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Asymptotics of the Solution to a Stationary Piecewise-Smooth Reaction-Diffusion-Advection Equation*

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Abstract A singularly perturbed boundary value problem for a piecewise-smooth nonlinear stationary equation of reaction-diffusion-advection type is studied. A new class of problems in the case when the discontinuous curve which separates the domain is monotone with respect to the time variable is considered. The existence of a smooth solution with an internal layer appearing in the neighborhood of some point on the discontinuous curve is studied. An efficient algorithm for constructing the point itself and an asymptotic representation of arbitrary-order accuracy to the solution is proposed. For sufficiently small parameter values, the existence theorem is proved by the technique of matching asymptotic expansions. An example is given to show the effectiveness of their method.

Keywords Reaction-Diffusion-Advection equation, Internal layer, Asymptotic method, Piecewise-Smooth dynamical system
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1 Introduction

In recent years, reaction-diffusion-advection equation with a small parameter that is also called singularly perturbed equation has attracted much attention due to their practical use as mathematical models in optical electronics, medical science, in-situ combustion problems, atmospheric science, etc (see [1–4]). Generally speaking, this kind of problem is studied by several methods such as geometric singular perturbation theory (see [5–10]), asymptotic method (see [11–15]), numerical algorithm (see [16–19]).

This paper investigates a boundary value problem for a stationary case of piecewise-smooth reaction-diffusion-advection equation. In this case, the nonlinear function on the right of differential equation is discontinuous on some monotone discontinuity curve. The main complexity of this problem is the existence of a smooth solution with a steep gradient in the neighborhood of some point on this curve, which is called the internal layer (see [20–26]). The biggest challenge is to determine the transition point itself and an asymptotic approximation of the smooth solution. Similarly stated problems have been considered in [27–35]. As studied in papers [30–35],

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the discontinuity line located in the domain of function on the right of differential equation is vertical to the time variable. Using a new method, we shall generalize the basic results in the case of problems with discontinuous time variables (see [35]) to the case of equations whose state variables are discontinuous. Moreover, our results can be used to develop efficient numerical algorithms for some models with discontinuous coefficients (see [36–37]).

1.1 Model problem

We will consider the singularly perturbed boundary value problem

$$\begin{cases} \mu^2 \dfrac{\mathrm{d}^2 u}{\mathrm{d}x^2} = f\Big(\mu \dfrac{\mathrm{d}u}{\mathrm{d}x}, u, x\Big), \quad 0 < x < 1, \\ u(0,\mu) = u^0, \quad u(1,\mu) = u^1, \end{cases} \tag{1.1}$$

where $\mu > 0$ is a small parameter, and u is an unknown scalar function.

Let

$$\mu \frac{\mathrm{d}u}{\mathrm{d}x} = v.$$

It is easy to see that problem (1.1) is equivalent to the following system of first-order differential equations:

$$\begin{cases} \mu \dfrac{\mathrm{d}u}{\mathrm{d}x} = v, \quad \mu \dfrac{\mathrm{d}u}{\mathrm{d}x} = f(v,u,x), \quad 0 < x < 1, \\ u(0,\mu) = u^0, \quad u(1,\mu) = u^1. \end{cases} \tag{1.2}$$

Suppose that the following conditions are satisfied.

Condition 1 The function $f(v,u,x)$ has the form

$$f(z,u,x) = \begin{cases} f^{(-)}\Big(\mu \dfrac{\mathrm{d}u}{\mathrm{d}x}, u, x\Big), \quad (u,x) \in D^{(-)}, \\ f^{(+)}\Big(\mu \dfrac{\mathrm{d}u}{\mathrm{d}x}, u, x\Big), \quad (u,x) \in D^{(+)}, \end{cases}$$

where

$$D^{(-)} = \{(u,x) \mid g(x) < u \le l,\, 0 \le x \le 1\},$$
$$D^{(+)} = \{(u,x) \mid -l \le u \le g(x),\, 0 \le x \le 1\},$$

here functions $f^{(\mp)}(v,u,x)$ are sufficiently smooth on the sets $\{v \mid -l_1 \le v \le l_1\} \times D^{(\mp)}$. Moreover, $g(x)$ is sufficiently smooth and monotonely nondecreasing in the interval $0 \le x \le 1$. As shown in Figure 1, the discontinuous curve $\Gamma : u = g(x),\ 0 \le x \le 1$ separates the domain $D = \{(u,x) \mid -l \le u \le l,\, 0 \le x \le 1\}$ into two subdomains $D^{(\mp)}$.

Condition 2 Assume that the degenerate equation $f^{(\mp)}(0,u,x) = 0$ has isolated roots $u = \varphi^{(\mp)}(x)$ on the subdomains $D^{(\mp)}$, and one has the inequalities:

(a) $f^{(-)}(0,g(x),x) \ne f^{(+)}(0,g(x),x),\ 0 \le x \le 1$,

(b) $f_u^{(\mp)}(0,\varphi^{(\mp)}(x),x) > 0,\ (\varphi^{(\mp)}(x),x) \in D^{(\mp)}$.

As shown in Figure 1, two curves $y = \varphi^{(-)}(x)$ and $y = \varphi^{(+)}(x)$ intersect the curve $y = g(x)$ at two points Q and P, whose abscissas are denoted by q and p, respectively, in the xy-plane. If $p < q$, then problem (1.1) may have a solution with a sharp internal layer in the

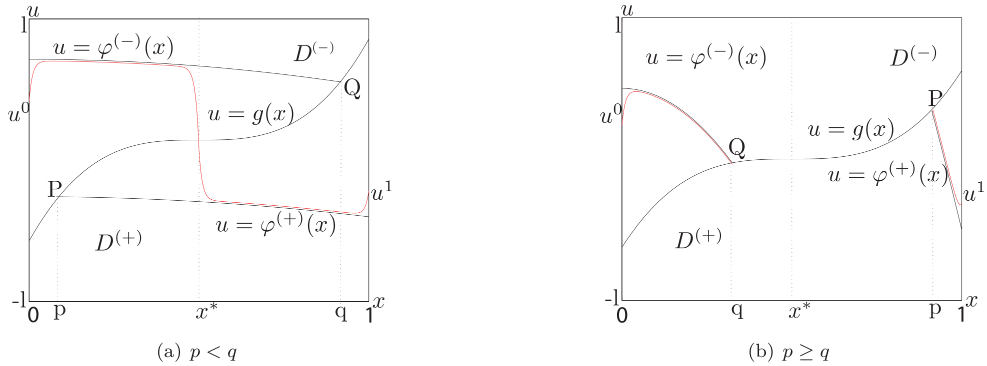

Figure 1 The solution of problem (1.1).

neighborhood of $x = x^*$ $(0 < x^* < 1)$. The transition point x^* where the solution passes through the monotone curve is unknown beforehand. Similarly, the case when the function $g(x)$ is monotonely nonincreasing in the interval $[0, 1]$ can also be considered.

1.2 Associated system

Consider the associated system (see [11]),

$$\frac{\mathrm{d}\widetilde{u}}{\mathrm{d}\tau} = \widetilde{v}, \quad \frac{\mathrm{d}\widetilde{v}}{\mathrm{d}\tau} = f(\widetilde{v}, \widetilde{u}, \overline{x}), \tag{1.3}$$

where $\overline{x} \in [0, 1]$ is a parameter.

By virtue of Conditions 1–2, the characteristic equation

$$\begin{vmatrix} 0 - \lambda & 1 \\ f_u(0, \varphi^{(\mp)}(\overline{x}), \overline{x}) & f_v(0, \varphi^{(\mp)}(\overline{x}), \overline{x}) - \lambda \end{vmatrix} = 0$$

or

$$\lambda^2 - f_v(0, \varphi^{(\mp)}(\overline{x}), \overline{x})\lambda - f_u(0, \varphi^{(\mp)}(\overline{x}), \overline{x}) = 0$$

has two roots of different signs. The reason behind this is that

$$\Delta = f_v^2(0, \varphi^{(\mp)}(\overline{x}), \overline{x}) + 4f_u(0, \varphi^{(\mp)}(\overline{x}), \overline{x}) > 0, \quad \lambda_1\lambda_2 = -f_u(0, \varphi^{(\mp)}(\overline{x}), \overline{x}) < 0.$$

In the phase plane $(\widetilde{u}, \widetilde{v})$, each of the equilibrium points $(\varphi^{(\mp)}(\overline{x}), 0)$ is a saddle point.

The associated system (1.3) defines equations

$$\widetilde{v}\frac{\mathrm{d}\widetilde{v}}{\mathrm{d}\widetilde{u}} = f(\widetilde{v}, \widetilde{u}, \overline{x}) \tag{1.4}$$

for the phase trajectories on the plane $(\widetilde{u}, \widetilde{v})$, for any $\overline{x} \in [0, 1]$.

Condition 3 Suppose that the separatrices which are described by (1.4) with the initial conditions

$$\widetilde{v}(\varphi^{(\mp)}(\overline{x})) = 0 \tag{1.5}$$

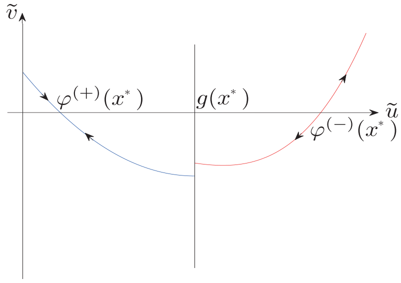

Figure 2 Graphic illustration of the separatrix $\Phi(\widetilde{v},\widetilde{u},x)=\Phi(0,\varphi^{(-)}(x^*),x^*)$ issuing from the saddle point $(\varphi^{(-)}(x^*),0)$ and the separatrix $\Phi(\widetilde{v},\widetilde{u},x)=\Phi(0,\varphi^{(+)}(x^*),x^*)$ entering the saddle point $(\varphi^{(+)}(x^*),0)$.

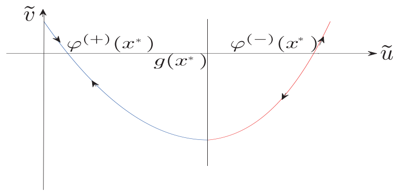

Figure 3 Graphic illustration of the heteroclinic orbit between saddle points $(\varphi^{(-)}(x^*),0)$ and $(\varphi^{(+)}(x^*),0)$.

have the form $\Phi(\widetilde{v},\widetilde{u},\overline{x})=c$ for any $\overline{x}\in[0,1]$.

By Conditions 1–3, in the plane $(\widetilde{u},\widetilde{v})$, for any fixed $\overline{x}$, there exist separatrixes $\Phi(\widetilde{v},\widetilde{u},\overline{x})=\Phi(0,\varphi^{(\mp)}(\overline{x}),\overline{x})$ passing through the saddle points $(\varphi^{(\mp)}(\overline{x}),0)$. In the course of determining the leading terms in the asymptotic expansion of boundary layer, the solvability of the following boundary value problems for system (1.3) taken at $\overline{x}=0$ and $\overline{x}=1$ plays an important role:

$$\begin{cases}\dfrac{d\widetilde{u}}{d\tau_0}=\widetilde{v}, \quad \dfrac{d\widetilde{v}}{d\tau_0}=f^{(-)}(\widetilde{v},\widetilde{u},0), \quad \tau_0=\dfrac{x}{\mu}\geq 0,\\ \widetilde{u}(0)=u^0, \quad \widetilde{u}(+\infty)=\varphi^{(-)}(0), \quad \widetilde{v}(+\infty)=0\end{cases} \tag{1.6}$$

and

$$\begin{cases}\dfrac{d\widetilde{u}}{d\tau_1}=\widetilde{v}, \quad \dfrac{d\widetilde{v}}{d\tau_1}=f^{(+)}(\widetilde{v},\widetilde{u},1), \quad \tau_1=\dfrac{x-1}{\mu}\leq 0,\\ \widetilde{u}(0)=u^1, \quad \widetilde{u}(-\infty)=\varphi^{(+)}(1), \quad \widetilde{v}(-\infty)=0.\end{cases} \tag{1.7}$$

According to Conditions 1–3, in the phase plane $(\widetilde{u},\widetilde{v})$, there exists a stable manifold $\Phi(\widetilde{v},\widetilde{u},x)=\Phi(0,\varphi^{(-)}(0),0)$ passing through the saddle point $(\varphi^{(-)}(0),0)$, and there exists a stable manifold $\Phi(\widetilde{v},\widetilde{u},x)=\Phi(0,\varphi^{(+)}(1),1)$ passing through the saddle point $(\varphi^{(+)}(1),0)$. Therefore, if boundary values u^0 and u^1 are located on the corresponding stable manifolds, then problems (1.6) and (1.7) are solvable.

Condition 4 Suppose that in the phase plane $(\widetilde{u},\widetilde{v})$, the vertical line $\widetilde{u}=u^0$ intersects the stable manifold $\Phi(\widetilde{v},\widetilde{u},x)=\Phi(0,\varphi^{(-)}(0),0)$ passing through the saddle point $(\varphi^{(-)}(0),0)$, and the vertical line $\widetilde{u} = u^1$ intersects the stable manifold $\Phi(\widetilde{v}, \widetilde{u}, x) = \Phi(0, \varphi^{(+)}(1), 1)$ passing through the saddle point $(\varphi^{(+)}(1), 0)$.

To determine the leading term in the asymptotic expansion of internal layer, it is necessary to consider the following boundary value problem for system (1.3) taken at $x = x^*$:

$$\begin{cases} \dfrac{\mathrm{d}\widetilde{u}}{\mathrm{d}\tau} = \widetilde{v}, \quad \dfrac{\mathrm{d}\widetilde{v}}{\mathrm{d}\tau} = f^{(\mp)}(\widetilde{v}, \widetilde{u}, x^*), \quad \tau = \dfrac{x - x^*}{\mu} \in \mathbf{R}, \\ \widetilde{u}(0) = g(x^*), \quad \widetilde{u}(\mp\infty) = \varphi^{(\mp)}(x^*), \quad \widetilde{v}(\mp\infty) = 0, \end{cases} \tag{1.8}$$

where $g(x^*) \in [\varphi^{(+)}(x^*), \varphi^{(-)}(x^*)]$. Then in the phase plane $(\widetilde{u}, \widetilde{v})$, there are two separatrixes in and out passing through each of saddle points $(\varphi^{(\mp)}(x^*), 0)$. Without loss of generality, we assume that these separatrixes lie in the lower half-plane $\widetilde{v} < 0$. Thus, one can obtain the following sufficient condition needed to guarantee that problem (1.8) is solvable.

Condition 5 Suppose that the separatrix $\Phi(\widetilde{v}, \widetilde{u}, x) = \Phi(0, \varphi^{(-)}(x^*), x^*)$ issuing from the saddle point $(\varphi^{(-)}(x^*), 0)$ intersects the vertical line $y = g(x^*)$, and the separatrix $\Phi(\widetilde{v}, \widetilde{u}, x) = \Phi(0, \varphi^{(+)}(x^*), x^*)$ entering the saddle point $(\varphi^{(+)}(x^*), 0)$ intersects the vertical line $y = g(x^*)$ in the phase plane $(\widetilde{u}, \widetilde{v})$ (see Figure 2).

As shown in Figure 1, the transition point x^* where the solution of problem (1.1) passes the monotone curve is unknown. To find x^*, for any $x \in [p, q]$, we introduce the function

$$\mathrm{H}(x) = \widetilde{v}^{(-)}(g(x)) - \widetilde{v}^{(+)}(g(x)), \tag{1.9}$$

where $\widetilde{v}^{(\mp)}(\widetilde{u})$ is the solution of $\Phi(\widetilde{v}, \widetilde{u}, x) = \Phi(0, \varphi^{(\mp)}(x), x)$. We require that the following condition is satisfied.

Condition 6 Assume that the equation $\mathrm{H}(x) = 0$ has a solution $x = x_0$, $x_0 \in [p, q]$ (see Figure 3), and the inequality $\mathrm{H}'(\mathrm{x})(x_0) \neq 0$ holds.

2 Formal Asymptotics

As shown in Figure 1, problem (1.1) may have a solution with an internal transition layer in the neighborhood of monotone curve $y = g(x)$. To construct an asymptotic approximation to the solution of problem (1.1) that has an internal layer separately on the intervals $[0, x^*]$ and $[x^*, 1]$, we consider two classical boundary value problems

$$\begin{cases} \mu\dfrac{\mathrm{d}u^{(-)}}{\mathrm{d}x} = v^{(-)}, \quad \mu\dfrac{\mathrm{d}u^{(-)}}{\mathrm{d}x} = f^{(-)}(v^{(-)}, u^{(-)}, x), \quad 0 < x < x^*, \\ u^{(-)}(0, \mu) = u^0, \quad u^{(-)}(x^*, \mu) = g(x^*) \end{cases} \tag{2.1}$$

and

$$\begin{cases} \mu\dfrac{\mathrm{d}u^{(+)}}{\mathrm{d}x} = v^{(+)}, \quad \mu\dfrac{\mathrm{d}u^{(+)}}{\mathrm{d}x} = f^{(+)}(v^{(+)}, u^{(+)}, x), \quad x^* < x < 1, \\ u^{(+)}(x^*, \mu) = g(x^*), \quad u^{(+)}(1, \mu) = u^1. \end{cases} \tag{2.2}$$

Then we match the asymptotic expansions of the solutions to problems (2.1), (2.2) at the transition point $(x^*, g(x^*))$ smoothly. Thus, a smooth asymptotic solution of the original problem

(1.1) with an internal layer in the neighborhood of discontinuous curve $y = g(x)$ is obtained. To this end, it is necessary to satisfy the following smoothness condition

$$v^{(-)}(x^*, \mu) = v^{(+)}(x^*, \mu), \tag{2.3}$$

where the unknown point $(x^*, g(x^*))$ shall be found by the matching asymptotic expansion method after constructing the asymptotics of solutions to problems (2.1), (2.2).

In order to describe the solution in a sharp internal layer region and in the neighborhood of endpoints of the interval $[0, 1]$, the stretched variables are introduced:

$$\tau_0 = \frac{x}{\mu}, \quad \tau = \frac{x - x^*}{\mu}, \quad \tau_1 = \frac{x - 1}{\mu}.$$

Note that the solution in region without steep gradient depends on the slow variable x. Then we write out the asymptotic representations of two auxiliary problems (2.1), (2.2):

$$\begin{cases} u^{(-)}(x, \mu) = \overline{u}^{(-)}(x, \mu) + Lu(\tau_0, \mu) + Q^{(-)}u(\tau, \mu), \\ v^{(-)}(x, \mu) = \overline{v}^{(-)}(x, \mu) + Lv(\tau_0, \mu) + Q^{(-)}v(\tau, \mu) \end{cases} \tag{2.4}$$

and

$$\begin{cases} u^{(+)}(x, \mu) = \overline{u}^{(+)}(x, \mu) + Q^{(+)}u(\tau, \mu) + Ru(\tau_1, \mu), \\ v^{(+)}(x, \mu) = \overline{v}^{(+)}(x, \mu) + Q^{(+)}v(\tau, \mu) + Rv(\tau_1, \mu), \end{cases} \tag{2.5}$$

where $\overline{u}^{(\mp)}(x, \mu)$, $\overline{v}^{(\mp)}(x, \mu)$ are the regular parts of asymptotic approximation to the solutions $u^{(\mp)}(x, \mu)$, $v^{(\mp)}(x, \mu)$, respectively, $Lu(\tau_0, \mu)$, $Lv(\tau_0, \mu)$ are left boundary layer parts at the endpoint $x = 0$, $Ru(\tau_1, \mu)$, $Rv(\tau_1, \mu)$ are right boundary layer parts at the endpoint $x = 1$, and $Q^{(\mp)}u(\tau, \mu), Q^{(\mp)}$
$v(\tau, \mu)$ are internal layer parts in the neighborhood of transition point $(x^*, g(x^*))$.

Each function of asymptotic representations (2.4), (2.5) shall be written in the forms of power series of small parameter μ:

$$\overline{u}^{(\mp)}(x, \mu) = \overline{u}_0^{(\mp)}(x) + \mu \overline{u}_1^{(\mp)}(x) + \cdots + \mu^k \overline{u}_k^{(\mp)}(x) + \cdots, \tag{2.6}$$
$$\overline{v}^{(\mp)}(x, \mu) = \overline{v}_0^{(\mp)}(x) + \mu \overline{v}_1^{(\mp)}(x) + \cdots + \mu^k \overline{v}_k^{(\mp)}(x) + \cdots, \tag{2.7}$$
$$Lu(\tau_0, \mu) = L_0u(\tau_0) + \mu L_1u(\tau_0) + \cdots + \mu^k L_ku(\tau_0) + \cdots, \tag{2.8}$$
$$Lv(\tau, \mu) = L_0v(\tau_0) + \mu L_1v(\tau_0) + \cdots + \mu^k L_kv(\tau_0) + \cdots, \tag{2.9}$$
$$Ru(\tau_1, \mu) = R_0u(\tau_1) + \mu R_1u(\tau_1) + \cdots + \mu^k R_ku(\tau_1) + \cdots, \tag{2.10}$$
$$Rv(\tau_1, \mu) = R_0v(\tau_1) + \mu R_1v(\tau_1) + \cdots + \mu^k R_kv(\tau_1) + \cdots, \tag{2.11}$$
$$Q^{(\mp)}u(\tau, \mu) = Q_0^{(\mp)}u(\tau) + \mu Q_1^{(\mp)}u(\tau) + \cdots + \mu^k Q_k^{(\mp)}u(\tau) + \cdots, \tag{2.12}$$
$$Q^{(\mp)}v(\tau, \mu) = Q_0^{(\mp)}v(\tau) + \mu Q_1^{(\mp)}v(\tau) + \cdots + \mu^k Q_k^{(\mp)}v(\tau) + \cdots, \tag{2.13}$$

where $Q_k^{(\mp)}u(\tau)$, $Q_k^{(\mp)}v(\tau)$, $L_ku(\tau_0)$, $L_kv(\tau_0)$, $R_ku(\tau_1)$, $R_kv(\tau_1)$ $(k \geq 0)$ are imposed the standard conditions at infinity:

$$Q_k^{(\mp)}u(\mp\infty) = 0, \quad L_ku(+\infty) = 0, \quad R_ku(-\infty) = 0, \tag{2.14}$$
$$Q_k^{(\mp)}v(\mp\infty) = 0, \quad L_kv(+\infty) = 0, \quad R_kv(-\infty) = 0. \tag{2.15}$$

In order to determine high-order asymptotic terms of internal layer and boundary layer $Q_k^{(\mp)}u(\tau)$, $Q_k^{(\mp)}v(\tau)$ and $L_k u(\tau_0)$, $L_k v(\tau_0)$, $R_k u(\tau_1)$, $R_k v(\tau_1)$ $(k \geq 1)$, the following condition is also needed.

Condition 7 Suppose that one has the inequalities

$$\begin{aligned}
&f_v^{(-)}(Q_0^{(-)}v, \varphi^{(-)}(x^*) + Q_0^{(-)}u, x^*) \leq 0,\\
&f_v^{(+)}(Q_0^{(+)}v, \varphi^{(+)}(x^*) + Q_0^{(+)}u, x^*) \geq 0,\\
&f_v^{(-)}(L_0 v, \varphi^{(-)}(0) + L_0 u, 0) \geq 0,\\
&f_v^{(+)}(R_0 v, \varphi^{(+)}(1) + R_0 u, 0) \leq 0.
\end{aligned}$$

2.1 Regular part

The equations for determining the regular terms $\overline{u}^{(\mp)}(x)$, $\overline{v}^{(\mp)}(x)$ take the form

$$\mu\frac{\mathrm{d}\overline{u}^{(\mp)}}{\mathrm{d}x} = \overline{v}^{(\mp)}, \quad \mu\frac{\mathrm{d}\overline{v}^{(\mp)}}{\mathrm{d}x} = f^{(\mp)}(\overline{v}^{(\mp)}, \overline{u}^{(\mp)}, x). \tag{2.16}$$

Substituting (2.6)–(2.7) into (2.16) and matching the coefficients of like powers of μ, we obtain the degenerate equations

$$\overline{v}_0^{(\mp)}(x) = 0, \quad f^{(\mp)}(0, u_0^{(\mp)}(x), x) = 0,$$

whose solution are

$$\overline{v}_0^{(\mp)}(x) = 0, \quad u_0^{(\mp)}(x) = \varphi^{(\mp)}(x)$$

by Condition 2.

By virtue of Condition 2, the remaining coefficients $\overline{u}_k^{(\mp)}(x)$, $\overline{v}_k^{(\mp)}(x)$ $(k > 0)$ are obtained by linear equations

$$\overline{v}_k^{(\mp)}(x) = \frac{\mathrm{d}\overline{u}_{k-1}^{(\mp)}(x)}{\mathrm{d}x}, \quad \overline{u}_k^{(\mp)}(x) = \frac{\overline{h}_k^{(\mp)}(x)}{f_u^{(\mp)}(0, u_0^{(\mp)}(x), x)},$$

where $\overline{h}_k^{(\mp)}(x)$ is known functions that depends on $\overline{u}_j^{(\mp)}(x)$ $(j < k)$. These equations are solvable. In particular,

$$\overline{h}_1^{(\mp)}(x) = -f_v^{(\mp)}(0, u_0^{(\mp)}(x), x)\frac{\mathrm{d}\overline{u}_0^{(\mp)}}{\mathrm{d}x}.$$

2.2 Internal layer functions

The problems for finding the internal layer functions $Q^{(\mp)}u(\tau)$, $Q^{(\mp)}v(\tau)$ at the transition point $(x^*, g(x^*))$ are as follows

$$\begin{cases}
\dfrac{\mathrm{d}Q^{(\mp)}u}{\mathrm{d}\tau} = Q^{(\mp)}v,\\
\dfrac{\mathrm{d}Q^{(\mp)}v}{\mathrm{d}\tau} = f^{(\mp)}(v^{(\mp)}(x^* + \mu\tau) + Q^{(\mp)}v, u^{(\mp)}(x^* + \mu\tau) + Q^{(\mp)}u, x^* + \mu\tau)\\
\qquad\qquad -f^{(\mp)}(v^{(\mp)}(x^* + \mu\tau), u^{(\mp)}(x^* + \mu\tau), x^* + \mu\tau),\\
Q^{(\mp)}u(0, \mu) = g(x^*) - \overline{u}^{(\mp)}(x^*, \mu), \quad Q^{(\mp)}u(\mp\infty, \mu) = 0,\\
Q^{(\mp)}z(\mp\infty, \mu) = 0.
\end{cases} \tag{2.17}$$

After substituting (2.12)–(2.13) into (2.17) and matching the coefficients of like powers of μ, one can obtain the problems for determining $Q_0^{(\mp)}u(\tau)$, $Q_0^{(\mp)}v(\tau)$,

$$\begin{cases} \dfrac{\mathrm{d}Q_0^{(\mp)}u}{\mathrm{d}\tau} = Q_0^{(\mp)}v, \\ \dfrac{\mathrm{d}Q_0^{(\mp)}v}{\mathrm{d}\tau} = f^{(\mp)}(Q_0^{(\mp)}v, \varphi^{(\mp)}(x^*) + Q_0^{(\mp)}u, x^*), \\ Q_0^{(\mp)}u(0) = g(x^*) - \varphi^{(\mp)}(x^*), \\ Q_0^{(\mp)}u(\mp\infty) = 0, \quad Q_0^{(\mp)}v(\mp\infty) = 0. \end{cases} \tag{2.18}$$

By means of variable substitution

$$\widetilde{u}(\tau) = \varphi^{(\mp)}(x^*) + Q_0^{(\mp)}u(\tau), \quad \widetilde{v}(\tau) = Q_0^{(\mp)}v(\tau), \tag{2.19}$$

problem (2.18) can be rewritten as (1.8). According to the discussion of associated system (1.8), problem (2.18) has solutions $Q_0^{(\mp)}u(\tau)$, $Q_0^{(\mp)}v(\tau)$, which satisfy the exponential estimates (see [11])

$$|Q_0^{(\mp)}u(\tau)| \le C_0^{(\mp)}\mathrm{e}^{\pm\kappa^{(\mp)}\tau}, \quad |Q_0^{(\mp)}v(\tau)| \le C_1^{(\mp)}\mathrm{e}^{\pm\kappa^{(\mp)}\tau}, \tag{2.20}$$

where $C_0^{(\mp)} > 0$, $C_1^{(\mp)} > 0$, $\kappa^{(\mp)} > 0$.

Since $Q_k^{(\mp)}u(\tau)$, $Q_k^{(\mp)}v(\tau)$, $k \ge 1$ can be determined in a similar way, here we consider $Q_1^{(\mp)}u(\tau)$, $Q_1^{(\mp)}v(\tau)$, which are defined from the following boundary layer problem

$$\begin{cases} \dfrac{\mathrm{d}Q_1^{(\mp)}u}{\mathrm{d}\tau} = Q_1^{(\mp)}v, \\ \dfrac{\mathrm{d}Q_1^{(\mp)}v}{\mathrm{d}\tau} = \widetilde{f}_v^{(\mp)}(\tau)Q_1^{(\mp)}v + \widetilde{f}_u^{(\mp)}(\tau)Q_1^{(\mp)}u + h_1^{(\mp)}(\tau), \\ Q_1^{(\mp)}u(0) = -\overline{u}_1^{(\mp)}(x^*), \\ Q_1^{(\mp)}u(\mp\infty) = 0, \quad Q_1^{(\mp)}v(\mp\infty) = 0, \end{cases} \tag{2.21}$$

where

$$h_1^{(\mp)}(\tau) = \widetilde{f}_v^{(\mp)}\varphi^{(\mp)\prime}(x^*) + \widetilde{f}_u[\varphi^{(\mp)\prime}(x^*)\tau + \overline{u}_1^{(\mp)}(x^*)] + \widetilde{f}_x^{(\mp)}\tau,$$

here $\widetilde{f}_v^{(\mp)}(\tau)$, $\widetilde{f}_u^{(\mp)}(\tau)$, $\widetilde{f}_x^{(\mp)}(\tau)$ are defined at the point $(Q_0^{(\mp)}v, \varphi^{(\mp)}(x^*) + Q_0^{(\mp)}u, x^*)$.

The equations of problem (2.21) can be rewritten as second-order differential equations

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}Q_1^{(\mp)}u = \widetilde{f}_v^{(\mp)}(\tau)\frac{\mathrm{d}}{\mathrm{d}\tau}Q_1^{(\mp)}u + \widetilde{f}_u^{(\mp)}(\tau)Q_1^{(\mp)}u + h_1^{(\mp)}(\tau). \tag{2.22}$$

Multiplying $p^{(\mp)}(\tau) = \mathrm{e}^{-\int_0^\tau \widetilde{f}_v^{(\mp)}(s)\,\mathrm{d}s}$ on both sides of these equations, it follows from Condition 7, (2.21) and Green formula that

$$Q_1^{(\mp)}v(\tau) = \frac{\int_{\mp\infty}^{\tau} \widetilde{v}^{(\mp)}(\eta)p^{(\mp)}(\eta)h_1^{(\mp)}(\eta)\,\mathrm{d}\eta + Q_1^{(\mp)}u(\tau)\widetilde{v}^{(\mp)\prime}(\tau)}{p^{(\mp)}(\tau)\widetilde{v}^{(\mp)}(\tau)}. \tag{2.23}$$

Thus, one can obtain the Cauchy problem

$$\begin{cases} \dfrac{\mathrm{d}Q_1^{(\mp)}u}{\mathrm{d}\tau} = \dfrac{\widetilde{v}^{(\mp)\prime}(\tau)}{p^{(\mp)}(\tau)\widetilde{v}^{(\mp)}(\tau)}Q_1^{(\mp)}u + \dfrac{\int_{\mp\infty}^{\tau} \widetilde{v}^{(\mp)}(\eta)p^{(\mp)}(\eta)h_1^{(\mp)}(\tau)\,\mathrm{d}\eta}{p^{(\mp)}(\tau)\widetilde{v}^{(\mp)}(\tau)}, \\ Q_1^{(\mp)}u(0) = -\overline{u}_1^{(\mp)}(x^*), \end{cases} \tag{2.24}$$

whose solutions can be represented in the forms

$$Q_1^{(\mp)}u(\tau) = Q_1^{(\mp)}u(0)M(\tau) + N(\tau), \tag{2.25}$$

where

$$M(\tau) = \exp\Big(\int_0^{\tau} \frac{\widetilde{v}^{(\mp)'}(s)}{p^{(\mp)}(s)\widetilde{v}^{(\mp)}(s)}\,\mathrm{d}s\Big),$$

$$N(\tau) = \int_0^{\tau} \frac{\int_{\mp\infty}^{\xi} \widetilde{v}^{(\mp)}(\eta)p^{(\mp)}(\eta)h_1^{(\mp)}(\eta)\,\mathrm{d}\eta}{p^{(\mp)}(\xi)\widetilde{v}^{(\mp)}(\xi)} \exp\Big(\int_{\xi}^{\tau} \frac{\widetilde{v}^{(\mp)'}(s)}{p^{(\mp)}(s)\widetilde{v}^{(\mp)}(s)}\,\mathrm{d}s\Big)\,\mathrm{d}\xi.$$

From (2.23) and (2.25), functions $Q_1^{(\mp)}u(\tau)$, $Q_1^{(\mp)}v(\tau)$ have the exponential estimates of the type (2.20).

2.3 Boundary layer terms

The equations and boundary value conditions for determining left boundary layer functions $Lu(\tau_0)$, $Lv(\tau_0)$ at the endpoint $x = 0$ have the forms

$$\begin{cases} \dfrac{\mathrm{d}Lu}{\mathrm{d}\tau_0} = Lv, \\ \dfrac{\mathrm{d}Lv}{\mathrm{d}\tau_0} = f^{(-)}(v^{(-)}(\mu\tau_0) + Lv, u^{(-)}(\mu\tau_0) + Lu, \mu\tau_0) \\ \qquad\quad -f^{(-)}(v^{(-)}(\mu\tau_0), u^{(\mp)}(\mu\tau_0), \mu\tau_0), \\ Lu(0,\mu) = u^0 - \overline{u}^{(-)}(0,\mu), \quad Lu(+\infty,\mu) = 0, \\ Lv(+\infty,\mu) = 0. \end{cases} \tag{2.26}$$

Likewise, the problems for finding the right boundary layer functions $Ru(\tau_1)$, $Rv(\tau_1)$ at the endpoint $x = 1$ can be obtained

$$\begin{cases} \dfrac{\mathrm{d}Ru}{\mathrm{d}\tau_1} = Rv, \\ \dfrac{\mathrm{d}Rv}{\mathrm{d}\tau_1} = f^{(+)}(v^{(+)}(1+\mu\tau_1) + Rv, u^{(+)}(1+\mu\tau_1) + Ru, 1+\mu\tau_1) \\ \qquad\quad -f^{(+)}(v^{(+)}(1-\mu\tau_1), u^{(+)}(1-\mu\tau_1), 1-\mu\tau_1), \\ Ru(0,\mu) = u^1 - \overline{u}^{(+)}(1,\mu), \quad Ru(-\infty,\mu) = 0, \\ Rv(-\infty,\mu) = 0. \end{cases} \tag{2.27}$$

For $L_0u(\tau_0)$, $L_0v(\tau_0)$ and $R_0u(\tau_1)$, $R_0v(\tau_1)$, we have

$$\begin{cases} \dfrac{\mathrm{d}L_0u}{\mathrm{d}\tau_0} = L_0v, \\ \dfrac{\mathrm{d}L_0v}{\mathrm{d}\tau_0} = f^{(-)}(L_0v, \varphi^{(-)}(0) + L_0u, 0), \\ L_0u(0) = u^0 - \varphi^{(-)}(0), \\ L_0u(+\infty) = 0, \quad L_0v(+\infty) = 0 \end{cases} \tag{2.28}$$

and

$$\begin{cases} \dfrac{\mathrm{d}R_0u}{\mathrm{d}\tau_1} = R_0v, \\ \dfrac{\mathrm{d}R_0v}{\mathrm{d}\tau_1} = f^{(+)}(R_0v, \varphi^{(+)}(1) + R_0u, 1), \\ R_0u(0) = u^1 - \varphi^{(+)}(1), \\ R_0u(-\infty) = 0, \quad R_0v(-\infty) = 0. \end{cases} \tag{2.29}$$

Set

$$\widetilde{u}^{(-)}(\tau_0) = \varphi^{(-)}(0) + L_0u(\tau_0), \quad \widetilde{v}^{(-)}(\tau_0) = L_0v(\tau_0)$$

and

$$\widetilde{u}^{(+)}(\tau_1) = \varphi^{(+)}(1) + R_0u(\tau_1), \quad \widetilde{v}^{(-)}(\tau_1) = R_0v(\tau_1).$$

Considering the discussions about auxiliary systems (1.6) and (1.7), problems (2.28) and (2.29) have solutions $L_0u(\tau_0)$, $L_0v(\tau_0)$ and $R_0u(\tau_1)$, $R_0v(\tau_1)$ accordingly. In addition, if Conditions 1–4 are satisfied, then functions $L_0u(\tau_0)$, $L_0v(\tau_0)$ and $R_0u(\tau_1)$, $R_0v(\tau_1)$ have exponential estimates of type (2.20).

The problems for determining $L_ku(\tau_0)$, $L_kv(\tau_0)$ and $R_ku(\tau_1)$, $R_kv(\tau_1)$ have the forms

$$\begin{cases} \dfrac{\mathrm{d}L_ku}{\mathrm{d}\tau_0} = L_kv, \\ \dfrac{\mathrm{d}L_kv}{\mathrm{d}\tau_0} = \widehat{f}_v^{(-)}(\tau_0)L_kv + \widehat{f}_u^{(-)}(\tau_0)L_ku + Lh_k(\tau_0), \\ L_ku(0) = -\overline{u}_k^{(-)}(0), \\ L_ku(+\infty) = 0, \quad L_kv(+\infty) = 0 \end{cases} \tag{2.30}$$

and

$$\begin{cases} \dfrac{\mathrm{d}R_ku}{\mathrm{d}\tau_1} = R_kv, \\ \dfrac{\mathrm{d}R_kv}{\mathrm{d}\tau_1} = \widehat{f}_v^{(+)}(\tau_1)R_kv + \widehat{f}_u^{(+)}(\tau_1)R_ku + Rh_k(\tau_1), \\ R_ku(0) = -\overline{u}_k^{(+)}(1), \\ R_ku(-\infty) = 0, \quad R_kv(-\infty) = 0, \end{cases} \tag{2.31}$$

where the functions $\widehat{f}_v^{(-)}(\tau_0)$, $\widehat{f}_u^{(-)}(\tau_0)$ are defined at the point $(L_0v, \varphi^{(-)}(0) + L_0u, 0)$, $\widehat{f}_v^{(+)}(\tau_1)$, $\widehat{f}_u^{(+)}(\tau_1)$ are defined at the point $(R_0v, \varphi^{(+)}(1) + R_0u, 0)$, Lh_k are known functions that depend on $\overline{u}_n^{(-)}(x)$ $(n \le k)$ and $L_nu(\tau_0)$, $R_nu(\tau_1)$ $(n < k)$, and the functions Rh_k are known functions depending on $\overline{u}_n^{(+)}(x)$ $(n \le k)$ and $L_nu(\tau_0)$, $R_nu(\tau_1)$ $(n < k)$.

By Condition 7, the solutions of problems (2.30) and (2.31) can be represented as

$$L_kv(\tau_0) = \frac{\int_{+\infty}^{\tau_0} \widetilde{v}^{(-)}(\eta)\widehat{p}^{(-)}(\eta)Lh_k(\eta)\mathrm{d}\eta + L_ku(\tau_0)\widetilde{v}^{(-)\prime}(\tau_0)}{\widehat{p}^{(-)}(\tau_0)\widetilde{v}^{(-)}(\tau_0)}, \tag{2.32}$$

$$L_ku(\tau_0) = -\overline{u}_k^{(-)}(0)M^{(-)}(\tau_0) + N^{(-)}(\tau_0), \tag{2.33}$$

where

$$\widehat{p}^{(-)}(\tau_0) = \mathrm{e}^{-\int_0^{\tau_0} \widehat{f}_v^{(-)}(s)\,\mathrm{d}s}, \quad M^{(-)}(\tau_0) = \exp\Big(\int_0^{\tau_0} \frac{\widetilde{v}^{(-)\prime}(s)}{\widehat{p}^{(-)}(s)\widetilde{v}^{(-)}(s)}\,\mathrm{d}s\Big),$$

$$N^{(-)}(\tau_0) = \int_0^{\tau_0} \frac{\int_{+\infty}^{\xi} \widetilde{v}^{(-)}(\eta)\widehat{p}^{(-)}(\eta)Lh_k(\eta)\,\mathrm{d}\eta}{\widehat{p}^{(-)}(\xi)\widetilde{v}^{(-)}(\xi)} \exp\Big(\int_{\xi}^{\tau_0} \frac{\widetilde{v}^{(-)\prime}(s)}{\widehat{p}^{(-)}(s)\widetilde{v}^{(-)}(s)}\,\mathrm{d}s\Big)\,\mathrm{d}\xi$$

and

$$R_k v(\tau_1) = \frac{\int_{-\infty}^{\tau_1} \widetilde{v}^{(+)}(\eta)\widehat{p}^{(+)}(\eta)Rh_k(\eta)\,\mathrm{d}\eta + R_k u(\tau_1)\widetilde{v}^{(+)\prime}(\tau_1)}{\widehat{p}^{(+)}(\tau_1)\widetilde{v}^{(+)}(\tau_1)}, \tag{2.34}$$

$$R_k u(\tau_1) = -\overline{u}_k^{(+)}(1)M^{(+)}(\tau_1) + N^{(+)}(\tau_1), \tag{2.35}$$

where

$$\widehat{p}^{(+)}(\tau_1) = \mathrm{e}^{-\int_0^{\tau_1} \widehat{f}_v^{(+)}\,\mathrm{d}s}, \quad M^{(+)}(\tau_1) = \exp\Big(\int_0^{\tau_1} \frac{\widetilde{v}^{(+)\prime}(s)}{\widehat{p}^{(+)}(s)\widetilde{v}^{(+)}(s)}\,\mathrm{d}s\Big),$$

$$N^{(+)}(\tau_1) = \int_0^{\tau_1} \frac{\int_{-\infty}^{\xi} \widetilde{v}^{(+)}(\eta)\widehat{p}^{(+)}(\eta)Rh_k(\eta)\,\mathrm{d}\eta}{\widehat{p}^{(+)}(\xi)\widetilde{v}^{(+)}(\xi)} \exp\Big(\int_{\xi}^{\tau_1} \frac{\widetilde{v}^{(+)\prime}(s)}{\widehat{p}^{(+)}(s)\widetilde{v}^{(+)}(s)}\,\mathrm{d}s\Big)\,\mathrm{d}\xi.$$

It follows from (2.32)–(2.35) that $L_k u(\tau_0)$, $L_k v(\tau_0)$ and $R_k u(\tau_1)$, $R_k v(\tau_1)$ also satisfy the exponential estimates similar to (2.20).

3 Existence of a Smooth Solution to the Original Problem (1.1)

Suppose that x^* exists, then solutions of two auxiliary problems (2.1) and (2.2) are defined in the regions $[0, x^*] \times [g(x^*), l]$ and $[x^*, 1] \times [-l, g(x^*)]$ respectively. Thus, it is easy to see that these two problems are classical singularly perturbed two point boundary value problems whose asymptotic solutions are smooth on both sides of monotone curve $y = g(x)$. In the following, the existence of x^* is proved and we justify the fact that composite solution obtained in Section 2 is smooth at the transition point $(x^*, g(x^*))$.

We represent x^* in the form of the sum

$$x^* = x_\delta := x_0 + \mu x_1 + \cdots + \mu^n x_n + \mu^{n+1}(x_{n+1} + \delta), \tag{3.1}$$

where δ is a parameter.

As proved in [11], problems (2.1), (2.2) have solutions $u^{(\mp)}(x, \mu, \delta)$, whose asymptotic representations are

$$\begin{cases} u^{(\mp)}(x, \mu, \delta) = U_n^{(\mp)}(x, \mu, \delta) + O(\mu^{n+1}), \\ v^{(\mp)}(x, \mu, \delta) = V_{n-1}^{(\mp)}(x, \mu, \delta) + O(\mu^n), \end{cases} \tag{3.2}$$

where

$$\begin{cases} U_n^{(-)}(x, \mu, \delta) = \sum_{k=0}^{n} \mu^k(\overline{u}_k^{(-)}(x) + Q_k^{(-)}u(\tau) + L_k u(\tau_0)), \\ U_n^{(+)}(x, \mu, \delta) = \sum_{k=0}^{n} \mu^k(\overline{u}_k^{(+)}(x) + Q_k^{(+)}u(\tau) + R_k u(\tau_1)), \\ V_{n-1}^{(-)}(x, \mu, \delta) = \sum_{k=0}^{n-1} \mu^k(\overline{v}_k^{(-)}(x) + Q_k^{(-)}v(\tau) + L_k v(\tau_0)), \\ V_{n-1}^{(+)}(x, \mu, \delta) = \sum_{k=0}^{n-1} \mu^k(\overline{v}_k^{(+)}(x) + Q_k^{(+)}v(\tau) + R_k v(\tau_1)), \end{cases}$$

here the internal layer functions $Q_k^{(\mp)}u(\tau)$, $Q_k^{(\mp)}v(\tau)$ depend on $\tau_\delta = \frac{x-x_\delta}{\mu}$ and the unknown parameter x^*. Taking account of $\overline{v}_0^{(\mp)}(x) = 0$ and expanding both sides of the smoothness condition (2.3) into power series of μ, one can obtain the matching conditions that make sure the asymptotic solution is smooth at the point $(x^*, g(x^*))$:

$$\begin{cases} Q_0^{(-)}v(0) = Q_0^{(+)}v(0), & (3.3) \\ \overline{v}_1^{(-)}(x_0) + Q_1^{(-)}v(0) = \overline{v}_1^{(+)}(x_0) + Q_1^{(+)}v(0), & (3.4) \\ \gamma_k^{(-)}(x_0, \cdots, x_{k-1}) + Q_k^{(-)}v(0) = \gamma_k^{(+)}(x_0, \cdots, x_{k-1}) + Q_k^{(+)}v(0), \quad k \geq 2, \end{cases}$$

where $\gamma_k^{(\mp)}(x_0, \cdots, x_{k-1})$ are known functions depending on $x_0, \cdots, x_{k-1}$. By virtue of Condition 6, there exists $x_0 \in (p, q)$ such that $\mathrm{H}(x_0) = 0$, thus, (3.3) is satisfied, and (3.3) can be rewritten as $\widetilde{v}^{(-)}(0) = \widetilde{v}^{(+)}(0)$. Therefore, the leading term of x^* is found. Below we will discuss how to find x_1.

Lemma 3.1 *Under Condition* 6, *x_1 is determined by the linear equation*

$$x_1 \mathrm{H}'(x_0)\widetilde{v}^{(\mp)}(0) + \int_{-\infty}^{0} \widetilde{v}^{(-)} p^{(-)} \widetilde{f}_x^{(-)} \eta \,\mathrm{d}\eta - \int_{+\infty}^{0} \widetilde{v}^{(+)} p^{(+)} \widetilde{f}_x^{(+)} \eta \,\mathrm{d}\eta = 0. \tag{3.5}$$

Proof Taking $\widetilde{v}^{(-)}(0) = \widetilde{v}^{(+)}(0)$ into consideration, and substituting (2.23), (2.25) into (3.4), we have

$$\widehat{H}(x_1) = 0, \tag{3.6}$$

where

$$\begin{aligned} \widehat{H}(x_1) = {} & \widetilde{v}^{(-)}(0)\varphi^{(-)'}(x_0) - \widetilde{v}^{(-)}(0)\varphi^{(+)'}(x_0) - \widetilde{v}^{(-)'}(0)\overline{u}_1^{(-)}(x_0) + \widetilde{v}^{(+)'}(0)\overline{u}_1^{(+)}(x_0) \\ & + [(g'(x_0) - \varphi^{(-)'}(x_0))\widetilde{v}^{(-)'}(0) - (g'(x_0) - \varphi^{(+)'}(x_0))\widetilde{v}^{(+)'}(0)]x_1 \\ & + \int_{-\infty}^{0} \widetilde{v}^{(-)}(\eta)p^{(-)}(\eta)h_1^{(-)}(\eta)\,\mathrm{d}\eta - \int_{+\infty}^{0} \widetilde{v}^{(+)}(\eta)p^{(+)}(\eta)h_1^{(+)}(\eta)\,\mathrm{d}\eta. \end{aligned}$$

Let

$$\widehat{J} = \int_{\mp\infty}^{0} \widetilde{v}^{(\mp)}(\eta)p^{(\mp)}(\eta)h_1^{(\mp)}(\eta)\mathrm{d}\eta,$$

here

$$\begin{aligned} \widehat{J} = {} & \varphi^{(\mp)'}(x_0)\int_{\mp\infty}^{0} [\widetilde{v}^{(\mp)}(\eta)p^{(\mp)}(\eta)\widetilde{f}_v^{(\mp)} + \widetilde{v}^{(\mp)}(\eta)p^{(\mp)}(\eta)\widetilde{f}_u\eta]\,\mathrm{d}\eta \\ & + x_1\int_{\mp\infty}^{0} [\widetilde{v}^{(\mp)}(\eta)p^{(\mp)}(\eta)\widetilde{f}_u(\eta)\varphi^{(\mp)'}(x_0) + \widetilde{v}^{(\mp)}(\eta)p^{(\mp)}(\eta)\widetilde{f}_x^{(\mp)}(\eta)]\,\mathrm{d}\eta + \\ & + \int_{\mp\infty}^{0} \widetilde{v}^{(\mp)}(\eta)p^{(\mp)}(\eta)\widetilde{f}_u(\eta)\overline{u}_1^{(\mp)}(x_0)\,\mathrm{d}\eta + \int_{\mp\infty}^{0} \widetilde{v}^{(\mp)}(\eta)p^{(\mp)}(\eta)\widetilde{f}_x^{(\mp)}(\eta)\eta\,\mathrm{d}\eta, \end{aligned}$$

which can be simplified as

$$\begin{aligned} \widehat{J} = {} & -\varphi^{(\mp)'}(x_0)\widetilde{u}^{(\mp)'}(0) + x_1\varphi^{(\mp)'}(x_0)\widetilde{u}^{(\mp)''}(0) + \widetilde{u}^{(\mp)''}(0)\overline{u}_1^{(\mp)}(x_0) \\ & + x_1\int_{\mp\infty}^{0} \widetilde{v}^{(\mp)}(\eta)p^{(\mp)}(\eta)\widetilde{f}_x^{(\mp)}(\eta)\,\mathrm{d}\eta + \int_{\mp\infty}^{0} \widetilde{v}^{(\mp)}(\eta)p^{(\mp)}(\eta)\widetilde{f}_x^{(\mp)}(\eta)\eta\,\mathrm{d}\eta \end{aligned}$$

by using the equality

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\Big(p^{(\mp)}(\tau)\frac{\mathrm{d}\widetilde{u}^{(\mp)'}}{\mathrm{d}\tau}\Big) - p^{(\mp)}(\tau)\widetilde{f}_u^{(\mp)}(\tau)\widetilde{u}^{(\mp)'} = 0.$$

Thus, $\widehat{H}(x_1)$ can be rewritten as

$$\begin{aligned}\widehat{H}(x_1) &= x_1\Big[g'(x_0)(\widetilde{v}^{(-)'}(0) - \widetilde{v}^{(+)'}(0)) + \int_{-\infty}^{0}\widetilde{v}^{(-)}(\eta)p^{(-)}(\eta)\widetilde{f}_x^{(-)}(\eta)\,\mathrm{d}\eta\Big]\\ &\quad - x_1\int_{+\infty}^{0}\widetilde{v}^{(+)}(\eta)p^{(+)}(\eta)\widetilde{f}_x^{(+)}(\eta)\,\mathrm{d}\eta + \int_{-\infty}^{0}\widetilde{v}^{(-)}(\eta)p^{(-)}(\eta)\widetilde{f}_x^{(-)}(\eta)\eta\,\mathrm{d}\eta\\ &\quad - \int_{+\infty}^{0}\widetilde{v}^{(+)}(\eta)p^{(+)}(\eta)\widetilde{f}_x^{(+)}(\eta)\eta\,\mathrm{d}\eta.\end{aligned}$$

In order to make sure that the coefficient of x_1 in the linear equation above is nonzero, it is necessary to obtain the representation of $\mathrm{H}'(x_0)$.

Let

$$\frac{\partial Q_0^{(\mp)}u}{\partial x} = \widetilde{w}^{(\mp)}, \quad \frac{\partial Q_0^{(\mp)}v}{\partial x} = \widetilde{z}^{(\mp)}.$$

Differentiating with respect to x for the equations of problem (1.8), we have

$$\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}\tau}\widetilde{w}^{(\mp)} = \widetilde{z}^{(\mp)},\\ \dfrac{\mathrm{d}}{\mathrm{d}\tau}\widetilde{w}^{(\mp)} = f_v^{(\mp)}(\tau)\widetilde{z}^{(\mp)} + f_u^{(\mp)}(\tau)(\widetilde{w}^{(\mp)} + \varphi^{(\mp)'}(x)) + f_x^{(\mp)}(\tau),\\ \widetilde{w}^{(\mp)}(0) = g'(x_0) - \varphi^{(\mp)'}(x_0), \quad \widetilde{w}^{(\mp)}(\mp\infty) = \widetilde{z}^{(\mp)}(\mp\infty) = 0.\end{cases} \tag{3.7}$$

Applying Green formula leads to

$$\begin{aligned}\mathrm{H}'(x_0) &= \widetilde{z}^{(-)}(0) - \widetilde{z}^{(+)}(0) = [\widetilde{v}^{(\mp)}(0)]^{-1}g'(x_0)(\widetilde{v}^{(-)'}(0) - \widetilde{v}^{(+)'}(0))\\ &\quad + [\widetilde{v}^{(\mp)}(0)]^{-1}\Big\{\int_{-\infty}^{0}\widetilde{v}^{(-)}p^{(-)}\widetilde{f}_x^{(-)}\,\mathrm{d}\eta - \int_{+\infty}^{0}\widetilde{v}^{(+)}p^{(+)}\widetilde{f}_x^{(+)}\,\mathrm{d}\eta\Big\}.\end{aligned}$$

By virtue of Condition 5, the separatrixes $\Phi(\widetilde{v},\widetilde{u},x) = \Phi(0,\varphi^{(-)}(x_0),x_0)$ do not intersect $\widetilde{u} = 0$ as $\tau = 0$. Thus, $\widetilde{v}^{(\mp)}(0) \neq 0$. Therefore, (3.6) can be rewritten as (3.5). By Condition 6, x_1 can be uniquely determined by the linear equation (3.5).

Similarly, x_k $(k = 2,3,\cdots)$ can be obtained by the same algorithm.

Denote

$$G(x_\delta,\mu) = \frac{\mathrm{d}}{\mathrm{d}x}u^{(-)}(x_\delta,\mu) - \frac{\mathrm{d}}{\mathrm{d}x}u^{(+)}(x_\delta,\mu).$$

Substituting the asymptotic representations (3.2) of $u^{(\mp)}(x_\delta,\mu)$ into $G(x_\delta,\mu)$, it follows from $\mathrm{H}(x_0) = 0$, Lemma 3.1 and the algorithm of constructing x_k $(k \geq 2)$ that

$$G(x_\delta,\mu) = \mu^n(\mathrm{H}'(x_0)\delta + O(\mu)).$$

By Condition 6, $\mathrm{H}'(x_0) \neq 0$, for sufficiently small μ, there exists $\delta = \overline{\delta}$ such that $G(x_{\overline{\delta}},\mu) = 0$. Thus, provided $\delta = \overline{\delta}$ in the representation (3.1), the obtained $x_{\overline{\delta}}$ will guarantee that the composite function

$$u(x,\mu) = \begin{cases}u^{(-)}(x,\mu,\overline{\delta}(\mu)), & 0 \leq x < x_{\overline{\delta}},\\ u^{(+)}(x,\mu,\overline{\delta}(\mu)), & x_{\overline{\delta}} \leq x \leq 1\end{cases}$$

is a smooth contrast structure solution to problem (1.1) in the neighborhood of $x = x_{\overline{\delta}}$. After replacing $x_{\overline{\delta}}$ by $\overline{x}$, the accuracy of the first equation of (3.2) remains constant. So the main theorem in this paper can be derived.

Theorem 3.1 *Under Conditions* 1–7, *for sufficiently small* $\mu > 0$, *the nonlinear boundary value problem* (1.1) *has a smooth contrast structure solution* $u(x, \mu)$, *whose asymptotic representations are as follows*

$$u(x,\mu) = \begin{cases} \sum\limits_{k=0}^{n} \mu^k (\overline{u}_k^{(-)}(x) + Q_k^{(-)} u(\tau) + L_k u(\tau_0)) + O(\mu^{n+1}), & 0 \le x < \overline{x}, \\ \sum\limits_{k=0}^{n} \mu^k (\overline{u}_k^{(+)}(x) + Q_k^{(+)} u(\tau) + R_k u(\tau_1)) + O(\mu^{n+1}), & \overline{x} \le x \le 1, \end{cases}$$

where

$$\overline{x} = x_0 + \mu x_1 + \cdots + \mu^{n+1}(x_{n+1} + \delta), \quad \tau = \frac{(x - \overline{x})}{\mu},$$

and an internal layer appears in the neighborhood of the discontinuous curve $y = g(x)$.

4 Example

We consider the second order nonlinear Dirichlet boundary value problem

$$\begin{cases} \mu^2 u'' = \begin{cases} (0.2 - x)\mu u' + u - 3, & (u,x) \in D^{(-)}, \\ (0.9 - x)\mu u' + u + 1, & (u,x) \in D^{(+)}, \end{cases} \\ u(0) = 0, \quad u(1) = 1, \end{cases} \tag{4.1}$$

where $g(x) = 8x - 4$.

Since the reaction-diffusion-advection equation with a small parameter is an important mathematical model in different research fields, this example has a profound practical background. Example (4.1) can be used to develop mathematical models in mechanics, electronics, biology, and other fields in the case when phenomena in inhomogeneous media with discontinuous characteristics occur. In [1], a multidimensional singularly perturbed reaction-diffusion-advection problem with wave functions of electrons and holes and Coulomb potential depending on the parameters of the layer is studied and the increase of tunnel barrier transparency leads to a transition from the dipolar electron-hole system (EHS for short) with a double-peak wave function of electrons to the spatially direct EHS. Furthermore, to describe an in-situ combustion process, the reaction-diffusion-advection mathematical model with an internal layer is proposed in [3]. Based on our theoretical results, the asymptotic behavior of a solution to example (4.1) shall be discussed.

Let $\mu = 0$, we have $\varphi^{(-)}(x) = 3$, $\varphi^{(+)}(x) = -1$ and $\overline{v}_0^{(\mp)}(x) = 0$.

It is easy to verify that Conditions 1–7 in Theorem 3.1 are satisfied. The function (1.9) can be rewritten as

$$\mathrm{H}(x) = \frac{0.2 - x + \sqrt{(0.2 - x)^2 + 4}}{2}(8x - 7) - \frac{0.9 - x - \sqrt{(0.9 - x)^2 + 4}}{2}(8x - 3),$$

and the equation $\mathrm{H}(x_0) = 0$ has a solution $x_0 = 0.6204$. The computation shows that $\mathrm{H}'(x_0) \neq 0$, so Condition 6 is also satisfied.

The problems for determining $Q_0^{(\mp)}u$, $Q_0^{(\mp)}v$ have the forms

$$\begin{cases} \dfrac{\mathrm{d}Q_0^{(-)}u}{\mathrm{d}\tau} = Q_0^{(-)}v, \quad \dfrac{\mathrm{d}Q_0^{(-)}v}{\mathrm{d}\tau} = (0.2 - x_0)Q_0^{(-)}v + Q_0^{(-)}u, \\ Q_0^{(-)}u(0) = 8x_0 - 7, \quad Q_0^{(-)}u(-\infty) = 0, \quad Q_0^{(-)}v(-\infty) = 0; \end{cases}$$

$$\begin{cases} \dfrac{\mathrm{d}Q_0^{(+)}u}{\mathrm{d}\tau} = Q_0^{(+)}v, \quad \dfrac{\mathrm{d}Q_0^{(+)}v}{\mathrm{d}\tau} = (0.9 - x_0)Q_0^{(+)}v + Q_0^{(+)}u, \\ Q_0^{(+)}u(0) = 8x_0 - 3, \quad Q_0^{(+)}u(+\infty) = 0, \quad Q_0^{(+)}v(+\infty) = 0, \end{cases}$$

whose solutions are

$$Q_0^{(-)}u(\tau) = -2.0368\mathrm{e}^{0.8116\tau}, \quad Q_0^{(-)}v(\tau) = -1.6531\mathrm{e}^{0.8116\tau};$$

$$Q_0^{(+)}u(\tau) = 1.9632\mathrm{e}^{-0.8699\tau}, \quad Q_0^{(+)}v(\tau) = -1.7078\mathrm{e}^{-0.8699\tau}.$$

For L_0u, R_0u, we have

$$\begin{cases} \dfrac{\mathrm{d}L_0u}{\mathrm{d}\tau_0} = L_0v, \quad \dfrac{\mathrm{d}L_0v}{\mathrm{d}\tau_0} = 0.2L_0v + L_0u, \\ L_0u(0) = -3, \quad L_0u(+\infty) = 0, \quad L_0v(+\infty) = 0 \end{cases}$$

and

$$\begin{cases} \dfrac{\mathrm{d}R_0u}{\mathrm{d}\tau_1} = R_0v, \quad \dfrac{\mathrm{d}R_0v}{\mathrm{d}\tau_1} = -0.1R_0v + R_0u, \\ R_0u(0) = 2, \quad R_0u(-\infty) = 0, \quad R_0v(-\infty) = 0, \end{cases}$$

whose solutions acquire the forms

$$L_0u = -3\mathrm{e}^{-0.9050\tau_0}, \quad R_0u = 2\mathrm{e}^{0.9513\tau_1}.$$

From the linear equation (3.5), one can obtain $x_1 = 0.6429$.

Applying Theorem 3.1, problem (4.1) has a smooth asymptotic solution $u(x,\mu)$, which can be represented as

$$u(x,\mu) = \begin{cases} 3 - 2.0368\mathrm{e}^{0.8116\tau} - 3\mathrm{e}^{-0.9050\tau_0} + O(\mu), & 0 \le x < \overline{x}, \\ -1 + 1.9632\mathrm{e}^{-0.8699\tau} + 2\mathrm{e}^{0.9513\tau_1} + O(\mu), & \overline{x} \le x \le 1, \end{cases}$$

where

$$\overline{x} = 0.6204 + 0.6429\mu, \quad \tau = \frac{(x - \overline{x})}{\mu}.$$

This problem has not an analytical solution, whose behavior can be described by an asymptotic solution obtained by our method. In addition, as μ takes sufficiently small value, the accuracy $O(\mu)$ of the obtained smooth asymptotic solution $y(x,\mu)$ is very small. As shown in Figure 4, our asymptotic solution is close to the numerical solution and there is an internal layer in the neighborhood of discontinuous curve $y = 8x - 4$.

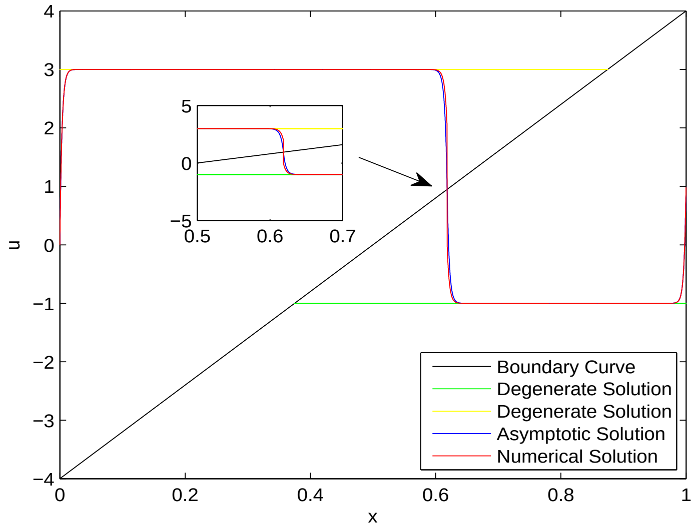

Figure 4 Numerical solution of problem (4.1) and its zero-order asymptotic approximation ($\mu = 0.003$).

5 Conclusion

This paper investigates a stationary problem for a reactive-advection-diffusion differential equation with discontinuous right-hand side. The sufficient conditions for the existence of a smooth solution with an internal layer in the neighborhood of a point on the discontinuous curve is given. Using theorems on existence of solutions to classical boundary value problems for singularly perturbed nonlinear equations and algorithm for constructing asymptotic expansions to these solutions, the existence of a smooth solution is proved. This work is an extension and further development of the results in [35]. What's more, our method can be generalized to multidimensional singularly perturbed reaction-diffusion-advection problems. The results can also be applied to propose an efficient numerical algorithm that uses the asymptotic solution to construct non-uniform meshes to describe the behavior of internal layer of the solution to similar problems stated in [36–37].

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