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Multiple Nontrivial Solutions for Superlinear Double Phase Problems Via Morse Theory*

Bin GE[1] Beilei ZHANG[2] Wenshuo YUAN[2]

Abstract The aim of this paper is the study of a double phase problems involving superlinear nonlinearities with a growth that need not satisfy the Ambrosetti-Rabinowitz condition. Using variational tools together with suitable truncation and minimax techniques with Morse theory, the authors prove the existence of one and three nontrivial weak solutions, respectively.

Keywords Double phase problems, Musielak-Orlicz space, Variational method, Critical groups, Nonlinear regularity, Multiple solution
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1 Introduction and Main Results

The study of differential equations and variational problems with double phase operator is a new and interesting topic. Such problems go back to Zhikov [1–3] who introduced such classes of operators to describe models of strongly anisotropic materials and also the monograph of Zhikov-Kozlov-Oleinik [4]. The main idea was the introduction of the functional

$$u \mapsto \int_\Omega (|\nabla u|^p + a(x)|\nabla u|^q)\mathrm{d}x, \tag{1.1}$$

where the integrand switches two different elliptic behaviors. More precisely, energies of the form (1.1) are used in the context of homogenization and elasticity and the modulating coefficient $a(\cdot)$ dictates the geometry of a composite of two different materials with distinct power hardening exponents p and q (see [4]). Significant progresses were recently achieved in the framework of regularity results for quasi-minimizer or minimizers of such functionals, see e.g., [5–12].

The purpose of this paper is to investigate the existence and multiplicity of solutions for the following double phase problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P}$$

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[1]Corresponding author. College of Mathematical Sciences, Harbin Engineering University, Harbin 150001, China. E-mail: gebin04523080261@163.com
[2]College of Mathematical Sciences, Harbin Engineering University, Harbin 150001, China.
E-mail: 18737010665@163.com yuanwenshuo@hrbeu.edu.cn
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where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a C^2-boundary $\partial\Omega$, $N \geq 2$, $1 < p < q < N$,

$$\frac{q}{p} < 1 + \frac{1}{N}, \quad a : \overline{\Omega} \mapsto [0, +\infty) \text{ is Lipschitz continuous,} \tag{1.2}$$

and $f : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is a Carathéodory function satisfying the following conditions:

(f_1) $f(x,t) = o(|t|^{q-2}t)$ as $t \to 0$ uniformly in $x \in \Omega$.

(f_2) There exist $q < r < p^*$ and some positive constant C such that

$$|f(x,t)| \leq C(1 + |t|^{r-1}),$$

where $p^* = \frac{Np}{N-p}$ is the critical exponent.

(f_3) $\lim\limits_{|t| \to +\infty} \frac{F(x,t)}{|t|^q} = +\infty$ uniformly in $x \in \Omega$, where $F(x,t) = \int_0^t f(x,s)\mathrm{d}s$.

(f_4) If $\mathcal{F}(x,t) = f(x,t)t - qF(x,t)$, then there exists $g \in L^1(\Omega)$ satisfying

$$\mathcal{F}(x,t) \leq \mathcal{F}(x,s) + g(x) \quad \text{for a.a. } x \in \Omega, \ \text{all } 0 < t < s \text{ or } s < t < 0.$$

The solution of (P) is understood in the weak sense, that is, $u \in W_0^{1,H}(\Omega)$ is a solution of problem (P) if it satisfies

$$\int_\Omega (|\nabla u|^{p-2}\nabla u \cdot \nabla v + a(x)|\nabla u|^{q-2}\nabla u \cdot \nabla v)\mathrm{d}x = \int_\Omega f(x,u)v\mathrm{d}x, \quad \forall v \in W_0^{1,H}(\Omega),$$

where $W_0^{1,H}(\Omega)$ will be defined in Section 2.

Note that energy functional associated to (P) is denoted by

$$\varphi(u) = \int_\Omega \Big(\frac{1}{p}|\nabla u|^p + \frac{a(x)}{q}|\nabla u|^q\Big)\mathrm{d}x - \int_\Omega F(x,u)\mathrm{d}x.$$

It is a well-known consequence of (f_1) and (f_2) that $\varphi \in C^1(W_0^{1,H}(\Omega), \mathbb{R})$ and the critical points of φ are weak solutions of (P).

Existence and multiplicity results for problems of type (P) have been discussed precisely by several authors. Especially Perera et al. [13] considered a double-phase problem with the q-superlinear reaction term, that is,

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u) = \lambda|u|^{p-2}u + |u|^{r-2}u + h(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P_1}$$

where $\lambda \in \mathbb{R}$ is a parameter, $q < r < p^*$ and h is a Carathéodory function on $\Omega \times \mathbb{R}$ satisfying

$$|h(x,t)| \leq C(|t|^{\rho-1} + |t|^{\sigma-1})$$

for some $p < \sigma < \rho < r$ and $C > 0$. In particular, applying the Morse theory, they proved the existence of a solution of problem (P_1). Following this, Liu-Dai [14] considered the same problems for more general reaction term, and proved existence and multiplicity results, also sign-changing solutions. Furthermore, we refer to a recent work [15] which shows the existence of at least three solutions of problem (P) by using strong maximum principle. A similar treatment has been recently done by Hou-Ge-Zhang-Wang [16] via the Nehari manifold method. Eigenvalue problems for double phase operators with Dirichlet boundary condition are also investigated in

[17], where the authors proved the existence and properties of related variational eigenvalues. For other existence results on elliptic equations with double phase operators we refer to the papers of Ge-Lv-Lu [18], Ge-Chen [19], Ge-Wang-Lu [20], Papageorgiou-Radulescu-Repovs [21–22], Radulescu [23], Cencelj-Radulescu-Repovs [24], Gasinski-Winkert [25–26], Zeng-Gasinski-Winkert-Bai [27] and the references therein.

Motivated by the aforementioned works, in the present paper we focus our attention on the existence and multiplicity of solutions to (P). Our approach uses minimax techniques coming from critical point theory and Morse theory combined with truncation arguments. Precisely, we obtain the following result.

Theorem 1.1 *Assume that* (f_1)*–*(f_4) *hold. Then problem* (P) *has at least one nontrivial weak solution* $u_0 \in C_0^1(\overline{\Omega})$.

Furthermore, we establish the existence of at least three nontrivial weak solutions, by using an additional assumption on the reaction term $f(x,t)$. Precisely, we have the following result.

Theorem 1.2 *Assume that* (f_1)*–*(f_4) *hold. In addition we will assume the following condition*:

(f_5) $f(x,t)t \geq 0$ *for a.a.* $x \in \Omega$, *all* $t \in \mathbb{R}$ *and the set* $\{x \in \Omega : f(x,t) = 0 \text{ for some } t \neq 0\}$ *has empty interior.*

Then problem (P) *has at least three nontrivial weak solutions* $u_0 \in N_+$ $(N_+$ *is defined in Section* 2), $v_0 \in -N_+$ *and* $w_0 \in C_0^1(\overline{\Omega})$.

The outline of this paper is as follows. In Section 2, we introduce the required preliminary knowledge on space $W_0^{1,H}(\Omega)$ and recall some necessary concepts and results in Morse theory. In Section 3, we obtain several preliminary lemmas which are needed for the proofs of our main results. The proofs of Theorem 1.1 and Theorem 1.2 will also be presented in Section 4.

2 Preliminaries

In this section, we first recall some necessary properties on Musielak-Orlicz-Sobolev space $W_0^{1,H}(\Omega)$ which will be used later, see [17, 28–31] for more details.

Denote by $N(\Omega)$ the set of all generalized N-function (see [29, p.82]). For $1 < p < q$ and $0 \leq a(\cdot) \in L^1(\Omega)$, we define

$$H(x,t) = t^p + a(x)t^q, \quad \forall (x,t) \in \Omega \times [0,+\infty).$$

It is clear that $H \in N(\Omega)$ is locally integrable and

$$H(x,2t) \leq 2^q H(x,t), \quad \forall (x,t) \in \Omega \times [0,+\infty),$$

which is known as the $(\triangle_2)$ (see [29, p.52]).

The Musielak-Orlicz space $L^H(\Omega)$ is defined by

$$L^H(\Omega) = \Big\{u : \Omega \to \mathbb{R} \text{ measurable } : \int_\Omega H(x,|u|)\mathrm{d}x < +\infty\Big\}$$

equipped with the Luxemburg norm

$$|u|_H = \inf\Big\{\lambda > 0 : \int_\Omega H\Big(x, \Big|\frac{u}{\lambda}\Big|\Big)\mathrm{d}x \leq 1\Big\}.$$

It is clear that the space $L^H(\Omega)$ is a uniformly convex, and hence reflexive Banach space. The related Sobolev space $W^{1,H}(\Omega)$ is defined by

$$W^{1,H}(\Omega) = \{u \in L^H(\Omega) : |\nabla u| \in L^H(\Omega)\}$$

with the norm

$$\|u\| = |u|_H + |\nabla u|_H. \tag{2.1}$$

We denote by $W_0^{1,H}(\Omega)$ the completion of $C_0^\infty(\Omega)$ in $W^{1,H}(\Omega)$. With these norms, the spaces $W_0^{1,H}(\Omega)$ and $W^{1,H}(\Omega)$ are uniformly convex and so, reflexive Banach spaces; see [17] for the details.

Proposition 2.1 (see [14, Proposition 2.1]) *Set $\rho_H(u) = \int_\Omega(|u|^p + a(x)|u|^q)\mathrm{d}x$. For $u \in L^H(\Omega)$, we have*

(i) *For $u \neq 0, |u|_H = \lambda \Leftrightarrow \rho_H(\frac{u}{\lambda}) = 1$.*

(ii) $|u|_H < 1(= 1; > 1) \Leftrightarrow \rho_H(u) < 1(= 1; > 1)$.

(iii) *If $|u|_H \geq 1$, then $|u|_H^p \leq \rho_H(u) \leq |u|_H^q$.*

(iv) *If $|u|_H \leq 1$, then $|u|_H^q \leq \rho_H(u) \leq |u|_H^p$.*

Proposition 2.2 (see [17, Proposition 2.15, Proposition 2.18]) (1) *If $1 \leq \vartheta \leq p^*$, then the embedding from $W_0^{1,H}(\Omega)$ to $L^\vartheta(\Omega)$ is continuous. In particular, if $\vartheta \in [1, p^*)$, then the embedding $W_0^{1,H}(\Omega) \hookrightarrow L^\vartheta(\Omega)$ is compact.*

(2) *Assume that (1.2) holds. Then the Poincare's inequality holds, that is, there exists a positive constant C_0 such that*

$$|u|_H \leq C_0|\nabla u|_H, \quad \forall u \in W_0^{1,H}(\Omega).$$

(3) *The embedding $L^H(\Omega) \hookrightarrow L^\vartheta(\Omega)$ and $W_0^{1,H}(\Omega) \hookrightarrow W_0^{1,\vartheta}(\Omega)$ are continuous for all $\vartheta \in [1, p]$.*

By the above Proposition 2.2(1), we know that there exists $c_\vartheta > 0$ such that

$$|u|_\vartheta \leq c_\vartheta\|u\|, \quad \forall u \in W_0^{1,H}(\Omega),$$

where $|u|_\vartheta$ denotes the usual norm in $L^\vartheta(\Omega)$ for all $1 \leq \vartheta < p^*$. Thanks to Proposition 2.2(2), we have an equivalent norm on $W_0^{1,H}(\Omega)$ given by $|\nabla u|_H$. We will use the equivalent norm in the following discussion and write $\|u\| = |\nabla u|_H$ for simplicity.

Remark 2.1 The Poincare's inequality has been proved also in [32] under the more general assumption

$$\Omega \text{ is quasiconvex and } a \in C^{0,\alpha}(\Omega) \text{ with } \frac{q}{p} \leq 1 + \frac{\alpha}{N} \quad \text{for some } \alpha \in (0, 1]. \tag{2.2}$$

Furthermore, we observe that, since $p^* > p\big(1 + \frac{1}{n}\big)$, both (1.2) and (2.2) imply $q < p^*$.

In order to discuss the problem (P), we need to define a functional in $W_0^{1,H}(\Omega)$:

$$J(u) = \int_\Omega \Big(\frac{1}{p}|\nabla u|^p + \frac{a(x)}{q}|\nabla u|^q\Big)\mathrm{d}x.$$

We know that, $J \in C^1(W_0^{1,H}(\Omega), \mathbb{R})$ and double phase operator $-\mathrm{div}(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u)$ is the derivative operator of J in the weak sense (see [33]). We denote $L = J' : W_0^{1,H}(\Omega) \to (W_0^{1,H}(\Omega))^*$, then

$$\langle L(u), v\rangle = \int_\Omega (|\nabla u|^{p-2}\nabla u \cdot \nabla v + a(x)|\nabla u|^{q-2}\nabla u \cdot \nabla v)\mathrm{d}x$$

for all $u, v \in W_0^{1,H}(\Omega)$. Here $(W_0^{1,H}(\Omega))^*$ denotes the dual space of $W_0^{1,H}(\Omega)$ and $\langle \cdot, \cdot \rangle$ denotes the pairing between $W_0^{1,H}(\Omega)$ and $(W_0^{1,H}(\Omega))^*$. Then, we have the following result.

Proposition 2.3 (see [14, Proposition 3.1]) *Set $E = W_0^{1,H}(\Omega)$, L is as above, then*

(1) *L is a continuous, bounded and strictly monotone operator.*

(2) *L is a mapping of type $(S)_+$, i.e., if $u_n \rightharpoonup u$ in E and $\limsup_{n\to+\infty}\langle L(u_n) - L(u), u_n - u\rangle \le 0$, implies $u_n \to u$ in E.*

(3) *L is a homeomorphism.*

The Banach space $C_0^1(\overline{\Omega})$ is an ordered Banach space with positive (order) cone given by

$$C_+ = \{u \in C_0^1(\overline{\Omega}) : u(x) \ge 0 \text{ for all } x \in \overline{\Omega}\}.$$

It has nonempty interior given by

$$N_+ = \operatorname{int} C_+ = \Big\{u \in C_0^1(\overline{\Omega}) : u(x) > 0 \text{ for all } x \in \Omega,\ \frac{\partial u}{\partial n}\Big|_{\partial\Omega} < 0\Big\}.$$

Now, we introduce some elements of critical point and Morse theories needed in the sequel. Let X be a Banach space and X^* be its topological dual. Let $\varphi \in C^1(X, \mathbb{R})$. We set $K_\varphi = \{u \in X : \varphi'(u) = 0\}$, $K_\varphi^c = \{u \in K_\varphi : \varphi(u) = c\}$ and, for every $c \in \mathbb{R}$, denote $\varphi^c = \{u \in X : \varphi(u) \le c\}$. Let (X_1, X_2) be a topological pair with $X_2 \subset X_1 \subset X$, then for every integer $k \ge 0$, we denote by $H_k(X_1, X_2)$ the kth-relative singular homology group with integer coefficients. Let $u_0 \in K_\varphi^c$ be isolated. Then the critical groups of φ at u_0 with $\varphi(u_0) = c$ are defined by

$$C_k(\varphi, u_0) = H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{u_0\}), \quad \forall k \ge 0,$$

where U is a neighborhood of u_0 such that $K_\varphi \cap \varphi^c \cap U = \{u_0\}$ (see [34]). The excision property of singular homology implies that above definition is independent of the particular neighborhood U. Assume that the $\varphi \in C^1(X, \mathbb{R})$ satisfies the (C_c)-condition and $\inf \varphi(K_\varphi) > -\infty$. Choosing $c < \inf \varphi(K_\varphi)$, the critical groups of φ at infinity are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c), \quad \forall k \ge 0$$

see [35] for details.

Let us recall the definition of Cerami condition [36].

Definition 2.1 *Let X be a Banach space. $\varphi \in C^1(X, \mathbb{R})$ is said to satisfy condition $(C)_c$ at the level $c \in \mathbb{R}$, if the following fact is true: For any sequence $\{u_k\} \subset X$ such that*

$$\varphi(u_k) \to c \quad \text{and} \quad (1 + \|u_k\|)\|\varphi'(u_k)\|_{X^*}, \quad \text{as } k \to \infty,$$

$\{u_k\}$ possesses a convergent subsequence.

The following mountain pass theorem obtained by Motreanu-Motreanu-Papageorgiou [37] will be used to seek the existence of solutions.

Theorem 2.1 *Let X be a Banach space, $\varphi \in C^1(X,\mathbb{R})$, and assume that φ satisfies the $(C)_c$-condition. Let $u_0, u_1 \in X$, $u_0 \neq u_1$, satisfy $\|u_1 - u_0\| > \rho > 0$, and assume that*

$$\max\{\varphi(u_0), \varphi(u_1)\} < \inf\{\varphi(u) : \|u - u_0\| = \rho\} = m_\rho, \quad c = \inf_{\gamma\in\Gamma} \max_{0\le t\le 1} \varphi(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = u_0, \gamma(1) = u_1\}$.

Then $c \geq m_\rho$ and c is a critical value of φ, that is, there exists $\widehat{u} \in X$ such that $\varphi'(\widehat{u}) = 0$ and $\varphi(\widehat{u}) = c$.

3 Some Preliminary Lemmas

In this section, we give some preliminary lemmas which are crucial for proving our results. Firstly, we show that Cerami condition holds.

Lemma 3.1 *If assumption $(f_2)-(f_4)$ hold, then the functional φ satisfies the $(C)_c$-condition for each $c > 0$.*

Proof Let $\{u_n\} \subset W_0^{1,H}(\Omega)$ be a $(C)_c$ sequence, that is,

$$c = \varphi(u_n) + c_n, \quad \langle\varphi'(u_n), u_n\rangle \to 0 \quad \text{as } n \to +\infty, \tag{3.1}$$

where $c_n \to 0$ as $n \to +\infty$.

First of all, we claim that the sequence $\{u_n\}$ is bounded in $W_0^{1,H}(\Omega)$. Indeed, arguing by contradiction, we suppose that $\|u_n\| \to +\infty$ as $n \to +\infty$. Define $v_n = \frac{u_n}{\|u_n\|}$, for any $n \in N$. It is clear that $\{v_n\} \subset E$ and $\|v_n\| = 1$ for any $n \in N$. Thus, going if necessary to a subsequence, we may assume that

$$\begin{aligned} &v_n \rightharpoonup v \quad \text{in } W_0^{1,H}(\Omega),\\ &v_n \to v \quad \text{in } L^s(\Omega),\ 1 \le s < p^*,\\ &v_n(x) \to v(x) \quad \text{a.e. on } \Omega. \end{aligned} \tag{3.2}$$

Set $\Omega_{\neq} := \{x \in \Omega : v(x) \neq 0\}$. If $x \in \Omega_{\neq}$, then it follows from (3.2) that

$$\lim_{n\to\infty} v_n(x) = \lim_{n\to\infty} \frac{u_n(x)}{\|u_n\|} = v(x) \neq 0,$$

which yields

$$|u_n(x)| = |v_n(x)|\|u_n\| \to +\infty \quad \text{a.e. in } \Omega_{\neq} \text{ as } n \to +\infty.$$

By the hypothesis (f_3), it follows that for each $x \in \Omega_{\neq}$ we have

$$\lim_{n\to\infty} \frac{F(x, u_n(x))}{|u_n(x)|^q} \frac{|u_n(x)|^q}{\|u_n\|^q} = \lim_{n\to\infty} \frac{F(x, u_n(x))}{|u_n(x)|^q} |v_n(x)|^q = +\infty. \tag{3.3}$$

Also by virtue of hypothesis (f_3), we can find $t_0 > 0$ such that

$$\frac{F(x,t)}{|t|^q} > 1, \quad \forall x \in \Omega \text{ and } |t| > t_0. \tag{3.4}$$

Moreover from hypothesis (f_2), we have that there exists a positive constant C_1 such that

$$|F(x,t)| \leq C_1, \quad \forall (x,t) \in \Omega \times [-t_0, t_0]. \tag{3.5}$$

Then, by (3.4)–(3.5), there is a constant $C_2 \in \mathbb{R}$ such that

$$F(x,t) \geq C_2, \quad \forall (x,t) \in \Omega \times \mathbb{R}, \tag{3.6}$$

which implies that

$$\frac{F(x,u_n) - C_2}{\|u_n\|^q} \geq 0, \quad \forall x \in \Omega,\ \forall n \in N,$$

that is,

$$\frac{F(x,u_n)}{|u_n(x)|^q}|v_n(x)|^q - \frac{C_2}{\|u_n\|^q} \geq 0, \quad \forall x \in \Omega,\ \forall n \in N. \tag{3.7}$$

Recalling $\|u_n\| > 1$ for n large, using (3.1) we have

$$\begin{aligned}
c &= \varphi(u_n) + c_n \\
&= \int_\Omega \Big(\frac{1}{p}|\nabla u_n|^p + \frac{a(x)}{q}|\nabla u_n|^q\Big)\mathrm{d}x - \int_\Omega F(x,u_n)\mathrm{d}x + c_n \\
&\geq \frac{1}{q}\|u_n\|^p - \int_\Omega F(x,u_n)\mathrm{d}x + c_n,
\end{aligned}$$

which shows that

$$\int_\Omega F(x,u_n)\mathrm{d}x \geq \frac{1}{q}\|u_n\|^p - c + c_n \to +\infty \quad \text{as } n \to +\infty. \tag{3.8}$$

Similarly, from (3.1), we deduce that that

$$\begin{aligned}
c &= \varphi(u_n) + c_n \\
&= \int_\Omega \Big(\frac{1}{p}|\nabla u_n|^p + \frac{a(x)}{q}|\nabla u_n|^q\Big)\mathrm{d}x - \int_\Omega F(x,u_n)\mathrm{d}x + c_n \\
&\leq \frac{1}{p}\|u_n\|^q - \int_\Omega F(x,u_n)\mathrm{d}x + c_n.
\end{aligned}$$

This combined with (3.8) yields

$$\|u_n\|^q \geq p\int_\Omega F(x,u_n)\mathrm{d}x + pc - c_n > 0 \tag{3.9}$$

for n large enough.

Next, we claim that $|\Omega_{\neq}| = 0$. In fact, if $|\Omega_{\neq}| \neq 0$, then from (3.3), (3.7), (3.9) and the Fatou's lemma, we obtain that

$$\begin{aligned}
+\infty &= \int_{\Omega_{\neq}} \lim_{n\to\infty} \frac{F(x,u_n(x))}{|u_n(x)|^q}|v_n(x)|^q\mathrm{d}x - \int_{\Omega_{\neq}} \lim_{n\to\infty} \frac{C_2}{\|u_n\|^q}\mathrm{d}x \\
&= \int_{\Omega_{\neq}} \lim_{n\to\infty} \Big(\frac{F(x,u_n(x))}{|u_n(x)|^q}|v_n(x)|^q - \frac{C_2}{\|u_n\|^q}\Big)\mathrm{d}x \\
&\leq \liminf_{n\to\infty} \int_{\Omega_{\neq}} \Big(\frac{F(x,u_n(x))}{|u_n(x)|^q}|v_n(x)|^q - \frac{C_2}{\|u_n\|^q}\Big)\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&\leq \liminf_{n\to\infty}\int_\Omega\Big(\frac{F(x,u_n(x))}{|u_n(x)|^q}|v_n(x)|^q-\frac{C_2}{\|u_n\|^q}\Big)\mathrm{d}x\\
&=\liminf_{n\to\infty}\int_\Omega\frac{F(x,u_n(x))}{|u_n(x)|^q}|v_n(x)|^q\mathrm{d}x-\limsup_{n\to\infty}\int_\Omega\frac{C_2}{\|u_n\|^q}\mathrm{d}x\\
&=\liminf_{n\to\infty}\int_\Omega\frac{F(x,u_n(x))}{\|u_n\|^q}\mathrm{d}x\\
&\leq \liminf_{n\to\infty}\int_\Omega\frac{F(x,u_n(x))}{p\int_\Omega F(x,u_n)\mathrm{d}x+pc-c_n}\mathrm{d}x. \qquad (3.10)
\end{aligned}
$$

Therefore, by (3.8) and (3.10), we conclude

$$+\infty\leq\frac{1}{p},$$

which is a contradiction. Therefore, $|\Omega_{\neq}|=0$ and $v(x)=0$ a.e. in Ω.

Since $\varphi(tu_n)$ is continuous in $t\in[0,1]$, for each n there exists $t_n\in[0,1]$, $n=1,2,\cdots$, such that

$$\varphi(t_nu_n):=\max_{t\in[0,1]}\varphi(tu_n). \qquad (3.11)$$

It is clear that $t_n>0$ and $\varphi(t_nu_n)\geq c>0=\varphi(0)=\varphi(0\cdot u_n)$. If $t_n<1$, then by using $\frac{\mathrm{d}}{\mathrm{d}t}\varphi(tu_n)|_{t=t_n}=0$, we deduce that

$$\langle\varphi'(t_nu_n),t_nu_n\rangle=0. \qquad (3.12)$$

If $t_n=1$, then it follows from (3.1) that

$$\langle\varphi'(u_n),u_n\rangle=c_n. \qquad (3.13)$$

Hence, from (3.12)–(3.13), we obtain

$$\langle\varphi'(t_nu_n),t_nu_n\rangle=c_n. \qquad (3.14)$$

On one hand, by (f_4), (3.1) and (3.11), for any $t\in[0,1]$, we achieve that

$$
\begin{aligned}
q\varphi(tu_n)&\leq q\varphi(t_nu_n)\\
&=\langle q\varphi(t_nu_n)-\varphi'(t_nu_n),t_nu_n\rangle+c_n\\
&=\int_\Omega\Big(\frac{q}{p}-1\Big)|\nabla t_nu_n|^p\mathrm{d}x-\int_\Omega qF(x,t_nu_n)\mathrm{d}x+\int_\Omega f(x,t_nu_n)t_nu_n\mathrm{d}x+c_n\\
&=\int_\Omega\Big(\frac{q}{p}-1\Big)|\nabla t_nu_n|^p\mathrm{d}x+\int_\Omega\mathcal{F}(x,t_nu_n)\mathrm{d}x+c_n\\
&\leq\int_\Omega\Big(\frac{q}{p}-1\Big)|\nabla u_n|^p\mathrm{d}x+\int_\Omega(\mathcal{F}(x,u_n)+g(x))\mathrm{d}x+c_n\\
&=q\varphi(u_n)-\langle\varphi'(u_n),u_n\rangle+|g|_1+c_n\\
&\to qc+|g|_1\quad\text{as } n\to+\infty. \qquad (3.15)
\end{aligned}
$$

Let $\{R_k\}_{k\in N}$ be a positive sequence of real numbers such that $R_k>1$ for any k and $R_k\to+\infty$ as $k\to+\infty$. Then

$$\|R_kv_n\|=R_k>1,\quad\forall k,n\in N.$$

Moreover, due to the fact that $v_n \to 0$ in $L^r(\Omega)$ and $v_n(x) \to 0$ a.e. $x \in \Omega$ as $n \to +\infty$, by (f_1) and the Lebesgue dominated convergence theorem, we have that for fixed $k \in N$ that

$$\int_\Omega F(x, R_k v_n)\mathrm{d}x \to 0 \quad \text{as } n \to +\infty. \tag{3.16}$$

Recall that $\|u_n\| \to +\infty$ as $n \to +\infty$. So, we have either $\|u_n\| > R_k$ or $\frac{R_k}{\|u_n\|} \in (0,1)$ for n large enough. Consequently, by (3.11) and (3.16), we deduce for fix $k \in N$ that

$$\varphi(t_n u_n) \geq \varphi\Big(\frac{R_k}{\|u_n\|}u_n\Big) = \varphi(R_k v_n) \geq \frac{1}{q}R_k^p - \int_\Omega F(x, R_k v_n)\mathrm{d}x \geq \frac{1}{2q}R_k^p, \tag{3.17}$$

for any n large enough. From (3.17), letting $n, k \to +\infty$ we have

$$\varphi(t_n u_n) \to +\infty \quad \text{as } n \to +\infty. \tag{3.18}$$

From (3.15) and (3.18) we obtain a contradiction. Therefore we infer that the sequence $\{u_n\}$ is bounded in $W_0^{1,H}(\Omega)$.

Finally, we turn to proving that any $(C)_c$ sequence has a convergent subsequence. Indeed, by the boundedness of $\{u_n\}$, passing to a subsequence if necessary, still denoted by $\{u_n\}$, we may assume that

$$u_n \rightharpoonup u_0 \quad \text{in } W_0^{1,H}(\Omega).$$

Using Proposition 2.2(1), we have

$$u_n \to u_0 \quad \text{in } L^s(\Omega),\ s \in [1, p^*).$$

It is easy to compute directly that

$$\begin{aligned}
&\int_\Omega |f(x,u_n) - f(x,u_0)||u_n - u_0|\mathrm{d}x \\
&\leq \int_\Omega (|f(x,u_n)| + |f(x,u_0)|)|u_n - u_0|\mathrm{d}x \\
&\leq \int_\Omega [C(1+|u_n|^{r-1}) + C(1+|u_0|^{r-1})]|u_n - u_0|\mathrm{d}x \\
&\leq 2C\int_\Omega |u_n - u_0|\mathrm{d}x + C\int_\Omega |u_n|^{r-1}|u_n - u_0|\mathrm{d}x + \int_\Omega |u_0|^{r-1}|u_n - u_0|\mathrm{d}x \\
&\leq 2C\int_\Omega |u_n - u_0|\mathrm{d}x + C\Big(\int_\Omega |u_n|^{(r-1)r'}\mathrm{d}x\Big)^{\frac{1}{r'}}\Big(\int_\Omega |u_n - u_0|^r\mathrm{d}x\Big)^{\frac{1}{r}} \\
&\quad + C\Big(\int_\Omega |u_0|^{(r-1)r'}\mathrm{d}x\Big)^{\frac{1}{r'}}\Big(\int_\Omega |u_n - u_0|^r\mathrm{d}x\Big)^{\frac{1}{r}} \\
&= 2C\int_\Omega |u_n - u_0|\mathrm{d}x + C\Big(\int_\Omega |u_n|^r\mathrm{d}x\Big)^{\frac{r-1}{r}}\Big(\int_\Omega |u_n - u_0|^r\mathrm{d}x\Big)^{\frac{1}{r}} \\
&\quad + C\Big(\int_\Omega |u_0|^r\mathrm{d}x\Big)^{\frac{r-1}{r}}\Big(\int_\Omega |u_n - u_0|^r\mathrm{d}x\Big)^{\frac{1}{r}} \\
&= 2C|u_n - u_0|_1 + C|u_n|_r^{r-1}|u_n - u_0|_r + C|u_0|_r^{r-1}|u_n - u_0|_r \\
&\to 0 \quad \text{as } n \to \infty,
\end{aligned} \tag{3.19}$$

where $\frac{1}{r} + \frac{1}{r'} = 1$.

Note that

$$\begin{aligned}&\langle L(u_n)-L(u_0),u_n-u_0\rangle\\&=\langle\varphi'(u_n)-\varphi'(u_0),u_n-u_0\rangle+\int_\Omega(f(x,u_n)-f(x,u_0))(u_n-u_0)\mathrm{d}x.\end{aligned}\tag{3.20}$$

Moreover, by (3.1), it is easy to see that

$$\lim_{n\to\infty}\langle\varphi'(u_n)-\varphi'(u_0),u_n-u_0\rangle=0.\tag{3.21}$$

Therefore, the combination of (3.19)–(3.21) implies

$$\lim_{n\to\infty}\langle L(u_n)-L(u_0),u_n-u_0\rangle=0.\tag{3.22}$$

Therefore, it follows that $u_n\to u_0$ in $W_0^{1,H}(\Omega)$ because L is a mapping of type $(S)_+$ (see Proposition 2.3). This ends the proof of lemma.

Our second result is the following lemma.

Lemma 3.2 *Assume that* (f_1)–(f_3) *hold. Then the following assertions hold*:
(a) *there exist* $\rho>0$ *and* $\delta>0$ *such that* $\varphi(u)\geq\delta$ *for each* $u\in W_0^{1,H}(\Omega)$ *with* $\|u\|=\rho$;
(b) *there exists* $v\in W_0^{1,H}(\Omega)$ *such that* $\varphi(v)<0$ *and* $\|v\|>\rho$.

Proof Verification of (a). Since $1<p<q<r<p^*$, by Proposition 2.2, we conclude that the embeddings $W_0^{1,H}(\Omega)\hookrightarrow L^q(\Omega)$ and $W_0^{1,H}(\Omega)\hookrightarrow L^r(\Omega)$ are continuous and so there exists a constant $C_1>0$ such that

$$|u|_q\leq C_1\|u\|\quad\text{and}\quad|u|_r\leq C_1\|u\|.\tag{3.23}$$

Using assumptions (f_1) and (f_2), we deduce that for any $\varepsilon>0$, there is a $C_\varepsilon>0$ such that

$$|f(x,t)|\leq\varepsilon|t|^{q-1}+C_\varepsilon|t|^{r-1},\quad|F(x,t)|\leq\varepsilon|t|^q+C_\varepsilon|t|^r\tag{3.24}$$

for all $(x,t)\in\Omega\times\mathbb{R}$, where $r\in[1,p^*)$ was given in (f_2).

Thus, for $u\in W_0^{1,H}(\Omega)$ with $\|u\|<1$ sufficiently small. By (3.24) and Proposition 2.2(2),

$$\begin{aligned}\varphi(u)&=\int_\Omega\Big(\frac{1}{p}|\nabla u|^p+\frac{a(x)}{q}|\nabla u|^q\Big)\mathrm{d}x-\int_\Omega F(x,u)\mathrm{d}x\\&\geq\frac{1}{q}\int_\Omega(|\nabla u|^p+a(x)|\nabla u|^q)\mathrm{d}x-\int_\Omega(\varepsilon|u|^q+C_\varepsilon|u|^r)\mathrm{d}x\\&\geq\frac{1}{q}\|u\|^q-\varepsilon C_1^q\|u\|^q-C_\varepsilon C_1^r\|u\|^r,\end{aligned}\tag{3.25}$$

and so there exist $\rho>0$ and $\delta>0$ such that $\varphi(u)\geq\delta$ for any $u\in W_0^{1,H}(\Omega)$ with $\|u\|=\rho$.

Verification of (b). By the assumption (f_3), for any $M>0$, there exists a constant $\delta_M>0$ such that

$$F(x,t)\geq M|t|^q$$

for $|t|>\delta_M$ and for almost all $x\in\Omega$. Also, by (f_2), for all $x\in\Omega$ and $|t|\leq\delta_M$, we have

$$|F(x,t)|\leq C(1+|\delta_M|^{r-1}).$$

The above two inequalities imply that there exists a constant $C_M > 0$ such that

$$F(x,t) \geq M|t|^q - C_M, \quad \forall x \in \Omega,\ \forall t \in \mathbb{R}. \tag{3.26}$$

Take $\phi \in W_0^{1,H}(\Omega)$ with $\phi > 0$ on Ω and $t > 1$. Then, the relation (3.26) implies that

$$\begin{aligned}\varphi(t\phi) &= \int_\Omega \Big(\frac{1}{p}|t\nabla\phi|^p + \frac{a(x)}{q}|t\nabla\phi|^q\Big)\mathrm{d}x - \int_\Omega F(x,t\phi)\mathrm{d}x \\ &\leq \frac{t^q}{p}\int_\Omega (|\nabla\phi|^p + a(x)|\nabla\phi|^q)\mathrm{d}x - t^q M\int_\Omega |\phi|^q\mathrm{d}x + C_M\mathrm{meas}(\Omega).\end{aligned} \tag{3.27}$$

If M is large enough that

$$\frac{1}{p}\int_\Omega (|\nabla\phi|^p + a(x)|\nabla\phi|^q)\mathrm{d}x - M\int_\Omega |\phi|^q\mathrm{d}x < 0.$$

This means that

$$\lim_{t\to+\infty}\varphi(t\phi) = -\infty.$$

Hence, there exists $v = t_0\phi \in W_0^{1,H}(\Omega)$ such that $\varphi(v) < 0$ and $\|u\| > \rho$.

Next we compute the critical groups of the energy functional φ at infinity.

Lemma 3.3 *Assume that $(f_1) - (f_4)$ hold. Then $C_k(\varphi,\infty) = 0$ for all $k \geq 0$.*

Proof Let $\partial B_1 = \{u \in W_0^{1,H}(\Omega) : \|u\| = 1\}$. From (f_1) and (f_2), for any $M > 0$, there exists $C_M > 0$, such that

$$F(x,t)) \geq M|t|^q - C_M|\Omega|, \quad \forall (x,t) \in \Omega\times\mathbb{R}.$$

Thus, for any $u \in \partial B_1$ and $t > 1$, we get

$$\varphi(tu) \leq t^q\Big[\int_\Omega \Big(\frac{1}{p}|\nabla u|^p + \frac{a(x)}{q}|\nabla u|^q\Big)\mathrm{d}x - M\int_\Omega |u|^q\mathrm{d}x\Big] + C_M|\Omega|.$$

By the arbitrariness of M, we have

$$\varphi(tu) \to -\infty, \quad \text{as } t \to +\infty.$$

Moreover, for $u \in \partial B_1$ and $t > 1$, by (f_4), one yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\varphi(tu) &= \langle\varphi'(tu), u\rangle = \frac{1}{t}\langle\varphi'(tu), tu\rangle \\ &= \frac{1}{t}\Big[\int_\Omega (|\nabla tu|^p + a(x)|\nabla tu|^q)\mathrm{d}x - \int_\Omega f(x,tu)tu\mathrm{d}x\Big] \\ &\leq \frac{1}{t}\Big[q\int_\Omega \Big(\frac{1}{p}|\nabla tu|^p + \frac{a(x)}{q}|\nabla tu|^q\Big)\mathrm{d}x - q\int_\Omega F(x,tu)\mathrm{d}x - \int_\Omega \mathcal{F}(x,u)\mathrm{d}x + |g|_1\Big] \\ &= \frac{1}{t}\Big[q\varphi(tu) - \int_\Omega \mathcal{F}(x,u)\mathrm{d}x + |g|_1\Big] \\ &\to -\infty, \quad \text{as } t \to +\infty.\end{aligned}$$

This shows that $\frac{\mathrm{d}}{\mathrm{d}t}\varphi(tu) < 0$ for all $t > 1$ big. Using the implicit function theorem, there exists a $t \in C(\partial B_1)$ such that $t > 0$ and $\varphi(t(u)u) = \rho_0$, where

$$q\rho_0 - \int_\Omega \mathcal{F}(x,u)\mathrm{d}x + |g|_1 < 0.$$

We extend $t(\cdot)$ on $W_0^{1,H}(\Omega)\backslash\{0\}$ by

$$t_0(u) = \frac{1}{\|u\|}t\Big(\frac{u}{\|u\|}\Big)$$

for all $u \in W_0^{1,H}(\Omega)\backslash\{0\}$. It is clear that $t_0 \in W_0^{1,H}(\Omega)\backslash\{0\}$ and $\varphi(t_0(u)u) = \rho_0$. Therefore, we have

$$\varphi(u) = \rho_0 \Rightarrow t_0(u) = 1. \tag{3.28}$$

Denoting by

$$\widehat{t_0}(u) = \begin{cases} 1, & \text{if } \varphi(u) \leq \rho_0, \\ t_0(u), & \text{if } \varphi(u) > \rho_0, \end{cases}$$

then we have $\widehat{t_0} \in C(W_0^{1,H}(\Omega)\backslash\{0\})$. Now, we introduce the deformation $h : [0,1]\times W_0^{1,H}(\Omega)\backslash\{0\} \to W_0^{1,H}(\Omega)\backslash\{0\}$ defined by

$$h(\lambda, u) = (1-\lambda)u + \lambda\widehat{t_0}(u)u$$

for all $\lambda \in [0,1]$ and all $u \in W_0^{1,H}(\Omega)$.

Due to the definition of $\widehat{t_0}$ and to (3.28), we necessarily have

$$\begin{aligned} &(1)\ h(0,u) = u, \quad \forall u \in W_0^{1,H}(\Omega)\backslash\{0\}, \\ &(2)\ h(1,u) = \widehat{t_0}(u)u + \varphi^{\rho_0}, \\ &(3)\ h(t,\cdot)|_{\varphi^{\rho_0}} = \mathrm{id}|_{\varphi^{\rho_0}}. \end{aligned}$$

The above facts imply that

$$\varphi^{\rho_0} \text{ is a strong deformation retractor of } W_0^{1,H}(\Omega)\backslash\{0\}. \tag{3.29}$$

Next, we consider the radial retraction $r : W_0^{1,H}(\Omega)\backslash\{0\} \to \partial B_1$ defined by

$$r(u) = \frac{u}{\|u\|}, \quad \forall u \in W_0^{1,H}(\Omega)\backslash\{0\}$$

and the deformation $\widehat{h} : [0,1] \times W_0^{1,H}(\Omega)\backslash\{0\} \to W_0^{1,H}(\Omega)\backslash\{0\}$ defined by

$$\widehat{h}(\lambda, u) = (1-\lambda)u + \lambda r(u), \quad \forall u \in W_0^{1,H}(\Omega)\backslash\{0\}.$$

On one hand, using this deformation we deduce that

$$W_0^{1,H}(\Omega)\backslash\{0\} \text{ is deformable into } \partial B_1. \tag{3.30}$$

On the other hand, using radial retraction $r(\cdot)$, we see that

$$\partial B_1 \text{ is a retractor of } W_0^{1,H}(\Omega)\backslash\{0\}. \tag{3.31}$$

Hence, by (3.30)–(3.31) and [38, Theorem 6.5], we conclude that

$$\partial B_1 \text{ is a deformation retractor of } W_0^{1,H}(\Omega)\backslash\{0\}. \tag{3.32}$$

Again from (3.29) and (3.32), we conclude that φ^{ρ_0} and ∂B_1 are homotopy equivalent. In view of [37, Proposition 6.11]

$$H_k(W_0^{1,H}(\Omega), \varphi^{\rho_0}) = H_k(W_0^{1,H}(\Omega), \partial B_1), \quad \forall k \geq 0. \tag{3.33}$$

Invoking Problems (4.154), (4.159) of Gasinski-Papageorgiou [39], we infer that ∂B_1 is contractible. Again from [37, P.147], we have $H_k(W_0^{1,H}(\Omega), \partial B_1) = 0$ for all $k \geq 0$. This, together with (3.33), shows $H_k(W_0^{1,H}(\Omega), \varphi^{\rho_0}) = 0$ for all $k \geq 0$. As usual we assume that K_φ is finite (or otherwise we already have an infinity of nontrivial solutions). Hence, if we choose ρ_0 such that $q\rho_0 - \int_\Omega \mathcal{F}(x,u)\mathrm{d}x + |g|_1 < 0$, then we have $C_k(\varphi, \infty) = H_k(W_0^{1,H}(\Omega), \varphi^{\rho_0}) = 0$ for all $k \geq 0$.

4 The Proof of Main Theorems

Proof of Theorem 1.1 Let $X = W_0^{1,H}(\Omega)$ and $u_0 = 0$. We know that φ satisfies the $(C)_c$-condition from Lemma 3.1 and $\varphi(0) = 0$. In view of Lemma 3.2(b), we get trivially that $u = 0$ is a local minimizer of φ. Thus, it follows from Lemmas 3.1–3.2 that all conditions of Theorem 2.1 are satisfied. Hence, problem (P) has at least one nontrivial weak solution u_0. Again using [40, Lemma 4.1], the solution u_0 is in $C_0^1(\overline{\Omega})$. Then the proof is completed.

Now, we are ready to prove Theorem 1.2.

Proof of Theorem 1.2 Firstly, we consider the functions $F_\pm : \Omega \times \mathbb{R} \to \mathbb{R}$ given as

$$F_+(x,t) = \int_0^t f(x, s^+)\mathrm{d}s, \quad F_-(x,t) = \int_0^t f(x, -s^-)\mathrm{d}s, \quad \forall (x,t) \in \Omega \times \mathbb{R}$$

and the functionals $\varphi_\pm : W_0^{1,H}(\Omega) \to \mathbb{R}$ given as

$$\varphi_\pm(u) = \int_\Omega \Big(\frac{1}{p}|\nabla u|^p + \frac{a(x)}{q}|\nabla u|^q\Big)\mathrm{d}x - \int_\Omega F_\pm(x,u)\mathrm{d}x.$$

Due to hypotheses $(f_1) - (f_4)$, we deduce that $\varphi_\pm \in C^1(W_0^{1,H}(\Omega), \mathbb{R})$ and

$$\begin{aligned}
\langle \varphi_+'(u), v\rangle &= \int_\Omega (|\nabla u|^{p-2}\nabla u \nabla v + a(x)|\nabla u|^{q-2}\nabla u\nabla v)\mathrm{d}x - \int_\Omega f(x,u^+)v\mathrm{d}x,\\
\langle \varphi_-'(u), v\rangle &= \int_\Omega (|\nabla u|^{p-2}\nabla u \nabla v + a(x)|\nabla u|^{q-2}\nabla u\nabla v)\mathrm{d}x - \int_\Omega f(x,-u^-)v\mathrm{d}x
\end{aligned}$$

for all $v \in W_0^{1,H}(\Omega)$.

Claim 1 The functional φ_+ satisfies the (C_c)-condition if and only if it satisfies the (C_c)-condition with respect to all the sequences $\{u_n\} \subset W_0^{1,H}(\Omega)$ such that $u_n(x) \geq 0$ for all $x \in \Omega$ and all $n \in N$.

In fact, if $\{u_n\} \subset W_0^{1,H}(\Omega)$ and $(1 + \|u_n\|)\varphi_+(u_n) \to 0$ as $n \to +\infty$. Then, there exists a sequence $\{\varepsilon_n\}$ of nonnegative real numbers such that $\varepsilon_n \to 0$ as $n \to +\infty$ and

$$|\langle \varphi_+'(u_n), v\rangle| \leq \frac{\varepsilon_n \|v\|}{1 + \|u_n\|}, \quad \forall n \in N, \ \forall u \in W_0^{1,H}(\Omega).$$

Moreover, by (f_5), one yields

$$f(x, u_n^+(x))v_n(x) = 0 \quad \text{for a.a. } x \in \Omega. \tag{4.1}$$

Take $v_n = \min\{0, u_n\}$. Then, the relation (4.1) and Proposition 2.1 imply that

$$\min\{\|v_n\|^p, \|v_n\|^q\} \le \int_\Omega (|\nabla v_n|^p + a(x)|\nabla v_n|^q)\mathrm{d}x \le \frac{\varepsilon_n\|v_n\|}{1+\|u_n\|}, \quad \forall n \in N. \tag{4.2}$$

This shows $\|v_n\| \to 0$ as $n \to +\infty$. The proof of Claim is complete.

Now, for all $u \in W_0^{1,H}(\Omega)$ such that $u(x) \ge 0$ for all $x \in \Omega$, one easily deduces

$$\begin{aligned}\varphi_+(u) &= \int_\Omega \Big(\frac{1}{p}|\nabla u|^p + \frac{a(x)}{q}|\nabla u|^q\Big)\mathrm{d}x - \int_\Omega F_+(x, u(x))\mathrm{d}x \\ &= \int_\Omega \Big(\frac{1}{p}|\nabla u|^p + \frac{a(x)}{q}|\nabla u|^q\Big)\mathrm{d}x - \int_\Omega F(x, u(x))\mathrm{d}x = \varphi(u),\end{aligned}$$

because $F_+(x, u(x)) = \int_0^{u(x)} f(x, t^+)\mathrm{d}t = \int_0^{u(x)} f(x, t)\mathrm{d}t = F(x, u(x))$ for all $x \in \Omega$.

Therefore, by Lemma 3.1, we deduce that the functional φ_+ satisfies the (C_c)-condition for all the sequences $\{u_n\} \subset W_0^{1,H}(\Omega)$ such that $u_n(x) \ge 0$ for all $x \in \Omega$, all $n \in N$. Clearly, Lemma 3.2 also holds for the functional φ_+. The above facts (by Theorem 2.1) imply that there exists a function $u_0 \in W_0^{1,H}(\Omega)$ such that

$$\int_\Omega (|\nabla u_0|^{p-2}\nabla u_0 \nabla v + a(x)|\nabla u_0|^{q-2}\nabla u_0 \nabla v)\mathrm{d}x = \int_\Omega f(x, u_0^+(x))v(x)\mathrm{d}x$$

for all $v \in W_0^{1,H}(\Omega)$.

Choosing $v = \min\{0, u_0\}$, we get that $v = 0$, because $f(x, u_0^+(x))v(x) = 0$ for a.a. $x \in \Omega$. It is obviously that $u_0(x) \ge 0$ for all $x \in \Omega$. Thus, $f(x, u_0^+(x)) = f(x, u_0(x))$ for all $x \in \Omega$. Hence,

$$\int_\Omega (|\nabla u_0|^{p-2}\nabla u_0 \nabla v + a(x)|\nabla u_0|^{q-2}\nabla u_0 \nabla v)\mathrm{d}x = \int_\Omega f(x, u_0(x))v(x)\mathrm{d}x$$

for all $v \in W_0^{1,H}(\Omega)$. This shows that u_0 is a nonnegative nontrivial weak solution of problem (P). By again using [40, Lemma 3.5], we deduce that $u_0 \in N_+$.

Claim 2 $C_k(\varphi_+, u_0) = \delta_{k,1}\mathbb{Z}$ for all $k \ge 0$, where $\delta_{k,m} = 1$ if $k = m$ and $\delta_{k,m} = 0$ if $k \ne m$.

Without loss of generality, we may assume that $K_{\varphi_+} = \{0, u_0\}$. Recall that $u = 0$ is a local minimizer of φ_+ and u_0 is a critical point of φ_+ of mountain pass type. So, by the similar proof of Lemma 3.2(1), we can show that there exists $\delta > 0$ such that

$$0 = \varphi_+(0) < \delta \le \varphi_+(u_0).$$

Let $\nu_- < 0 < \nu_+ < \delta$ and consider the inclusions $\varphi_+^{\nu_-} \subset \varphi_+^{\nu_+} \subset W_0^{1,H}(\Omega)$. Consider the following corresponding long exact sequence of singular homology groups (see [38, p.143]):

$$\cdots \to H_k(W_0^{1,H}(\Omega), \varphi_+^{\nu_-}) \xrightarrow{i_\sharp} H_k(W_0^{1,H}(\Omega), \varphi_+^{\nu_+}) \xrightarrow{j_\sharp} H_{k-1}(\varphi_+^{\nu_-}, \varphi_+^{\nu_+}) \to \cdots, \tag{4.3}$$

where $i_\sharp, j_\sharp$ are induced by inclusions. It follows that $i_\sharp$ and $i_\sharp$ are isomorphisms. Thus, due to the fact that $K_{\varphi_+} = \{0, u_0\}$ and $\nu_- < 0 = \varphi_+(0)$, by using Lemma 3.3, we deduce that

$$H_k(W_0^{1,H}(\Omega), \varphi_+^{\nu_-}) = C_k(\varphi_+, \infty) = 0, \quad \forall k \ge 0. \tag{4.4}$$

Moreover, it follows from $0 = \varphi_+(0) < \nu_+$ that

$$H_k(W_0^{1,H}(\Omega), \varphi_+^{\nu_+}) = C_k(\varphi_+, u_0), \quad \forall k \geq 0. \tag{4.5}$$

Similarly, we also deduce that

$$H_{k-1}(\varphi_+^{\nu_+}, \varphi_+^{\nu_-}) = C_{k-1}(\varphi_+, 0) = \delta_{k-1,0}\mathbb{Z} = \delta_{k,1}\mathbb{Z}, \quad \forall k \geq 0. \tag{4.6}$$

As a consequence of (4.4)–(4.6) and taking into account (4.3), we infer that only the tail of that chain (i.e., $k = 1$) is nontrivial. Consequently, by using the rank theorem, (4.3)–(4.4) and (4.6), we obtain that

$$\mathrm{Rank} H_1(W_0^{1,H}(\Omega), \varphi_+^{\nu_+}) = \mathrm{Rank}\ \ker j_\sharp + \mathrm{Rank}\ \mathrm{im} j_\sharp = \mathrm{Rank}\ \mathrm{im} i_\sharp + \mathrm{Rank}\ \mathrm{im} j_\sharp \leq 1. \tag{4.7}$$

Using the fact that u_0 is a critical point of φ_+ of mountain pass type, we get that

$$C_1(\varphi_+, u_0) \neq 0. \tag{4.8}$$

Finally, due to the fact that only for $k = 1$ the chain (4.3) is nontrivial and using again (4.5)–(4.8), we observe that

$$C_k(\varphi_+, u_0) = \delta_{k,1}\mathbb{Z}, \quad \forall k \geq 0.$$

Claim 3 $C_k(\varphi, u_0) = C_k(\varphi_+, u_0)$ for all $k \geq 0$.

First of all, we introduce the homotopy function $h : [0, 1] \times W_0^{1,H}(\Omega) \to \mathbb{R}$ defined as

$$h(\lambda, u) = (1 - \lambda)\varphi(u) + \lambda\varphi_+(u).$$

Assume that there exist $\{\lambda_n\} \subset [0, 1]$ and $\{u_n\} \subset W_0^{1,H}(\Omega)$ such that

$$\lambda_n \to \lambda_0,\ u_n \to u_0 \quad \text{in } W_0^{1,H}(\Omega) \text{ and } h'_u(\lambda_n, u_n) = 0, \quad \forall n \in N, \tag{4.9}$$

which yields that

$$\begin{aligned}
&\int_\Omega (|\nabla u_n|^{p-2}\nabla u_n \nabla v + a(x)|\nabla u_n|^{q-2}\nabla u_n \nabla v)\mathrm{d}x \\
&= (1 - \lambda_n)\int_\Omega f(x, u_n)v\mathrm{d}x + \lambda_n \int_\Omega f(x, u_n^+)v\mathrm{d}x \\
&= \int_\Omega f(x, u_n^+)v\mathrm{d}x + (1 - \lambda_n)\int_\Omega f(x, -u_n^-)v\mathrm{d}x
\end{aligned}$$

for all $v \in W_0^{1,H}(\Omega)$. Then u_n is a weak solution to

$$\begin{cases} -\mathrm{div}(|\nabla u_n|^{p-2}\nabla u_n + a(x)|\nabla u_n|^{q-2}\nabla u_n) = f_n(x, u_n) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases}$$

where we have set $f_n(x, t) = f(x, t^+) + (1 - \lambda_n)f(x, -t^-)$.

By the assumption (f_1), for any $\varepsilon > 0$, there exists a constant $\delta_\varepsilon \in (0, 1)$ such that

$$|f(x, t)| \leq \varepsilon|t|^{q-1} \leq \varepsilon|t|^{p-1}$$

for $|t| \le \delta_\varepsilon$ and for almost all $x \in \Omega$. Also, by (f_2), for all $x \in \Omega$ and $|t| \ge \delta_\varepsilon$, we have

$$|f(x,t)| \le C(1+|t|^{r-1}) \le C\Big(\Big|\frac{t}{\delta_\varepsilon}\Big| + |t|^{r-1}\Big)$$
$$\le C\Big(\Big|\frac{t}{\delta_\varepsilon}\Big|^{r-1} + |t|^{r-1}\Big) = C\Big(\frac{1}{\delta_\varepsilon^{r-1}} + 1\Big)|t|^{r-1}.$$

The above two inequalities imply that there exists a constant $C_\varepsilon > 0$ such that

$$|f_n(x,t)| \le 2\varepsilon|t|^{p-1} + 2C_\varepsilon|t|^{r-1}, \quad \forall x \in \Omega,\ \forall t \in \mathbb{R}. \tag{4.10}$$

Then $u_n \in L^\infty(\Omega)$ (see [17, Sec. 3.2]) with L^∞-bounded independent of n. This implies that there exists a constant $d > 0$, such that

$$|u_n|_\infty \le d, \quad \forall n \in N. \tag{4.11}$$

Let $h_n(x) := f(x, u_n^+(x)) + (1-\lambda_n)f(x, -u_n^-(x))$, $n \in N, x \in \Omega$. Then from (4.10)–(4.11), we get

$$|h_n|_\infty \le 2\varepsilon|u_n|_\infty^{p-1} + 2C_\varepsilon|u_n|_\infty^{r-1} \le 2\varepsilon d^{p-1} + 2C_\varepsilon d^{r-1} < +\infty, \quad \forall n \in N.$$

Again from Lemma 3.3 of Fukagai-Narukawa [40, p.545], we conclude that there exist $\alpha \in (0,1)$ and $M > 0$ such that

$$u_n \in C^{1,\alpha}(\overline{\Omega}) \text{ and } |u_n|_{C^{1,\alpha}(\overline{\Omega})} \le M, \quad \forall n \in N.$$

Using the compactness of the embedding $C^{1,\alpha}(\overline{\Omega}) \hookrightarrow C^1(\overline{\Omega})$ together with (4.9), it follows that $u_n \to u_0$ in $C^1(\overline{\Omega})$.

Recall that $u_0 \in N_+$ (see Claim 1). Therefore, $u_n \in N_+$ for $n \ge 1$ large, which implies that there exists a $n_0 \in N$ such that $u_n \in N_+$ for all $n \ge n_0$. Then $\{u_n : n \ge n_0\}$ are distinct positive solutions of (P), which leads to contradiction as K_{φ_+} must be finite. Consequently, (4.9) can not happen and hence we obtain that $C_k(\varphi, u_0) = C_k(\varphi_+, u_0)$ for all $k \ge 0$ (it is a direct consequence of the homotopy invariance of critical groups, see [41, Theorem 5.2]). This proves Claim 3.

By reasoning in a similar way as above, Lemma 3.2 also holds for the functional φ_-. Thus, there is another function $v_0 \in W_0^{1,H}(\Omega)$ that is a nonpositive nontrivial weak solution of problem (P). Similar to the proof of Claims 1–3, we can obtain that

$$v_0 \in -N_+ \text{ and } C_k(\varphi, v_0) = C_k(\varphi_-, v_0) \quad \text{for all } k \ge 0.$$

Hence, we retrieve the two constant sign solutions $u_0 \in N_+$ and $v_0 \in -N_+$. If we assume $K_\varphi = \{0, u_0, v_0\}$ which means that u_0 and v_0 are the only nontrivial solutions of (P), then by Claim 2 we have

$$C_k(\varphi, u_0) = C_k(\varphi, v_0) = \delta_{k,1}\mathbb{Z}, \quad \forall k \ge 0. \tag{4.12}$$

Moreover, we recall that $u = 0$ is local minimizer of φ. So that

$$C_k(\varphi, 0) = \delta_{k,0}\mathbb{Z}, \quad \forall k \ge 0. \tag{4.13}$$

Then from (4.12)–(4.13), Lemma 3.3 and the Morse relation, we may write

$$\sum_{u\in K_\varphi}\sum_{k\geq 0}\mathrm{rank}C_k(\varphi,u)t^k=\sum_{k\geq 0}\mathrm{rank}C_k(\varphi,\infty)t^k+(1+t)\sum_{k\geq 0}\beta_k t^k,\quad \forall t\in\mathbb{R},$$

where $\beta_k \in N$. Assume that 0, u_0 and v_0 are the only critical points of φ. Then the Morse inequality becomes $2(-1)^1+(-1)^0+(-1)^k=(-1)^k$. This is impossible. Thus φ must have at least one more critical point w_0. So (P) has at least third nontrivial solution. This completes the proof of Theorem 1.1.

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