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Left-Invariant Minimal Unit Vector Fields on the Solvable Lie Group*

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Abstract Božek (1980) has introduced a class of solvable Lie groups $\mathrm{G_n}$ with arbitrary odd dimension to construct irreducible generalized symmetric Riemannian space such that the identity component of its full isometry group is solvable. In this article, the authors provide the set of all left-invariant minimal unit vector fields on the solvable Lie group $\mathrm{G_n}$, and give the relationships between the minimal unit vector fields and the geodesic vector fields, the strongly normal unit vectors respectively.

Keywords Solvable Lie groups, Lagrangian multiplier method, Minimal unit vector fields, Geodesic vector fields, Strongly normal unit vectors
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1 Introduction

Let (M, g) be a Riemannian manifold and (T_1M, g_s) be the unit tangent sphere bundle equipped with the Sasaki metric. Every smooth unit vector field determines a mapping between (M, g) and (T_1M, g_s), embedding M into its tangent unit sphere bundle T_1M. Every smooth unit vector field X on M can be viewed as a submanifold of T_1M, then if the manifold M is compact and orientable, we can define the volume of X as the volume of the immersion.

Gluck and Ziller firstly considered the problem of determining unit vector fields which have minimal volume in [8]. They proved that the unit vector fields of minimum volume on the unit sphere S^3 are precisely the Hopf vector fields. However, this is no longer true for the higher dimensional sphere S^{2n+1}, $n \geq 2$ (see [10–11, 13–14]). In [7], the authors proved that a unit vector field V is a critical point of the volume functional restricted to the set of unit vector fields $\mathfrak{X}^1(M)$ if and only if $V : M \to T_1M$ is a minimal immersion. So such unit vector fields are called minimal even though the manifold is not compact.

Some examples of Lie groups equipped with minimal unit vector fields are provided in [4, 7, 9, 15–18]. For three dimensional Lie groups, Tsukada and Vanhecke gave all the left invariant

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minimal unit vector fields in [16]. Yi in [18] obtained all the left-invariant minimal unit vector fields on the semi-direct product $\mathbb{R}^n \rtimes_P \mathbb{R}$, where P is a nonsingular diagonal matrix. But for most of examples of Lie groups, it is difficult to determine all the left invariant minimal unit vector fields, there are just some special minimal unit vector fields.

Božek introduced a class of important solvable Lie groups $\mathrm{G_n}$ with arbitrary odd dimension to construct irreducible generalized symmetric Riemannian space such that the identity component of its full isometry group is solvable in [5]. In rencent years, a great deal of mathematical effort has been devoted to the study of the solvable Lie group $\mathrm{G_n}$. In [6], Calvaruso, Kowalski and Marinosci studied homogeneous geodesic of solvable Lie groups $\mathrm{G_n}$. Aghasi and Nasehi in [3] generalized this study to the Randers setting of Douglas type, they proved that these homogeneous Randers spaces are locally projectively flat Finsler spaces. In [1], the authors studied some other geometrical properties on these spaces with dimension five, and extended those geometrical properties for an arbitrary odd dimension in both Riemannian and Lorentzian cases in [2].

Thus, it is an interesting question to determine the left invariant minimal unit vector fields on Lie groups $\mathrm{G_n}$. The study of this problem will deepen our understanding of this kind of Lie groups undoubtedly. In this paper, the aim is to provide the set of all left invariant minimal unit vector fields on these Lie groups $\mathrm{G_n}$ by Lagrange multiplier method. For an integer $n \geq 2$, a unimodular solvable Lie group $\mathrm{G_n}$ is defined as follows:

$$\mathrm{G_n} = \begin{pmatrix} \mathrm{e}^{u_0} & 0 & \cdots & 0 & x_0 \\ 0 & \mathrm{e}^{u_1} & \cdots & 0 & x_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \mathrm{e}^{u_n} & x_n \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

where $(x_0, x_1, \cdots, x_n, u_1, \cdots, u_n) \in \mathbb{R}^{2n+1}$ and $u_0 = -\sum_{i=1}^{n} u_i$. Considering the following left invariant vector fields

$$U_\alpha = \frac{\partial}{\partial u_\alpha}, \quad \alpha = 1, \cdots, n$$

and

$$X_i = \mathrm{e}^{u_i} \frac{\partial}{\partial x_i}, \quad i = 0, 1, \cdots, n.$$

By [2], we can equip $\mathrm{G_n}$ with a left invariant Riemannian metric as follow

$$g = \sum_{i=0}^{n} \mathrm{e}^{-2u_i} (\mathrm{d}x_i)^2 + \sum_{\alpha=1}^{n} (\mathrm{d}u_\alpha)^2.$$

The inner product which is induced by this metric shows that the set $\{X_0, X_1, \cdots, X_n, U_1, \cdots, U_n\}$ is an orthonormal frame field for the Lie algebra $\mathfrak{g}_n$ of G_n.

The main result can be shown as follows.

Theorem 1.1 *For Lie group* $\mathrm{G_n}$, $n \geq 2$, *the set of left invariant minimal unit vector fields is*

$$\{\pm X_0\} \bigcup_{i=1}^{n} \{\pm X_i\} \bigcup_{i=1}^{n} \{\pm U_i\}$$
$$\bigcup_{l=2}^{n} \Big\{ \sum_{\alpha=1}^{l} (k_{i_\alpha} X_{i_\alpha} + k_{n+i_\alpha} U_{i_\alpha}) \mid {k_{n+i_\alpha}}^2 + {k_{i_\alpha}}^2 = \frac{1}{l},\ \sum_{\beta=1}^{n} k_{n+\beta} = 0 \Big\}$$
$$\cup \Big\{ \sum_{i=1}^{n} \frac{1}{\sqrt{n}} U_i \Big\} \cup \Big\{ \sum_{i=1}^{n} -\frac{1}{\sqrt{n}} U_i \Big\}.$$

This paper is organized as follows. We recall some basic notions and facts about minimal unit vector fields in Section 2. In Section 3, we give all left invariant minimal unit vector fields on G_n, i.e., Theorem 1.1. Finally we devote Section 4 to discuss the relationships between the minimal unit vector and the geodesic vector, the strongly normal vector in Theorems 4.1 and 4.2, respectively.

2 Preliminaries

Let (M, g) be a n-dimensional smooth Riemannian manifold and $\mathfrak{X}(M)$ be the set of all vector fields on M. Furthermore, $\triangledown$ denotes the Levi-Civita connection.

Assume that $\mathfrak{X}^1(M)$ is the non-empty set of unit vector fields. For $V \in \mathfrak{X}^1(M)$, we define a positive definite symmetric tensor field L_V by

$$L_V = I + (\triangledown V)^* \triangledown V, \tag{2.1}$$

where I, $(\triangledown V)^*$ denote the identity map and the adjoint operator of $(\triangledown V)$, respectively. And let $f(V) = (\det L_V)^{\frac{1}{2}}$, then for a compact closed oriented manifold M, the volume functional of vector fields $\mathrm{Vol} : \mathfrak{X}^1(M) \to \mathbb{R}$ is given by

$$\mathrm{Vol}(V) = \int_M f(V) \mathrm{d}v,$$

where $\mathrm{d}v$ is the volume form on (M, g).

Now we give a $(1,1)$-tensor field K_V and a 1-form ω_V associated to V. They are defined as

$$K_V = f(V) {L_V}^{-1} (\triangledown V)^*, \quad \omega_V = C_1^1(\triangledown K_V).$$

We can easily get

$$\omega_V(X) = \mathrm{tr}(Z \mapsto (\triangledown_Z K_V)(X)).$$

Let $\mathcal{H}^V$ denote the distribution consisting of the tangent vectors orthogonal to V. In [7] it is proved that V is a critical point for the volume functional Vol on $\mathfrak{X}^1(M)$ if and only if ω_V vanishes on $\mathcal{H}^V$.

For an orthonormal basis $\{E_1, E_2, \cdots, E_n\}$ of the tangent space, $\omega_V(X)$ can be written as

$$\omega_V(X) = \sum_{i=1}^{n} g((\nabla_{E_i} K_V)(X), E_i).$$

Besides, it is shown that V is critical if and only if the submanifold of (T_1M, g_s) determined by V is minimal (see [7]), where g_s is the Sasaki metric.

Definition 2.1 (see [18]) *A unit vector field V on a Riemannian manifold (M, g) is called minimal if $\omega_V(X) = 0$ for all $X \in \mathcal{H}^V$.*

Now we consider left invariant unit vector fields on Lie groups. Let G be a n-dimensional connected Lie group equipped with a left invariant metric, and $\mathbf{g}$ be the Lie algebra of G. Then the left invariant metric on G determines the inner product $\langle \cdot, \cdot \rangle$ on $\mathbf{g}$. Furthermore, let $\mathcal{S}$ be the unit sphere of $\mathbf{g}$. By the left invariance, the function f can be considered as a function on $\mathcal{S}$. The distribution $\mathcal{H}^V$ is invariant with respect to the left translation and can be equal to orthogonal complement $V^\perp$ of V in $\mathbf{g}$. So $V^\perp$ is identified with the tangent space $T_V\mathcal{S}$ of the unit sphere $\mathcal{S}$ at V. Therefore we have the following lemma.

Lemma 2.1 (see [16]) *A left invariant unit vector field V on a Lie group G is minimal if and only if the linear map ω_V on $\mathbf{g}$ vanishes on $V^\perp \cong T_V\mathcal{S}$.*

Now we compute the differential $\mathrm{d}f$ of the function f on $\mathcal{S}$ at V. And we have the following proposition:

Proposition 2.1 (see [17]) *For $X \in T_V\mathcal{S}$, we have*

$$\omega_V(X) = -\mathrm{d}f_V(X) - \mathrm{tr}\, ad_{K_V X}$$

and V is minimal if and only if

$$\mathrm{d}f_V(X) = -\mathrm{tr}\, ad_{K_V X}$$

for all $X \in T_V\mathcal{S}$.

If the Lie group G is a unimodular Lie group, that is $\mathrm{tr}\, ad_X = 0$ for all $X \in \mathbf{g}$ (see [12]), we can easily get the following corollary.

Corollary 2.1 *A left invariant unit vector field V on a unimodular Lie group G is minimal if and only if V is a critical point of the function f on $\mathcal{S}$.*

3 Left-Invariant Minimal Unit Vector Fields on $\mathrm{G_n}$

For any $n \geq 1$, the unimodular solvable Lie group $\mathrm{G_n}$ is as follows:

$$\mathrm{G_n} = \begin{pmatrix} \mathrm{e}^{u_0} & 0 & \cdots & 0 & x_0 \\ 0 & \mathrm{e}^{u_1} & \cdots & 0 & x_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \mathrm{e}^{u_n} & x_n \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

where $(x_0, x_1, \cdots, x_n, u_1, \cdots, u_n) \in \mathbb{R}^{2n+1}$ and $u_0 = -\sum_{i=1}^{n} u_i$. Considering the following left invariant vector fields

$$U_\alpha = \frac{\partial}{\partial u_\alpha}, \quad \alpha = 1, \cdots, n$$

and

$$X_i = \mathrm{e}^{u_i} \frac{\partial}{\partial x_i}, \quad i = 0, 1, \cdots, n.$$

By [2], we can equip $\mathrm{G_n}$ with a left invariant Riemannian metric as following

$$g = \sum_{i=0}^{n} \mathrm{e}^{-2u_i} (\mathrm{d}x_i)^2 + \sum_{\alpha=1}^{n} (\mathrm{d}u_\alpha)^2.$$

The inner product which is induced by this metric shows that the set $\{X_0, X_1, \cdots, X_n, U_1, \cdots, U_n\}$ is an orthonormal frame field for the Lie algebra $\mathfrak{g}_n$ of G_n and the Lie bracket is introduced as follows:

$$[X_0, U_\alpha] = X_0, \quad [X_\alpha, U_\beta] = -\delta_{\alpha\beta} X_\alpha, \quad [X_i, X_j] = [U_\alpha, U_\beta] = 0.$$

By Koszul's formula in [12],

$$2g(\nabla_{e_i} e_j, e_k) = g([e_i, e_j], e_k) - g([e_j, e_k], e_i) + g([e_k, e_i], e_j),$$

the non-vanishing Riemannian connection components are given by

$$\nabla_{X_0} U_\alpha = X_0, \quad \nabla_{X_0} X_0 = -\sum_{i=1}^{n} U_i,$$
$$\nabla_{X_i} U_i = -X_i, \quad \nabla_{X_i} X_i = U_i,$$

where $i, j, \alpha, \beta = 1, \cdots, n$.

For $V = \sum_{i=0}^{n} k_i X_i + \sum_{i=1}^{n} k_{n+i} U_i$, where $\sum_{i=0}^{2n} {k_i}^2 = 1$, we have

$$\nabla V = \begin{pmatrix} \sum_{i=1}^{n} k_{n+i} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & -k_{n+1} & \cdots & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -k_{2n} & 0 & \cdots & 0 \\ -k_0 & k_1 & \cdots & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -k_0 & 0 & \cdots & k_n & 0 & \cdots & 0 \end{pmatrix},$$

and by (2.1), we can get

$$L_V = \begin{pmatrix} 1 + \big(\sum_{i=1}^{n} k_{n+i}\big)^2 + n{k_0}^2 & -k_0 k_1 & \cdots & -k_0 k_n & 0 & \cdots & 0 \\ -k_0 k_1 & 1 + {k_{n+1}}^2 + {k_1}^2 & \cdots & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -k_0 k_n & 0 & \cdots & 1 + {k_{2n}}^2 + {k_n}^2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

So we can easily get

$$\det L_V = \prod_{i=1}^{n}(1 + {k_{n+i}}^2 + {k_i}^2)\Big[1 + \Big(\sum_{i=1}^{n} k_{n+i}\Big)^2 + n{k_0}^2 - \sum_{i=1}^{n} \frac{{k_0}^2{k_i}^2}{1 + {k_{n+i}}^2 + {k_i}^2}\Big]. \tag{3.1}$$

Proof of Theorem 1.1 By Corollary 2.1, for a unimodular Lie group G, a left invariant unit vector field V is minimal if and only if V is a critical point of the function f on $\mathcal{S}$.

Let $H(k_0, k_1, \cdots, k_{2n}) \triangleq \det L_V$, we will find the set of critical points of the function $H(k_0, k_1, \cdots, k_{2n})$ with the constraint

$$g(k_0, k_1, \cdots, k_{2n}) = \sum_{i=0}^{2n} {k_i}^2 - 1 = 0. \tag{3.2}$$

Using the Lagrange multiplier method, we need to solve the following system of equations:

$$\begin{cases} \nabla H = \lambda \nabla g, \\ g = 0. \end{cases}$$

Let $\Omega = \prod\limits_{j\neq i, j=1}^{n} (1 + {k_{n+j}}^2 + {k_j}^2)$. From the first equation above, we have the following:

$$2k_0\Big(n - \sum_{j=1}^{n} \frac{{k_j}^2}{1 + {k_{n+j}}^2 + {k_j}^2}\Big)\Omega(1 + {k_{n+i}}^2 + {k_i}^2) = 2\lambda k_0, \tag{3.3}$$

$$2k_i\Omega\Big(1 + \Big(\sum_{j=1}^{n} k_{n+j}\Big)^2 + (n-1){k_0}^2 - \sum_{j\neq i, j=1}^{n} \frac{{k_0}^2{k_j}^2}{1 + {k_{n+j}}^2 + {k_j}^2}\Big) = 2\lambda k_i, \tag{3.4}$$

$$\begin{aligned} &2\Omega\Big[\Big(\sum_{j=1}^{n} k_{n+j}\Big)(1 + {k_{n+i}}^2 + {k_i}^2) + k_{n+i}\Big(1 + \Big(\sum_{j=1}^{n} k_{n+j}\Big)^2 + n{k_0}^2 \\ &\quad - \sum_{j\neq i, j=1}^{n} \frac{{k_0}^2{k_j}^2}{1 + {k_{n+j}}^2 + {k_j}^2}\Big)\Big] = 2\lambda k_{n+i}, \end{aligned} \tag{3.5}$$

where $i = 1, 2, \cdots, n$.

For the case $n = 1$, the result is already known in [16] by Tsukada and Vanhecke. So we will restrict ourselves to $n \geq 2$. Firstly, we have the following assertion.

Claim 3.1 $\lambda \neq 0$.

Proof It is easy to find that

$$\Big(n - \sum_{j=1}^{n} \frac{{k_j}^2}{1 + {k_{n+j}}^2 + {k_j}^2}\Big) \prod_{j=1}^{n}(1 + {k_{n+j}}^2 + {k_j}^2) > 0,$$

$$\prod_{j\neq i, j=1}^{n} (1 + {k_{n+j}}^2 + {k_j}^2)\Big(1 + \Big(\sum_{j=1}^{n} k_{n+j}\Big)^2 + (n-1){k_0}^2 - \sum_{j\neq i, j=1}^{n} \frac{{k_0}^2{k_j}^2}{1 + {k_{n+j}}^2 + {k_j}^2}\Big) > 0.$$

If $\lambda = 0$, by (3.3)–(3.4) we have

$$k_0 = k_1 = \cdots = k_n = 0.$$

Then we can simplify (3.5) to

$$\Big(\sum_{j=1}^{n} k_{n+j}\Big)(1+{k_{n+i}}^2)+k_{n+i}\Big[1+\Big(\sum_{j=1}^{n} k_{n+j}\Big)^2\Big]=0, \tag{3.6}$$

if $k_{n+i}=0$, we obtain $\sum\limits_{j=1}^{n} k_{n+j}=0$, then we get $k_{n+1}=\cdots=k_{2n}=0$. This contradicts to (3.2).

So all k_{n+j} $(j=1,\cdots,n)$ are not equal to 0, by (3.6), we have

$$\frac{1}{k_{n+1}}+k_{n+1}=\cdots=\frac{1}{k_{2n}}+k_{2n},$$

by (3.2), we obtain $k_{n+1}=\cdots=k_{2n}=\pm\frac{1}{\sqrt{n}}$. This contradicts to (3.6). Therefore, we have the conclusion: $\lambda\neq 0$.

Case 3.1 $k_0\neq 0$.

Now, by (3.3) we can get

$$\lambda=\Big(n-\sum_{j=1}^{n}\frac{{k_j}^2}{1+{k_{n+j}}^2+{k_j}^2}\Big)\prod_{j=1}^{n}(1+{k_{n+j}}^2+{k_j}^2). \tag{3.7}$$

If $k_1\neq 0$, according to (3.4), we have

$$\lambda=\prod_{j=2}^{n}(1+{k_{n+j}}^2+{k_j}^2)\Big(1+\Big(\sum_{j=1}^{n} k_{n+j}\Big)^2+(n-1){k_0}^2-\sum_{j=2}^{n}\frac{{k_0}^2{k_j}^2}{1+{k_{n+j}}^2+{k_j}^2}\Big). \tag{3.8}$$

Then with the help of Mathematica, we solve the system of (3.2), (3.7)–(3.8), the solution set is empty. So we have $k_1=0$. In the same way, we can get $k_2=\cdots=k_n=0$.

If $k_{n+1}\neq 0$, from (3.5) we obtain

$$\lambda=\prod_{j=2}^{n}(1+{k_{n+j}}^2)\Big[\Big(\sum_{j=1}^{n} k_{n+j}\Big)\frac{1+k_{n+1}^2}{k_{n+1}}+\Big(1+\Big(\sum_{j=1}^{n} k_{n+j}\Big)^2+n{k_0}^2\Big)\Big]. \tag{3.9}$$

Since $k_1=\cdots=k_n=0$, (3.7) can be reduced to

$$\lambda=n\prod_{j=1}^{n}(1+{k_{n+j}}^2).$$

So we have

$$n(1+{k_{n+1}}^2)=\Big(\sum_{j=1}^{n} k_{n+j}\Big)\frac{1+k_{n+1}^2}{k_{n+1}}+\Big(1+\Big(\sum_{j=1}^{n} k_{n+j}\Big)^2+n{k_0}^2\Big). \tag{3.10}$$

Solving the system of (3.2), (3.10) by Mathematica, the solution set is empty. We have $k_{n+1}=0$. Similarly, $k_{n+2}=\cdots=k_{2n}=0$.

In conciusion, we have $k_0=\pm 1$, so $\pm X_0$ are minimal unit vector fields.

Case 3.2 $k_0=0$.

Case 3.2.1 $\exists\, l(1 \leq l \leq n)$ such that $k_{i_1} \neq 0, \cdots, k_{i_l} \neq 0$, $k_{n+1} = \cdots = k_{2n} = 0$, where $1 \leq i_1 < \cdots < i_l \leq n$. In this case, for $k_{i_\alpha}(1 \leq \alpha \leq l)$, from (3.4), we have

$$\prod_{j \neq i_\alpha, j=1}^{n} (1 + {k_j}^2) = \lambda$$

according to (3.2), we have $k_{i_\alpha} = \pm\frac{1}{\sqrt{l}}$, $1 \leq \alpha \leq l$.

Thus, the set of left invariant minimal vector fields is

$$\bigcup_{l=1}^{n} \Big\{ \sum_{\alpha=1}^{l} \pm\frac{1}{\sqrt{l}} X_{i_\alpha} \,\Big|\, 1 \leq i_1 < \cdots < i_l \leq n \Big\}. \tag{3.11}$$

Case 3.2.2 $\exists\, l\ (1 \leq l \leq n)$ such that $k_{i_1} \neq 0, \cdots, k_{i_l} \neq 0$, where $1 \leq i_1 < \cdots < i_l \leq n$, and $k_{n+j}\ (1 \leq j \leq n)$ are not equal to 0 simultaneously.

Assume $k_i \neq 0$, (3.4) can be written as

$$\lambda = \prod_{j \neq i, j=1}^{n} (1 + {k_{n+j}}^2 + {k_j}^2)\Big(1 + \Big(\sum_{j=1}^{n} k_{n+j}\Big)^2\Big), \tag{3.12}$$

so if $\exists\, \beta \neq \mu$, $k_\beta \neq 0, k_\mu \neq 0$, we have

$$1 + {k_{n+\beta}}^2 + {k_\beta}^2 = 1 + {k_{n+\mu}}^2 + {k_\mu}^2. \tag{3.13}$$

Then for k_{n+i}, according to (3.5), we have

$$k_{n+i}\lambda = \Big(\sum_{j=1}^{n} k_{n+j}\Big) \prod_{j=1}^{n} (1 + {k_{n+j}}^2 + {k_j}^2) + k_{n+i} \prod_{j \neq i, j=1}^{n} (1 + {k_{n+j}}^2 + {k_j}^2)\Big[1 + \Big(\sum_{j=1}^{n} k_{n+j}\Big)^2\Big].$$

If $k_{n+i} = 0$, we can obtain $\sum\limits_{j=1}^{n} k_{n+j} = 0$, if $k_{n+i} \neq 0$, we also have $\sum\limits_{j=1}^{n} k_{n+j} = 0$. Then we can easily get at least two $k_{n+j} \neq 0$.

Thus for $\forall\, k_{n+\alpha}\ (1 \leq \alpha \leq n)$, (3.5) can be reduced to

$$k_{n+\alpha}\lambda = k_{n+\alpha} \prod_{j \neq \alpha, j=1}^{n} (1 + {k_{n+j}}^2 + {k_j}^2). \tag{3.14}$$

If $k_{n+\alpha} \neq 0$, solving the system of (3.12), (3.14), we can get

$$1 + {k_{n+\alpha}}^2 + {k_\alpha}^2 = 1 + {k_{n+i}}^2 + {k_i}^2. \tag{3.15}$$

According to (3.15), (3.2) and $\sum\limits_{j=1}^{n} k_{n+j} = 0$, we obtain

$${k_{n+i_\beta}}^2 + {k_{i_\beta}}^2 = \frac{1}{l}, \quad \sum_{\alpha=1}^{n} k_{n+\alpha} = 0, \tag{3.16}$$

where $k_{n+i_\beta} \neq 0$ or $k_{i_\beta} \neq 0$, $\beta = 1, \cdots, l$.

Thus, the set of left invariant minimal vector field is

$$\bigcup_{l=2}^{n}\Big\{\sum_{\alpha=1}^{l}(k_{i_\alpha}X_{i_\alpha}+k_{n+i_\alpha}U_{i_\alpha})\,\Big|\,{k_{n+i_\alpha}}^2+{k_{i_\alpha}}^2=\frac{1}{l},\exists\,k_{n+i_\beta}\neq 0,\sum_{\beta=1}^{n}k_{n+\beta}=0\Big\}. \tag{3.17}$$

Case 3.2.3 $k_1=\cdots=k_n=0$.

In this case, (3.5) can be reduced to

$$\Big(\sum_{j=1}^{n}k_{n+j}\Big)\prod_{j=1}^{n}(1+{k_{n+j}}^2)+k_{n+i}\prod_{j\neq i,j=1}^{n}(1+{k_{n+j}}^2)\Big[1+\Big(\sum_{j=1}^{n}k_{n+j}\Big)^2\Big]=\lambda k_{n+i}. \tag{3.18}$$

If $k_{n+\alpha}\neq 0$, the equation above can be written as

$$\prod_{j\neq\alpha,j=1}^{n}(1+{k_{n+j}}^2)\Big[\Big(\sum_{j=1}^{n}k_{n+j}\Big)\frac{(1+{k_{n+\alpha}}^2)}{k_{n+\alpha}}+1+\Big(\sum_{j=1}^{n}k_{n+j}\Big)^2\Big]=\lambda. \tag{3.19}$$

• If $\exists\,k_{n+i}=0$, by (3.18), we have $\sum\limits_{j=1}^{n}k_{n+j}=0$. Then we can simplify the equation above to

$$\lambda=\prod_{j\neq\alpha,j=1}^{n}(1+{k_{n+j}}^2). \tag{3.20}$$

Therefore if $k_{n+\alpha},k_{n+\beta}\neq 0$, $\alpha,\beta=1,\cdots,n$, we have ${k_{n+\alpha}}^2={k_{n+\beta}}^2$.

According to

$$\begin{cases}{k_{n+\alpha}}^2={k_{n+\beta}}^2\\ \sum\limits_{j=1}^{n}k_{n+j}=0,\end{cases}$$

we obtain that the set of minimal unit vector fields is

$$\bigcup_{l=1}^{n-1}\Big\{\sum_{\alpha=1}^{l}k_{n+i_\alpha}U_{i_\alpha}\,\Big|\,k_{n+i_\alpha}=\pm\frac{1}{\sqrt{l}},\sum_{i=1}^{n}k_{n+i_\alpha}=0,1\le i_1<\cdots<i_l\le n\Big\}. \tag{3.21}$$

• If $k_{n+1}\neq 0,\cdots,k_{2n}\neq 0$, according to (3.1), we have

$$H=\prod_{i=1}^{n}(1+{k_{n+i}}^2)\Big[1+\Big(\sum_{i=1}^{n}k_{n+i}\Big)^2\Big].$$

Then taking natural logarithm, we have $H'=\ln H=\sum\limits_{i=1}^{n}\ln(1+{k_{n+i}}^2)+\ln\big[1+\big(\sum\limits_{i=1}^{n}k_{n+i}\big)^2\big]$. Applying Lagrange multiplier method, we get

$$\lambda=\frac{1}{1+{k_{n+i}}^2}+\frac{\sum\limits_{i=1}^{n}k_{n+i}}{\big[1+\big(\sum\limits_{i=1}^{n}k_{n+i}\big)^2\big]k_{n+i}},\quad \forall i=1,\cdots,n. \tag{3.22}$$

For $\alpha \neq \beta$, we can obtain

$$-\frac{(k_{n+\alpha}+k_{n+\beta})(k_{n+\alpha}-k_{n+\beta})}{(1+{k_{n+\alpha}}^2)(1+{k_{n+\beta}}^2)}=\frac{\big(\sum\limits_{i=1}^{n}k_{n+i}\big)(k_{n+\alpha}-k_{n+\beta})}{\big[1+\big(\sum\limits_{i=1}^{n}k_{n+i}\big)^2\big]k_{n+\alpha}k_{n+\beta}}. \tag{3.23}$$

If $k_{n+\alpha}-k_{n+\beta}\neq 0$, we have

$$-\frac{(k_{n+\alpha}+k_{n+\beta})}{(1+{k_{n+\alpha}}^2)(1+{k_{n+\beta}}^2)}=\frac{\big(\sum\limits_{i=1}^{n}k_{n+i}\big)}{\big[1+\big(\sum\limits_{i=1}^{n}k_{n+i}\big)^2\big]k_{n+\alpha}k_{n+\beta}}. \tag{3.24}$$

With the help of Mathematica, when $k_{n+\alpha}+k_{n+\beta}\neq 0$ and $\sum\limits_{i=1}^{n}k_{n+i}\neq 0$, there are no solutions.

Therefore, we have

$$(k_{n+\alpha}+k_{n+\beta})(k_{n+\alpha}-k_{n+\beta})=\Big(\sum_{i=1}^{n}k_{n+i}\Big)(k_{n+\alpha}-k_{n+\beta})=0. \tag{3.25}$$

According to the constraint condition $g=\sum\limits_{i=1}^{n}{k_{n+i}}^2-1=0$, we can get the following results.

When n is odd, $k_{n+1}=\cdots=k_{2n}=\pm\frac{1}{\sqrt{n}}$.

When n is even, $k_{n+1}=\cdots=k_{2n}=\pm\frac{1}{\sqrt{n}}$ or $k_{n+i}=\pm\frac{1}{\sqrt{n}}$, $\sum\limits_{i=1}^{n}k_{n+i}=0$, $i=1,\cdots,n$.

So we obtain the set of minimal vector fields is

$$\begin{cases}\Big\{\sum\limits_{i=1}^{n}\dfrac{1}{\sqrt{n}}U_i\Big\}\cup\Big\{\sum\limits_{i=1}^{n}-\dfrac{1}{\sqrt{n}}U_i\Big\}, \quad n=2m+1,\ m\in\mathbb{N}_+,\\ \Big\{\sum\limits_{i=1}^{n}\dfrac{1}{\sqrt{n}}U_i\Big\}\cup\Big\{\sum\limits_{i=1}^{n}-\dfrac{1}{\sqrt{n}}U_i\Big\}\cup\Big\{\sum\limits_{i=1}^{n}k_{n+i}U_i\mid k_{n+i}=\pm\dfrac{1}{\sqrt{n}},\sum\limits_{i=1}^{n}k_{n+i}=0\Big\}.\end{cases} \tag{3.26}$$

In conclusion, combining with (3.11), (3.17), (3.21), (3.26) and case 3.1, we obtain all left invariant minimal unit vector fields on the solvable Lie group $\mathrm{G_n}$ ($n\geq 2$) as follows:

$$\begin{aligned}&\{\pm X_0\}\bigcup_{i=1}^{n}\{\pm X_i\}\bigcup_{i=1}^{n}\{\pm U_i\}\\ &\bigcup_{l=2}^{n}\Big\{\sum_{\alpha=1}^{l}(k_{i_\alpha}X_{i_\alpha}+k_{n+i_\alpha}U_{i_\alpha})\mid {k_{n+i_\alpha}}^2+{k_{i_\alpha}}^2=\frac{1}{l},\sum_{\beta=1}^{n}k_{n+\beta}=0\Big\}\\ &\cup\Big\{\sum_{i=1}^{n}\frac{1}{\sqrt{n}}U_i\Big\}\cup\Big\{\sum_{i=1}^{n}-\frac{1}{\sqrt{n}}U_i\Big\}.\end{aligned}$$

This completes the proof of Theorem 1.1.

4 Geodesic Vector Fields and Strongly Normal Unit Vectors on $\mathrm{G_n}$

In this section, firstly we determine all geodesic vector fields on $\mathrm{G_n}$ and obtain the relationship between geodesic vector fields and minimal unit vector fields. Then we study the strongly normal unit vectors on $\mathrm{G_n}$.

Definition 4.1 *A unit vector field V on a Riemannian manifold (M,g) is called a geodesic vector field if $\nabla_V V=0$.*

The set of all geodesic vector fields on G_n is given as follows.

Theorem 4.1 *For $n\geq 2$, the set of all left invariant geodesic vector fields on the solvable Lie group G_n is $\{\sum\limits_{i=1}^{n}k_{n+i}U_i \,\big|\, \sum\limits_{i=1}^{n}{k_{n+i}}^2=1\}\cup\{\sum\limits_{i=0}^{n}k_iX_i \,\big|\, k_i=\pm\frac{1}{\sqrt{n+1}}\}$.*

Proof Let $V=\sum\limits_{i=0}^{n}k_iX_i+\sum\limits_{i=1}^{n}k_{n+i}U_i$, since the Lie brackets of $\mathfrak{g}_n$ are as follows:

$$[X_0,U_\alpha]=X_0,\quad [X_\alpha,U_\beta]=-\delta_{\alpha\beta}X_\alpha,\quad [X_i,X_j]=[U_\alpha,U_\beta]=0,$$

and the non-vanishing Riemannian connection components are given by

$$\nabla_{X_0}U_\alpha=X_0,\quad \nabla_{X_0}X_0=-\sum_{i=1}^{n}U_i,$$
$$\nabla_{X_i}U_i=-X_i,\quad \nabla_{X_i}X_i=U_i,$$

where $i,j,\alpha,\beta=1,\cdots,n$.

Then we have

$$\nabla V=\begin{pmatrix}\sum\limits_{i=1}^{n}k_{n+i} & 0 & \cdots & 0 & 0 & \cdots & 0\\ 0 & -k_{n+1} & \cdots & 0 & 0 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots & -k_{2n} & 0 & \cdots & 0\\ -k_0 & k_1 & \cdots & 0 & 0 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ -k_0 & 0 & \cdots & k_n & 0 & \cdots & 0\end{pmatrix}.$$

So

$$\begin{aligned}\nabla_V V&=k_0\Big[\Big(\sum_{i=1}^{n}k_{n+i}\Big)X_0-\sum_{i=1}^{n}k_0U_i\Big]+\sum_{i=1}^{n}k_i(-k_{n+i}X_i+k_iU_i)\\&=k_0\Big(\sum_{i=1}^{n}k_{n+i}\Big)X_0-\sum_{i=1}^{n}k_ik_{n+i}X_i+\sum_{i=1}^{n}({k_i}^2-{k_0}^2)U_i.\end{aligned}$$

.

If V is a geodesic vector field, by $\nabla_V V=0$, we have

$$\begin{cases}k_0\Big(\sum\limits_{i=1}^{n}k_{n+i}\Big)=0,\\ k_ik_{n+i}=0,\\ {k_i}^2-{k_0}^2=0,\quad i=1,\cdots,n.\end{cases}$$

If $k_0=0$, then $k_i=0, i=1,\cdots,n$, so the geodesic vector field $V=\sum\limits_{i=1}^{n}k_{n+i}U_i$, where $\sum\limits_{i=1}^{n}{k_{n+i}}^2=1$; If $k_0\neq 0$, then $k_i\neq 0$ and $k_{n+i}=0$, $i=1,\cdots,n$, so the geodesic vector field $V=\sum\limits_{i=0}^{n}k_iX_i$, where $k_i=\pm\frac{1}{\sqrt{n+1}}$.

Therefore the set of all left invariant geodesic vector fields on the solvable Lie group G_n is

$$\Big\{\sum_{i=1}^{n} k_{n+i}U_i \,\Big|\, \sum_{i=1}^{n} {k_{n+i}}^2 = 1\Big\} \cup \Big\{\sum_{i=0}^{n} k_i X_i \,\Big|\, k_i = \pm\frac{1}{\sqrt{n+1}}\Big\}.$$

By Theorem 1.1 and Theorem 4.1, we can easily obtain the sets of vector fields which are both the minimal unit vector fields and the geodesic vector fields on the Lie group G_n, $n \geq 2$ as follows:

$$\begin{aligned}&\bigcup_{i=1}^{n}\{\pm U_i\}\bigcup_{l=2}^{n}\Big\{\sum_{\alpha=1}^{l}\pm\frac{1}{\sqrt{l}}U_{i_\alpha}\,\Big|\,\sum_{i=1}^{n}k_{n+i}=0\Big\}\\ &\cup\Big\{\sum_{i=1}^{n}\frac{1}{\sqrt{n}}U_i\Big\}\cup\Big\{\sum_{i=1}^{n}-\frac{1}{\sqrt{n}}U_i\Big\}.\end{aligned}$$

Definition 4.2 *A unit vector field V on a Riemannian manifold (M, g) is called strongly normal if $g(\nabla_X A_V Y, Z) = 0$, $\forall X, Y, Z \in \mathcal{H}^V$, where $A_V = -\nabla V$.*

It is difficult to calculate all strongly normal unit vectors on the solvable Lie group G_n. So we study the set of vector fields which are both the minimal unit vector fields and the strongly normal vector fields as follows.

Theorem 4.2 *For $n \geq 2$, The set of vector fields which are both the minimal unit vector fields and the strongly normal vector fields on the solvable Lie group G_n is*

$$\{\pm X_0\}\bigcup_{i=1}^{n}\{\pm X_i\}.$$

Proof Let $X = \sum\limits_{i=0}^{n} a_i X_i + \sum\limits_{i=1}^{n} a_{n+i}U_i$, $Y = \sum\limits_{i=0}^{n} b_i X_i + \sum\limits_{i=1}^{n} b_{n+i}U_i$, $Z = \sum\limits_{i=0}^{n} c_i X_i + \sum\limits_{i=1}^{n} c_{n+i}U_i$ and $V = \sum\limits_{i=0}^{n} k_i X_i + \sum\limits_{i=1}^{n} k_{n+i}U_i$. Assume that X, Y, Z, V satisfy the conditions

$$\|X\| = \|Y\| = \|Z\| = \|V\| = 1$$

and

$$g(X, V) = g(Y, V) = g(Z, V) = 0.$$

According to the non-vanishing Riemannian connection components

$$\begin{aligned}&\nabla_{X_0}U_\alpha = X_0, \quad \nabla_{X_0}X_0 = -\sum_{i=1}^{n}U_i,\\ &\nabla_{X_i}U_i = -X_i, \qquad \nabla_{X_i}X_i = U_i,\end{aligned}$$

where $i, j, \alpha, \beta = 1, \cdots, n$. Then

$$\nabla_X A_V Y = -\nabla_X(\nabla_Y V) + \nabla_{\nabla_X Y}V$$

$$= a_0\Big[\Big(\sum_{i=1}^{n} b_{n+i}\Big)\Big(\sum_{i=1}^{n} k_{n+i}\Big) + \sum_{i=1}^{n}(k_0b_0 - b_ik_i)\Big]X_0$$
$$+ \sum_{i=1}^{n} a_i(b_{n+i}k_{n+i} + b_ik_i - b_0k_0)X_i$$
$$+ \sum_{i=1}^{n}\Big[a_0b_0\Big(\sum_{j=1}^{n} k_{n+j}\Big) + a_ib_ik_{n+i} - a_0k_0\Big(\sum_{j=1}^{n} b_{n+j}\Big) - a_ik_ib_{n+i}\Big]U_i.$$

If V is a strongly normal unit vector, then

$$g(\triangledown_X A_V Y, Z) = 0.$$

If $V = \pm X_0$, that is $k_0 = \pm 1$, then $a_0 = b_0 = c_0 = 0$, according to the above equation, we can easily get $\triangledown_X A_V Y = 0$, so $V = \pm X_0$ are the strongly normal unit vectors. Similarly, $\pm X_i$, $i = 1, \cdots, n$ also are the strongly normal unit vectors.

If $V = \{\pm U_i\}$, without loss of generality, let $V = U_1$, that is $k_{n+1} = 1, k_0 = \cdots = k_n = k_{n+2} = \cdots = k_{2n} = 0$, then $a_{n+1} = b_{n+1} = c_{n+1} = 0$, we have

$$\triangledown_X A_V Y = a_0\Big(\sum_{i=1}^{n} b_{n+i}\Big)X_0 + (a_0b_0 + a_1b_1)U_1 + a_0b_0\sum_{i=2}^{n} U_i.$$

We can easily give a counter-example that V is not a strongly normal unit vector. Let $X = X_0$, $Y = X_0$, $Z = U_2$, then $g(\triangledown_X A_V Y, Z) = 1 \neq 0$. In the same way, we can prove that $\bigcup_{l=2}^{n}\big\{\sum_{\alpha=1}^{l}(k_{i_\alpha}X_{i_\alpha} + k_{n+i_\alpha}U_{i_\alpha}) \mid {k_{n+i_\alpha}}^2 + {k_{i_\alpha}}^2 = \frac{1}{l}, \sum_{\beta=1}^{n} k_{n+\beta} = 0\big\} \cup \big\{\sum_{i=1}^{n}\frac{1}{\sqrt{n}}U_i\big\} \cup \big\{\sum_{i=1}^{n} -\frac{1}{\sqrt{n}}U_i\big\}$ are not the strongly normal unit vectors.

Therefore, The set of vector fields which are both the minimal unit vector fields and the strongly normal vector fields on the Lie group G_n, $n \geq 2$ is $\{\pm X_0\}\bigcup_{i=1}^{n}\{\pm X_i\}$.

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