch of the proof, which is similar to that of [6, Theorem 4.8].

Let $U = U(t, x)$ be the solution to system (1.3)–(1.5), which realizes the exact boundary synchronization by p-groups at the time $t = T$. Taking $(\widehat{U}_0, \widehat{U}_1) = (0, 0)$, by Theorem 2.1, the linear mapping $F: \ H \to (U, U_t)$ is continuous from $\mathcal{H}$ to $(L^2_{\mathrm{loc}}(0, +\infty; H^1(0, L) \times L^2(0, L)))^n$, where $\mathcal{H}$ is given by (2.1). By linearity, the Fréchet derivative

$$\widehat{U} \stackrel{\triangle}{=} F'(0)H \tag{4.11}$$

satisfies also system (1.3)–(1.5) with $t = 0: \widehat{U} = \widehat{U}_t = 0$. Since $(U, \varepsilon_i)(i = 1, \cdots, p)$ are independent of applied boundary controls, we have

$$(\widehat{U}, \varepsilon_i) \equiv 0, \quad i = 1, \cdots, p \tag{4.12}$$

for any given $H \in \mathcal{H}$.

Since $(\varepsilon_1, \cdots, \varepsilon_p, C_p^{\mathrm{T}})$ constitutes a basis in $\mathbb{R}^n$, we have

$$A^{\mathrm{T}}\varepsilon_i = \sum_{j=1}^{p} a_{ij}\varepsilon_j + C_p^{\mathrm{T}} P_i, \quad i = 1, \cdots, p, \tag{4.13}$$

$$B^{\mathrm{T}}\varepsilon_i = \sum_{j=1}^{p} b_{ij}\varepsilon_j + C_p^{\mathrm{T}} Q_i, \quad i = 1, \cdots, p, \tag{4.14}$$

where a_{ij} and $b_{ij}(i,j=1,\cdots,p)$ are constants, P_i and $Q_i \in \mathbb{R}^{n-p}(i=1,\cdots,p)$. Then, multiplying $\varepsilon_i(i=1,\cdots,p)$ on system (1.3) of $\widehat{U}$, it follows from (4.12)–(4.13) that

$$0=(A\widehat{U}_t,\varepsilon_i)=\Big(\widehat{U}_t,\sum_{j=1}^{p}a_{ij}\varepsilon_j\Big)+(\widehat{U}_t,C_p^{\mathrm{T}}P_i)=(C_p\widehat{U}_t,P_i),\quad i=1,\cdots,p. \tag{4.15}$$

Let $W\stackrel{\text{def.}}{=}C_p\widehat{U}$. W satisfies the reduced system (3.12)–(3.14). By [16, Proposition 2.21], when -1 is not an eigenvalue of B, -1 is not an eigenvalue of $\overline{B}_p$. Then, when $\operatorname{rank}(C_pD)=n-p$, by Remark 3.2, system (3.12)–(3.14) is in fact exactly controllable. Thus the value of $C_p\widehat{U}_t$ at the time $t=T$ can be arbitrarily chosen. Then we have $P_i=0(i=1,\cdots,p)$. Similarly, noting $H\equiv 0$ as $t\geq T$, we can prove that $Q_i=0(i=1,\cdots,p)$. Moreover, noting [6, Lemma 4.7], the value of H on $(0,t_0)$ can be chosen arbitrarily for $t_0>0$ small enough, we can prove $D^T\varepsilon_i=0(i=1,\cdots,p)$, thus $\varepsilon_i\in\operatorname{Ker}(D^{\mathrm{T}})(i=1,\cdots,p)$. The proof is complete.

Theorem 4.4 *Assume that A satisfies the condition of C_p-compatibility* (3.9)*. Assume furthermore that -1 is not an eigenvalue of B, and B satisfies the condition of C_p-compatibility* (3.10) *in case* (1.5c)*. Assume finally that system* (1.3)–(1.5) *is exactly synchronizable by p-groups under condition* $\operatorname{rank}(C_pD)=n-p$ *with $H(t)$ satisfying* (3.11)*. Then the exactly synchronizable state by p-groups $(\widetilde{u}_1,\cdots,\widetilde{u}_p)^{\mathrm{T}}$ satisfies the following estimate*:

$$\|(\widetilde{u}_i-\phi_i,\widetilde{u}_{it}-\phi_{it})(T)\|_{H^1(0,L)\times L^2(0,L)}\leq c\|C_p(\widehat{U}_0,\widehat{U}_1)\|_{(H^1(0,L)\times L^2(0,L))^{n-p}} \tag{4.16}$$

for $i=1,\cdots,p$, where $\phi_i(i=1,\cdots,p)$ satisfy problem (4.6)–(4.9)*.*

Proof The proof is similar to that of [9, Theorem 8.4], we only give a sketch here. Let U be the solution to problem (1.3)–(1.6), which realizes the exact boundary synchronization by p-groups at the time $t=T$, and let $z_i=(\varepsilon_i,U)(i=1,\cdots,p)$ with $\varepsilon_i(i=1,\cdots,p)$ given by (4.4). It is easy to prove that $A^{\mathrm{T}}\varepsilon_i-\sum\limits_{j=1}^{p}\alpha_{ji}\varepsilon_j\in\operatorname{Im}(C_p^{\mathrm{T}})$ and $B^{\mathrm{T}}\varepsilon_i-\sum\limits_{j=1}^{p}\beta_{ji}\varepsilon_j\in\operatorname{Im}(C_p^{\mathrm{T}})$ for $i=1,\cdots,p$, where α_{ij} and $\beta_{ij}(i,j=1,\cdots,p)$ are given by (3.9) and (3.10), respectively. Then there exist P_i and $Q_i\in\mathbb{R}^{n-p}(i=1,\cdots,p)$, such that $A^{\mathrm{T}}\varepsilon_i-\sum\limits_{j=1}^{p}\alpha_{ji}\varepsilon_j=C_p^{\mathrm{T}}P_i$ and $B^{\mathrm{T}}\varepsilon_i-\sum\limits_{j=1}^{p}\beta_{ji}\varepsilon_j=C_p^{\mathrm{T}}Q_i$ for $i=1,\cdots,p$. Thus we have

$$(\varepsilon_i,AU_t)=\sum_{j=1}^{p}\alpha_{ji}z_{jt}+(P_i,C_pU_t),\quad i=1,\cdots,p \tag{4.17}$$

and

$$(\varepsilon_i,BU_t(t,L))=\sum_{j=1}^{p}\beta_{ji}z_{jt}(t,L)+(Q_i,C_pU_t(t,L)),\quad i=1,\cdots,p. \tag{4.18}$$

Multiplying $\varepsilon_i(i=1,\cdots,p)$ on both sides of problem (1.3)–(1.6), noting (4.17)–(4.18), for $i=1,\cdots,p$, we have

$$z_{itt}-z_{ixx}+\sum_{j=1}^{p}\alpha_{ji}z_{jt}=-(P_i,C_pU_t),\quad t\in(0,+\infty),\ x\in(0,L), \tag{4.19}$$

$$x=0:\ z_i=0,\quad t\in(0,+\infty) \tag{4.20}$$

and any one of

$$x = L: \ z_i = (H, D^{\mathrm{T}}\varepsilon_i), \quad t \in (0, +\infty), \tag{4.21a}$$

$$x = L: \ z_{ix} = (H, D^{\mathrm{T}}\varepsilon_i), \quad t \in (0, +\infty), \tag{4.21b}$$

$$x = L: \ z_{ix} + \sum_{j=1}^{p} \beta_{ji} z_{jt} = (H, D^{\mathrm{T}}\varepsilon_i) - (Q_i, C_p U_t(t, L)), \quad t \in (0, +\infty) \tag{4.21c}$$

with

$$t = 0: \ z_i = (\widehat{U}_0, \varepsilon_i), \quad z_{it} = (\widehat{U}_1, \varepsilon_i), \quad x \in (0, L). \tag{4.22}$$

Let $y_i = z_i - \phi_i (i = 1, \cdots, p)$, where $\phi_i(t, x)(i = 1, \cdots, p)$ is the solution to problem (4.6)–(4.9). For $i = 1, \cdots, p$, we have

$$y_{itt} - y_{ixx} + \sum_{j=1}^{p} \alpha_{ji} y_{jt} = -(P_i, C_p U_t), \quad t \in (0, +\infty), \ x \in (0, L), \tag{4.23}$$

$$x = 0: \ y_i = 0, \quad t \in (0, +\infty) \tag{4.24}$$

and any one of

$$x = L: \ y_i = (H, D^{\mathrm{T}}\varepsilon_i), \quad t \in (0, +\infty), \tag{4.25a}$$

$$x = L: \ y_{ix} = (H, D^{\mathrm{T}}\varepsilon_i), \quad t \in (0, +\infty), \tag{4.25b}$$

$$x = L: \ y_{ix} + \sum_{j=1}^{p} \beta_{ji} y_{jt} = (H, D^{\mathrm{T}}\varepsilon_i) - (Q_i, C_p U_t(t, L)), \quad t \in (0, +\infty) \tag{4.25c}$$

with

$$t = 0: \ y_i = y_{it} = 0, \quad x \in (0, L). \tag{4.26}$$

According to the theory of first order hyperbolic systems, by [20, Theorem 3.3], with $\mathcal{H}$ given by (2.1) we have the following estimate for $i = 1, \cdots, p$:

$$\|(y_i, y_{it})(T)\|_{H^1(0,L)\times L^2(0,L)} \leq c(\|C_p U_t\|_{(L^2(0,T;L^2(0,L)))^{n-p}} + \|H\|_{\mathcal{H}}) \tag{4.27}$$

in cases (4.25a) and (4.25b); while

$$\begin{aligned} &\|(y_i, y_{it})(T)\|_{H^1(0,L)\times L^2(0,L)} \\ \leq\ & c(\|C_p U_t\|_{(L^2(0,T;L^2(0,L)))^{n-p}} + \|H\|_{\mathcal{H}} + \|C_p U_t(t, L)\|_{(L^2(0,T))^{n-p}}) \end{aligned} \tag{4.28}$$

in case (4.25c), where $W \overset{\text{def.}}{=} C_p U$ satisfies the reduced system (3.12)–(3.14) with

$$t = 0: \ W = C_p \widehat{U}_0, \quad W_t = C_p \widehat{U}_1, \quad x \in (0, L). \tag{4.29}$$

By Theorem 2.1, noting (3.11), we have

$$\begin{aligned} &\|C_p U_t\|_{(L^2(0,T;L^2(0,L)))^{n-p}} = \|W_t\|_{(L^2(0,T;L^2(0,L)))^{n-p}} \\ \leq\ & c(\|C_p(\widehat{U}_0, \widehat{U}_1)\|_{(H^1(0,L)\times L^2(0,L))^{n-p}} + \|H\|_{\mathcal{H}}) \leq c\|C_p(\widehat{U}_0, \widehat{U}_1)\|_{(H^1(0,L)\times L^2(0,L))^{n-p}} \end{aligned} \tag{4.30}$$

and

$$\begin{aligned}&\|C_pU_t(t,L)\|_{(L^2(0,T))^{n-p}}=\|W_t(t,L)\|_{(L^2(0,T))^{n-p}}\\ \le\ & c(\|C_p(\widehat{U}_0,\widehat{U}_1)\|_{(H^1(0,L)\times L^2(0,L))^{n-p}}+\|H\|_{\mathcal{H}})\le c\|C_p(\widehat{U}_0,\widehat{U}_1)\|_{(H^1(0,L)\times L^2(0,L))^{n-p}}.\end{aligned}\tag{4.31}$$

Therefore, it follows from (3.11), (4.27)–(4.28) and (4.30)–(4.31) that

$$\begin{aligned}&\|(z_i-\phi_i,z_{it}-\phi_{it})(T)\|_{H^1(0,L)\times L^2(0,L)}=\|(y_i,y_{it})(T)\|_{H^1(0,L)\times L^2(0,L)}\\ \le\ & c\|C_p(\widehat{U}_0,\widehat{U}_1)\|_{(H^1(0,L)\times L^2(0,L))^{n-p}},\quad i=1,\cdots,p.\end{aligned}\tag{4.32}$$

On the other hand, noting (3.5) and that $\mathrm{Span}\{\varepsilon_1,\cdots,\varepsilon_p\}$ and $\mathrm{Ker}(C_p)=\mathrm{Span}\{\epsilon_1,\cdots,\epsilon_p\}$ are bi-orthonormal, we have

$$t\ge T:\ z_i=(\varepsilon_i,U)=\Big(\varepsilon_i,\sum_{j=1}^{p}\widetilde{u}_j\epsilon_j\Big)=\widetilde{u}_i,\quad i=1,\cdots,p.\tag{4.33}$$

Substituting (4.33) into (4.32), we get (4.16).

Remark 4.2 [4] and [18] discussed the exact boundary synchronization by groups for the coupled system of 1-D wave equations (1.1) with various types of boundary conditions but in the framework of classical solutions. It was proved that when D in (1.5) is the identity matrix, we can find $(n-p)$ boundary controls so that system (1.1) and (1.4)–(1.5) is exactly synchronization by p-groups. In this paper we extend the corresponding result to system (1.3)–(1.5) in the framework of weak solutions, for which not only corresponding exactly synchronizable states by p-groups are further determined and estimated, but also the necessary rank conditions of Kalman type can be obtained in Section 6.

5 Necessity of the Conditions of C_p-Compatibility

Assume that system (1.3)–(1.5) is exactly synchronizable by p-groups, namely, we have (3.5), then, multiplying C_p on (1.3), we have

$$t\ge T:\ C_pAU_t=C_pA\sum_{i=1}^{p}\widetilde{u}_{it}\epsilon_i=\sum_{i=1}^{p}C_pA\epsilon_i\widetilde{u}_{it}=0,\tag{5.1}$$

where $\epsilon_i(i=1,\cdots,p)$ are given by (3.4). If $C_pA\epsilon_i=0(i=1,\cdots,p)$, then, noting (3.4), we have the condition of C_p-compatibility (3.9) for A. Otherwise, $\widetilde{u}_{1t},\cdots,\widetilde{u}_{pt}$ are linearly dependent, without loss of generality, we may assume that

$$t\ge T:\ \widetilde{u}_{pt}=\sum_{i=1}^{p-1}\delta_i\widetilde{u}_{it},\tag{5.2}$$

where $\delta_i(i=1,\cdots,p-1)$ are constants.

Let

$$\boldsymbol{\Theta}=\begin{pmatrix} I_{n_1} & & & & \mathbf{0}\\ & I_{n_2} & & & \mathbf{0}\\ & & \ddots & & \vdots\\ & & & I_{n_{p-1}} & \mathbf{0}\\ \frac{\delta_1}{n_1}\widetilde{e}_1^{\mathrm{T}} & \frac{\delta_2}{n_2}\widetilde{e}_2^{\mathrm{T}} & \cdots & \frac{\delta_{p-1}}{n_{p-1}}\widetilde{e}_{p-1}^{\mathrm{T}} & \\ \vdots & \vdots & & \vdots & -I_{n_p}\\ \frac{\delta_1}{n_1}\widetilde{e}_1^{\mathrm{T}} & \frac{\delta_2}{n_2}\widetilde{e}_2^{\mathrm{T}} & \cdots & \frac{\delta_{p-1}}{n_{p-1}}\widetilde{e}_{p-1}^{\mathrm{T}} & \end{pmatrix}$$

be an invertible matrix, and let

$$\overline{U} = \begin{pmatrix} \overline{U}_1 \\ \vdots \\ \overline{U}_{p-1} \\ \overline{U}_p \end{pmatrix} = \mathbf{\Theta} U = \begin{pmatrix} U_1 \\ \vdots \\ U_{p-1} \\ \Big(\sum\limits_{i=1}^{p-1} \frac{\delta_i}{n_i} \widetilde{e}_i^{\mathrm{T}} U_i\Big) \widetilde{e}_p - U_p \end{pmatrix}, \tag{5.3}$$

where $\widetilde{e}_i (i = 1, \cdots, p)$ are given by (3.3). Noting (3.6) and (5.2), as $t \geq T$ we have

$$\overline{U}_i = U_i = \widetilde{u}_i \widetilde{e}_i, \quad i = 1, \cdots, p-1, \tag{5.4}$$

$$\overline{U}_p = \Big(\sum_{i=1}^{p-1} \frac{\delta_i}{n_i} \widetilde{e}_i^{\mathrm{T}} U_i\Big) \widetilde{e}_p - U_p = \Big(\sum_{i=1}^{p-1} \delta_i \widetilde{u}_i\Big) \widetilde{e}_p - \widetilde{u}_p \widetilde{e}_p = \Big(\sum_{i=1}^{p-1} \delta_i \widetilde{u}_i - \widetilde{u}_p\Big) \widetilde{e}_p \tag{5.5}$$

and

$$\overline{U}_{pt} = \Big(\sum_{i=1}^{p-1} \delta_i \widetilde{u}_{it} - \widetilde{u}_{pt}\Big) \widetilde{e}_p = 0. \tag{5.6}$$

On the other hand, noting (1.7) and (5.3), let

$$\begin{aligned} \overline{V} = (\overline{V}^-, \overline{V}^+)^{\mathrm{T}} &= \begin{pmatrix} \mathbf{\Theta} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Theta} \end{pmatrix} V = \begin{pmatrix} \mathbf{\Theta} V^- \\ \mathbf{\Theta} V^+ \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \mathbf{\Theta}(U_t + U_x) \\ \mathbf{\Theta}(U_t - U_x) \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} \overline{U}_t + \overline{U}_x \\ \overline{U}_t - \overline{U}_x \end{pmatrix} \end{aligned} \tag{5.7}$$

and let $\overline{V}^- = (\overline{V}_1^-, \cdots, \overline{V}_p^-)^{\mathrm{T}}$ and $\overline{V}^+ = (\overline{V}_1^+, \cdots, \overline{V}_p^+)^{\mathrm{T}}$ with

$$\overline{V}_i^- = \frac{1}{2}(\overline{U}_{it} + \overline{U}_{ix}) \quad \text{and} \quad \overline{V}_i^+ = \frac{1}{2}(\overline{U}_{it} - \overline{U}_{ix}), \quad i = 1, \cdots, p. \tag{5.8}$$

Noting (5.4)–(5.6) and (5.8), as $t \geq T$ we have

$$\overline{V}_i^- = \frac{1}{2}(\widetilde{u}_{it} + \widetilde{u}_{ix}) \widetilde{e}_i, \quad \overline{V}_i^+ = \frac{1}{2}(\widetilde{u}_{it} - \widetilde{u}_{ix}) \widetilde{e}_i, \quad i = 1, \cdots, p-1 \tag{5.9}$$

and

$$\overline{V}_p^- = -\overline{V}_p^+ = \frac{1}{2} \overline{U}_{px} = \frac{1}{2} \Big(\sum_{i=1}^{p-1} \delta_i \widetilde{u}_{ix} - \widetilde{u}_{px}\Big) \widetilde{e}_p. \tag{5.10}$$

Let

$$\widetilde{\mathbf{\Theta}} = \begin{pmatrix} I_{n_1} & & & & & & \mathbf{0} \\ & \ddots & & & & & \vdots \\ & & I_{n_p} & & & & \mathbf{0} \\ & & & I_{n_1} & & & \mathbf{0} \\ & & & & \ddots & & \vdots \\ & & & & & I_{n_{p-1}} & \mathbf{0} \\ \mathbf{0} & \cdots & I_{n_p} & \mathbf{0} & \cdots & \mathbf{0} & I_{n_p} \end{pmatrix}$$

and let $\overline{\overline{\mathbf{\Theta}}} = \widetilde{\mathbf{\Theta}} \begin{pmatrix} \mathbf{\Theta} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Theta} \end{pmatrix}$. Noting (5.9)–(5.10), we have

$$t \geq T: \ \overline{\overline{V}} \overset{\text{def.}}{=} \overline{\overline{\mathbf{\Theta}}} V = \widetilde{\mathbf{\Theta}} \overline{V} = \begin{pmatrix} \overline{V}_1^- \\ \vdots \\ \overline{V}_{p-1}^- \\ \overline{V}_p^- \\ \overline{V}_1^+ \\ \vdots \\ \overline{V}_{p-1}^+ \\ \overline{V}_p^- + \overline{V}_p^+ \end{pmatrix} = \frac{1}{2} \begin{pmatrix} (\widetilde{u}_{1t} + \widetilde{u}_{1x})\widetilde{e}_1 \\ \vdots \\ ((\widetilde{u}_{p-1})_t + (\widetilde{u}_{p-1})_x)\widetilde{e}_{p-1} \\ (\sum\limits_{i=1}^{p-1} \delta_i \widetilde{u}_{ix} - \widetilde{u}_{px})\widetilde{e}_p \\ (\widetilde{u}_{1t} - \widetilde{u}_{1x})\widetilde{e}_1 \\ \vdots \\ ((\widetilde{u}_{p-1})_t - (\widetilde{u}_{p-1})_x)\widetilde{e}_{p-1} \\ 0\widetilde{e}_p \end{pmatrix},$$

namely, by an invertible linear transformation, one group of the components of system (1.8) and (1.10)–(1.11) is in fact exactly null controllable, while any one of the other groups is exactly synchronizable. From the perspective of first order hyperbolic systems, we should exclude this situation (see [6, Theorem 6.1]). Hence, we have the following theorem.

Theorem 5.1 *If system* (1.3)–(1.5) *is exactly synchronizable by* p*-groups, but the derivatives of the exactly synchronizable states with respect to* t*,* $\widetilde{u}_{1t}, \cdots, \widetilde{u}_{pt}$ *are not linearly dependent, then* A *satisfies the condition of* C_p*-compatibility* (3.9)*.*

In particular, when $p = 1$, by [6, Theorems 6.1–6.2] we have the following corollary.

Corollary 5.1 *If system* (1.3)–(1.5) *is exactly synchronizable but not exactly null controllable, then* A *must satisfy the condition of* C_1*-compatibility* (3.9) *with* $p = 1$*, moreover,* B *must satisfy the condition of* C_1*-compatibility* (3.10) *with* $p = 1$ *in case* (1.5c)*, where* C_1 *is given by* (3.2)*.*

Remark 5.1 The discussions on the exact boundary synchronization by p-groups and corresponding exactly synchronizable states by p-groups for system (1.1) and for system (1.3) are quite similar. In the study of the necessity of the conditions of C_p-compatibility for the coupling matrix A in Theorem 5.1, noting the form of coupling AU_t in system (1.3), we need to check the linear independence of $\widetilde{u}_{1t}, \cdots, \widetilde{u}_{pt}$. In comparison, for system (1.1) with coupling AU, we need to check the linear independence of $\widetilde{u}_1, \cdots, \widetilde{u}_p$, instead. Similarly for that of B on the boundary.

Specifically, for system (1.1), the coupling of displacements AU can be regarded as a compact perturbation. Because of this compactness, we can prove that $\widetilde{u}_1, \cdots, \widetilde{u}_p$ is linearly independent in the domain or on the boundary, then we get the necessity of the conditions of C_p-compatibility for the coupling matrices (see more details in [22]). However, the coupling of velocities AU_t in system (1.3) is not compact, we can not prove the linear independence of $\widetilde{u}_{1t}, \cdots, \widetilde{u}_{pt}$ in the same way, the necessity of the conditions of C_p-compatibility for both A and B for the exact boundary synchronization by p-groups for system (1.3)–(1.5) is still an open problem.

6 Kalman's Criterion

In this section, we give some necessary conditions for the coupling matrices.

Theorem 6.1 *If system* (1.3)–(1.4) *with* (1.5a) (*resp.* (1.5b)) *is exactly null controllable, then we necessarily have*

$$\operatorname{rank}(D, AD, \cdots, A^{n-1}D) = n. \tag{6.1}$$

Proof The result can be proved by applying [6, Theorem 7.1] to system (1.8), (1.10) and (1.11a) (resp. (1.11b)). We only need to point out that the rank condition (6.1) holds if and only if A^{T} doesn't have any invariant subspace that is contained in $\operatorname{Ker}(D^{\mathrm{T}})$ (see [16, Proposition 2.12]).

Corollary 6.1 *Assume that A satisfies the condition of C_p-compatibility* (3.9)*. If system* (1.3)–(1.4) *with* (1.5a) (*resp.* (1.5b)) *is exactly synchronizable by p-groups, then we necessarily have*

$$\operatorname{rank}(C_pD, C_pAD, \cdots, C_pA^{n-1}D) = n - p. \tag{6.2}$$

Proof Under the condition of C_p-compatibility (3.9) for A, the exact boundary synchronization by p-groups for system (1.3)–(1.4) with (1.5a) (resp. (1.5b)) can be equivalently transformed into the exact boundary null controllability for the reduced system (3.12)–(3.13) with (3.14a) (resp. (3.14b)). Then, applying Theorem 6.1 to system (3.12)–(3.13) with (3.14a) (resp. (3.14b)), we have

$$\operatorname{rank}(C_pD, \overline{A}_pC_pD, \cdots, \overline{A}_p^{n-p-1}C_pD) = n - p. \tag{6.3}$$

Noting (3.9), (6.2) follows from (6.3).

For system (1.3)–(1.4) with (1.5c), the Kalman's criterion, which is similar to that in [17], is different because there is another coupling matrix B on the boundary. Let

$$\mathcal{R}_{(p,q,\cdots,r,s)} = A^pB^q \cdots A^rB^sD$$

be an $n \times M$ matrix for any given non-negative integers $p, q, \cdots, r, s \geq 0$, and let

$$\mathcal{R} = (\mathcal{R}_{(p,q,\cdots,r,s)}, \mathcal{R}_{(p',q',\cdots,r',s')}, \cdots) \tag{6.4}$$

be an enlarged matrix by the matrices $\mathcal{R}_{(p,q,\cdots,r,s)}$ for all possible $(p, q, \cdots, r, s)$.

Lemma 6.1 (see [17, Lemma 2.1]) $\operatorname{Ker}(\mathcal{R}^{\mathrm{T}})$ *is the largest subspace of all the subspaces which are contained in* $\operatorname{Ker}(D^{\mathrm{T}})$ *and invariant for A^{T} and B^{T}.*

Lemma 6.2 *Assume that -1 is not an eigenvalue of the coupling matrix B. For any given $k \times n$ matrix C, there exists a matrix $\overline{B}$ of order k, such that*

$$CB = \overline{B}C \tag{6.5}$$

if and only if there exists a matrix $\overline{G}_1$ of order k, such that

$$CG_1 = \overline{G}_1C, \tag{6.6}$$

where G_1 is given by (1.12).

Proof Assume first that (6.5) holds, then we have

$$C(I_n+B)=C+CB=C+\overline{B}C=(I_k+\overline{B})C. \tag{6.7}$$

Since -1 is not an eigenvalue of B, by [16, Proposition 2.21], -1 is not an eigenvalue of $\overline{B}$, then it follows from (6.7) that

$$(I_k+\overline{B})^{-1}C=C(I_n+B)^{-1}. \tag{6.8}$$

Let $\overline{G}_1=(I_k+\overline{B})^{-1}(I_k-\overline{B})$. Noting (1.12), (6.5) and (6.8), we have

$$\overline{G}_1C=(I_k+\overline{B})^{-1}(I_k-\overline{B})C=(I_k+\overline{B})^{-1}C(I_n-B)=C(I_n+B)^{-1}(I_n-B)=CG_1. \tag{6.9}$$

Inversely, assume that (6.6) holds. We claim that -1 is not an eigenvalue of G_1. Otherwise, noting (1.12), there exists a non-trivial $\xi\in\mathbb{R}^n$, such that

$$G_1\xi=(I_n+B)^{-1}(I_n-B)\xi=-\xi,$$

then we have $(I_n-B)\xi=-(I_n+B)\xi$, which leads to $\xi=-\xi$, a contradiction. Thus, by [16, Proposition 2.21], -1 is not an eigenvalue of $\overline{G}_1$. Then, similarly to (6.8), we have

$$(I_k+\overline{G}_1)^{-1}C=C(I_n+G_1)^{-1}. \tag{6.10}$$

Noting (1.12), we have

$$B=(I_n-G_1)(I_n+G_1)^{-1}. \tag{6.11}$$

Let $\overline{B}=(I_k-\overline{G}_1)(I_k+\overline{G}_1)^{-1}$. Noting (6.6) and (6.10)–(6.11), we have

$$\begin{aligned}\overline{B}C&=(I_k-\overline{G}_1)(I_k+\overline{G}_1)^{-1}C\\&=(I_k-\overline{G}_1)C(I_n+G_1)^{-1}=C(I_n-G_1)(I_n+G_1)^{-1}=CB.\end{aligned} \tag{6.12}$$

Theorem 6.2 *Assume that both -1 and 1 are not eigenvalues of B. If system* (1.3)–(1.4) *with* (1.5c) *is exactly null controllable, then we necessarily have* $\text{rank}(\mathcal{R})=n$.

Proof Assume by contradiction that $\text{rank}(\mathcal{R})=n-d$ with $d>0$, then $\dim \text{Ker}(\mathcal{R}^{\text{T}})=d$. By Lemma 6.1, $\text{Ker}(\mathcal{R}^{\text{T}})$ is contained in $\text{Ker}(D^{\text{T}})$ and invariant for A^{T} and B^{T}. Since $\text{Ker}(\mathcal{R}^{\text{T}})$ is invariant for B^{T}, by Lemma 6.2, $\text{Ker}(\mathcal{R}^{\text{T}})$ is also invariant for G_1^T. Noting that 1 is not an eigenvalue of B, G_1 is invertible. Moreover, since $\text{Ker}(\mathcal{R}^{\text{T}})$ is contained in $\text{Ker}(D^{\text{T}})$, it is easy to see that $\text{Ker}(\mathcal{R}^{\text{T}})$ is contained in $\text{Ker}(D^T(I_n+B)^{-\text{T}})$. Thus the desired result can be proved by applying [6, Theorem 7.1] to system (1.8), (1.10) and (1.11c).

Corollary 6.2 *Assume that A and B satisfy the conditions of C_p-compatibility* (3.9) *and* (3.10)*, respectively. Assume furthermore that -1 is not an eigenvalue of B, and 1 is not an eigenvalue of $\overline{B}_p$ given by* (3.10)*. If system* (1.3)–(1.4) *with* (1.5c) *is exactly synchronizable by p-groups, then we necessarily have*

$$\text{rank}(C_p\mathcal{R})=n-p. \tag{6.13}$$

Proof Under the conditions of C_p-compatibility for A and B, the exact boundary synchronization by p-groups for system (1.3)–(1.4) with (1.5c) can be equivalently transformed into the exact boundary null controllability for the reduced system (3.12)–(3.13) with (3.14c). Then, applying Theorem 6.2 to this reduced system (3.12)–(3.13) with (3.14c) and noting (3.9)–(3.10), we have (6.13).

Remark 6.1 By [16, Proposition 2.21], if 1 is not an eigenvalue of B, then 1 is not an eigenvalue of $\overline{B}_p$.

7 Remark

The preceding method can be used to consider the corresponding problems for the following 1-D coupled system of wave equations

$$U_{tt} - U_{xx} + AU_t + \overline{A}U_x = 0, \quad t \in (0, +\infty), \ x \in (0, L) \tag{7.1}$$

with the same boundary conditions (1.4) and (1.5) and the same initial data (1.6), where both A and $\overline{A}$ are matrices of order n with real constant elements.

Let V be defined by (1.7). We have still system (1.8) and (1.10)–(1.11) with

$$\mathbb{A} = \frac{1}{2}\begin{pmatrix} A + \overline{A} & A - \overline{A} \\ A + \overline{A} & A - \overline{A} \end{pmatrix}. \tag{7.2}$$

Suppose furthermore that the matrix $\overline{A}$ also satisfies the condition of C_p-compatibility

$$\begin{aligned} &\overline{A}\mathrm{Ker}(C_p) \subseteq \mathrm{Ker}(C_p), \quad \text{namely,} \quad C_p\overline{A} = \overline{\overline{A}}_p C_p, \\ &\text{or} \quad \overline{A}\epsilon_i = \sum_{j=1}^{p} \overline{\alpha}_{ij}\epsilon_j \quad \text{for } i = 1, \cdots, p, \end{aligned} \tag{7.3}$$

in which $\overline{\overline{A}}_p$ is a matrix of order $(n-p)$, and $\overline{\alpha}_{ij} (i, j = 1, \cdots, p)$ are constants, we can similarly get all the results mentioned above, in which the exactly synchronizable state by p-groups $(\widetilde{u}_1, \cdots, \widetilde{u}_p)^{\mathrm{T}}$ satisfies

$$\widetilde{u}_{itt} - \widetilde{u}_{ixx} + \sum_{j=1}^{p} \alpha_{ji}\widetilde{u}_{jt} + \sum_{j=1}^{p} \overline{\alpha}_{ji}\widetilde{u}_{jx} = 0, \quad t \in (T, +\infty), \ x \in (0, L) \tag{7.4}$$

with (4.2)–(4.3) for $i = 1, \cdots, p$, and the exactly synchronizable state by p-groups $(\widetilde{u}_1, \cdots, \widetilde{u}_p)^{\mathrm{T}}$ satisfies the estimate (4.16), where $\phi_i (i = 1, \cdots, p)$ satisfy the system

$$\phi_{itt} - \phi_{ixx} + \sum_{j=1}^{p} \alpha_{ji}\phi_{jt} + \sum_{j=1}^{p} \overline{\alpha}_{ji}\phi_{jx} = 0, \quad t \in (0, +\infty), \ x \in (0, L) \tag{7.5}$$

with (4.7)–(4.9), where α_{ij}, β_{ij} and $\overline{\alpha}_{ij} (i, j = 1, \cdots, p)$ are given by (3.9), (3.10) and (7.3), respectively.

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