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Digital Cofibration and Digital Lusternik-Schnirelmann Category in the Sense of Subdivision*

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Abstract In this paper, using the notion of subdivision, the authors generalize the definition of cofibration in digital topology and show that this kind of cofibration is injective in the sense of subdivision. Meanwhile, they give the necessary condition under which a digital map is a cofibration. Furthermore, they consider the Lusternik-Schnirelmann category of digital maps in the sense of subdivision and give several fundamental homotopy properties about it.

Keywords Digital topology, Digital cofibration, Lusternik-Schnirelmann category
2000 MR Subject Classification 55M30, 54A99

1 Introduction

In digital topology, the basic object is a digital image. A digital image is a finite set of integer lattice points in an ambient Euclidean space with an adjacency relation between points. In our daily life, the sets which can be handled on computers are discrete sets or finite sets. Studying digital topology is helpful to develop artificial intelligence, image recognition and some others technology. Many topologists have made some progress along this direction. In [3, 5], some basic concepts of digital topology were introduced and some results of digital topology via techniques from classical topology were obtained. In [3], Boxer constructed the digital fundamental group of a digital image based on the notions of digitally continuous functions and digital homotopy. In [5], it was shown that a digital image $X \subseteq \mathbb{Z}^n (n \geq 3)$ admits a continuous analog $C(X) \subseteq \mathbb{R}^n$ such that the digital fundamental group of X is isomorphic to that of $C(X)$. However, the definition of digital fundamental group in [3, 5] can not greatly resemble the classical construction of the fundamental group of a topological space.

In [4], Ege-Karaca gave the definition of fibration in digital topology with analogy of the definition of fibration in classical algebraic topology and showed that the composition or the product of fibrations is still fibration. However, the defined digital fibrations in [4] do not coincide with classical fibrations in topology in aspect of some basic properties. In [1–2], Borat-Vergili defined the notion of Lusternik-Schnirelmann category (LS category for short) of a digital

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space and a digital map, respectively. In [1], it was shown that the digital LS category of a digital space is a digital homotopy invariant. In [2], Borat-Vergili showed how the digital LS category of a digital map behaves after changing the adjacency relation.

According to the above statements, we see that the former literatures just simply directly translate the topological notions into the digital topology. This leads to no more results can be obtained as the classical topology. Considering this shortcoming, in [7], Lupton-Oprea-Scoville introduced the idea of subdivision and redefined some concepts of digital topology such as cofibrations and LS category. By using this new notion of digital cofibrations, they established some basic examples such as the inclusion of one or both endpoints into an interval being a cofibration. In [6], Lupton-Oprea-Scoville redefined the fundamental group for digital images and showed that this kind of fundamental group can be preserved in the sense of subdivision. In [8] Lupton-Oprea-Scoville mainly discussed the subdivision of digital maps and established some related results about the subdivision of digital maps with one or two dimensional domains.

We note that Lupton-Oprea-Scoville [7] only gave the definition of digital cofibrations as inclusion maps without considering the injective map. Hence by virtue of the notion of subdivision, we define a generalized digital cofibration and show that the generalized cofibration is injective in the sense of subdivision. Meanwhile, we define LS category of a digital map in the sense of subdivision and give some basic properties of this notion under some restricted conditions. Our main results are stated as follows.

Theorem 1.1 *If a digital map $f\colon A \to X$ is a digital cofibration, then f is injective in the sense of subdivision.*

Theorem 1.2 *For a given digital map $f\colon X \to Y$, suppose that the subdivision $S(f,k)$ exists for any $k \geq 0$. Then*

(1) *for any digital map $g\colon Y \to Z$, there is $\mathrm{Dcat}(g \circ f) \leq \min\{\mathrm{Dcat}(f), \mathrm{Dcat}(g)\}$,*

(2) *for any digital map $f'\colon X' \to Y'$, there is $\mathrm{Dcat}(f \times f') \leq \mathrm{Dcat}(f) \cdot \mathrm{Dcat}(f')$,*

where $\mathrm{Dcat}(f)$ denotes the digital LS category of the digital map f.

This paper is organized as follows. In Section 2, we introduce some notions in the sense of subdivision defined in [7] and recall some basic properties about these notions. In Section 3, we give the proof of Theorem 1.1 and give a condition under which a digital map is a digital cofibration. In Section 4, we define the notion of digital LS category of a digital map and then give the proof of Theorem 1.2.

2 Some Preliminaries on Digital Topology

In this section, we introduce some basic notions in digital topology.

Definition 2.1 (see [7]) *A digital image X is a finite subset $X \subseteq \mathbb{Z}^n$ with a particular adjacency relation inherited from that of $\mathbb{Z}^n$. Two points $x = (x_1, \cdots, x_n) \in X \subseteq \mathbb{Z}^n$ and $y = (y_1, \cdots, y_n) \in X \subseteq \mathbb{Z}^n$ are adjacent if their coordinates satisfy $|x_i - y_i| \leq 1$ for each $i = 1, \cdots, n$, denoted by $x \sim_X y$.*

Definition 2.2 (see [7]) *For digital images $X \subseteq \mathbb{Z}^n$ and $Y \subseteq \mathbb{Z}^m$, a function $f\colon X \to Y$ is called continuous if $f(x) \sim_Y f(y)$ whenever $x \sim_X y$. By a map of digital images, we mean a continuous function.*

Definition 2.3 (see [7]) *An isomorphism between two digital images is a continuous bijection $f\colon X \to Y$ which admits a continuous inverse $g\colon Y \to X$.*

If $f\colon X \to Y$ is an isomorphism, then X is isomorphic to Y, being denoted by $X \cong Y$.

Definition 2.4 (see [7]) *A digital interval of length N is the set $\{0, 1, \cdots, N\}$, denoted by I_N.*

Definition 2.5 (see [7]) *The product of digital images X with Y is the Cartesian product of sets $X \times Y$ with the adjacency relation $(x, y) \sim_{X\times Y} (x', y')$ when $x \sim_X x'$ and $y \sim_Y y'$.*

Definition 2.6 (see [7]) *Given maps of digital images $f_i\colon X_i \to Y_i$ $(i = 1, 2)$, we define their product in the usual way as*

$$f_1 \times f_2\colon X_1 \times X_2 \to Y_1 \times Y_2$$

by $(f_1 \times f_2)(x_1, x_2) = (f_1(x_1), f_2(x_2))$.

The product of maps is obviously a continuous map.

Definition 2.7 (see [7]) *For a given digital image $X \in \mathbb{Z}^n$ and each $k \geq 2$, a k-fold subdivision of X, denoted by $S(X, k)$, is defined by $S(X, k) = \bigcup\limits_{x\in X} S(x, k)$, where*

$$S(x, k) = \{(kx_1 + r_1, \cdots, kx_n + r_n) \mid 0 \leq r_i \leq k - 1\}$$

for any $x = (x_1, \cdots, x_n) \in X$. A projection $\rho_k\colon S(X, k) \to X$ is defined by $\rho_k(y) = \left(\left\lfloor \frac{y_1}{k} \right\rfloor, \cdots, \left\lfloor \frac{y_n}{k} \right\rfloor\right)$ for any $y = (y_1, \cdots, y_n) \in S(X, k)$.

Proposition 2.1 (see [7]) *For any digital images $X, Y \subseteq Z^m$ and any $k \geq 2$, there is*

$$S(X \times Y, k) \cong S(X, k) \times S(Y, k)$$

and the standard projection $\rho_k\colon S(X \times Y, k) \to X \times Y$ can be indentified with

$$\rho_k = \rho_k \times \rho_k : S(X, k) \times S(Y, k) \to X \times Y.$$

Proposition 2.2 (see [7]) *For any $k, l \in \mathbb{Z}$ and digital image $X \subseteq \mathbb{Z}^n$, there is $S(S(X, k), l) \cong S(X, kl)$.*

Definition 2.8 (see [7]) *Let $f, g\colon X \to Y$ be two digital maps. We say that f and g are homotopic, denoted by $f \simeq g$, if for some $N \geq 1$, there is a map*

$$H\colon X \times I_N \to Y$$

such that $H(x, 0) = f(x), H(x, N) = g(x)$ for any $x \in Y$. In this case, H is called a homotopy from f to g.

Proposition 2.3 (see [7]) *Homotopy relation is an equivalence relation on the set of all maps from X to Y.*

Proposition 2.4 (see [7]) *For digital maps $g, h\colon X \to Y$ and $f\colon Y \to Z$, if $g \simeq h$, then $f \circ g \simeq f \circ h$.*

Definition 2.9 (see [7]) *A digital image X is called contractible if there are some $x_0 \in X$ and some N such that we have a homotopy $H\colon X \times I_N \to X$ satisfying $H(x,0) = x$ and $H(x,N) = x_0$.*

Definition 2.10 *A digital map $f\colon X \to Y$ is called null homotopic if there are some $y_0 \in Y$ and some N such that we have a homotopy $H\colon X \times I_N \to Y$ satisfying $H(x,0) = f(x)$ with $H(x,N) = y_0$.*

Definition 2.11 (see [6]) *A digital image X is called subdivision-contractible if for some subdivision $S(X,k)$ of X, some $x_0 \in X$ and some N, there is a homotopy $H\colon S(X,k) \times I_N \to X$ such that $H(x,0) = \rho_k(x)$ and $H(x,N) = x_0$ for any $x \in X$.*

Definition 2.12 *For two digital images $X, Y \subseteq Z^n$. Two digital maps $f\colon S(X,k) \to Y$ and $f\colon S(X,l) \to Y$ are called subdivision-homotopic if for some k', l' with $kk' = ll' = m$, such that we have a homotopy $f \circ \rho_{k'} \simeq g \circ \rho_{l'}$. In particular, if g is a constant map, then f is called subdivision-null-homotopic.*

Definition 2.13 (see [7]) *By an inclusion of digital images $i\colon A \to X \subseteq \mathbb{Z}^n$ we mean that A is a subset of X.*

It is obvious that given an inclusion of digital images of the same dimension $i\colon A \to X \subseteq \mathbb{Z}^n$, we have an obvious corresponding continuous inclusion of subdivisions $S(i,k)\colon S(A,k) \to S(X,k)$ such that the following commutative diagram holds:

$$\begin{array}{ccc} S(A,k) & \xrightarrow{S(i,k)} & S(X,k) \\ \downarrow{\scriptstyle \rho_k} & & \downarrow{\scriptstyle \rho_k} \\ A & \xrightarrow{\quad i \quad} & X. \end{array}$$

In this case, We say that the map $S(i,k)$ covers the map i. For any $a = (a_1, \cdots, a_n) \in A$ and $t = (t_1, \cdots, t_n)$ $(0 \le t_1, \cdots, t_n \le k-1) \in (I_{k-1})^n$, the points $S(a,k) \subseteq S(A,k)$ can be written as

$$S(a,k) = \{ka + t \mid t \in (I_{k-1})^n\} = \{(ka_1 + t_1, \cdots, ka_n + t_n) \mid 0 \le t_1, \cdots, t_n \le k-1\}$$

with $\rho_k(ka+t) = a$ for all $t \in (I_{k-1})^n$. Thus $S(i,k) : S(A,k) \to S(X,k)$ can be written as

$$S(i,k)(ka+t) = ki(a) + t,$$

where $i(a) = (a_1, \cdots, a_n) \in X$.

Remark 2.1 For two inclusion of digital images $i\colon A \to X$ and $j\colon B \to A$, we obviously have $S(i \circ j, k) = S(i,k) \circ S(j,k)$.

Remark 2.2 It is known from [7] that a general map $f\colon X \to Y$ may not induce a subdivision map $S(f,k)\colon S(X,k) \to S(Y,k)$ which makes the following diagram

$$\begin{array}{ccc} S(X,k) & \xrightarrow{S(f,k)} & S(Y,k) \\ \downarrow{\scriptstyle \rho_k} & & \downarrow{\scriptstyle \rho_k} \\ X & \xrightarrow{\quad f \quad} & Y \end{array}$$

commute. In [8], Lupton-Oprea-Scoville gave a full discussion on $S(f,k)$.

3 Digital Cofibrations

In classical topology, a cofibration $i\colon A \to X$ is a map satisfying the homotopy extension property. A map $i\colon A \to X$ is a cofibration if for any maps

$$f\colon X \to Y$$

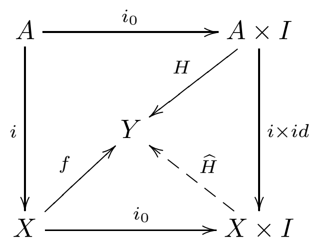
and

$$H\colon A \times I \to Y$$

satisfying $H(a,0) = f \circ i(a)$, there is a map

$$\widehat{H}\colon X \times I \to Y$$

such that the diagram



commutes. The inclusion $i\colon \{0\} \to I$ is a classical cofibration in classical toplogy.

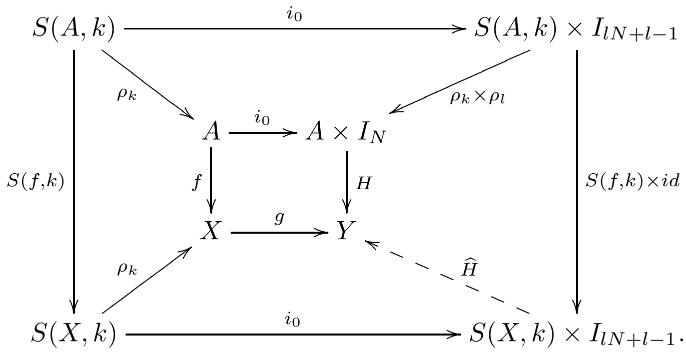
As we know, cofibration as well as fibration are two fundamental notions in homotopy theory of various categories. They are the starting point of research on homotopy theory. The cofibraiton may help us to understand the notion of homology via the cofibre sequence. Thus in order to construct the system of digital homotopy theory, it is meaningful for us to understand the digital cofibration in digital topology. In [7], it was shown that it is difficult to repeat this definition in the digital setting due to the fact that the inclusion $i\colon \{0\} \to I_M$ fails to be a cofibration. In order to make the inclusion $i\colon \{0\} \to I_M$ be a digital cofibration, Lupton-Oprea-Scoville [7] redefined the digital cofibration in mapping space in a less rigid way. In what follows, we give another definition of digital cofibration which is equivalent to that in [7] in a dual way. In the category of topological spaces, Spanier [9] gave a general definition of cofibration which needs not to be an inclusion. Inspired by the above definition of the digital cofibration, we also give a general definition of digital cofibration as follows.

Definition 3.1 *A digital map $f\colon A \to X \subseteq \mathbb{Z}^n$ is a cofibration if for a given commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{i_0} & A \times I_N \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle H} \\ X & \xrightarrow{g} & Y, \end{array}$$

there are subdivisions $S(X,k)$ and $S(I_N,l)$ as well as maps $S(f,k)\colon S(A,k) \to S(X,k)$ and

$\widehat{H}\colon S(X,k)\times S(I_N,l)\to Y$ such the following commutative diagram holds:

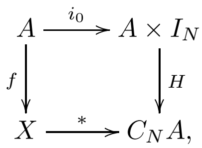


Remark 3.1 Definition 3.1 is equivalent to the one in [7] in the special case of inclusion of digital images.

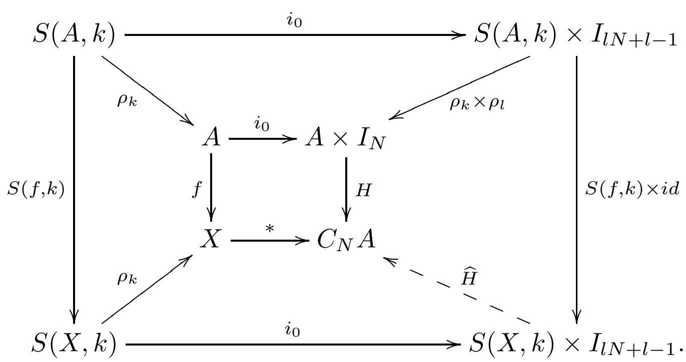
It is known that a cofibration is injective in the category of topological spaces [9]. In digital topology, we also have an analogous result in the digital category as follows.

Theorem 3.1 (see Theorem 1.1) *If a digital map $f\colon A\to X$ is a digital cofibration, then f is injective in the sense of subdivision.*

Proof Let $C_NA = A\times I_N/A\times\{0\}$. We have a commutative diagram



where $H(a,0)=*, H(a,t)=[(a,t)]$. Since f is a digital cofibration, there are subdivisions $S(X,k)$, $S(I_N,l)$, and maps $S(f,k)\colon S(A,k)\to S(X,k)$ and $\widehat{H}\colon S(X,k)\times S(I_N,l)\to C_NA$ such that the following commutative diagram holds:



For any $a\in S(A,k)$, we have

$$\widehat{H}(S(f,k)(a), lN+l-1) = H(\rho_k(a), \rho_l(lN+l-1)) = H(\rho_k(a), N) = \rho_k(a),$$

which implies $S(f,k)(a_1)\neq S(f,k)(a_2)$ whenever $\rho_k(a_1)\neq\rho_k(a_2)$.

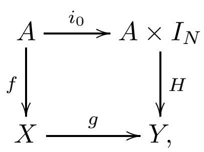
Moreover, if $\rho_k(a_1) \neq \rho_k(a_2)$ and $\rho_k(S(f,k)(a_1)) \neq \rho_k(S(f,k)(a_2))$, then it means that $S(f,k)(a_1)$ and $S(f,k)(a_2)$ do not belong to the same $S(x,k)$ $(x \in X)$. Thus we have

$$f(\rho_k(a_1)) = \rho_k(S(f,k)(a_1)) \neq \rho_k(S(f,k)(a_2)) = f(\rho_k(a_2)),$$

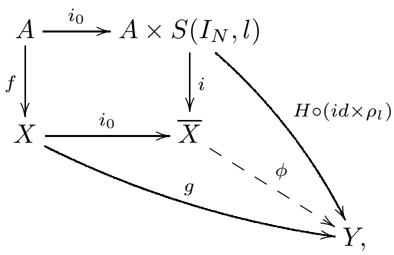
which follows that f is injective.

In [8], it is shown that if $A \subseteq \mathbb{Z}^n$ and k is odd, a digital map $f\colon A \to X$ can always induce a map $S(f,k)\colon S(A,k) \to S(X,k)$ satisfying $\rho_k(S(f,k)(a_1)) \neq \rho_k(S(f,k)(a_2))$ when $\rho_k(a_1) \neq \rho_k(a_2)$.

Lemma 3.1 *For any digital map $f\colon A \to X$, if we have a diagram*



then for any subdivision $S(I_N,l)$ $(l \geq 2)$, there is a digital map $\phi\colon A \times S(I_N,l) \cup X \times \{0\} \to Y$ such that the following commutative diagram holds:



where $\overline{X} = A \times S(I_N,l) \cup X \times \{0\}/\overset{f}{\sim}$ with the relation $(a,0)\overset{f}{\sim}(x,0)$ if and only if $f(a) = x$.

Proof For any $x \in S(X,k)$, $a \in S(A,k)$ and $t \in S(I_N,l)$, define

$$\phi([x,0]) = g(x) \quad \text{and} \quad \phi([a,t]) = H \circ (id \times \rho_l)(a,t).$$

In order to show that ϕ is continuous, we just need to show that $[(a,1)] \sim [x,0](a \in A, x \in X)$ implies $\phi([a,1]) \sim \phi([x,0])$. This directly follows from

$$\phi([a,1]) = H(a,\rho_l(1)) = H(a,0) = gf(a) \sim gf(x) = \phi([x,0]).$$

In [7], Lupton-Oprea-Scoville gave a digital version of the necessary condition for an inclusion of digital images being a cofibration. Similarly, under the condition that the subdivision map $S(f,k)$ exists, we give a digital version of the necessary condition for a general digital map being a cofibration.

Theorem 3.2 (1) *If a digital map $f\colon A \to X$ is a digital cofibration, then for any I_N, there are subdivisions $S(X,k)$, $S(I_N,l)$, and a map*

$$R\colon S(X,k) \times S(I_N,lm) \to \overline{X}$$

for some $m \in \mathbb{N}$ *such that*

$$R(x,0) = [\rho_k(x), 0] \quad \text{and} \quad R(S(f,k)(a), t) = [\rho_k(a), \rho_m(t)]$$

for any $a \in S(A,k)$.

(2) *A digital map* $f\colon A \to X$ *is a digital cofibration if for any* $k \in \mathbb{N}$ *and* I_N*, there are a map* $S(f,k)\colon S(A,k) \to S(X,k)$ *satisfying*

$$\begin{array}{ccc} S(A,k) & \xrightarrow{S(f,k)} & S(X,k) \\ \downarrow{\scriptstyle \rho_k} & & \downarrow{\scriptstyle \rho_k} \\ A & \xrightarrow{f} & X \end{array}$$

and a map $R\colon S(X,k) \times S(I_N, lm) \to \overline{X}$ *such that*

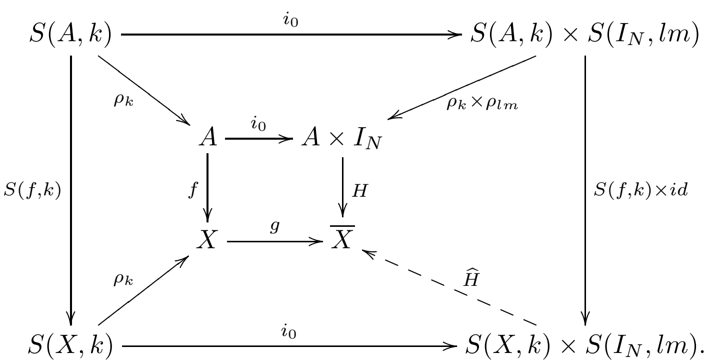
$$R(x,0) = [\rho_k(x), 0] \quad \text{and} \quad R(S(f,k)(a), t) = [\rho_k(a), \rho_m(t)]$$

for any $a \in S(A,k)$.

Proof (1) We define the digital maps $g\colon X \to \overline{X}$ by $g(x) = [x,0]$ and $H\colon A \times S(I_N, l) \to \overline{X}$ by $H(a,t) = [a,t]$, respectively. It follows the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{i_0} & A \times I_N \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle H} \\ X & \xrightarrow{g} & \overline{X}. \end{array}$$

Since $f\colon A \to X$ is a digital cofibration, there are subdivisions $S(X,k)$ and $S(I_N, lm)$, with maps $S(f,k)\colon S(A,k) \to S(X,k)$ and $\widehat{H}\colon S(X,k) \times S(I_N, lm) \to Y$ such that the following diagram holds:



Let $R = \widehat{H}$. Thus we have

$$R(x,0) = g \circ \rho_k(x) = [\rho_k(x), 0], \quad R(S(f,k)(a), t) = H(\rho_k(a), \rho_m(t)) = [\rho_k(a), \rho_m(t)].$$

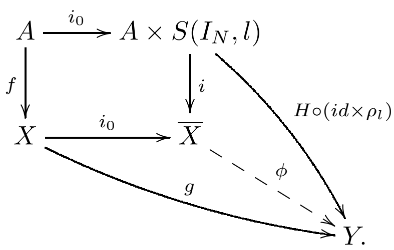
(2) Suppose that there are two digital maps $g\colon X \to Y$ and $H\colon A \times I_N \to Y$ such that the following commutative diagram holds:

$$\begin{array}{ccc} A & \xrightarrow{i_0} & A \times I_N \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle H} \\ X & \xrightarrow{g} & Y. \end{array}$$

We define a map $R\colon S(X,k)\times S(I_N,lm)\to \overline{X}$ by

$$R(x,0)=[\rho_k(x),0] \quad \text{and} \quad R(S(f,k)(a),t)=[\rho_k(a),\rho_m(t)], \quad \text{where } a\in S(A,k).$$

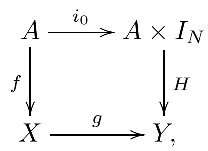
According to Lemma 3.1, there is a digital map $\phi\colon A\times S(I_N,l)\cup X\times\{0\}\to Y$ such that the following commutative diagram holds:



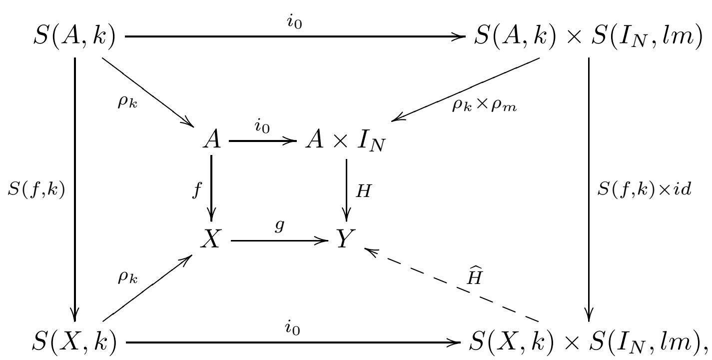
Let $\widehat{H}=\phi\circ R\colon S(X,k)\times S(I_N,lm)\to Y$. Thus for any $x\in S(X,k), a\in S(A,k)$ and $t\in S(I_N,lm)$, we have $\widehat{H}(x,0)=\phi\circ R(x,0)=\phi([\rho_k(x),0])=g\circ\rho_k(x)$ and

$$\begin{aligned}\widehat{H}(S(f,k)(a),t)&=\phi\circ R(S(f,k)(a),t)=\phi([\rho_k(a),\rho_m(t)])\\&=H\circ(id\times\rho_l)([\rho_k(a),\rho_m(t)])=H\circ(id\times\rho_{lm})(a,t).\end{aligned}$$

Hence, for two digital maps $g\colon X\to Y$ and $H\colon A\times I_N\to Y$ satisfying the commutative diagram



we have the following commutative diagram



which shows that $f\colon A\to X$ is a digital cofibration.

By applying the above results, we can give an example of injective map which is a digital cofibration but not an inclusion.

Example 3.1 For the digital images $X=\{(0,2,1),(0,1,0),(1,1,0),(2,1,0),(2,0,1)\}$ and $Y=I_2\times I_2$, we define a digital map $f\colon X\to Y$ by $f(0,2,1)=(0,2)$, $f(0,1,0)=(0,1)$,

$f(1,1,0)=(1,1)$, $f(2,1,0)=(2,1)$, $f(2,0,1)=(2,0)$. It is easy to check that f is injective. We define a digital function $D\colon S(Y,2)\to S(f(X),2)$ by

$$D(x,y)=\begin{cases}(1,4), & (x,y)=(2,5),\\ (1,3), & (x,y)=(2,4),(3,5),\\ (2,3), & (x,y)=(3,4),(4,5),\\ (3,3), & (x,y)=(4,4),(5,5),\\ (4,3), & (x,y)=(5,4),\\ (1,2), & (x,y)=(0,1),\\ (2,2), & (x,y)=(0,0),(1,1),\\ (3,2), & (x,y)=(1,0),(2,1),\\ (4,2), & (x,y)=(2,0),(3,1),\\ (4,1), & (x,y)=(3,0),\\ (x,y), & \text{other else}\end{cases}$$

and a digital function $D'\colon S(Y,2)\times S(I_1,2)\to S(f(X),2)\times S(I_1,2)\cup S(Y,2)\times\{0\}$ by

$$D'((x,y),t)=\begin{cases}D((x,y),t), & t=1,2,3,\\ (x,y), & t=0.\end{cases}$$

It is routine to check that D and D' are not continuous but $(\rho_2\times\rho_2)\circ D'$ is continuous. Also we have a digital map $S(f,2)\colon S(X,2)\to S(Y,2)$ defined by

$$S(f,2)(p,q,r)=\begin{cases}(1,4), & (p,q,r)\in S((0,2,1),2),\\ (1,3), & (p,q,r)\in S((0,1,0),2),\\ (2,3), & (p,q,r)=(i,j,0), i=2,3, j=2,3,\\ (3,3), & (p,q,r)=(i,j,1), i=2,3, j=2,3,\\ (4,3), & (p,q,r)=(i,j,0), i=4,5, j=2,3,\\ (4,2), & (p,q,r)=(i,j,1), i=4,5, j=2,3,\\ (4,1), & (p,q,r)\in S((2,0,1),2),\end{cases}$$

which makes the following diagram commutes:

$$\begin{array}{ccc} S(X,2) & \xrightarrow{S(f,2)} & S(Y,2)\\ \downarrow{\scriptstyle\rho_2} & & \downarrow{\scriptstyle\rho_2}\\ X & \xrightarrow{\quad f\quad} & Y. \end{array}$$

Furthermore, we have an obvious isomorphic digital map

$$g\colon f(X)\times I_1\cup Y\times\{0\}\to X\times I_1\cup Y\times\{0\}/\overset{f}{\sim}.$$

Let $R=g\circ((\rho_2\times\rho_2)\circ D')\colon S(Y,2)\times S(I_1,2)\to X\times I_1\cup Y\times\{0\}/\overset{f}{\sim}$. Thus we have

$$R(x,0)=[\rho_2(x),0]\quad\text{and}\quad R(S(f,2)(x),t)=[\rho_2(x),\rho_2(t)].$$

It follows that $f\colon X\to Y$ is a digital cofibration according to Theorem 3.2(2).

4 Lusternik-Schnirelmann Category of Digital Map

In [7], Lupton-Oprea-Scoville redefined the LS category in digital topology via the notion of subdivision. Inspired by the new definition in [7], in this section we give a definition of LS category for the digital map via the notion of subdivision and give some fundamental properties about this notion.

Definition 4.1 (see [7]) *Let* $i\colon A \to X$ *be an inclusion and we say* A *is categorical in* X *if there are* $x_0 \in X$, I_N *and a map* $H\colon A \times I_N \to X$ *such that* $H(a,0) = i(a)$, $H(a,N) = x_0$ *for any* $a \in A$.

Definition 4.2 (see [7]) *Let* $i\colon A \to X$ *be an inclusion. We say that* A *is subdivision-categorical in* X *if there are subdivision* $S(X,k)$, $x_0 \in X$, I_N *and a map* $H\colon S(A,k) \times I_N \to X$ *such that* $H(a',0) = i \circ \rho_k(a')$, $H(a',N) = x_0$ *for any* $a' \in S(A,k)$.

Definition 4.3 (see [7]) *The digital category of* X*, denoted by* $\mathrm{Dcat}(X)$*, is the smallest number* $n \geq 0$ *for which there is a covering of* X *by* $n+1$ *subsets that are subdivision-categorical in* X.

Definition 4.4 *The digital category of a digital map* $f\colon X \to Y$*, denoted by* $\mathrm{Dcat}(f)$*, is the smallest number* $n \geq 0$ *for which there is a covering of* X *by* $n+1$ *subsets* $\{A_1, \cdots, A_{n+1}\}$ *that each* $f(A_j)$ $(j = 1, \cdots, n)$ *is subdivision-contractible in* Y*, i.e., there are* k_j *and* N*, with the map* $H_j\colon S(A_j,k_j) \times I_N \to Y$ *such that* $H_j(a',0) = f|_{A_j} \circ \rho_{k_j}(a')$ *and* $H_j(a',N) = y_j$ *for any* $a' \in S(A_j,k_j)$.

Remark 4.1 When we take $f = id_X\colon X \to X$, then there is $\mathrm{Dcat}(X) = \mathrm{Dcat}(id_X)$.

In what follows, we assume that for any digital map and any $k \geq 0$, there is a subdivision map $S(f,k)\colon S(X,k) \to S(Y,k)$ such that the following commutative diagram holds:

$$\begin{array}{ccc} S(X,k) & \xrightarrow{S(f,k)} & S(Y,k) \\ \downarrow{\scriptstyle \rho_k} & & \downarrow{\scriptstyle \rho_k} \\ X & \xrightarrow{\quad f \quad} & Y. \end{array}$$

Proposition 4.1 *If* $f\colon X \to Y$ *is a digital map, then* $\mathrm{Dcat}(f) \leq \min\{\mathrm{Dcat}(X), \mathrm{Dcat}(Y)\}$.

Proof Assume that $\mathrm{Dcat}(X) = n$ and $\{A_1, \cdots, A_{n+1}\}$ is a subdivision-contractible covering on X. For every inclusion $i_j\colon A_j \to X$, there is a subdivision $S(A_j,k_j)$, and a map $H_j\colon S(A_j,k_j) \times I_N \to X$ such that

$$H_j(a,0) = i_j \circ \rho_{k_j}(a) \quad \text{and} \quad H_j(a,N) = a_j \text{ for any } a \in S(A_j,k_j).$$

Let $F_j = f \circ H_j\colon S(A_j,k_j) \times I_N \to Y$. It is routine to check that $f(A_j)$ $(j = 1, \cdots, n)$ are subdivision-contractible in Y and thus we have $\mathrm{Dcat}(f) \leq \mathrm{Dcat}(X)$.

Assume that $\mathrm{Dcat}(Y) = m$ and $\{V_1, \cdots, V_{m+1}\}$ is a subdivision-contractible covering on Y which implies that $\{f^{-1}(V_1), \cdots, f^{-1}(V_{m+1})\}$ is a covering on X. For every inclusion $i'_l\colon V_l \to Y$, there is a subdivision $S(V_l,k'_l)$ and a map $G_l\colon S(V_l,k'_l) \times I_M \to Y$ such that

$$G_l(v,0) = i'_l \circ \rho_{k'_l}(v) \quad \text{and} \quad G_l(v,N) = y_l.$$

For the map $f\colon X \to Y$, there is a subdivision $S(f,k_l')\colon S(X,k_l') \to S(Y,k_l')$ such that the following commutative diagram holds:

$$\begin{array}{ccc} S(X,k_l') & \xrightarrow{S(f,k_l')} & S(Y,k_l') \\ \Big\downarrow{\scriptstyle \rho_{k_l'}} & & \Big\downarrow{\scriptstyle \rho_{k_l'}} \\ X & \xrightarrow{\quad f \quad} & Y. \end{array}$$

Define $G_l'\colon S(f^{-1}(V_l),k_l') \times I_N \to Y$ by $G_l'(x,t) = G_l(S(f,k_l')|_{S(f^{-1}(V_l),k_l')}(x),t)$. Then we have

$$G_l'(x,0) = i_l' \circ \rho_{k_l'} \circ S(f,k_l')|_{S(V_l,k_l')}(x) = f|_{f^{-1}(V_l)} \circ \rho_{k_l'}(x) \quad \text{and} \quad G_l'(x,N) = y_l,$$

which implies $\mathrm{Dcat}(f) \le \mathrm{Dcat}(Y)$. Thus we have $\mathrm{Dcat}(f) \le \min\{\mathrm{Dcat}(X), \mathrm{Dcat}(Y)\}$.

Proposition 4.2 *If $f \simeq g\colon X \to Y$, then there is* $\mathrm{Dcat}(f) = \mathrm{Dcat}(g)$.

Proof We just need to show that $\mathrm{Dcat}(f) \le \mathrm{Dcat}(g)$. Assume that $\mathrm{Dcat}(f) = n$ and $\{A_1, \cdots, A_{n+1}\}$ is a covering of X with $f(A_j)$ $(1 \le j \le n+1)$ being subdivision-contractible in Y. Then there is a subdivision $S(A_j,k_j)$ and a map $H_j\colon S(A_j,k_j) \times I_N \to Y$ such that

$$H_j(a',N) = y_j \text{ for some } y_j \in Y \quad \text{and} \quad H_j(a',0) = f|_{A_j} \circ \rho_{k_j}(a') \text{ for any } a' \in S(A_j,k_j).$$

We define $e_j\colon S(A_j,k_j) \to Y$ by $e_j(a') = y_j$ for any $a' \in S(A_j,k_j)$. It follows $f|_{A_j} \circ \rho_{k_j} \simeq e_j$. Since $f \simeq g$, we have $f|_{A_j} \simeq g|_{A_j}$ and $f|_{A_j} \circ \rho_{k_j} \simeq g|_{A_j} \circ \rho_{k_j}$. Thus $g|_{A_j} \circ \rho_{k_j} \simeq e_j$ and $g(A_j)$ $(1 \le j \le n)$ is subdivision-contractible in Y. It follows $\mathrm{Dcat}(f) \le \mathrm{Dcat}(g)$.

In what follows we give the proof of Theorem 1.2.

Proof of Theorem 1.2 (1) Assume that $\mathrm{Dcat}(f) = n$, $\{A_1, \cdots, A_{n+1}\}$ is a covering on X with $f(A_j)$ $(1 \le j \le n+1)$ being subdivision-contractible in Y. Then there is a subdivision $S(A_j,k_j)$ and a map $H_j\colon S(A_j,k_j) \times I_N \to Y$ such that

$$H_j(a',0) = y_j \quad \text{for some } y_j \in Y \quad \text{and} \quad H_j(a',N) = f|_{A_j} \circ \rho_{k_j}(a') \quad \text{for all } a' \in S(A_j,k_j).$$

Let $H_j' = g \circ H_j\colon S(A_j,k_j) \times I_N \to Z$. Then we have

$$H_j'(a',0) = g \circ H_j(a',0) = g(y_j)$$

and

$$H_j'(a',N) = g \circ H_j(a',N) = g \circ f|_{A_j} \circ \rho_{k_j}(a') = (g \circ f)|_{A_j} \circ \rho_{k_j}(a').$$

It follows $\mathrm{Dcat}(g \circ f) \le n = \mathrm{Dcat}(f)$.

Assume $\mathrm{Dcat}(g) = m$, $\{V_1, \cdots, V_{m+1}\}$ is a covering of Y with $g(V_l)$ $(1 \le j \le m+1)$ being subdivision-contractible in Z. Then there is a subdivision $S(V_l,k_l')$ and a map $G_l\colon S(V_l,k_l') \times I_M \to Z$ such that

$$G_l(v',0) = z_l \quad \text{for some } z_l \in Z \quad \text{and} \quad G_l(v',N) = g|_{V_l} \circ \rho_{k_l'}(v') \quad \text{for all } v' \in S(V_j,k_l').$$

For $f\colon X \to Y$, there is a subdivision $S(f,k_l')\colon S(X,k_l') \to S(Y,k_l')$ such that the following commutative diagram holds:

$$\begin{array}{ccc} S(X,k_l') & \xrightarrow{S(f,k_l')} & S(Y,k_l') \\ \Big\downarrow{\scriptstyle \rho_{k_l'}} & & \Big\downarrow{\scriptstyle \rho_{k_l'}} \\ X & \xrightarrow{\quad f \quad} & Y. \end{array}$$

We define $G'_l\colon S(f^{-1}(V_l), k_l) \times I_M \to Z$ by

$$G'_l(x,t) = G_l(S(f,k'_l)(x),t)$$

for any $x \in S(f^{-1}(V_l), k'_l)$ and $t \in I_M$. Then we have $G'_l(x,0) = z_l$ and

$$\begin{aligned} G'_l(x,M) &= G_l(S(f,k'_l)(x),M) = g|_{V_l} \circ \rho_{k'_l} \circ S(f,k'_l)|_{S(f^{-1}(V_l),k'_l)}(x) \\ &= g|_{V_l} \circ f|_{f^{-1}(V_l)} \circ \rho_{k'_l}(x) = (g \circ f)|_{f^{-1}(V_l)} \circ \rho_{k'_l}(x). \end{aligned}$$

It follows $\mathrm{Dcat}(g \circ f) \leq m = \mathrm{Dcat}(g)$ and thus $\mathrm{Dcat}(g \circ f) \leq \min\{\mathrm{Dcat}(f), \mathrm{Dcat}(g)\}$.

(2) Assume that $\mathrm{Dcat}(f) = n$ and $\{A_1, \cdots, A_{n+1}\}$ is a covering of X with $f(A_j)$ $(1 \leq j \leq n+1)$ being subdivision-contractible in Y. Then there is a subdivision $S(A_j, k_j)$ and a map $H_j\colon S(A_j,k_j) \times I_N \to Y$ such that

$$H_j(a',0) = y_j \quad \text{and} \quad H_j(a',N) = f|_{A_j} \circ \rho_{k_j}(a')$$

for any $a' \in S(A_j, k_j)$. Assume that $\mathrm{Dcat}(f') = m$ and $\{B_1, \cdots, B_{m+1}\}$ is a covering of X' with $g(B_i)$ $(1 \leq i \leq m+1)$ being subdivision-contractible in Y'. Then there are a subdivision $S(B_i, l_i)$ and a map $G_i\colon S(B_i,l_i) \times I_M \to Y_2$ such that

$$G_i(b',0) = y_{l_i} \quad \text{and} \quad G_i(b',M) = f'|_{B_i} \circ \rho_{l_i}(b')$$

for any $b' \in S(B_i, l_i)$. For each $A_j \times B_i$, we let $K = \max\{N, M\}$ and define

$$H'_j\colon S(A_j,k_j) \times I_K \to Y \quad \text{and} \quad G'_l\colon S(B_i,l_i) \times I_K \to Y'$$

by

$$H'_j(a,t) = \begin{cases} H_j(a,t), & 0 \leq t \leq N, \\ f|_{A_j} \circ \rho_{k_j}(a), & N \leq t \leq K \end{cases}$$

and

$$G'_l(b,t) = \begin{cases} G_l(b,t), & 0 \leq t \leq M, \\ f'|_{B_i} \circ \rho_{l_i}(b), & M \leq t \leq K. \end{cases}$$

Thus we can define a map $F_i^j\colon S(A_j \times B_i, k_j l_i) \times I_K \to Y \times Y'$ by

$$F_i^j((a',b'),t) = (H'_j(\rho_{l_i}(a'),t), G'_j(\rho_{k_j}(b'),t)).$$

It follows $F_i^j((a',b'),0) = (y_{k_j}, y_{l_i})$ and

$$F_i^j((a',b'),N_i^j) = (f|_{A_j} \circ \rho_{k_j}(\rho_{l_i}(a')), f'|_{B_i} \circ \rho_{l_i}(\rho_{k_j}(b'))) = (f \times f'|_{A_j \times B_i}) \circ \rho_{k_j l_i}(a',b'),$$

which implies that $(f \times f')(A_j \times B_i)$ is subdivision-contractible in $Y \times Y'$. Since the set $\{A_j \times B_i\}$ $(1 \leq j \leq n+1, 1 \leq i \leq m+1)$ is a covering of $X \times X'$, then we have

$$\mathrm{Dcat}(f \times f') \leq (n+1)(m+1) = \mathrm{Dcat}(f) \cdot \mathrm{Dcat}(f').$$

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References

[1] Borat, A. and Vergili, T., Digital Lusternik-Schnirelmann category, *Turkish Jour. Math.*, **42**, 2018, 1845–1852.

[2] Borat, A. and Vergili, T., Digital Lusternik-Schnirelmann category of digital functions, *Hacettepe Jour. Math. and Stat.*, **49**, 2020, 1414–1422.

[3] Boxer, L., A classical construction for the digital fundamental group, *Math. Imaging Vision*, **10**, 1999, 51–62.

[4] Ege, O. and Karaca, I., Digital fibrations, *Proc. Nat. Acad. Sci. India Sect. A*, **87**, 2017, 109–114.

[5] Kong, T. Y., Roscoe, A. W. and Rosenfeld, A., Concepts of digital topology, *Topology Appl.*, **46**, 1992, 219–262.

[6] Lupton, G., Oprea, J. and Scoville, N., A fundamental group for digital images, *Jour. Appl. Comput. Topology*, **5**, 2021, 249–311.

[7] Lupton, G., Oprea, J. and Scoville, N., Homotopy theory in digital topology, *Disc. Comput. Geometry*, **67**, 2022, 112–165.

[8] Lupton, G., Oprea, J. and Scoville, N., Subdivision of maps of digital images, *Disc. Comput. Geometry*, **67**, 2022, 698–742.

[9] Spanier, E., Algebraic Topology, McGraw-Hill, New York, 1966.