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Finite Abelian Groups of $K3$ Surfaces with Smooth Quotient

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Abstract The quotient space of a $K3$ surface by a finite group is an Enriques surface or a rational surface if it is smooth. Finite groups where the quotient space are Enriques surfaces are known. In this paper, by analyzing effective divisors on smooth rational surfaces, the author will study finite groups which act faithfully on $K3$ surfaces such that the quotient space are smooth. In particular, he will completely determine effective divisors on Hirzebruch surfaces such that there is a finite Abelian cover from a $K3$ surface to a Hirzebrunch surface such that the branch divisor is that effective divisor. Furthermore, he will decide the Galois group and give the way to construct that Abelian cover from an effective divisor on a Hirzebruch surface. Subsequently, he studies the same theme for Enriques surfaces.

Keywords $K3$ surface, Finite Abelian group, Abelian cover of a smooth rational surface
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1 Introduction

In this paper, we work over $\mathbb{C}$. A $K3$ surface X is a smooth surface with $h^1(\mathcal{O}_X) = 0$ and $\mathcal{O}_X(K_X) \cong \mathcal{O}_X$, where K_X is the canonical divisor of X. In particular, a $K3$ surface is simply connected. Finite groups acting faithfully on $K3$ surfaces are well studied. Let ω be a non-degenerated two holomorphic form. An automorphism f of a $K3$ surface is called symplectic if $f^*\omega = \omega$. A finite subgroup G of automorphisms of a $K3$ surface is called symplectic if G is generated by symplectic automorphisms. The minimal resolution X_m of the quotient space X/G is one of a $K3$ surface, an Enriques surface and a rational surface. The surface X_m is a $K3$ surface if and only if G is a symplectic group. Symplectic groups are classified (see [10, 13, 16]). If the quotient space of X/G is smooth, then it is an Enriques surface or a rational surface. The quotient space X/G is an Enriques surface if and only if G is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ as a group and the fixed locus of G is an empty set. It is not well-known what kind of rational surface is realized as the quotient space of a $K3$ surface by a finite subgroup of $\mathrm{Aut}(X)$. In this paper, we will consider the case where X/G is a smooth rational surface. The minimal model of smooth rational surfaces is the projective plane $\mathbb{P}^2$ or a Hirzebruch surfaces $\mathbb{F}_n$ where $n \neq 1$, and $\mathbb{F}_1$ is isomorphic to $\mathbb{P}^2$ blow-up at a point. In other words, all smooth rational surfaces which are not minimal are $\mathbb{F}_1$ or given by blowups of $\mathbb{F}_n$ for $0 \leq n$. Therefore, if X/G is not

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$\mathbb{P}^2$, then there is a birational morphism $f : X/G \to \mathbb{F}_n$. Our first main results are to analyze the quotient space X/G and G when X/G is smooth.

Theorem 1.1 *Let X be a $K3$ surface and G be a finite subgroup of* $\mathrm{Aut}(X)$ *such that X/G is smooth. For a birational morphism $f : X/G \to \mathbb{F}_n$ from the quotient space X/G to a Hirzebruch surface $\mathbb{F}_n$, we get that $n = 0, 1, 2, 3, 4, 6, 8$ or 12. Furthermore, if $n = 6, 8, 12$, then f is an isomorphism.*

Let X be a $K3$ surface, and ω be a non-degenerated holomorphic two form of X. For a finite group G of $\mathrm{Aut}(X)$, we write G_s as a set of symplectic automorphisms of G. Then there is a short exact sequence: $1 \to G_s \to G \xrightarrow{\varphi} C_n \to 1$, where C_n is a cyclic group of order n, and $\varphi(g) := \xi_g \in \mathbb{C}^*$ such that $g^*\omega = \xi_g\omega$ in $\mathrm{H}^{2,0}(X)$ for $g \in G$.

Theorem 1.2 *Let X be a $K3$ surface, G be a finite subgroup of* $\mathrm{Aut}(X)$ *such that X/G is smooth. Then the above exact sequence is split, i.e., there is a purely non-symplectic automorphism $g \in G$ such that G is the semidirect product $G_s \rtimes \langle g\rangle$ of G_s and $\langle g\rangle$.*

Next, we will classify finite Abelian groups which act faithfully on $K3$ surfaces and the quotient space is smooth.

Definition 1.1 *We will use the following notations*:

$$\mathcal{A}G := \left\{\begin{array}{l} \mathbb{Z}/2\mathbb{Z}^{\oplus a},\ \mathbb{Z}/3\mathbb{Z}^{\oplus b},\ \mathbb{Z}/4\mathbb{Z}^{\oplus c},\ \mathbb{Z}/2\mathbb{Z}^{\oplus d} \oplus \mathbb{Z}/3\mathbb{Z}^{\oplus e},\ \mathbb{Z}/2\mathbb{Z}^{\oplus f} \oplus \mathbb{Z}/4\mathbb{Z}^{\oplus g}, \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}^{\oplus h} \oplus \mathbb{Z}/4\mathbb{Z},\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z} \\ : 1 \le a \le 5,\ 1 \le b, c \le 3, \\ (d, e) = (1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 1), (3, 2), \\ (f, g) = (1, 1), (1, 2), (2, 1), (3, 1),\ h = 1, 2 \end{array}\right\},$$

$$\mathcal{A}G_\infty := \left\{\begin{array}{l} \mathbb{Z}/2\mathbb{Z}^{\oplus a},\ \mathbb{Z}/4\mathbb{Z}^{\oplus c},\ \mathbb{Z}/2\mathbb{Z}^{\oplus d} \oplus \mathbb{Z}/3\mathbb{Z}^{e},\ \mathbb{Z}/2\mathbb{Z}^{\oplus 2} \oplus \mathbb{Z}/4\mathbb{Z} \\ : a = 1, 2, 3, 4, 5,\ c = 1 \text{ or } 3,\ (d, e) = (1, 1), (1, 2) \text{ or } (3, 2) \end{array}\right\},$$

$$\mathcal{A}G_0 := \left\{\begin{array}{l} \mathbb{Z}/2\mathbb{Z}^{\oplus a},\ \mathbb{Z}/3\mathbb{Z}^{\oplus b},\ \mathbb{Z}/2\mathbb{Z}^{\oplus f} \oplus \mathbb{Z}/4\mathbb{Z}^{\oplus g} \\ : a = 1, 2, 3, 4, 5,\ b = 1, 2, 3,\ (f, g) = (1, 1), (1, 2), (2, 1), (3, 1) \end{array}\right\},$$

$$\mathcal{A}G_1 := \left\{\begin{array}{l} \mathbb{Z}/2\mathbb{Z}^{\oplus a},\ \mathbb{Z}/4\mathbb{Z}^{\oplus 2},\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}^{\oplus e},\ \mathbb{Z}/2\mathbb{Z}^{\oplus f} \oplus \mathbb{Z}/4\mathbb{Z}, \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}^{\oplus 2} \oplus \mathbb{Z}/4\mathbb{Z},\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z} \\ : a = 1, 2, 3, 4, 5,\ e = 1, 2, 3,\ f = 1, 2, 3 \end{array}\right\},$$

$$\mathcal{A}G_2 := \left\{\begin{array}{l} \mathbb{Z}/2\mathbb{Z}^{\oplus a},\ \mathbb{Z}/3\mathbb{Z}^{b},\ \mathbb{Z}/2\mathbb{Z}^{2} \oplus \mathbb{Z}/3\mathbb{Z}^{\oplus 2},\ \mathbb{Z}/2\mathbb{Z}^{\oplus f} \oplus \mathbb{Z}/4\mathbb{Z}^{\oplus g} \\ : a = 1, 2, 3, 4,\ b = 1, 2, 3,\ (f, g) = (1, 1), (1, 2), (2, 1), (3, 1) \end{array}\right\},$$

$$\mathcal{A}G_3 := \left\{\begin{array}{l} \mathbb{Z}/2\mathbb{Z}^{\oplus d} \oplus \mathbb{Z}/3\mathbb{Z}^{\oplus e},\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \\ : (d, e) = (1, 1), (1, 2), (3, 1) \end{array}\right\},$$

$$\mathcal{A}G_4 := \left\{\begin{array}{l} \mathbb{Z}/2\mathbb{Z}^{\oplus a},\ \mathbb{Z}/4\mathbb{Z},\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}^{\oplus 2},\ \mathbb{Z}/2\mathbb{Z}^{\oplus f} \oplus \mathbb{Z}/4\mathbb{Z} \\ : a = 1, 2, 3,\ f = 1, 2 \end{array}\right\},$$

$$\mathcal{A}G_6 := \left\{\mathbb{Z}/3\mathbb{Z}^{\oplus b},\ \mathbb{Z}/2\mathbb{Z}^{\oplus 2} \oplus \mathbb{Z}/3\mathbb{Z} : b = 1, 2\right\},$$
$$\mathcal{A}G_8 := \left\{\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}\right\},$$
$$\mathcal{A}G_{12} := \left\{\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}\right\}.$$

Notice that $\mathcal{A}G = \bigcup_{n=0,1,2,3,4,6,8,12,\infty} \mathcal{A}G_n$. In [15], Uludağ classified finite Abelian groups for the case X/G is $\mathbb{P}^2$. Furthermore, he gave the way to construct the pair (X, G) where X is a $K3$ surface and G is a finite subgroup of $\mathrm{Aut}(X)$ such that $X/G \cong \mathbb{P}^2$. We have the following theorem.

Theorem 1.3 (see [15]) *Let X be a $K3$ surface and G be a finite Abelian subgroup of $\mathrm{Aut}(X)$ such that the quotient space X/G is isomorphic to $\mathbb{P}^2$. Then G is one of $\mathcal{AG}_\infty$ as a group. Conversely, for every $G \in \mathcal{AG}_\infty$, there is a $K3$ surface X' and a finite Abelian subgroup G' of $\mathrm{Aut}(X')$ such that $X'/G' \cong \mathbb{P}^2$ and $G' \cong G$ as a group.*

By analyzing the irreducible components of the branch locus of the quotient map $p : X \to X/G$, we will study a pair (X, G) consisting of a $K3$ surface X and a finite Abelian subgroup G of $\mathrm{Aut}(X)$ such that the quotient space X/G is smooth. More precisely, the preimage of the branch locus of p is $\bigcup_{g \in G \setminus \{\mathrm{id}_X\}} \mathrm{Fix}(g)$ where $\mathrm{Fix}(g) := \{x \in X : g(x) = x\}$. Recall that for an automorphism f of finite order of a $K3$ surface, if $\mathrm{Fix}(f)$ contains a curve, then f is non-symplectic. The fixed locus of a non-symplectic automorphism is well-known, e.g. [1–2, 14]. By analyzing the fixed locus of non-symplectic automorphisms of G from the branch divisor of the quotient map, we will reconstruct G from the branch divisor of the quotient map. In Section 4, we will investigate the relationship between a branch divisor and exceptional divisors of blowups. Based on the above results, we will obtain our second main result.

Theorem 1.4 *Let X be a $K3$ surface and G be a finite Abelian subgroup of $\mathrm{Aut}(X)$ such that the quotient space X/G is smooth. Then G is one of $\mathcal{AG}$ as a group. Conversely, for every $G \in \mathcal{AG}$, there is a $K3$ surface X' and a finite Abelian subgroup G' of $\mathrm{Aut}(X')$ such that X'/G' is smooth and $G' \cong G$ as a group.*

Furthermore, in Section 3, for a Hirzebruch surface $\mathbb{F}_n$ and an effective divisor B on $\mathbb{F}_n$, we will give a necessary and sufficient condition for the existence of a finite Abelian cover $f : X \to \mathbb{F}_n$ such that X is a $K3$ surface and the branch divisor of f is B. In other words, we will solve a part of the Fenchel's problem for Hirzebruch surfaces. In addition, we will decide the Galois group and give the way to construct $f : X \to \mathbb{F}_n$ from the pair $\mathbb{F}_n$ and B.

Theorem 1.5 *Let X be a $K3$ surface and G be a finite Abelian subgroup of $\mathrm{Aut}(X)$ such that the quotient space X/G is isomorphic to $\mathbb{F}_n$. Then G is one of $\mathcal{AG}_n$ as a group. Conversely, for every $G \in \mathcal{AG}_n$, there is a $K3$ surface X' and a finite Abelian subgroup G' of $\mathrm{Aut}(X')$ such that X'/G' is isomorphic to $\mathbb{F}_n$ and $G' \cong G$ as a group.*

Subsequently, we will get a similar result for Enriques surfaces.

Definition 1.2 *We use the following notations*:

$$\mathcal{AG}(E) := \left\{ \begin{array}{l} \mathbb{Z}/2\mathbb{Z}^{\oplus a},\ \mathbb{Z}/4\mathbb{Z}^{\oplus 2},\ \mathbb{Z}/2\mathbb{Z}^{\oplus f} \oplus \mathbb{Z}/4\mathbb{Z},\ \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z} \\ : a = 2, 3, 4,\ f = 1, 2 \end{array} \right\},$$
$$\mathcal{AG}_\infty(E) := \left\{ \mathbb{Z}/2\mathbb{Z}^{\oplus a} :\ a = 2, 3, 4 \right\},$$
$$\mathcal{AG}_0(E) := \left\{ \begin{array}{l} \mathbb{Z}/2\mathbb{Z}^{\oplus a},\ \mathbb{Z}/4\mathbb{Z}^{\oplus 2},\ \mathbb{Z}/2\mathbb{Z}^{\oplus f} \oplus \mathbb{Z}/4\mathbb{Z} \\ : a = 2, 3, 4,\ f = 1, 2 \end{array} \right\},$$
$$\mathcal{AG}_1(E) := \left\{ \begin{array}{l} \mathbb{Z}/2\mathbb{Z}^{\oplus a},\ \mathbb{Z}/2\mathbb{Z}^{\oplus f} \oplus \mathbb{Z}/4\mathbb{Z},\ \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z} \\ : a = 2, 3, 4,\ f = 1, 2 \end{array} \right\},$$
$$\mathcal{AG}_2(E) := \left\{ \mathbb{Z}/2\mathbb{Z}^{\oplus a},\ \mathbb{Z}/4\mathbb{Z}^{\oplus 2},\ \mathbb{Z}/2\mathbb{Z}^{\oplus 2} \oplus \mathbb{Z}/4\mathbb{Z} :\ a = 2, 3 \right\},$$
$$\mathcal{AG}_4(E) := \left\{ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \right\}.$$

Then $\mathcal{A}G(E) = \bigcup_{n=0,1,2,4,\infty} \mathcal{A}G_n(E)$. Let E be an Enriques surface and H be a finite Abelian subgroup of $\mathrm{Aut}(E)$ such that E/H is smooth. Let X be the $K3$-cover of E, and $G := \{s \in \mathrm{Aut}(X) : s \text{ is a lift of some } h \in H\}$. Then G is a finite Abelian subgroup of $\mathrm{Aut}(X)$, G has a non-symplectic involution whose fixed locus is empty, and $X/G = E/H$. The case of $E/H \cong \mathbb{P}^2$ was studied in [7]. By analyzing the groups of Theorem 1.4, we get the following theorems.

Theorem 1.6 *Let E be an Enriques surface and H be a finite subgroup of* $\mathrm{Aut}(E)$ *such that the quotient space E/H is smooth. If there is a birational morphism from E/H to a Hirzebruch surface $\mathbb{F}_n$, then $0 \leq n \leq 4$. In particular, if the quotient space E/H is a Hirzebruch surface $\mathbb{F}_n$, then $n = 0, 1, 2, 4$.*

Theorem 1.7 *Let E be an Enriques surface and H be a finite Abelian subgroup of* $\mathrm{Aut}(E)$ *such that the quotient space E/H is isomorphic to $\mathbb{F}_n$. Then H is one of $\mathcal{A}G_n(E)$ as a group. Conversely, for every $H' \in \mathcal{A}G_n(E)$, there is an Enriques surface E' and a finite Abelian subgroup H' of* $\mathrm{Aut}(E')$ *such that E'/H' is smooth and $H' \cong H$ as a group.*

Theorem 1.8 *Let E be an Enriques surface and H be a finite Abelian subgroup of* $\mathrm{Aut}(E)$ *such that the quotient space E/H is smooth. Then H is one of $\mathcal{A}G(E)$ as a group. Conversely, for every $H \in \mathcal{A}G(E)$, there is an Enriques surface E' and a finite Abelian subgroup H' of* $\mathrm{Aut}(E')$ *such that E'/H' is smooth and $H' \cong H$ as a group.*

Section 2 is preliminaries. In Subsection 3.1, we will give examples for pairs (X', G') described in Theorem 1.4. In other words, we will show that for each $G \in \mathcal{A}G_n$ where $n = 0, 1, 2, 3, 4, 6, 8, 12$, there is a pair (X', G'), where X' is a $K3$ surface and G' is a finite Abelian subgroup of $\mathrm{Aut}(X')$ such that $G \cong G'$ as a group and $X'/G' \cong \mathbb{F}_n$. Furthermore, we will give the way to construct (X', G'), and we will show that the way to construct (X', G') is uniquely determined up to isomorphism from the branch divisor of the quotient map $p : X' \to X'/G'$. In Subsection 3.2, we will describe branch divisors and Abelian groups for the case where the quotient space is a Hirzebruch surface. In Section 4, first, we will show Theorems 1.1–1.2. Next, we will show that for a pair (X, G) where X is a $K3$ surface and G is a finite Abelian subgroup, if X/G is smooth, then G is isomorphic to one of $\mathcal{A}G$ as a group. In Section 5, we will show Theorems 1.6–1.8.

2 Preliminaries

We recall the properties of the Galois cover.

Definition 2.1 *Let $f : X \to M$ be a branched covering, where M is a complex manifold and X is a normal complex space. We call $f : X \to M$ the Galois cover if there is a subgroup G of* $\mathrm{Aut}(X)$ *such that $X/G \cong M$ and $f : X \to M$ is isomorphic to the quotient map $p : X \to X/G \cong M$. We call G the Galois group of $f : X \to M$. Furthermore, if G is an Abelian group, then we call $f : X \to M$ the Abelian cover.*

Definition 2.2 *Let $f : X \to M$ be a finite branched covering, where M is a complex manifold and X is a normal complex space and Δ be the branch locus of f. Let $B_1, \cdots, B_s$ be irreducible hypersurfaces of M and positive integers $b_1, \cdots, b_s$, where $b_i \geq 2$ for $i = 1, \cdots, s$. If $\Delta = B_1 \cup \cdots \cup B_s$ and for every j and for any irreducible component D of $f^{-1}(B_j)$ the* *ramification index at D is b_j, then we call an effective divisor $B := \sum_{i=1}^{s} b_i B_i$ the branch divisor of f.*

Let X be a normal projective variety and G be a finite subgroup of $\mathrm{Aut}(X)$. Let $Y := X/G$ be the quotient space and $p : X \to Y$ be the quotient map. The branch locus, denoted by Δ is a subset of Y given by $\Delta := \{y \in Y \mid |p^{-1}(y)| < |G|\}$. It is known that Δ is an algebraic subset of dimension $\dim(X) - 1$ if Y is smooth (see [19]). Let $\{B_i\}_{i=1}^{r}$ be the irreducible components of Δ whose dimension is 1. Let D be an irreducible component of D of $p^{-1}(B_j)$ and $G_D := \{g \in G : g_{|D} = \mathrm{id}_D\}$. Then the ramification index at D is $b_j := |G_D|$, and the positive integer b_j is independent of an irreducible component of $p^{-1}(B_j)$. Then $b_1B_1+\cdots+b_rB_r$ is the branch divisor of G. We state the facts (Theorems 2.1–2.2) of the Galois cover theory which we need.

Theorem 2.1 (see [12]) *For a complex manifold M and an effective divisor B on M, if there is a branched covering map $f : X \to M$ where X is a simply connected complex manifold X and the branch divisor of f is B, then there is a subgroup G of* $\mathrm{Aut}(X)$ *such that $X/G \cong M$ and $f : X \to M$ is isomorphic to the quotient map $p : X \to X/G \cong M$. Furthermore, a pair (X, G) is a unique up to isomorphism.*

Theorem 2.2 (see [12]) *For a complex manifold M and an effective divisor $B := \sum_{i=1}^{n} b_i B_i$ on M, where B_i is an irreducible hypersurface for $i = 1, \cdots, n$. Let $f : X \to M$ be a branched cover whose branch divisor is B and where X is a simply connected complex manifold. Then for a branched cover $g : Y \to M$ whose branch divisor is $\sum_{j=1}^{m} b'_j B_j$ and b'_j is divisible by b_i and $m \leq n$, there is a branched cover $h : X \to Y$ such that $f = g \circ h$.*

Let X be a $K3$ surface and G be a finite subgroup of $\mathrm{Aut}(X)$ such that X/G is smooth. Since $K3$ surfaces are simply connected, G is determined by the branch divisor of the quotient map $p : X \to X/G$ from Theorem 2.1. In order to classify finite Abelian groups G which act on $K3$ surfaces and the quotient space is smooth, we will search a smooth rational surface S and an effective divisor B on S such that there is a $K3$ surface and a finite subgroup G of $\mathrm{Aut}(X)$ such that $X/G \cong S$ and the branch divisor of the quotient map $p : X \to X/G$ is B. There is the problem which is called Fenchel's problem.

Problem 2.1 Let M be a projective manifold. Give a necessary and sufficient condition on an effective divisor D on M for the existence of a finite Galois (resp. Abelian) cover $\pi : X \to M$ whose branch divisor is D.

The Fenchel's problem was originally for compact Riemann surfaces and was answered by Bundgaard-Nielsen [4] and Fox [5].

Theorem 2.3 (see [4–5]) *Let $k \geq 1$ and let $D := \sum_{i=1}^{k} m_i x_i$ be a divisor on a compact Riemann surface M where $x_i \in M$ and $m_i \in \mathbb{Z}$ for $i = 1, \cdots, k$. Then there is a finite Galois cover $p : X \to M$ such that the branch divisor of p is D except for*

(i) *$M = \mathbb{P}^1$ and $k = 1$, and*

(ii) *$M = \mathbb{P}^1$, $k = 2$ and $m_1 \neq m_2$.*

Furthermore, for the case $M = \mathbb{P}^1$, there exists a finite Abelian cover $\mathbb{P}^1 \to \mathbb{P}^1$ whose branch *divisor is D if and only if*

(i) $k = 2$ *and* $m_1 = m_2$ *or*

(ii) $k = 3$ *and* $m_1 = m_2 = m_3 = 2$.

In order to study the cover of the Galois cover $X \to X/G$, the following theorem is useful.

Theorem 2.4 *Let X be a smooth projective variety, and G be a finite subgroup of* $\mathrm{Aut}(X)$ *such that X/G is smooth. Let $p : X \to X/G$ be the quotient map, and $B := b_1B_1 + \cdots + b_rB_r$ be the branch divisor of p. Then*

$$K_X = p^*K_{X/G} + \sum_{i=1}^{r} \frac{b_i - 1}{b_i} p^*B_i,$$

where K_X (resp. $K_{X/G}$) is the canonical divisor of X (resp. X/G).

Let X be a $K3$ surface and G be a finite subgroup of $\mathrm{Aut}(X)$ such that X/G is smooth, and B be the branch divisor of the quotient map $p : X \to X/G$. The canonical line bundle of a $K3$ surface is trivial. By Theorem 2.4, the branch divisor is restricted in the Picard group of the smooth rational surface X/G, i.e., B must satisfy

$$K_{X/G} + \sum_{i=1}^{r} \frac{b_i - 1}{b_i} B_i = 0 \quad \text{in } \mathrm{Pic}_{\mathbb{Q}}(X/G).$$

In Subsection 3.1, we will show that for a Hirzebruch surface $\mathbb{F}_n$, if $\mathbb{F}_n$ has an effective divisor $B = \sum_{i=1}^{k} b_iB_i$, where B_i is an irreducible curve and $b_i \geq 2$ for $i = 1, \cdots, k$, such that $\sum_{i=1}^{k} \frac{b_i-1}{b_i} B_i + K_S = 0$ in $\mathrm{Pic}_{\mathbb{Q}}(\mathbb{F}_n)$, then $0 \leq n \leq 12$. In Section 4, we will show Theorem 1.1 by using Theorem 2.4.

The following theorem is important for checking the structure of G from the branch divisor.

Theorem 2.5 (see [17]) *For a $K3$ surface X and a finite subgroup G of* $\mathrm{Aut}(X)$ *such that X/G is smooth. Let $B := \sum_{i=1}^{k} b_iB_i$ be the branch divisor of the quotient map $p : X \to X/G$. We put $p^*B_i = \sum_{j=1}^{l} b_iC_{i,j}$ where $C_{i,j}$ is an irreducible curve for $j = 1, \cdots, l$. Let $G_{C_{i,j}} := \{g \in G : g_{|C_{i,j}} = \mathrm{id}_{C_{i,j}}\}$, and G_i be a subgroup of G, which is generated by $G_{C_{i,1}}, \cdots, G_{C_{i,l}}$, and $I \subset \{1, \cdots, k\}$ be a subset. Then, the following holds.*

(i) *If $(X/G) \setminus \cup_{i \in I} B_i$ is simply connected, then G is generated by $\{G_j\}_{j \in \{1,\cdots,k\} \setminus I}$.*

(ii) *$G_{C_{i,j}} \cong \mathbb{Z}/b_i\mathbb{Z}$ and $G_{C_{i,j}}$ is generated by a purely non-symplectic automorphism of order b_i.*

(iii) *If G is Abelian, then there is an automorphism $g \in G$ such that $\bigcup_{j=1}^{l} C_{i,j} \subset \mathrm{fix}(g)$, and hence $C_{i,j}$ are pairwise disjoint.*

(iv) *If the self-intersection number $(B_i \cdot B_i)$ of B_i is positive, then $l = 1$, and hence G_i is generated by a purely non-symplectic automorphism of order b_i.*

Proof We will show (i). We assume that $(X/G) \setminus \bigcup_{i \in I} B_i$ is simply connected. Let H be the subgroup of G which is generated by $\{G_j\}_{j \in \{1,\cdots,k\} \setminus I}$, and $X_0 := X \setminus \bigcup_{i \in I} p^{-1}(B_i)$. Then G and H act on X_0. We assume that $G \neq H$. Let $Y := X_0/H$ be the quotient space, and $G' := G/H$.

Then G' acts faithfully on Y, $Y/G' \cong (X/G)\setminus \bigcup_{i\in I} B_i$, and the branch locus of $Y \to Y/G'$ is a finite set. Since $(X/G)\setminus \bigcup_{i\in I} B_i$ is smooth and simply connected, this is a contradiction. Therefore, G is generated by $\{G_j\}_{j\in\{1,\cdots,k\}\setminus I}$.

Since X is a $K3$ surface, an automorphism whose fixed locus contains a curve can only be purely non-symplectic. Therefore, by the definition of the ramification index b_i, we get (ii).

We will show (iii) and (iv). Since B_i is contained in the branch locus, we get $p^{-1}(B_i) = \bigcup_{j=1}^{l} C_{i,j} \subset \bigcup_{g\in G} \mathrm{fix}(g)$. Since G is finite, for each j, there is $s_j \in G$ such that $C_{i,j} \subset \mathrm{fix}(s_j)$. Since B_i is irreducible, we get that $p(C_{i,j}) = p(C_{i,k})$ for $1 \leq j < k \leq l$. Therefore, there is $t \in G$ such that $t(C_{i,j}) = C_{i,k}$. Since $C_{i,j} \subset \mathrm{fix}(s_j)$ and $t(C_{i,j}) = C_{i,k}$, we obtain that $C_{i,k} \subset \mathrm{fix}(t\circ s_j \circ t^{-1})$. Since G is Abelian, we have $s_j = t\circ s_j\circ t^{-1}$. We get (iii). If the self intersection number $(B_i \cdot B_i)$ of B_i is positive, then by Hodge index theorem, we get $l = 1$. By (ii), $G_i \cong \mathbb{Z}/b_i\mathbb{Z}$ is generated by a purely non-symplectic automorphism of order b_i.

Let X be a $K3$ surface and G be a finite Abelian subgroup of $\mathrm{Aut}(X)$ such that X/G is smooth and $B := \sum_{i=1}^{k} b_i B_i$ be the branch divisor of the quotient map $p : X \to X/G$. If $k = 1$, then by Theorem 2.5, $G = G_{B_1} \cong \mathbb{Z}/b_1\mathbb{Z}$. We assume that $k = 2$. By Theorem 2.5, G is generated by $G_{B_1} \cong \mathbb{Z}/b_1\mathbb{Z}$ and $G_{B_2} \cong \mathbb{Z}/b_2\mathbb{Z}$. Moreover, we assume that the intersection $B_1 \cap B_2$ of B_1 and B_2 is not an empty set. Since $B_1 \cap B_2 \neq \emptyset$, $p^{-1}(B_1) \cap p^{-1}(B_2) \neq \emptyset$. Since the fixed locus of an automorphism is a pairwise disjoint set of points and curves, we get $G_{B_1} \cap G_{B_2} = \{\mathrm{id}_X\}$. Therefore, $G = G_{B_1} \oplus G_{B_2}$, but in the case of $k \geq 3$ it is not necessary $G = \bigoplus_{i=1}^{k} G_{B_i}$ even if $B_i \cap B_j \neq \emptyset$ for $1 \leq i < j \leq k$.

For an irreducible component B_i of B we write $p^*B_i = \sum_{j=1}^{l} b_i C_j$ where C_j is a smooth curve for $j = 1, \cdots, l$. Since the degree of p is $|G|$, by (iv) of Theorem 2.5, we get that $|G|(B_i \cdot B_i) = b_i^2 l (C_j \cdot C_j)$ for $j = 1, \cdots, l$. If the self-intersection number $(B_i)^2$ of B_i is positive, then by (iv) of Theorem 2.5, we get that $l = 1$ and the genus of C_1 is 2 or more. If $(B_i)^2$ is zero, then $C_1, \cdots, C_l$ are elliptic curves. If $(B_i)^2$ is negative, then $C_1, \cdots, C_l$ are rational curves. Recall that there is $g \in G$ such that g is a non-symplectic automorphism of order b_i and $C_1, \cdots, C_l$ are contained in $\mathrm{Fix}(g)$. There are many results on the number of curves, the genus of curves, and the number of isolated points of the fixed locus of a non-symplectic automorphism. We use them to search B such that there is a Galois cover $f : X \to S$ such that X is a $K3$ surface and the branch divisor of f is B and we use them to restore G from B. Here S is a smooth rational surface and B is an effective divisor on S.

3 Abelian Groups of $K3$ Surfaces with Hirzebruch Surfaces

Here, we give the list of a numerical class of an effective divisor $B = \sum_{i=1}^{k} b_i B_i$ on $\mathbb{F}_n$ such that B_i is a smooth curve for each $i = 1, \cdots, k$ and $K_{\mathbb{F}_n} + \sum_{i=1}^{k} \frac{b_i - 1}{b_i} B_i = 0$ in $\mathrm{Pic}_{\mathbb{Q}}(\mathbb{F}_n)$.

Definition 3.1 *For a Hirzebruch surface $\mathbb{F}_n$ where $n \in \mathbb{Z}_{\geq 0}$, we take two irreducible curves C and F such that* $\mathrm{Pic}(\mathbb{F}_n) = \mathbb{Z}C \oplus \mathbb{Z}F$, $(C \cdot F) = 1$, $(F \cdot F) = 0$, $(C \cdot C) = -n$ *and*

$K_{\mathbb{F}_n} = -2C - (n+2)F$ *in* $\mathrm{Pic}(\mathbb{F}_n) = \mathbb{Z}C \oplus \mathbb{Z}F$. *Notice that for* $n = 0$, $C = pr_1^*\mathcal{O}_{\mathbb{P}^1}(1)$ *and* $F = pr_2^*\mathcal{O}_{\mathbb{P}^1}(1)$, *and for* $n \geq 1$, C *is the unique curve on* $\mathbb{F}_n$ *such that the self-intersection number is negative, and* F *is the fibre class of the conic bundle of* $\mathbb{F}_n$.

Lemma 3.1 *Let* $\mathbb{F}_n$ *be a Hirzebruch surface where* $n \neq 0$ *and* $C' \subset \mathbb{F}_n$ *be an irreducible curve. Then one of the following holds*:
(1) $C' = C$.
(2) $C' = F$ *in* $\mathrm{Pic}(\mathbb{F}_n)$.
(3) $C' = aC + bF$ *where* $a \geq 1$ *and* $b \geq na$.

Definition 3.2 *Let* X *be a* $K3$ *surface and* G *be a finite subgroup of* $\mathrm{Aut}(X)$ *such that* $X/G \cong \mathbb{F}_n$. *Let* $B := \sum_{i=1}^{l} b_i B_i$ *be the branch divisor of the quotient map* $p : X \to X/G$. *For each* B_i, *there are integers* α_i, β_i *such that* $B_i = \alpha_i C + \beta_i F$ *in* $\mathrm{Pic}(\mathbb{F}_n)$. *We call*

$$\sum_{i=1}^{l} b_i(\alpha_i C + \beta_i F)$$

as the numerical class of B.

Proposition 3.1 *Let* X *be a* $K3$ *surface and* G *be a finite subgroup of* $\mathrm{Aut}(X)$ *such that* $X/G \cong \mathbb{F}_n$. *Then* $0 \leq n \leq 12$.

Proof We assume that $X/G \cong \mathbb{F}_n$ where $n \geq 1$. Let B be the branch divisor of the quotient map $p : X \to X/G$. We write $B := \sum_{i=1}^{k} b_i B_i + \sum_{j=1}^{l} b'_j B'_j$ such that $B_i \neq F$ and $B'_j = F$ in $\mathrm{Pic}(\mathbb{F}_n)$ for $i = 1, \cdots, k$ and $j = 1, \cdots, l$. Since the canonical line bundle of a $K3$ surface is trivial and $\mathrm{Pic}(\mathbb{F}_n)$ is torsion free, by Theorem 2.4, we get that

$$0 = K_{\mathbb{F}_n} + \sum_{i=1}^{k} \frac{b_i - 1}{b_i} B_i + \sum_{j=1}^{l} \frac{b'_j - 1}{b'_j} B'_j \quad \text{in} \quad \mathrm{Pic}(\mathbb{F}_n).$$

Since B_i is an irreducible curve for $i = 1, \cdots, k$, there are integers c_i, d_i such that $B_i = c_i C + d_i F$ in $\mathrm{Pic}(\mathbb{F}_n)$ and $(c_i, d_i) = (1, 0)$ or $d_i \geq nc_i > 0$. By $K_{\mathbb{F}_n} = -2C - (n+2)F$ in $\mathrm{Pic}(\mathbb{F}_n) = \mathbb{Z}C \oplus \mathbb{Z}F$, we get that

$$\begin{cases} 2 = \sum_{i=1}^{k} \frac{b_i - 1}{b_i} c_i, \\ n + 2 = \sum_{i=1}^{k} \frac{b_i - 1}{b_i} d_i + \sum_{j=1}^{l} \frac{b'_j - 1}{b'_j}. \end{cases}$$

Since $b_i \geq 2$, $\frac{1}{2} \leq \frac{b_i - 1}{b_i} < 1$. Since $2 = \sum_{i=1}^{k} \frac{b_i - 1}{b_i} c_i$, $\sum_{i=1}^{k} c_i = 3$ or 4. By a simple calculation, we get that (i) $\sum_{i=1}^{k} c_i = 4$ if and only if $b_1 = \cdots = b_k = 2$, and (ii) if $\sum_{i=1}^{k} c_i = 3$, then $(b_1, \cdots, b_k; c_1, \cdots, c_k)$ where $c_1 \leq \cdots \leq c_k$ is one of $(3; 3)$, $(2, 4; 1, 2)$, $(3, 3; 1, 2)$, $(2, 3, 6; 1, 1, 1)$, $(2, 4, 4; 1, 1, 1)$ and $(3, 3, 3; 1, 1, 1)$.

We assume that $(c_i, d_i) \neq (1, 0)$ for $i = 1, \cdots, k$, i.e., C is not an irreducible component of B. Since $d_i \geq nc_i$ for $i = 1, \cdots, k$, by $2 = \sum_{i=1}^{k} \frac{b_i - 1}{b_i} c_i$ and $n + 2 = \sum_{i=1}^{k} \frac{b_i - 1}{b_i} d_i + \sum_{j=1}^{l} \frac{b'_j - 1}{b'_j}$, we get that $n+2 \geq 2n + \sum_{j=1}^{l} \frac{b'_j-1}{b'_j}$. Since $\frac{b'_i-1}{b'_i} \geq 0$, we get $0 \leq n \leq 2$.

We assume that $(c_i, d_i) = (1, 0)$ for some $1 \leq i \leq k$, i.e., C is an irreducible component of B. For simplify, we assume that $i = 1$. In the same way as above, we get that $n+2 \geq n\big(2 - \frac{b_1-1}{b_1}\big)$. Since $2 \leq b_1 \leq 6$, we obtain $0 \leq 12 \leq n$.

Notice that by simple calculations, there are not a $K3$ surface X and a finite subgroup G of $\mathrm{Aut}(X)$ such that $X/G \cong \mathbb{F}_l$ for $l = 10, 11$.

In Section 6, we will give the list of a numerical class of an effective divisor $B = \sum_{i=1}^{k} b_i B_i$ on $\mathbb{F}_n$ such that B_i is a smooth curve for each $i = 1, \cdots, k$ and $K_{\mathbb{F}_n} + \sum_{i=1}^{k} \frac{b_i-1}{b_i} B_i = 0$ in $\mathrm{Pic}(\mathbb{F}_n)$.

3.1 Abelian covers of a Hirzebruch surface by a *K*3 surface

Let X be a $K3$ surface, G be a finite Abelian subgroup of $\mathrm{Aut}(X)$ such that X/G is a Hirzebruch surface $\mathbb{F}_n$, and B be the branch divisor of the quotient map $p : X \to X/G$. In this section, we will decide the numerical class of B. Notice that since G is Abelian and the quotient space X/G is smooth, the support of B and that of p^*B are simple normal crossing.

Furthermore, we will show that the structure as a group of G depends only on the numerical class of B by Theorem 2.5, and we will give the way to construct X and G which depends only on the numerical class of B by Theorem 2.1 and the cyclic cover. As a result the following will follow. For each $G \in \mathcal{A}G_n$ where $n = 0, 1, 2, 3, 4, 6, 8, 12$, there is a pair (X, G') where X is a $K3$ surface and G' is a finite Abelian subgroup of $\mathrm{Aut}(X)$ such that $G \cong G'$ as a group and $X/G' \cong \mathbb{F}_n$. In [9], the case where $G \cong \mathbb{Z}/2\mathbb{Z}$ is studied.

Theorem 3.1 (see [3, Chapter I, Section 17]) *Let M be a smooth projective variety, and D be a smooth effective divisor on M. Then if the class $\mathcal{O}_M(D)/n \in$Pic(M), then there is the Galois cover $f : X \to M$ whose branch divisor is nD and the Galois group is isomorphic to $\mathbb{Z}/n\mathbb{Z}$ as a group.*

For $n \geq 0$, a Hirzebruch surface $\mathbb{F}_n$ is isomorphic to a variety $\mathcal{F}_n$ in $\mathbb{P}^1 \times \mathbb{P}^2$,

$$\mathcal{F}_n := \{([X_0 : X_1], [Y_0 : Y_1 : Y_2]) \in \mathbb{P}^1 \times \mathbb{P}^2 : \ X_0^n Y_0 = X_1^n Y_1\}.$$

From here, we assume that $\mathbb{F}_n = \mathcal{F}_n$. The first projection gives the fibre space structure $f : \mathbb{F}_n \to \mathbb{P}^1$ such that the numerical class of the fibre of f is F, and

$$C = \{([X_0 : X_1], [Y_0 : Y_1 : Y_2]) \in \mathbb{F}_n : \ Y_0 = Y_1 = 0\}$$

is the unique irreducible curve on $\mathbb{F}_n$ such that the self-intersection number is negative. Let a and b be positive integers such that $b \geq na$. Furthermore, we put

$$F(X_0, X_1, Y_0, Y_1, Y_2) := \sum_{0 \leq i \leq b-na, 0 \leq j,k \leq a, j+k \leq a} t_{i,j,k} X_0^i X_1^{b-na-i} Y_0^j Y_1^k Y_2^{a-j-k},$$

where $t_{i,j,k} \in \mathbb{C}$, and

$$B_F := \{([X_0 : X_1], [Y_0 : Y_1 : Y_2]) \in \mathbb{F}_n : \ F(X_0, X_1, Y_0, Y_1, Y_2) = 0\}.$$

If B_F is an irreducible curve of $\mathbb{F}_n$, then $B_F = aC + bF$ in $\mathrm{Pic}(\mathbb{F}_n)$.

Let g_1 and g_m be automorphisms of $\mathbb{P}^1$ which are induced by matrixes

$$g_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad g_m := \begin{pmatrix} 1 & 0 \\ 0 & \zeta_m \end{pmatrix},$$

where ζ_m is an m-th root of unity $m \geq 2$. Then $\langle g_1, g_2 \rangle \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 2}$, and $\langle g_m \rangle \cong \mathbb{Z}/m\mathbb{Z}$ for $m \geq 2$. Here for a subset S of group G, $\langle S \rangle$ is the subgroup of G which is generated by S. Then

$$\mathbb{P}^1 \cong \mathbb{P}^1/\langle g_1, g_2 \rangle \quad \text{and} \quad \mathbb{P}^1 \cong \mathbb{P}^1/\langle g_m \rangle,$$

and the quotient maps are isomorphic to

$$\mathbb{P}^1 \ni [z_0 : z_1] \mapsto [(z_0^2 + z_1^2)^2 : (z_0^2 - z_1^2)^2] \in \mathbb{P}^1 \quad \text{and} \quad \mathbb{P}^1 \ni [z_0 : z_1] \mapsto [z_0^m : z_1^m] \in \mathbb{P}^1$$

for $m \geq 2$, and the branch divisors are

$$2x_0 + 2x_1 + 2x_2 \quad \text{and} \quad mx_0 + mx_1,$$

where $x_0 := [1 : 0]$, $x_1 := [0 : 1]$ and $x_2 := [1 : 1]$.

The above Galois covers $\mathbb{P}^1 \to \mathbb{P}^1/\langle g_1, g_2 \rangle \cong \mathbb{P}^1$ and $\mathbb{P}^1 \to \mathbb{P}^1/\langle g_m \rangle \cong \mathbb{P}^1$ naturally induce the Galois covers of $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{F}_n$ whose Galois groups are induced by g_m for $m \geq 2$. We will explain in a bit more detail for $\mathbb{F}_n$. For $\mathbb{P}^1 \to \mathbb{P}^1/\langle g_1, g_2 \rangle$, let $\mathbb{P}^1 \times_{\mathbb{P}^1} \mathbb{F}_n$ be the fibre product of $\mathbb{P}^1 \to \mathbb{P}^1/\langle g_1, g_2 \rangle$ and $f : \mathbb{F}_n \to \mathbb{P}^1$. Let $p : \mathbb{P}^1 \times_{\mathbb{P}^1} \mathbb{F}_n \to \mathbb{F}_n$ be the natural projection of the fibre product. Then

$$\mathbb{P}^1 \times_{\mathbb{P}^1} \mathbb{F}_n \cong \mathbb{F}_{4n},$$

and $p : \mathbb{P}^1 \times_{\mathbb{P}^1} \mathbb{F}_n \to \mathbb{F}_n$ is the Galois cover such that the branch divisor of p is

$$2F + 2F + 2F \quad \text{in } \mathrm{Pic}(\mathbb{F}_n),$$

and the Galois group is isomorphic to $\mathbb{Z}/2\mathbb{Z}^{\oplus 2}$ as a group, which is induced by $\langle g_1, g_2 \rangle$. Let C_m be the irreducible curve on $\mathbb{F}_m$ such that the self-intersection number is negative and F_m is the numerical class of the fibre $\mathbb{F}_m \to \mathbb{P}^1$ for $m \geq 1$. Then

$$p^* C_n = C_{4n} \quad \text{and} \quad p^* F_n = 4F_{4n} \quad \text{in } \mathrm{Pic}(\mathbb{F}_{4n}).$$

For $\mathbb{P}^1 \to \mathbb{P}^1/\langle g_m \rangle$, let $\mathbb{P}^1 \times_{\mathbb{P}^1} \mathbb{F}_n$ be the fibre product of $\mathbb{P}^1 \to \mathbb{P}^1/\langle g_m \rangle$ and $f : \mathbb{F}_n \to \mathbb{P}^1$. Let $p : \mathbb{P}^1 \times_{\mathbb{P}^1} \mathbb{F}_n \to \mathbb{F}_n$ be the natural projection of the fibre product. Then

$$\mathbb{P}^1 \times_{\mathbb{P}^1} \mathbb{F}_n \cong \mathbb{F}_{mn},$$

$p : \mathbb{P}^1 \times_{\mathbb{P}^1} \mathbb{F}_n \to \mathbb{F}_n$ is the Galois cover such that the branch divisor of p is

$$mF + mF \quad \text{in } \mathrm{Pic}(\mathbb{F}_n),$$

and the Galois group is isomorphic to $\mathbb{Z}/m\mathbb{Z}$ as a group, which is induced by $\langle g_m \rangle$, and

$$p^* C_n = C_{mn} \quad \text{and} \quad p^* F_n = mF_{mn} \quad \text{in } \mathrm{Pic}(\mathbb{F}_{mn}).$$

Definition 3.3 *From here, we use the notation that* $B_{i,j}^k$ *(or simply* $B_{i,j}$*) is a smooth curve on* $\mathbb{F}_n$ *such that* $B_{i,j}^k = iC + jF$ *in* $\mathrm{Pic}(\mathbb{F}_n)$ *for* $n \geq 0$*, where* $k \in \mathbb{N}$.

Proposition 3.2 *For each numerical classes* (6.1)–(6.3) *of the list in Section* 6*, there is a $K3$ surface X and a finite Abelian subgroup G of* $\mathrm{Aut}(X)$ *such that $X/G \cong \mathbb{P}^1 \times \mathbb{P}^1$ and the numerical class of the branch divisor B of the quotient map $p : X \to X/G$ is* (6.1)–(6.3).

Furthermore, for a pair (X, G) of a $K3$ surface X and a finite Abelian subgroup G of $\mathrm{Aut}(X)$*, if $X/G \cong \mathbb{P}^1 \times \mathbb{P}^1$ and the numerical class of the branch divisor B of the quotient map $p : X \to X/G$ is* (6.1)–(6.3)*, then G is isomorphic to $\mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}^{\oplus 2}$, $\mathbb{Z}/3\mathbb{Z}^{\oplus 3}$, in order, as a group.*

Proof Let $B_{3,3}$ be a smooth curve on $\mathbb{P}^1 \times \mathbb{P}^1$. Then the numerical class of $3B_{3,3}$ is (6.1). By Theorem 3.1, there is the Galois cover $p : X \to \mathbb{P}^1 \times \mathbb{P}^1$ such that the branch divisor is $3B_{3,3}$ and the Galois group is $\mathbb{Z}/3\mathbb{Z}$ as a group. By Theorem 2.4, the canonical divisor of X is a numerically trivial. By [18], X is not a bi-ellitptic surface. By [8], X is not an Abelian surface. If X is an Enriques surface, then there is the Galois cover $q : X' \to \mathbb{P}^1 \times \mathbb{P}^1$ such that X' is a $K3$ surface, the Galois group is $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ as a group, and the branch divisor is $3B_{3,3}$. By Theorem 2.5, this is a contradiction. Therefore, X is a $K3$ surface.

In addition, let (X', G') be a pair of a $K3$ surface X' and a finite Abelian subgroup G' of $\mathrm{Aut}(X')$ such that $X'/G' \cong \mathbb{P}^1 \times \mathbb{P}^1$ and the numerical class of the branch divisor B' of the quotient map $p' : X' \to X'/G'$ is (6.1). By Theorem 2.5, $G' \cong \mathbb{Z}/3\mathbb{Z}$ as a group. Since the support of B' is smooth, there is a smooth curve $B'_{3,3}$ such that $B' = 3B'_{3,3}$. Then by the above discussion, there is the Galois cover $f : X \to \mathbb{P}^1 \times \mathbb{P}^1$ such that X is a $K3$ surface, the branch divisor is B', and the Galois group G is $\mathbb{Z}/3\mathbb{Z}$ as a group. Since a $K3$ surface is simply connected, by Theorem 2.1, the pair (X', G') is isomorphic to the pair (X, G).

Let $B^1_{1,0}$, $B^2_{1,0}$ and $B_{1,3}$ be smooth curves on $\mathbb{P}^1 \times \mathbb{P}^1$ such that $B^1_{1,0} + B^2_{1,0} + B_{1,3}$ is simple normal crossing. Then the numerical class of $3B^1_{1,0} + 3B^2_{1,0} + 3B_{1,3}$ is (6.2). Let $p : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ be the Galois cover such that the branch divisor is $3B^1_{1,0} + 3B^2_{1,0}$, and the Galois group is $\mathbb{Z}/3\mathbb{Z}$ as a group, which is induced by the Galois cover $\mathbb{P}^1 \ni [z_0 : z_1] \mapsto [z_0^3 : z_1^3] \in \mathbb{P}^1$. Since $B^1_{1,0} + B^2_{1,0} + B_{1,3}$ is simple normal crossing, $p^*B_{1,3}$ is a reduced divisor on $\mathbb{P}^1 \times \mathbb{P}^1$ such that whose support is a union of pairwise disjoint smooth curves, and $p^*B_{1,3} = (3, 3)$ in $\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)$. As for the case of (6.1), there is the Galois cover $q : X \to \mathbb{P}^1 \times \mathbb{P}^1$ such that X is a $K3$ surface, the Galois group is $\mathbb{Z}/3\mathbb{Z}$ as a group, and the branch divisor is $3p^*B_{1,3}$. Then the branched cover $p \circ q : X \to \mathbb{P}^1 \times \mathbb{P}^1$ has $3B^1_{1,0} + 3B^2_{1,0} + 3B_{1,3}$ as the branch divisor. Since X is simply connected, by Theorem 2.1, $p \circ q$ is the Galois cover. Since the degree of $p \circ q$ is 9, by Theorem 2.5, the Galois group of $p \circ q$ is $\mathbb{Z}/3\mathbb{Z}^{\oplus 2}$ as a group.

Conversely, for a $K3$ surface X and a finite Abelian subgroup G of $\mathrm{Aut}(X)$ such that $X/G \cong \mathbb{P}^1 \times \mathbb{P}^1$ and the numerical class of the branch divisor B of the quotient map $p : X \to X/G$ is (6.2). By the above discussion, G isomorphic to $\mathbb{Z}/3\mathbb{Z}^{\oplus 2}$ as a group, and $X \to X/G$ is given by the composition of the Galois cover $X \to \mathbb{P}^1 \times \mathbb{P}^1$ whose numerical class of the branch divisor is (6.1) and the Galois cover $p : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ which is induced by the Galois cover $\mathbb{P}^1 \ni [z_0 : z_1] \mapsto [z_0^3 : z_1^3] \in \mathbb{P}^1$.

As for the case of (6.2), we get the claim for (6.3). In this case, the Galois group is $\mathbb{Z}/3\mathbb{Z}^{\oplus 3}$ as a group. Furthermore, let X be a $K3$ surface and G be a finite Abelian subgroup of $\mathrm{Aut}(X)$ such that $X/G \cong \mathbb{P}^1 \times \mathbb{P}^1$ and the numerical class of the branch divisor B of G is (6.3). As for the case of (6.2), $X \to X/G$ is given by the composition of the Galois cover $X \to \mathbb{P}^1 \times \mathbb{P}^1$ whose numerical class of the branch divisor is (6.1) and the Galois cover $p : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ which is isomorphic to the Galois cover $p : \mathbb{P}^1 \times \mathbb{P}^1 \ni ([z_0 : z_1], [w_0 : w_1]) \mapsto ([z_0^3 : z_1^3], [w_0^3 : w_1^3]) \in \mathbb{P}^1 \times \mathbb{P}^1$.

For (6.1), we obtain an example if we use a curve $B_{3,3}$ in $\mathbb{P}^1 \times \mathbb{P}^1$ given by the equation

$$B_{3,3} : z_0^3 w_0^3 + z_0^3 w_1^3 + z_1^3 w_0^3 + 2 z_1^3 w_1^3 = 0.$$

For (6.2), we obtain an example if we use curves $B_{1,0}^1, B_{1,0}^2, B_{1,3}$ in $\mathbb{P}^1 \times \mathbb{P}^1$ given by the equations

$$B_{1,0}^1 : z_0 = 0, \quad B_{1,0}^2 : z_1 = 0, \quad B_{1,3} : z_0 w_0^3 + z_0 w_1^3 + z_1 w_0^3 + 2 z_1 w_1^3 = 0.$$

For (6.3), we obtain an example if we use curves $B_{1,0}^1, B_{1,0}^2, B_{1,1}, B_{0,1}^1, B_{0,1}^2$ in $\mathbb{P}^1 \times \mathbb{P}^1$ given by the equations

$$B_{1,0}^1 : z_0 = 0, \quad B_{1,0}^2 : z_1 = 0, \quad B_{1,1} : z_0 w_0 + z_0 w_1 + z_1 w_0 + 2 z_1 w_1 = 0,$$

$$B_{0,1}^1 : w_0 = 0, \quad B_{0,1}^2 : w_1 = 0.$$

Corollary 3.1 *For each numerical classes* (6.194), (6.83) *and* (6.302), (6.251), (6.201), (6.84) *of the list in Section* 6, *there is a $K3$ surface X and a finite Abelian subgroup G of* $\mathrm{Aut}(X)$ *such that $X/G \cong \mathbb{F}_n$ and the numerical class of the branch divisor B of the quotient map $p : X \to X/G$ is* (6.194), (6.83) *and* (6.302), (6.251), (6.201), (6.84).

Furthermore, for a pair (X, G) of a $K3$ surface X and a finite Abelian subgroup G of $\mathrm{Aut}(X)$, *if $X/G \cong \mathbb{F}_n$ and the numerical class of the branch divisor B of the quotient map $p : X \to X/G$ is* (6.194), (6.83) *and* (6.302), (6.251), (6.201), (6.84), *then G is* $\mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}^{\oplus 2}$, $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}^{\oplus 2}$, *in order, as a group.*

Proof In the same way as Proposition 3.2, we get this corollary. More specifically, let X be a $K3$ surface, G be a finite Abelian subgroup of $\mathrm{Aut}(X)$ such that $X/G \cong \mathbb{F}_n$, and B be the branch divisor of the quotient map $p : X \to X/G$. Then we get the following.

i) If the numerical class of B is one of (6.194), (6.302), then $X \to X/G$ is given by Theorem 3.1.

ii) If the numerical class of B is (6.83), then $X \to X/G$ is given by the composition of the Galois cover $X' \to \mathbb{F}_2$ whose numerical class of the branch divisor is (6.194) and the Galois cover $\mathbb{F}_2 \to \mathbb{F}_1$ which is induced by the Galois cover $\mathbb{P}^1 \to \mathbb{P}^1$ of degree 2.

iii) If the numerical class of B is one of (6.251), (6.201), (6.84), then $X \to X/G$ is given by the composition of the Galois cover $X' \to \mathbb{F}_6$ whose numerical class of the branch divisor is (6.302) and the Galois cover $\mathbb{F}_6 \to \mathbb{F}_m$ which is induced by the Galois cover $\mathbb{P}^1 \to \mathbb{P}^1$ of degree $\frac{6}{m}$.

For (6.194), we obtain an example if we use a curve $B_{3,6}$ in $\mathbb{F}_2$ given by the equation

$$B_{3,6} : Y_0^3 + Y_1^3 + Y_2^3 = 0.$$

For (6.83), we obtain an example if we use curves $B_{3,3}, B_{0,1}^1, B_{0,1}^2$ in $\mathbb{F}_1$ given by the equations

$$B_{3,3} : Y_0^3 + Y_1^3 + Y_2^3 = 0, \quad B_{0,1}^1 : X_0 = 0, \quad B_{0,1}^2 : X_1 = 0.$$

For (6.302), we obtain an example if we use a section C and a curve $B_{2,12}$ in $\mathbb{F}_6$ given by the equation

$$B_{2,12} : Y_0^2 + Y_1^2 + Y_2^2 = 0.$$

For (6.251), we obtain an example if we use a section C and curves $B_{2,6}, B^1_{0,1}, B^2_{0,1}$ in $\mathbb{F}_3$ given by the equations

$$B_{2,6} : Y_0^2 + Y_1^2 + Y_2^2 = 0, \quad cB^1_{0,1} : X_0 = 0, \quad B^2_{0,1} : X_1 = 0.$$

For (6.201), we obtain an example if we use a section C and curves $B_{2,4}, B^1_{0,1}, B^2_{0,1}$ in $\mathbb{F}_2$ given by the equations

$$B_{2,4} : Y_0^2 + Y_1^2 + Y_2^2 = 0, \quad B^1_{0,1} : X_0 = 0, \quad B^2_{0,1} : X_1 = 0.$$

For (6.84), we obtain an example if we use a section C and curves $B_{2,2}, B^1_{0,1}, B^2_{0,1}$ in $\mathbb{F}_1$ given by the equations

$$B_{2,2} : Y_0^2 + Y_1^2 + Y_2^2 = 0, \quad B^1_{0,1} : X_0 = 0, \quad B^2_{0,1} : X_1 = 0.$$

Proposition 3.3 *For each numerical classes* (6.4)–(6.13) *of the list in Section* 6*, there are a* $K3$ *surface* X *and a finite Abelian subgroup* G *of* $\mathrm{Aut}(X)$ *such that* $X/G \cong \mathbb{F}_n$ *and the numerical class of the branch divisor* B *of the quotient map* $p : X \to X/G$ *is* (6.4)–(6.13)*.*

Furthermore, for a pair (X, G) *of a* $K3$ *surface* X *and a finite Abelian subgroup* G *of* $\mathrm{Aut}(X)$*, if* $X/G \cong \mathbb{F}_n$ *and the numerical class of the branch divisor* B *of the quotient map* $p : X \to X/G$ *is* (6.4)–(6.13)*, then* G *is* $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}^{\oplus 2}$, $\mathbb{Z}/2\mathbb{Z}^{\oplus 3}$, $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}^{\oplus 2}$, $\mathbb{Z}/2\mathbb{Z}^{\oplus 2} \oplus \mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}^{\oplus 3}$, $\mathbb{Z}/2\mathbb{Z}^{\oplus 5}$, $\mathbb{Z}/2\mathbb{Z}^{\oplus 4}$, $\mathbb{Z}/2\mathbb{Z}^3 \oplus \mathbb{Z}/4\mathbb{Z}$*, in order, as a group.*

Proof In the same way as Proposition 3.2, we get this proposition. More specifically, let X be a $K3$ surface, G be a finite Abelian subgroup of $\mathrm{Aut}(X)$ such that $X/G \cong \mathbb{F}_n$, and B be the branch divisor of the quotient map $p : X \to X/G$. Then we get the following.

(i) If the numerical class of B is (6.4), then $X \to X/G$ is given by Theorem 3.1.

(ii) If the numerical class of B is one of (6.5)–(6.13), then $X \to X/G$ is given by the composition of the Galois cover $X \to \mathbb{P}^1 \times \mathbb{P}^1$ whose numerical class of the branch divisor is (6.4) and the Galois cover $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ which is induced by the Galois cover $\mathbb{P}^1 \to \mathbb{P}^1$.

For (6.4), we obtain an example if we use a curve $B_{4,4}$ in $\mathbb{P}^1 \times \mathbb{P}^1$ given by the equation

$$B_{4,4} : (z_0^4 + z_1^4)(w_0^4 + w_1^4) + 2z_0^2 z_1^2 w_0^2 w_1^2 = 0.$$

For (6.5), we obtain an example if we use curves $B^1_{1,0}, B^2_{1,0}, B_{2,4}$ in $\mathbb{P}^1 \times \mathbb{P}^1$ given by the equations

$$B^1_{1,0} : z_0 = 0, \quad B^2_{1,0} : z_1 = 0, \quad B_{2,4} : (z_0^2 + z_1^2)(w_0^4 + w_1^4) + 2z_0 z_1 w_0^2 w_1^2 = 0.$$

For (6.6), we obtain an example if we use curves $B^1_{1,0}, B^2_{1,0}, B_{2,2}, B^1_{0,1}, B^2_{0,1}$ in $\mathbb{P}^1 \times \mathbb{P}^1$ given by the equations

$$B^1_{1,0} : z_0 = 0, \quad B^2_{1,0} : z_1 = 0, \quad B_{2,2} : (z_0^2 + z_1^2)(w_0^2 + w_1^2) + 2z_0 z_1 w_0 w_1 = 0,$$

$$B^1_{0,1} : w_0 = 0, \quad B^2_{0,1} : w_1 = 0.$$

For (6.7), we obtain an example if we use curves $B^1_{1,0}, B^2_{1,0}, B_{2,4}$ in $\mathbb{P}^1 \times \mathbb{P}^1$ given by the equations

$$B^1_{1,0} : z_0 = 0, \quad B^2_{1,0} : z_1 = 0, \quad B_{2,4} : (z_0 + z_1)(w_0^4 + w_1^4) + (z_0 - z_1) w_0^2 w_1^2 = 0.$$

For (6.8), we obtain an example if we use curves $B^1_{1,0}, B^2_{1,0}, B_{1,1}, B^1_{0,1}, B^2_{0,1}$ in $\mathbb{P}^1 \times \mathbb{P}^1$ given by the equations

$$B^1_{1,0} : z_0 = 0, \quad B^2_{1,0} : z_1 = 0, \quad B_{1,1} : (z_0 + z_1)(w_0 + w_1) + 2(z_0 - z_1)(w_0 - w_1) = 0,$$

$$B^1_{0,1} : w_0 = 0, \quad B^2_{0,1} : w_1 = 0.$$

For (6.9), we obtain an example if we use curves $B^1_{1,0}, B^2_{1,0}, B_{1,2}, B^1_{0,1}, B^2_{0,1}$ in $\mathbb{P}^1 \times \mathbb{P}^1$ given by the equations

$$B^1_{1,0} : z_0 = 0, \quad B^2_{1,0} : z_1 = 0, \quad B_{1,2}(z_0 + z_1)(w_0^2 + w_1^2) + (z_0 - z_1)w_0 w_1,$$

$$B^1_{0,1} : w_0 = 0, \quad B^2_{0,1} : w_1 = 0.$$

For (6.10), we obtain an example if we use curves $B^1_{1,0}, B^2_{1,0}, B^3_{1,0}, B_{1,4}$ in $\mathbb{P}^1 \times \mathbb{P}^1$ given by the equations

$$B^1_{1,0} : z_0 = 0, \quad B^2_{1,0} : z_1 = 0, \quad B^3_{1,0} : z_0 - z_1 = 0,$$

$$B_{1,4} : (z_0 + z_1)(w_0^4 + w_1^4) + 2(z_0 - z_1)(w_0^4 - w_1^4) = 0.$$

For (6.11), we obtain an example if we use curves $B^1_{1,0}, B^2_{1,0}, B^3_{1,0}, B_{1,1}, B^1_{0,1}, B^2_{0,1}, B^3_{0,1}$ in $\mathbb{P}^1 \times \mathbb{P}^1$ given by the equations

$$B^1_{1,0} : z_0 = 0, \quad B^2_{1,0} : z_1 = 0, \quad B^3_{1,0} : z_0 - z_1 = 0,$$

$$B_{1,1} : (z_0 - 2z_1)w_0 + (2z_0 + z_1)w_1 = 0,$$

$$B^1_{0,1} : w_0 = 0, \quad B^2_{0,1} : w_1 = 0, \quad B^3_{0,1} : w_0 - w_1 = 0,$$

For (6.12), we obtain an example if we use curves $B^1_{1,0}, B^2_{1,0}, B^3_{1,0}, B_{1,2}, B^1_{0,1}, B^2_{0,1}$ in $\mathbb{P}^1 \times \mathbb{P}^1$ given by the equations

$$B^1_{1,0} : z_0 = 0, \quad B^2_{1,0} : z_1 = 0, \quad B^3_{1,0} : z_0 - z_1 = 0,$$

$$B_{1,2} : (z_0 - 2z_1)w_0^2 + (2z_0 + z_1)w_1^2 = 0, \quad B^1_{0,1} : w_0 = 0, \quad B^2_{0,1} : w_1 = 0.$$

For (6.13), we obtain an example if we use curves $B^1_{1,0}, B^2_{1,0}, B^3_{1,0}, B_{1,1}, B^1_{0,1}, B^2_{0,1}$ in $\mathbb{P}^1 \times \mathbb{P}^1$ given by the equations

$$B^1_{1,0} : z_0 = 0, \quad B^2_{1,0} : z_1 = 0, \quad B^3_{1,0} : z_0 - z_1 = 0,$$

$$B_{1,2} : (z_0 - 2z_1)w_0 + (2z_0 + z_1)w_1 = 0, \quad B^1_{0,1} : w_0 = 0, \quad B^2_{0,1} : w_1 = 0.$$

Corollary 3.2 *For each numerical classes* (6.79) *and* (6.195), (6.85) *and* (6.277), (6.202), (6.86), (6.87) *of the list in Section* 6, *there is a* $K3$ *surface* X *and a finite Abelian subgroup* G *of* $\mathrm{Aut}(X)$ *such that* $X/G \cong \mathbb{F}_n$ *and the numerical class of the branch divisor* B *of the quotient map* $p : X \to X/G$ *is* (6.79) *and* (6.195), (6.85) *and* (6.277), (6.202), (6.86), (6.87).

Furthermore, for a pair (X, G) *of a* $K3$ *surface* X *and a finite Abelian subgroup* G *of* $\mathrm{Aut}(X)$, *if* $X/G \cong \mathbb{F}_n$ *and the numerical class of the branch divisor* B *of the quotient map* $p : X \to X/G$ *is* (6.79) *and* (6.195), (6.85) *and* (6.277), (6.202), (6.86), (6.87), *then* G *is* $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}^{\oplus 2}$, $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}^{\oplus 2}$, $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}^{\oplus 3}$, *in order, as a group.*

Proof In the same way as Proposition 3.2, we get this corollary. More specifically, let X be a $K3$ surface, G be a finite Abelian subgroup of $\mathrm{Aut}(X)$ such that $X/G \cong \mathbb{F}_n$, and B be the branch divisor of the quotient map $p : X \to X/G$. Then we get the following.

i) If the numerical class of B is one of (6.79), (6.195), (6.277), then $X \to X/G$ is given by Theorem 3.1.

ii) If the numerical class of B is one of (6.85), then $X \to X/G$ is given by the composition of the Galois cover $X \to \mathbb{F}_2$ whose numerical class of the branch divisor is (6.195) and the Galois cover $\mathbb{F}_2 \to \mathbb{F}_1$ which is induced by the Galois cover $\mathbb{P}^1 \to \mathbb{P}^1$ of degree 2.

iii) If the numerical class of B is one of (6.202) , (6.86), (6.87), then $X \to X/G$ is given by the composition of the Galois cover $X \to \mathbb{F}_4$ whose numerical class of the branch divisor is (6.277) and the Galois cover $\mathbb{F}_4 \to \mathbb{F}_m$ which is induced by the Galois cover $\mathbb{P}^1 \to \mathbb{P}^1$ of degree $\frac{4}{m}$.

For (6.79), we obtain an example if we use a curve $B_{4,6}$ in $\mathbb{F}_1$ given by the equation

$$B_{4,6} : X_0^2 Y_1^4 + X_1^2 Y_0^4 + X_0 X_1 Y_2^4 = 0.$$

For (6.195), we obtain an example if we use a curve $B_{4,8}$ in $\mathbb{F}_2$ given by the equation

$$B_{4,8} : Y_0^4 + Y_1^4 + Y_2^4 = 0.$$

For (6.85), we obtain an example if we use curves $B_{4,4}, B_{0,1}^1, B_{0,1}^2$ in $\mathbb{F}_1$ given by the equations

$$B_{4,4} : Y_0^4 + Y_1^4 + Y_2^4 = 0, \quad B_{0,1}^1 : X_0 = 0, \quad B_{0,1}^2 : X_1 = 0.$$

For (6.277), we obtain an example if we use a section C and a curve $B_{3,12}$ in $\mathbb{F}_4$ given by the equation

$$B_{3,12} : Y_0^3 + Y_1^3 + Y_2^3 = 0.$$

For (6.202), we obtain an example if we use a section C and curves $B_{3,6}, B_{0,1}^1, B_{0,1}^2$ in $\mathbb{F}_2$ given by the equations

$$B_{3,6} : Y_0^3 + Y_1^3 + Y_2^3 = 0, \quad B_{0,1}^1 : X_0 = 0, \quad B_{0,1}^2 : X_1 = 0.$$

For (6.86), we obtain an example if we use a section C and curves $B_{3,3}, B_{0,1}^1, B_{0,1}^2$ in $\mathbb{F}_1$ given by the equations

$$B_{3,3} : Y_0^3 + Y_1^3 + Y_2^3 = 0, \quad B_{0,1}^1 : X_0 = 0, \quad B_{0,1}^2 : X_1 = 0.$$

For (6.87), we obtain an example if we use a section C and curves $B_{3,3}, B_{0,1}^1, B_{0,1}^2, B_{0,1}^3$ in $\mathbb{F}_1$ given by the equations

$$B_{3,3} : Y_0^3 + Y_1^3 + Y_2^3 = 0, \quad B_{0,1}^1 : X_0 = 0, \quad B_{0,1}^2 : X_1 = 0, \quad B_{0,1}^3 : X_0 - X_1 = 0.$$

Proposition 3.4 *For each numerical classes* (6.14)–(6.16) *of the list in Section* 6*, there are a $K3$ surface X and a finite Abelian subgroup G of* $\mathrm{Aut}(X)$ *such that $X/G \cong \mathbb{F}_n$ and the numerical class of the branch divisor B of the quotient map $p : X \to X/G$ is* (6.14)–(6.16)*.*

Furthermore, for a pair (X, G) of a $K3$ surface X and a finite Abelian subgroup G of $\mathrm{Aut}(X)$*, if $X/G \cong \mathbb{F}_n$ and the numerical class of the branch divisor B of the quotient map $p : X \to X/G$ is* (6.14)–(6.16)*, then G is $\mathbb{Z}/2\mathbb{Z}^{\oplus 2}$, $\mathbb{Z}/2\mathbb{Z}^{\oplus 3}$, $\mathbb{Z}/2\mathbb{Z}^{\oplus 4}$, in order, as a group.*

Proof Let $B^1_{2,2}, B^2_{2,2}$ be smooth curves on $\mathbb{P}^1 \times \mathbb{P}^1$ such that $B^1_{2,2} + B^2_{2,2}$ is simple normal crossing. Then the numerical class of $2B^1_{2,2} + 2B^2_{2,2}$ is (6.14). Since $B^i_{2,2} = (2C + 2F)$ in $\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)$, by Theorem 3.1, there are the Galois covers $p_i : X_i \to \mathbb{P}^1 \times \mathbb{P}^1$ such that the branch divisor of p_i is $2B^i_{2,2}$ for $i = 1, 2$ and the Galois group of p_i is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ as a group for $i = 1, 2$. Since $B^1_{2,2} + B^2_{2,2}$ is simple normal crossing, the fibre product $X := X_1 \times_{\mathbb{P}^1 \times \mathbb{P}^1} X_2$ of p_1 and p_2 is smooth. Therefore, there is the Galois cover $p : X \to \mathbb{P}^1 \times \mathbb{P}^1$ such that X is a $K3$ surface, the Galois group is isomorphic to $\mathbb{Z}/2\mathbb{Z}^{\oplus 2}$ as a group, and the branch divisor is $2B^1_{2,2} + 2B^2_{2,2}$. The rest of this proposition is proved in the same way as Proposition 3.2. More specifically, let X be a $K3$ surface, G be a finite Abelian subgroup of $\mathrm{Aut}(X)$ such that $X/G \cong \mathbb{F}_n$, and B be the branch divisor of G. Then we get the following.

(i) If the numerical class of B is (6.14), then $X \to X/G$ is given by Theorem 3.1 and the fibre product.

(ii) If the numerical class of B is one of (6.15)–(6.16), then $X \to X/G$ is given by the composition of the Galois cover $X \to \mathbb{P}^1 \times \mathbb{P}^1$ whose numerical class of the branch divisor is (6.14) and the Galois cover $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ which is induced by the Galois cover $\mathbb{P}^1 \to \mathbb{P}^1$.

For (6.14), we obtain an example if we use curves $B^1_{2,2}, B^2_{2,2}$ in $\mathbb{P}^1 \times \mathbb{P}^1$ given by the equations

$$B^1_{2,2} : z_0^2 w_0^2 + z_1^2 w_1^2 = 0, \quad B^2_{2,2} : z_0^2 w_1^2 + z_1^2 w_0^2 = 0.$$

For (6.15), we obtain an example if we use curves $B^1_{1,0}, B^2_{1,0}, B^1_{1,2}, B^2_{1,2}$ in $\mathbb{P}^1 \times \mathbb{P}^1$ given by the equations

$$B^1_{1,0} : z_0 = 0, \quad B^2_{1,0} : z_1 = 0, \quad B^1_{1,2} : z_0 w_0^2 + z_1 w_1^2 = 0, \quad B^2_{1,2} z_0 w_1^2 + z_1 w_0^2 = 0.$$

For (6.16), we obtain an example if we use curves $B^1_{1,0}, B^2_{1,0}, B^1_{1,1}, B^2_{1,1}, B^1_{0,1}, B^2_{0,1}$ in $\mathbb{P}^1 \times \mathbb{P}^1$ given by the equations

$$B^1_{1,0} : z_0 = 0, \quad B^2_{1,0} z_1 = 0, \quad B^1_{1,1} : (z_0 - 2z_1)w_0 + (2z_0 + z_1)w_1 = 0,$$

$$B^2_{1,1} : z_0(w_0 - 2w_1) + z_1(2w_0 + w_1) = 0, \quad B^1_{0,1} : w_0 = 0, \quad B^2_{0,1} : w_1 = 0.$$

Corollary 3.3 *For each numerical classes* (6.80) *and* (6.196)*,* (6.89) *and* (6.197)*,* (6.88) *and* (6.279)*,* (6.203)*,* (6.90)*,* (6.91) *of the list in Section* 6*, there is are a $K3$ surface X and a finite Abelian subgroup G of* $\mathrm{Aut}(X)$ *such that $X/G \cong \mathbb{F}_n$ and the numerical class of the branch divisor B of the quotient map $p : X \to X/G$ is* (6.80) *and* (6.196)*,* (6.89) *and* (6.197)*,* (6.88) *and* (6.279)*,* (6.203)*,* (6.90)*,* (6.91)*.*

Furthermore, for a pair (X, G) of a $K3$ surface X and a finite Abelian subgroup G of $\mathrm{Aut}(X)$*, if $X/G \cong \mathbb{F}_n$ and the numerical class of the branch divisor B of the quotient map $p : X \to X/G$ is* (6.80) *and* (6.196)*,* (6.89) *and* (6.197)*,* (6.88) *and* (6.279)*,* (6.203)*,* (6.90)*,* (6.91)*, then G is* $\mathbb{Z}/2\mathbb{Z}^{\oplus 2}$*,* $\mathbb{Z}/2\mathbb{Z}^{\oplus 2}$*,* $\mathbb{Z}/2\mathbb{Z}^{\oplus 3}$*,* $\mathbb{Z}/2\mathbb{Z}^{\oplus 2}$*,* $\mathbb{Z}/2\mathbb{Z}^{\oplus 3}$*,* $\mathbb{Z}/2\mathbb{Z}^{\oplus 2}$*,* $\mathbb{Z}/2\mathbb{Z}^{\oplus 3}$*,* $\mathbb{Z}/2\mathbb{Z}^{\oplus 2} \oplus \mathbb{Z}/4\mathbb{Z}$*,* $\mathbb{Z}/2\mathbb{Z}^{\oplus 4}$*, in order, as a group.*

Proof In the same way as Proposition 3.2, we get this corollary. More specifically, let X be a $K3$ surface, G be a finite Abelian subgroup of $\mathrm{Aut}(X)$ such that $X/G \cong \mathbb{F}_n$, and B be the branch divisor of the quotient map $p : X \to X/G$. Then we get the following.

(i) If the numerical class of B is one of (6.80), (6.196), (6.197), (6.279), then $X \to X/G$ is given by Theorem 3.1 and the fibre product.

(ii) If the numerical class of B is (6.89), then $X \to X/G$ is given by the composition of the Galois cover $X \to \mathbb{F}_2$ whose numerical class of the branch divisor is (6.196) and the Galois cover $\mathbb{F}_2 \to \mathbb{F}_1$ which is induced by the Galois cover $\mathbb{P}^1 \to \mathbb{P}^1$ of degree 2.

(iii) If the numerical class of B is (6.88), then $X \to X/G$ is given by the composition of the Galois cover $X \to \mathbb{F}_2$ whose numerical class of the branch divisor is (6.197) and the Galois cover $\mathbb{F}_2 \to \mathbb{F}_1$ which is induced by the Galois cover $\mathbb{P}^1 \to \mathbb{P}^1$ of degree 2.

(iv) If the numerical class of B is one of (6.203), (6.90), (6.91), then $X \to X/G$ is given by the composition of the Galois cover $X \to \mathbb{F}_4$ whose numerical class of the branch divisor is (6.279) and the Galois cover $\mathbb{F}_4 \to \mathbb{F}_1$ which is induced by the Galois cover $\mathbb{P}^1 \to \mathbb{P}^1$ of degree 4.

For (6.80), we obtain an example if we use curves $B_{2,4}, B_{2,2}$ in $\mathbb{F}_1$ given by the equations

$$B_{2,4} : X_0^2Y_1^2 + X_1^2Y_0^2 + X_0X_1Y_2^2 = 0, \quad B_{2,2} : Y_0^2 + Y_1^2 + Y_2^2 = 0.$$

For (6.196), we obtain an example if we use curves $B^1_{2,4}, B^2_{2,4}$ in $\mathbb{F}_2$ given by the equations

$$B^1_{2,4} : 2Y_0^2 + Y_1^2 + Y_2^2 = 0, \quad B^2_{2,4} : Y_0^2 + Y_1^2 + 2Y_2^2 = 0.$$

For (6.89), we obtain an example if we use curves $B^1_{2,2}, B^2_{2,2}, B^1_{0,1}, B^2_{0,1}$ in $\mathbb{F}_1$ given by the equations

$$B^1_{2,2} : 2Y_0^2 + Y_1^2 + Y_2^2 = 0, \quad B^2_{2,2} : Y_0^2 + Y_1^2 + 2Y_2^2 = 0,$$

$$B^1_{0,1} : X_0 = 0, \quad B^2_{0,1} : X_1 = 0.$$

For (6.197), we obtain an example if we use a section C and curves $B_{1,2}, B_{2,6}$ in $\mathbb{F}_2$ given by the equations

$$B_{1,2} : Y_0 + Y_2 = 0, \quad B_{2,6} : X_0^2Y_1^2 + X_1^2Y_0^2 + (X_0^2 + 2X_1^2)Y_2^2 = 0.$$

For (6.88), we obtain an example if we use a section C and curves $B_{1,1}, B_{2,3}, B^1_{0,1}, B^2_{0,1}$ in $\mathbb{F}_1$ given by the equations

$$B_{1,1} : Y_0 + Y_2 = 0, \quad B_{2,3} : X_0Y_1^2 + X_1Y_0^2 + (X_0 + 2X_1)Y_2^2 = 0,$$

$$B^1_{0,1} : X_0 = 0, \quad B^2_{0,1} : X_1 = 0.$$

For (6.279), we obtain an example if we use a section C and curves $B_{1,4}, B_{2,8}$ in $\mathbb{F}_4$ given by the equations

$$B_{1,4} : Y_0 + Y_2 = 0, \quad B_{2,8} : Y_0^2 + Y_1^2 + Y_2^2 = 0.$$

For (6.203), we obtain an example if we use a section C and curves $B_{1,2}, B_{2,4}, B^1_{0,1}, B^2_{0,1}$ in $\mathbb{F}_2$ given by the equations

$$B_{1,2} : Y_0 + Y_2 = 0, \quad B_{2,4} : Y_0^2 + Y_1^2 + Y_2^2 = 0,$$

$$B^1_{0,1} : X_0 = 0, \quad B^2_{0,1} : X_1 = 0.$$

For (6.90), we obtain an example if we use a section C and curves $B_{1,1}, B_{2,2}, B^1_{0,1}, B^2_{0,1}$ in $\mathbb{F}_1$ given by the equations

$$B_{1,1} : Y_0 + Y_2 = 0, \quad B_{2,2} : Y_0^2 + Y_1^2 + Y_2^2 = 0,$$

$$B^1_{0,1} : X_0 = 0, \quad B^2_{0,1} : X_1 = 0.$$

For (6.91), we obtain an example if we use a section C and curves $B_{1,1}, B_{2,2}, B^1_{0,1}, B^2_{0,1}, B^3_{0,1}$ in $\mathbb{F}_1$ given by the equations

$$B_{1,1} : Y_0 + Y_2 = 0, \quad B_{2,2} : Y_0^2 + Y_1^2 + Y_2^2 = 0,$$

$$B^1_{0,1} : X_0 = 0, \quad B^2_{0,1} : X_1 = 0, \quad B^3 : X_0 - X_1 = 0.$$

A lattice is a pair (L, b) of a free Abelian group $L := Z^{\oplus n}$ of rank n and a symmetric non-degenerate bilinear form $b : L \times L \to \mathbb{Z}$ taking values in $\mathbb{Z}$. The discriminant group of L is $L^\vee/L$, where the dual $L^\vee := \{m \in L \otimes \mathbb{Q} \mid b(m, l) \in \mathbb{Z} \text{ for all } l \in L\}$ (here we denote by b the $\mathbb{Q}$ linear extension of b). Let U be the hyperbolic lattice, and A_n and let E_n be the negative definite lattices of rank n associated to the corresponding root systems.

Proposition 3.5 *For each classes* (6.17)–(6.18) *of the list in Section* 6*, there is a* $K3$ *surface* X *and a finite Abelian subgroup* G *of* $\mathrm{Aut}(X)$ *such that* $X/G \cong \mathbb{F}_n$ *and the numerical class of the branch divisor* B *of the quotient map* $p : X \to X/G$ *is* (6.17)–(6.18)*.*

Furthermore, for a pair (X, G) *of a* $K3$ *surface* X *and a finite Abelian subgroup* G *of* $\mathrm{Aut}(X)$*, if* $X/G \cong \mathbb{F}_n$ *and the numerical class of the branch divisor* B *of the quotient map* $p : X \to X/G$ *is* (6.17)–(6.18)*, then* G *is* $\mathbb{Z}/3\mathbb{Z}^{\oplus 2}$*,* $\mathbb{Z}/3\mathbb{Z}^{\oplus 2}$*, in order, as a group.*

Proof Let $B^1_{1,1}$, $B^2_{1,1}$ and $B^3_{1,1}$ be smooth curves such that $B^1_{1,1} + B^2_{1,1} + B^3_{1,1}$ is simple normal crossing. Since $B^1_{1,1} + B^2_{1,1} + B^3_{1,1} = (3C + 3F)$ in $\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)$, by Theorem 3.1, there is the Galois cover $p' : X' \to \mathbb{P}^1 \times \mathbb{P}^1$ such that the branch divisor is $3B^1_{1,1} + 3B^2_{1,1} + 3B^3_{1,1}$ and the Galois group is isomorphic to $\mathbb{Z}/3\mathbb{Z}$ as a group. Since $B^1_{1,1} + B^2_{1,1} + B^3_{1,1}$ is simple normal crossing, singular points of X' are rational double points. More precisely, the singular locus of X' consists of six A_2 points. Let $p_m : X'_m \to X'$ be the minimal resolution of X'. Then the canonical divisor of X'_m is numerical trivial. Since X'_m has a curve such that the self-intersection number is negative, X'_m is a $K3$ surface or Enriques surface. Since X'_m has an automorphism s of order 3 such that the curves of $\mathrm{Fix}(s)$ are three rational curves C_i for $i = 1, 2, 3$, by [11], X'_m is a $K3$ surface. By [1, Theorem 2.8 and Proposition 3.2] or [14, Table 2], we get that

$$\mathrm{Pic}(X'_m)^{s^*} := \{\alpha \in \mathrm{Pic}(X'_m) : \ s^*\alpha = \alpha\} \cong U \oplus E_6 \oplus A_2^3.$$

Let $z_1, \cdots, z_6$ be singular points of X', and $e_1, \cdots, e_{12}$ be the exceptional divisors of p_m, where $z_i = p_m(e_{2i-1}) = p_m(e_{2i})$ for $i = 1, \cdots, 6$. Notice that $(e_{2i-1} \cdot e_{2i}) = 1$, $(e_{2i-1} \cdot e_{2i-1}) = -2$ and $(e_{2i} \cdot e_{2i}) = -2$. Since $C_i \subset \mathrm{Fix}(s)$ for $i = 1, 2, 3$, we get that $(e_{2i-1} \cup e_{2i}) \cap \mathrm{Fix}(s)$ contains at least 2 points. Since $s(e_{2i-1} \cup e_{2i}) = (e_{2i-1} \cup e_{2i})$ and $e_{2i-1} \cap e_{2i}$ is one point, we get that $e_{2i-1} \cap e_{2i} \subset \mathrm{Fix}(s)$. Therefore, $s(e_{2i-1}) = e_{2i-1}$ and $s(e_{2i}) = e_{2i}$, and hence $e_{2i-1}, e_{2i} \in \mathrm{Pic}(X'_m)^{s^*}$ for $i = 1, \cdots, 6$. Since $\mathrm{Pic}(X'_m)^{s^*}$ is a primitive sublattice, the minimal primitive sublattice which contains $(p' \circ p_m)^*\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)$ and $e_1, \cdots, e_{12}$ of $\mathrm{Pic}(X'_m)$ is $\mathrm{Pic}(X'_m)^{s^*}$.

Let $f := p' \circ p_m : X'_m \to \mathbb{P}^1 \times \mathbb{P}^1$. Since $f_*C_i = B^i_{1,1}$, we get $(C_i \cdot f^*F) = ((C + F) \cdot F) = 1$ for $i = 1, 2, 3$. Let

$$C'_1 := C_1 + \sum_{i=1}^{6} \frac{(C_1 \cdot e_{2i-1})}{2} e_{2i-1} + \sum_{i=1}^{6} \frac{(C_1 \cdot (e_{2i-1} + 2e_{2i}))}{6} (e_{2i-1} + 2e_{2i}).$$

Then $(C_1' \cdot e_i) = 0$ for $i = 1, \cdots, 12$. Since $(e_{2i-1} \cdot e_{2i-1}) = -2$, $(e_{2i-1} \cdot e_{2i-1} + 2e_{2i}) = 0$ and $(e_{2i-1} + 2e_{2i} \cdot e_{2i-1} + 2e_{2i}) = -6$, we get $6C_1' \in \mathrm{Pic}(X_m')$. Therefore, the minimal primitive sublattice K of $\mathrm{Pic}(X_m')^{s^*}$, which contains f^*C and $6C_1'$ is a unimodular lattice. Let M be the minimal primitive sublattice of $\mathrm{Pic}(X_m')$, which contains the curves $e_1, \cdots, e_{12}$. Then $M \subset U^{\perp}$. Since U is a unimodular lattice and M and U are sublattice of $\mathrm{Pic}(X_m')^{s^*}$, we get $U \oplus M = \mathrm{Pic}(X_m')^{s^*}$. Therefore, the rank of M is 12 and $M^{\vee}/M \cong \mathbb{Z}/3\mathbb{Z}^{\oplus 4}$. Thus, by [6,Theorem 5.2] there is a $K3$ surface X and a symplectic automorphism t of order 3 of X such that $X' = X/\langle t\rangle$, and hence there is a finite Abelian subgroup $G \subset \mathrm{Aut}(X)$ such that $X/G \cong \mathbb{P}^1 \times \mathbb{P}^1$, $G \cong \mathbb{Z}/3\mathbb{Z}^{\oplus 2}$, and the branch divisor is $3B_{1,1}^1 + 3B_{1,1}^2 + 3B_{1,1}^3$. In the same way, we get the claim for (6.18).

More specifically, let X be a $K3$ surface X, G be a finite Abelian subgroup G of $\mathrm{Aut}(X)$ such that $X/G \cong \mathbb{P}^1 \times \mathbb{P}^1$, and the numerical class of the branch divisor B of G is (6.17) or (6.18). By Theorem 3.1, there is the Galois cover $p' : X' \to \mathbb{P}^1 \times \mathbb{P}^1$ such that the branch divisor is B and the Galois group is isomorphic to $\mathbb{Z}/3\mathbb{Z}$ as a group. Then we get that X is the universal cover of X' of degree 3.

For (6.17), we obtain an example if we use curves $B_{1,1}^1, B_{1,1}^2, B_{1,1}^3$ in $\mathbb{P}^1 \times \mathbb{P}^1$ given by the equations

$$B_{1,1}^1 : z_0w_0 + z_1w_1 = 0, \quad B_{1,1}^2 : z_0w_0 - z_1w_1 = 0, \quad B_{1,1}^3 : z_0w_1 + z_1w_0 = 0.$$

For (6.18), we obtain an example if we use curves $B_{1,0}, B_{1,1}, B_{1,2}$ in $\mathbb{P}^1 \times \mathbb{P}^1$ given by the equations

$$B_{1,0} : z_0 = 0, \quad B_{1,1} : z_0w_1 + z_1w_0 = 0, \quad B_{1,2} : z_0w_1^2 + z_1w_0^2 + z_1w_1^2 = 0.$$

Corollary 3.4 *For each numerical classes* (6.198), (6.92) *and* (6.204) *and* (6.303), (6.252), (6.205), (6.93) *of the list in Section* 6, *there are a $K3$ surface X and a finite Abelian subgroup G of* $\mathrm{Aut}(X)$ *such that $X/G \cong \mathbb{F}_n$ and the numerical class B of the branch divisor of the quotient map $p : X \to X/G$ is* (6.198), (6.92) *and* (6.204) *and* (6.303), (6.252), (6.205), (6.93).

Furthermore, for a pair (X, G) of a $K3$ surface X and a finite Abelian subgroup G of $\mathrm{Aut}(X)$, *if $X/G \cong \mathbb{F}_n$ and the numerical class of the branch divisor B of the quotient map $p : X \to X/G$ is* (6.198), (6.92) *and* (6.204) *and* (6.303), (6.252), (6.205), (6.93), *then G is* $\mathbb{Z}/3\mathbb{Z}^{\oplus 2}$, $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}^{\oplus 2}$, $\mathbb{Z}/3\mathbb{Z}^{\oplus 2}$, $\mathbb{Z}/3\mathbb{Z}^{\oplus 2}$, $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}^{\oplus 2}$, $\mathbb{Z}/3\mathbb{Z}^{\oplus 3}$, $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}^{\oplus 3}$, *in order, as a group.*

Proof In the same way as Proposition 3.5, we get this corollary. More specifically, let X be a $K3$ surface X, G be a finite Abelian subgroup G of $\mathrm{Aut}(X)$ such that $X/G \cong \mathbb{F}_n$, and B be the branch divisor of the quotient map $p : X \to X/G$. Let $p' : X' \to \mathbb{P}^1 \times \mathbb{P}^1$ be the Galois cover such that the branch divisor is B and which is given by Theorem 3.1. Then we get the following.

(i) If the numerical class of B is one of (6.198), (6.204), (6.303), then X is the universal cover of X' of degree 3.

(ii) If the numerical class of B is (6.92), then $X \to X/G$ is given by the composition of the Galois cover $X' \to \mathbb{F}_2$ whose numerical class of the branch divisor is (6.92) and the Galois cover $\mathbb{F}_2 \to \mathbb{F}_1$ which is induced by the Galois cover $\mathbb{P}^1 \to \mathbb{P}^1$ of degree 2.

(iii) If the numerical class of B is one of (6.252), (6.205), (6.93), then $X \to X/G$ is given by the composition of the Galois cover $X' \to \mathbb{F}_6$ whose numerical class of the branch divisor is

(6.303) and the Galois cover $\mathbb{F}_6 \to \mathbb{F}_m$ which is induced by the Galois cover $\mathbb{P}^1 \to \mathbb{P}^1$ of degree $\frac{6}{m}$.

For (6.198), we obtain an example if we use curves $B^1_{1,2}, B^2_{1,2}, B^3_{1,2}$ in $\mathbb{F}_2$ given by the equations

$$B^1_{1,2} : Y_0 + Y_2 = 0, \quad B^2_{1,2} : Y_1 + Y_2 = 0, \quad B^3_{1,2} : Y_0 + Y_1 + Y_2 = 0.$$

For (6.92), we obtain an example if we use curves $B^1_{1,1}, B^2_{1,1}, B^3_{1,1}, B^1_{0,1}, B^2_{0,1}$ in $\mathbb{F}_1$ given by the equations

$$B^1_{1,1} : Y_0 + Y_2 = 0, \quad B^2_{1,1} : Y_1 + Y_2 = 0, \quad B^3_{1,1} : Y_0 + Y_1 + Y_2 = 0,$$

$$B^1_{0,1} : X_0 = 0, \quad B^2_{0,1} : X_1 = 0.$$

For (6.204), we obtain examples if we use a section C and curves $B^1_{1,3}, B^2_{1,3}$ in $\mathbb{F}_2$ given by the equations

$$B^1_{1,3} : X_0Y_0 + X_0Y_1 + X_1Y_2 = 0, \quad B^2_{1,3} : X_1Y_0 + X_1Y_1 + 2X_0Y_2 = 0.$$

For (6.303), we obtain examples if we use a section C and curves $B^1_{1,6}, B^2_{1,6}$ in $\mathbb{F}_6$ given by the equations

$$B^1_{1,6} : Y_0 + 2Y_2 = 0, \quad B^2_{1,6} : Y_1 + 2Y_2 = 0.$$

For (6.252), we obtain examples if we use a section C and curves $B^1_{1,3}, B^2_{1,3}, B^1_{0,1}, B^2_{0,1}$ in $\mathbb{F}_3$ given by the equations

$$B^1_{1,3} : Y_0 + 2Y_2 = 0, \quad B^2_{1,3} : Y_1 + 2Y_2 = 0,$$

$$B^1_{0,1} : X_0 = 0, \quad B^2_{0,1} : X_1 = 0.$$

For (6.205), we obtain examples if we use a section C and curves $B^1_{1,2}, B^2_{1,2}, B^1_{0,1}, B^2_{0,1}$ in $\mathbb{F}_2$ given by the equations

$$B^1_{1,2} : Y_0 + 2Y_2 = 0, \quad B^2_{1,2} : Y_1 + 2Y_2 = 0,$$

$$B^1_{0,1} : X_0 = 0, \quad B^2_{0,1} : X_1 = 0.$$

For (6.93), we obtain examples if we use a section C and curves $B^1_{1,1}, B^2_{1,1}, B^1_{0,1}, B^2_{0,1}$ in $\mathbb{F}_1$ given by the equations

$$B^1_{1,1} : Y_0 + 2Y_2 = 0, \quad B^2_{1,1} : Y_1 + 2Y_2 = 0,$$

$$B^1_{0,1} : X_0 = 0, \quad B^2_{0,1} : X_1 = 0.$$

Proposition 3.6 *For each numerical classes* (6.19)–(6.20) *of the list in Section* 6*, there is a* $K3$ *surface* X *and a finite Abelian subgroup* G *of* $\mathrm{Aut}(X)$ *such that* $X/G \cong \mathbb{F}_n$ *and the numerical class of the branch divisor* B *of the quotient map* $p : X \to X/G$ *is* (6.19)–(6.20).

Furthermore, for a pair (X, G) *of a* $K3$ *surface* X *and a finite Abelian subgroup* G *of* $\mathrm{Aut}(X)$*, if* $X/G \cong \mathbb{F}_n$ *and the numerical class of the branch divisor* B *of the quotient map* $p : X \to X/G$ *is* (6.19)–(6.20)*, then* G *is* $\mathbb{Z}/2\mathbb{Z}^{\oplus 3}$*,* $\mathbb{Z}/2\mathbb{Z}^{\oplus 3}$*, in order, as a group.*

Proof Let $B^i_{1,1}$ be a smooth curve on $\mathbb{P}^1 \times \mathbb{P}^1$ for $i = 1, 2, 3, 4$ such that $\sum_{i=1}^{4} B^i_{1,1}$ is simple normal crossing. Then the numerical class of $\sum_{i=1}^{4} 2B^i_{1,1}$ is (6.19). We set $\{x_1, x_2\} := B^1_{1,1} \cap B^2_{1,1}$

and $\{x_3, x_4\} := B^3_{1,1} \cap B^3_{1,1}$. Let $Z := \mathrm{Blow}_{\{x_1,x_2,x_3,x_4\}}\mathbb{P}^1 \times \mathbb{P}^1$. Let E_i be the exceptional divisor for $i = 1, 2, 3, 4$. Then $\mathrm{Pic}(Z) = \mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) \bigoplus_{i=1}^{4} \mathbb{Z}E_i$. Let C_i be the proper transform of $B^i_{1,1}$ for $i = 1, 2, 3, 4$. Then for $i = 1, 2$, $j = 3, 4$,

$$C_i = (C + F) - E_1 - E_2 \quad \text{and} \quad C_j = (C + F) - E_3 - E_4 \quad \text{in } \mathrm{Pic}(Z).$$

By Theorem 3.1, there are the Galois covers $p_1 : Y_1 \to Z$ and $p_2 : Y_2 \to Z$ such that the branch divisor of p_1 is $2C_1 + 2C_2$, and that of p_2 is $2C_3 + 2C_4$. Since $C_1 \cap C_2$ and $C_3 \cap C_4$ are empty sets, Y_1 and Y_2 are smooth. Since $\sum_{i=1}^{4} C^i_{1,1}$ is simple normal crossing, $Y := Y_1 \times_Z Y_2$ is smooth and a $K3$ surface. Therefore, there is the Galois cover $f : Y \to Z$ whose branch divisor is $\sum_{i=1}^{4} 2C_i$ and Galois group is $\mathbb{Z}/2\mathbb{Z}^{\oplus 2}$ as a group. Let C'_i be a smooth curve on Y such that $f^*C_i = 2C'_i$ for $i = 1, 2, 3, 4$. Then

$$C'_1 = f^*\Big(\Big(\frac{C}{2}, \frac{F}{2}\Big) - \frac{1}{2}E_1 - \frac{1}{2}E_2\Big) \quad \text{and} \quad C'_3 = f^*\Big(\Big(\frac{C}{2}, \frac{F}{2}\Big) - \frac{1}{2}E_3 - \frac{1}{2}E_4\Big) \quad \text{in } \mathrm{Pic}(Y).$$

Thus, we get

$$\sum_{i=1}^{4} f^*E_i = 2f^*(C + F) - 2C'_1 - 2C'_2 \quad \text{in } \mathrm{Pic}(Y).$$

By Theorem 3.1, there is the Galois cover $g : W \to Y$ whose branch divisor is $\sum_{i=1}^{4} 2f^*E_i$. Let E'_i be a smooth curve on W such that $g^*f^*E_i = 2E'_i$. Since $(f^*E_i \cdot f^*E_i) = -2$, $(E'_i \cdot E'_i) = -1$ for $i = 1, 2, 3, 4$. Let $f : W \to X$ be a contraction of $E'_1, \cdots, E'_4$. Since Y is a $K3$ surface, X is a $K3$ surface. Since W is a double cover of Y, there is a symplectic involution s of X such that $X/\langle s\rangle \to \mathbb{P}^1 \times \mathbb{P}^1$ is a Galois cover whose branch divisor is $2B^1_{1,1} + 2B^2_{1,1} + 2B^3_{1,1} + 2B^4_{1,1}$. Therefore, there is a finite Abelian subgroup $G \subset \mathrm{Aut}(X)$ such that $X/G \cong \mathbb{P}^1 \times \mathbb{P}^1$, $G \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 3}$, and the branch divisor is $2B^1_{1,1} + 2B^2_{1,1} + 2B^3_{1,1} + 2B^4_{1,1}$.

Next, let $B_{1,0}, B_{1,2}, B^1_{1,1}, B^2_{1,1}$ be smooth curves on $\mathbb{P}^1 \times \mathbb{P}^1$ such that $B_{1,0} + B_{1,2} + B^1_{1,1} + B^2_{1,1}$ is simple normal crossing. Then the numerical class of $2B_{1,0} + 2B_{1,2} + 2B^1_{1,1} + 2B^2_{1,1}$ is (6.20). We set $\{x_1, x_2\} := B_{1,0} \cap B_{1,2}$ and $\{x_3, x_4\} := B^1_{1,1} \cap B^2_{1,1}$. Let $Z := \mathrm{Blow}_{\{x_1,x_2,x_3,x_4\}}\mathbb{P}^1 \times \mathbb{P}^1$. Let E_i be the exceptional divisor for $i = 1, 2, 3, 4$. Then $\mathrm{Pic}(Z) = \mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) \bigoplus_{i=1}^{4} \mathbb{Z}E_i$. Let $C_{1,0}, C_{1,2}, C^1_{1,1}, C^2_{1,1}$ be the proper transform of $B_{1,0}$, $B_{1,2}$, $B^1_{1,1}$, $B^2_{1,1}$ in order. Then

$$C_{1,0} = C - E_1 - E_2 \quad \text{and} \quad C_{1,2} = (C + F) - E_1 - E_2 \quad \text{in } \mathrm{Pic}(Z)$$

and

$$C^1_{1,1} = (C + F) - E_3 - E_4 \quad \text{and} \quad C^2_{1,1} = (C + F) - E_3 - E_4 \quad \text{in } \mathrm{Pic}(Z).$$

Let $p_1 : Y_1 \to Z$ be a cyclic cover whose branch divisor is $2C_{1,0} + 2C_{1,2}$, and $p_2 : Y_2 \to Z$ be a cyclic cover whose branch divisor is $2C^1_{1,1} + 2C^2_{1,1}$. Then as for the case of (6.19), $Y := Y_1 \times_Z Y_2$ is a $K3$ surface, and there is the Galois cover $f : Y \to Z$ whose branch divisor is $\sum_{i=1}^{4} 2C_i$ and Galois group is to $\mathbb{Z}/2\mathbb{Z}^{\oplus 2}$ as a group. Since $\frac{f^*C_{1,0}}{2} \in \mathrm{Pic}(Y)$ and $\frac{f^*C_{1,2}}{2} \in \mathrm{Pic}(Y)$, we get

$\frac{f^*(C_{1,2}-C_{1,1})}{2} = f^*(0,\frac{1}{2}) \in \mathrm{Pic}(Y)$. As for the case of (6.19), we get $\frac{\sum_{i=1}^{4} f^*E_i}{2} \in \mathrm{Pic}(Y)$, and hence we get the claim for (6.20).

More specifically, let X be a $K3$ surface X, G be a finite Abelian subgroup G of $\mathrm{Aut}(X)$ such that $X/G \cong \mathbb{P}^1 \times \mathbb{P}^1$, and the numerical class of the branch divisor B of the quotient map $p : X \to X/G$ is (6.19) or (6.20). By Theorem 3.1 and the fibre product, there is the Galois cover $p' : X' \to \mathbb{P}^1 \times \mathbb{P}^1$ such that the branch divisor is B and the Galois group is $\mathbb{Z}/2\mathbb{Z}^{\oplus 2}$ as a group. Then we get that X is the universal cover of X' of degree 2.

For (6.19), we obtain an example if we use curves $B^1_{1,1}, B^2_{1,1}, B^3_{1,1}, B^4_{1,1}$ in $\mathbb{P}^1 \times \mathbb{P}^1$ given by the equations

$$B^1_{1,1} : z_0w_0 + z_1w_1 = 0, \quad B^2_{1,1} : z_0w_0 - z_1w_1 = 0,$$

$$B^3_{1,1} : z_0w_1 + z_1w_0 = 0, \quad B^4_{1,1} : z_0w_1 - z_1w_0 = 0.$$

For (6.20), we obtain an example if we use curves $B_{1,0}, B^1_{1,1}, B^2_{1,1}, B_{1,2}$ in $\mathbb{P}^1 \times \mathbb{P}^1$ given by the equations

$$B_{1,0} : z_0 = 0, \quad B^1_{1,1} : z_0w_0 + z_1w_1 = 0,$$

$$B^2_{1,1} : z_0w_1 + z_1w_0 = 0, \quad B_{1,2} : z_0w_1^2 + 3z_1w_0^2 = 0.$$

Corollary 3.5 *For each numerical classes* (6.81) *and* (6.82) *and* (6.199), (6.94) *and* (6.200), (6.96) *and* (6.282), (6.206), (6.97), (6.98) *of the list in Section 6, there is a $K3$ surface X and a finite Abelian subgroup G of* $\mathrm{Aut}(X)$ *such that $X/G \cong \mathbb{F}_n$ and the numerical class of the branch divisor B of the quotient map $p : X \to X/G$ is* (6.81) *and* (6.82) *and* (6.199), (6.94) *and* (6.200), (6.96) *and* (6.282), (6.206), (6.97), (6.98).

Furthermore, for a pair (X, G) of a $K3$ surface X and a finite Abelian subgroup G of $\mathrm{Aut}(X)$, *if $X/G \cong \mathbb{F}_n$ and the numerical class of the branch divisor B of the quotient map $p : X \to X/G$ is* (6.81) *and* (6.82) *and* (6.199), (6.94) *and* (6.200), (6.96) *and* (6.282), (6.206), (6.97), (6.98), *then G is* $\mathbb{Z}/2\mathbb{Z}^{\oplus 3}$, $\mathbb{Z}/2\mathbb{Z}^{\oplus 3}$, $\mathbb{Z}/2\mathbb{Z}^{\oplus 3}$, $\mathbb{Z}/2\mathbb{Z}^{\oplus 4}$, $\mathbb{Z}/2\mathbb{Z}^{\oplus 3}$, $\mathbb{Z}/2\mathbb{Z}^{\oplus 4}$, $\mathbb{Z}/2\mathbb{Z}^{\oplus 3}$, $\mathbb{Z}/2\mathbb{Z}^{\oplus 4}$, $\mathbb{Z}/2\mathbb{Z}^{\oplus 3} \oplus \mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}^{\oplus 5}$, *in order, as a group.*

Proof In the same way as Proposition 3.6, we get this corollary. More specifically, let X be a $K3$ surface X, G be a finite Abelian subgroup G of $\mathrm{Aut}(X)$ such that $X/G \cong \mathbb{F}_n$, and B be the branch divisor of the quotient map $p : X \to X/G$. Then we get the following.

(i) We assume that the numerical class of B is one of (6.81), (6.82), (6.199), (6.200), (6.282). By Theorem 3.1 and the fibre product, there is the Galois cover $p' : X' \to \mathbb{F}_n$ such that the branch divisor is B and the Galois group is $\mathbb{Z}/2\mathbb{Z}^{\oplus 2}$ as a group. Then X is the universal cover of X' of degree 2.

(ii) If the numerical class of B is (6.94), then $X \to X/G$ is given by the composition of the Galois cover $X \to \mathbb{F}_2$ whose numerical class of the branch divisor is (6.199) and the Galois cover $\mathbb{F}_2 \to \mathbb{F}_1$ which is induced by the Galois cover $\mathbb{P}^1 \to \mathbb{P}^1$ of degree 2.

(iii) If the numerical class of B is (6.96), then $X \to X/G$ is given by the composition of the Galois cover $X \to \mathbb{F}_2$ whose numerical class of the branch divisor is (6.200) and the Galois cover $\mathbb{F}_2 \to \mathbb{F}_1$ which is induced by the Galois cover $\mathbb{P}^1 \to \mathbb{P}^1$ of degree 2.

(iv) If the numerical class of B is one of (6.206), (6.98), (6.97), then $X \to X/G$ is given by the composition of the Galois cover $X \to \mathbb{F}_4$ whose numerical class of the branch divisor is

(6.303) and the Galois cover $\mathbb{F}_4 \to \mathbb{F}_m$ which is induced by the Galois cover $\mathbb{P}^1 \to \mathbb{P}^1$ of degree $\frac{4}{m}$.

For (6.81), we obtain an example if we use a section C and curves $B_{1,2}^1, B_{1,2}^2, B_{1,2}^3$ in $\mathbb{F}_1$ given by the equations

$$B_{1,2}^1 : X_0Y_1 + X_1Y_0 + (X_0 + X_1)Y_2 = 0, \quad B_{1,2}^2 : X_0Y_1 + 2X_1Y_0 + (2X_0 + X_1)Y_2 = 0,$$

$$B_{1,2}^3 : 2X_0Y_1 + X_1Y_0 + (X_0 + 2X_1)Y_2 = 0.$$

For (6.82), we obtain an example if we use curves $B_{1,3}, B_{1,1}^1, B_{1,1}^2, B_{1,1}^3$ in $\mathbb{F}_1$ given by the equations

$$B_{1,3} : X_0^2Y_1 + X_1^2Y_0 + X_0X_1Y_2 = 0, \quad B_{1,1}^1 : Y_0 + Y_1 + Y_2 = 0,$$

$$B_{1,1}^2 : Y_0 + 2Y_1 + Y_2 = 0, \quad B_{1,1}^3 : 2Y_0 + Y_1 + Y_2 = 0.$$

For (6.199), we obtain an example if we use a section C and curves $B_{2,4}, B_{1,2}^1, B_{1,2}^2$ in $\mathbb{F}_2$ given by the equations

$$B_{2,4} : X_0^2Y_1 + (X_0^2 + X_1^2)Y_2 = 0, \quad B_{1,2}^1 : Y_0 + Y_2 = 0, \quad B_{1,2}^2 : 2Y_0 + 2Y_1 = 0.$$

For (6.94), we obtain an example if we use a section C and curves $B_{1,2}, B_{1,1}^1, B_{1,1}^2, B_{0,1}^1, B_{0,1}^2$ in $\mathbb{F}_1$ given by the equations

$$B_{1,2} : X_0Y_1 + (X_0 + X_1)Y_2 = 0, \quad B_{1,1}^1 : Y_0 + Y_2 = 0, \quad B_{1,1}^2 : 2Y_0 + 2Y_1 = 0,$$

$$B_{0,1}^1 : X_0 = 0, \quad B_{0,1}^2 : X_1 = 0.$$

For (6.200), we obtain an example if we use curves $B_{1,2}^1, B_{1,2}^2, B_{1,2}^3, B_{1,2}^4$ in $\mathbb{F}_2$ given by the equations

$$B_{1,2}^1 : Y_0 + 2Y_2 = 0, \quad B_{1,2}^2 : Y_1 + 2Y_2 = 0,$$

$$B_{1,2}^3 : 3Y_0 + Y_1 + Y_2 = 0, \quad B_{1,2}^4 : Y_0 + Y_1 + 3Y_2 = 0.$$

For (6.96), we obtain an example if we use curves $B_{1,1}^1, B_{1,1}^2, B_{1,1}^3, B_{1,1}^4, B_{0,1}^1, B_{0,1}^2$ in $\mathbb{F}_1$ given by the equations

$$B_{1,1}^1 : Y_0 + 2Y_2 = 0, \quad B_{1,1}^2 : Y_1 + 2Y_2 = 0,$$

$$B_{1,1}^3 : 3Y_0 + Y_1 + Y_2 = 0, \quad B_{1,1}^4 : Y_0 + Y_1 + 3Y_2 = 0,$$

$$B_{0,1}^1 : X_0 = 0, \quad B_{0,1}^2 : X_1 = 0.$$

For (6.282), we obtain an example if we use a section C and curves $B_{1,4}^1, B_{1,4}^2, B_{1,4}^3$ in $\mathbb{F}_4$ given by the equations

$$B_{1,4}^1 : Y_0 + 2Y_2 = 0, \quad B_{1,4}^2 : Y_1 + 2Y_2 = 0, \quad B_{1,4}^3 : 3Y_0 + Y_1 + Y_2 = 0.$$

For (6.206), we obtain examples if we use a section C and curves $B_{1,2}^1, B_{1,2}^2, B_{1,2}^3, B_{0,1}^1, B_{0,1}^2$ in $\mathbb{F}_2$ given by the equations

$$B_{1,2}^1 : Y_0 + 2Y_2 = 0, \quad B_{1,2}^2 : Y_1 + 2Y_2 = 0, \quad B_{1,2}^3 : 3Y_0 + Y_1 + Y_2 = 0.$$

$$B_{0,1}^1 : X_0 = 0, \quad B_{0,1}^2 : X_1 = 0.$$

For (6.97), we obtain examples if we use a section C and curves $B^1_{1,1}$, $B^2_{1,1}$, $B^3_{1,1}$, $B^1_{0,1}, B^2_{0,1}$ in $\mathbb{F}_1$ given by the equations

$$B^1_{1,1} : Y_0 + 2Y_2 = 0, \quad B^2_{1,1} : Y_1 + 2Y_2 = 0, \quad B^3_{1,1} : 3Y_0 + Y_1 + Y_2 = 0.$$

$$B^1_{0,1} : X_0 = 0, \quad B^2_{0,1} : X_1 = 0.$$

For (6.98), we obtain an example if we use a section C and curves $B^1_{1,1}$, $B^2_{1,1}$, $B^3_{1,1}$, $B^1_{0,1}$, $B^2_{0,1}$, $B^3_{0,1}$ in $\mathbb{F}_1$ given by the equations

$$B^1_{1,1:}Y_0 + 2Y_2 = 0, \quad B^2_{1,1} : Y_1 + 2Y_2 = 0, \quad B^3_{1,1} : 3Y_0 + Y_1 + Y_2 = 0, \quad B^4_{1,1} : Y_0 + Y_1 + 3Y_2 = 0,$$

$$B^1_{0,1} : X_0 = 0, \quad B^2_{0,1} : X_1 = 0, \quad B^3_{0,1} : X_0 - X_1 = 0.$$

Proposition 3.7 *For numerical classes* (6.278)*,* (6.207)*,* (6.99)*,* (6.100) *of the list in Section* 6*, there is a $K3$ surface X and a finite Abelian subgroup G of* $\mathrm{Aut}(X)$ *such that $X/G \cong \mathbb{F}_n$ and the numerical class of the branch divisor B of the quotient map $p : X \to X/G$ is* (6.278)*,* (6.207)*,* (6.99)*,* (6.100)*.*

Furthermore, for a pair (X, G) of a $K3$ surface X and a finite Abelian subgroup G of $\mathrm{Aut}(X)$*, if $X/G \cong \mathbb{F}_n$ and the numerical class of the branch divisor B of the quotient map $p : X \to X/G$ is* (6.278)*,* (6.207)*,* (6.99)*,* (6.100)*, then G is $\mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}^{\oplus 2}$, $\mathbb{Z}/2\mathbb{Z}^2 \oplus \mathbb{Z}/4\mathbb{Z}$, in order, as a group.*

Proof Let $B_{2,8}$ be a smooth curve on $\mathbb{F}_4$. Then the numerical class of $2C+4B_{2,8}$ is (6.278). Since $B_{2,8} = 2C + 8F$ in $\mathrm{Pic}(\mathbb{F}_4)$, by Theorem 3.1, there is the Galois cover $p_1 : X_1 \to \mathbb{F}_4$ such that the branch divisor is $2B_{2,8}$ and the Galois group is $\mathbb{Z}/2\mathbb{Z}$ as a group. Let $E_{2,8}$ be a smooth curve on X_1 such that $p_1^*B_{2,8} = 2E_{2,8}$. Since $C + B_{2,8}$ is simple normal crossing, p_1^*C is a reduced divisor on X_1, whose support is a union of pairwise disjoint smooth curves. Since $p_1^*C + E_{2,8} = p_1^*(2C + 4F) = 2p_1^*(C + 2F)$ in $\mathrm{Pic}(X_1)$, by Theorem 3.1, there is a Galois cover $p_2 : X_2 \to X_1$ such that the branch divisor is $p_1^*C + E_{2,8}$ and the Galois group is $\mathbb{Z}/2\mathbb{Z}$ as a group. Then $p := p_1 \circ p_2 : X_2 \to \mathbb{F}_4$ is the branched cover such that p has $2C + 4B_{2,8}$ as the branch divisor. In the same way of Proposition 3.2, X is a $K3$ surface, and $p : X \to \mathbb{F}_4$ is the Galois cover whose Galois group is $\mathbb{Z}/4\mathbb{Z}$ as a group. In the same way of Proposition 3.2, we get the claim for (6.207), (6.99), (6.100).

More specifically, let X be a $K3$ surface, G be a finite Abelian subgroup of $\mathrm{Aut}(X)$ such that $X/G \cong \mathbb{F}_n$, and B be the branch divisor of the quotient map $p : X \to X/G$. Then we get the following.

(i) If the numerical class of B is (6.278), then $X \to X/G$ is given by the above way.

(ii) If the numerical class of B is one of (6.207), (6.99), (6.100), then $X \to X/G$ is given by the composition of the Galois cover $X' \to \mathbb{F}_4$ whose numerical class of the branch divisor is (6.278) and the Galois cover $\mathbb{F}_4 \to \mathbb{F}_m$ which is induced by the Galois cover $\mathbb{P}^1 \to \mathbb{P}^1$ of degree $\frac{m}{4}$.

For (6.278), we obtain an example if we use a section C and a curve $B_{2,8}$ in $\mathbb{F}_4$ given by the equation

$$B_{2,8} : Y_0^2 + Y_1^2 + Y_2^2 = 0.$$

For (6.207), we obtain examples if we use a section C and curves $B_{2,4}$, $B^1_{0,1}$, $B^2_{0,1}$ in $\mathbb{F}_2$ given by the equations

$$B_{2,4} : Y_0^2 + Y_1^2 + Y_2^2 = 0, \quad B^1_{0,1} : X_0 = 0, \quad B^2_{0,1} : X_1 = 0.$$

For (6.99), we obtain examples if we use a section C and curves $B_{2,2}$, $B^1_{0,1}$, $B^2_{0,1}$ in $\mathbb{F}_1$ given by the equations

$$B_{2,2} : Y_0^2 + Y_1^2 + Y_2^2 = 0, \quad B^1_{0,1} : X_0 = 0, \quad B^2_{0,1} : X_1 = 0.$$

For (6.100), we obtain examples if we use a section C and curves $B_{2,2}$, $B^1_{0,1}$, $B^2_{0,1}$, $B^3_{0,1}$ in $\mathbb{F}_1$ given by the equations

$$B_{2,2} : Y_0^2 + Y_1^2 + Y_2^2 = 0, \quad B^1_{0,1} : X_0 = 0, \quad B^2_{0,1} : X_1 = 0, \quad B^3_{0,1} : X_0 - X_1 = 0.$$

Proposition 3.8 *For numerical classes* (6.280), (6.208) *of the list in Section* 6, *there is a* $K3$ *surface* X *and a finite Abelian subgroup* G *of* $\mathrm{Aut}(X)$ *such that* $X/G \cong \mathbb{F}_n$ *and the numerical class of the branch divisor* B *of the quotient map* $p : X \to X/G$ *is* (6.280), (6.208).

Furthermore, for a pair (X, G) *of a* $K3$ *surface* X *and a finite Abelian subgroup* G *of* $\mathrm{Aut}(X)$, *if* $X/G \cong \mathbb{F}_n$ *and the numerical class of the branch divisor* B *of the quotient map* $p : X \to X/G$ *is* (6.280), (6.208), *then* G *is* $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}^{\oplus 2} \oplus \mathbb{Z}/4\mathbb{Z}$, *in order, as a group.*

Proof Let $B_{1,6}$ and $B_{1,4}$ be smooth curves on $\mathbb{F}_4$ such that $C + B_{1,6} + B_{1,4}$ is simple normal crossing. Then the numerical class of $4C + 2B_{1,6} + 4B_{1,4}$ is (6.280). Since $C + B_{1,4} = 2C + 2F$ in $\mathrm{Pic}(\mathbb{F}_8)$, by Theorem 3.1, there is the Galois cover $p_1 : X_1 \to \mathbb{F}_4$ such that the branch divisor is $2C + 2B_{1,4}$ and the Galois group is $\mathbb{Z}/2\mathbb{Z}$ as a group. Let $E_C, E_{1,4}$ be two smooth curves on X_1 such that $p_1^*C = 2E_C$ and $p_1^*B_{1,4} = 2E_{1,4}$. Since $C + B_{1,6} + B_{1,4}$ is simple normal crossing, $p_1^*B_{1,6}$ is a reduced divisor on X_1, whose support is a union of pairwise disjoint smooth curves. Since $p_1^*B_{1,6} = p_1^*(C + 6F) = p_1^*(C + 4F) + p_1^*(2F) = 2E_{1,4} + 2p_1^*F$ in $\mathrm{Pic}(X_1)$, by Theorem 3.1, there is the Galois cover $p_2 : X_2 \to X_1$ such that the branch divisor is $2p_1^*B_{1,6}$ and the Galois group is $\mathbb{Z}/2\mathbb{Z}$. Notice that $\frac{p_2^*p_1^*B_{1,6}}{2} \in \mathrm{Pic}(X_2)$. Since $C + B_{1,6} + B_{1,4}$ is simple normal crossing, $p_2^*E_C$ and $p_2^*E_{1,4}$ are reduced divisors on X_2, whose support are unions of pairwise disjoint smooth curves. Since $p_2^*(E_C + E_{1,4}) = p_2^*p_1^*(C + 2F) = p_2^*p_1^*(C + 6F) - p_2^*p_1^*4F = p_2^*p_1^*B_{1,6} - 4p_2^*p_1^*F$ in $\mathrm{Pic}(X_2)$ and $\frac{p_2^*p_1^*B_{1,6}}{2} \in \mathrm{Pic}(X_2)$, by Theorem 3.1, there is the Galois cover $p_3 : X \to X_2$ such that the branch divisor is $p_2^*(E_C + E_{1,4})$ and the Galois group is $\mathbb{Z}/2\mathbb{Z}$. Then $p := p_1 \circ p_2 \circ p_3 : X \to \mathbb{F}_4$ is the branched cover such that p has $4C + 2B_{1,6} + 4B_{1,4}$ as the branch divisor. In the same way of Proposition 3.2, X is a $K3$ surface, and $p : X \to \mathbb{F}_4$ is the Galois cover whose Galois group is $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ as a group. In the same way of Proposition 3.2, we get the claim for (6.208).

More specifically, let X be a $K3$ surface, G be a finite Abelian subgroup of $\mathrm{Aut}(X)$ such that $X/G \cong \mathbb{F}_n$, and B be the branch divisor of the quotient map $p : X \to X/G$. Then we get the following.

(i) If the numerical class of B is (6.280), then $X \to X/G$ is given by the above way.

(ii) If the numerical class of B is (6.208), then $X \to X/G$ is given by the composition of the Galois cover $X' \to \mathbb{F}_4$ whose numerical class of the branch divisor is (6.280) and the Galois cover $\mathbb{F}_4 \to \mathbb{F}_2$ which is induced by the Galois cover $\mathbb{P}^1 \to \mathbb{P}^1$ of degree 2.

For (6.280), we obtain an example if we use a section C and curves $B_{1,6}$, $B_{1,4}$ in $\mathbb{F}_4$ given by the equations

$$B_{1,6} : X_0^2Y_1 + X_1^2Y_0 + (X_0^2 + 2X_1^2)Y_2 = 0, \quad B_{1,4} : 2Y_0 + Y_2 = 0.$$

For (6.208), we obtain an example if we use a section C and curves $B_{1,3}$, $B_{1,2}$, $B^1_{0,1}$, $B^2_{0,1}$ in $\mathbb{F}_2$ given by the equations

$$B_{1,3} : X_0Y_1 + X_1Y_0 + (X_0 + 2X_1)Y_2 = 0, \quad B_{1,2} : 2Y_0 + Y_2 = 0,$$

$$B^1_{0,1} : X_0 = 0, \quad B^2_{0,1} : X_1 = 0.$$

Corollary 3.6 *For each numerical classes* (6.311), (6.281), (6.210), (6.209), (6.101) *of the list in Section* 6, *there is a* $K3$ *surface* X *and a finite Abelian subgroup* G *of* $\mathrm{Aut}(X)$ *such that* $X/G \cong \mathbb{F}_n$ *and the numerical class of the branch divisor* B *of the quotient map* $p : X \to X/G$ *is* (6.311), (6.281), (6.210), (6.209), (6.101).

Furthermore, for a pair (X, G) *of a* $K3$ *surface* X *and a finite Abelian subgroup* G *of* $\mathrm{Aut}(X)$, *if* $X/G \cong \mathbb{F}_n$ *and the numerical class of the branch divisor* B *of the quotient map* $p : X \to X/G$ *is* (6.311), (6.281), (6.210), (6.209), (6.101), *then* G *is* $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}^{\oplus 2} \oplus \mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}^{\oplus 3} \oplus \mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}^{\oplus 2}$, $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$, *in order, as a group.*

Proof In the same way Proposition 3.8, we get the claim. More specifically, let X be a $K3$ surface, G be a finite Abelian subgroup of $\mathrm{Aut}(X)$ such that $X/G \cong \mathbb{F}_n$, and B be the branch divisor of the quotient map $p : X \to X/G$. Then we get the following.

(i) If the numerical class of B is (6.311), then $X \to X/G$ is given by the above way.

(ii) If the numerical class of B is one of (6.101), (6.209), (6.210), (6.281), then $X \to X/G$ is given by the composition of the Galois cover $X \to \mathbb{F}_8$ whose numerical class of the branch divisor is (6.311) and the Galois cover $\mathbb{F}_8 \to \mathbb{F}_m$ which is induced by the Galois cover $\mathbb{P}^1 \to \mathbb{P}^1$ of degree $\frac{8}{m}$.

For (6.311), we obtain examples if we use a section C and curves $B^1_{1,8}$, $B^2_{1,8}$ in $\mathbb{F}_8$ given by the equations

$$B^1_{1,8} : Y_0 + Y_1 + Y_2 = 0, \quad B^2_{1,8} : Y_0 + Y_1 + 2Y_2 = 0.$$

For (6.281), we obtain examples if we use a section C and curves $B^1_{1,4}$, $B^2_{1,4}$, $B^1_{0,1}$, $B^2_{0,1}$ in $\mathbb{F}_4$ given by the equations

$$B^1_{1,4} : Y_0 + Y_1 + Y_2 = 0, \quad B^2_{1,4} : Y_0 + Y_1 + 2Y_2 = 0,$$

$$B^1_{0,1} : X_0 = 0, \quad B^2 : X_1 = 0.$$

For (6.209), we obtain examples if we use a section C and curves $B^1_{1,2}$, $B^2_{1,2}$, $B^1_{0,1}$, $B^2_{0,1}$ in $\mathbb{F}_2$ given by the equations

$$B^1_{1,2} : Y_0 + Y_1 + Y_2 = 0, \quad B^2_{1,2} : Y_0 + Y_1 + 2Y_2 = 0,$$

$$B^1_{0,1} : X_0 = 0, \quad B^2 : X_1 = 0.$$

For (6.101), we obtain examples if we use a section C and curves $B^1_{1,1}$, $B^2_{1,1}$, $B^1_{0,1}$, $B^2_{0,1}$ in $\mathbb{F}_1$ given by the equations

$$B^1_{1,1} : Y_0 + Y_1 + Y_2 = 0, \quad B^2_{1,1} : Y_0 + Y_1 + 2Y_2 = 0,$$

$$B^1_{0,1} : X_0 = 0, \quad B^2 : X_1 = 0.$$

For (6.210), we obtain an example if we use a section C and curves $B^1_{1,2}$, $B^2_{1,2}$, $B^1_{0,1}$, $B^2_{0,1}$, $B^3_{0,1}$ in $\mathbb{F}_2$ given by the equations

$$B^1_{1,2} : Y_0 + Y_1 + Y_2 = 0, \quad B^2_{1,2} : Y_0 + Y_1 + 2Y_2 = 0,$$

$$B^1_{0,1} : X_0 = 0, \quad B^2_{0,1} : X_1 = 0, \quad B^3_{0,1} : X_0 - X_1 = 0.$$

Proposition 3.9 *For each numerical classes* (6.316), (6.304), (6.283), (6.254), (6.253), (6.211), (6.95) *of the list in Section* 6, *there is a* $K3$ *surface* X *and a finite Abelian subgroup* G *of* $\mathrm{Aut}(X)$ *such that* $X/G \cong \mathbb{F}_n$ *and the numerical class of the branch divisor* B *of the quotient map* $p : X \to X/G$ *is* (6.316), (6.304), (6.283), (6.254), (6.253), (6.211), (6.95).

Furthermore, for a pair (X, G) *of a* $K3$ *surface* X *and a finite Abelian subgroup* G *of* $\mathrm{Aut}(X)$, *if* $X/G \cong \mathbb{F}_n$ *and the numerical class of the branch divisor* B *of the quotient map* $p : X \to X/G$ *is* (6.316), (6.304), (6.283), (6.254), (6.253), (6.211), (6.95), *then* G *is* $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}^{\oplus 2} \oplus \mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}^{\oplus 2}$, $\mathbb{Z}/2\mathbb{Z}^{\oplus 3} \oplus \mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}^{\oplus 2} \oplus \mathbb{Z}/3\mathbb{Z}^{\oplus 2}$, $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}^{\oplus 2} \oplus \mathbb{Z}/4\mathbb{Z}$, *in order, as a group.*

Proof Let $B^i_{1,12}$ be a smooth curve on $\mathbb{F}_{12}$ for $i = 1, 2$ such that $C + B^1_{1,12} + B^2_{1,12}$ is simple normal crossing. Then the numerical class of $6C + 2B^1_{1,12} + 3B^2_{1,12}$ is (6.316). Since $C + B^1_{1,12} = 2C + 12F$ in $\mathrm{Pic}(\mathbb{F}_{12})$, by Theorem 3.1, there is the Galois cover $p_1 : X_1 \to \mathbb{F}_{12}$ such that the branch divisor is $2C + 2B^1_{1,12}$ and the Galois group is $\mathbb{Z}/2\mathbb{Z}$ as a group. Since $C + B^1_{1,12} + B^2_{1,12}$ is simple normal crossing, $p_1^* B^2_{1,12}$ is a reduced divisor on X_1, whose support is a union of pairwise disjoint smooth curves. Since C and $B^1_{1,12}$ are smooth curves, there are smooth curves E_C, $E^1_{1,12}$ on X_1 such that $p_1^* C = 2E_C$ and $p_1^* B^1_{1,12} = 2E^1_{1,12}$. Since $E_C + p_1^* B^2_{1,12} = E_C + p_1^*(C + 12F) = E_C + p_1^* C + 12p_1^* F = 3E_C + 12p_1^* F$ in $\mathrm{Pic}(X_1)$, by Theorem 3.1, there is the Galois cover $p_2 : X \to X_1$ such that the branch divisor is $3E_C + 3p_1^* B^2_{1,12}$ and the Galois group is $\mathbb{Z}/3\mathbb{Z}$ as a group. Then $p := p_1 \circ p_2 : X \to \mathbb{F}_{12}$ is the branched cover such that p has $6C + 2B^1_{1,12} + 3B^2_{1,12}$ as the branch divisor. In the same way as Proposition 3.2, X is a $K3$ surface, and $p : X \to \mathbb{F}_{12}$ is the Galois cover whose Galois group is $G \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ as a group.

More specifically, let X be a $K3$ surface, G be a finite Abelian subgroup of $\mathrm{Aut}(X)$ such that $X/G \cong \mathbb{F}_n$, and B be the branch divisor of the quotient map $p : X \to X/G$. Then we get the following.

(i) If the numerical class of B is (6.316), then $X \to X/G$ is given by the above way.

(ii) If the numerical class of B is one of (6.304), (6.283), (6.254), (6.253), (6.211), (6.95), then $X \to X/G$ is given by the composition of the Galois cover $X' \to \mathbb{F}_{12}$ whose numerical class of the branch divisor is (6.316) and the Galois cover $\mathbb{F}_{12} \to \mathbb{F}_m$ which is induced by the Galois cover $\mathbb{P}^1 \to \mathbb{P}^1$ of degree $\frac{12}{m}$.

For (6.316), we obtain an example if we use a section C and curves $B^1_{1,12}$, $B^2_{1,12}$ in $\mathbb{F}_{12}$ given by the equations

$$B^1_{1,12} : Y_0 + 2Y_2 = 0, \quad B^2_{1,12} : Y_1 + 2Y_2 = 0.$$

For (6.304), we obtain examples if we use a section C and curves $B^1_{1,6}$, $B^2_{1,6}$, $B^1_{0,1}$, $B^2_{0,1}$ in $\mathbb{F}_6$ given by the equations

$$B^1_{1,6} : Y_0 + 2Y_2 = 0, \quad B^2_{1,6} : Y_1 + 2Y_2 = 0,$$

$$B^1_{0,1} : X_0 = 0, \quad B^2_{0,1} : X_1 = 0.$$

For (6.283), we obtain examples if we use a section C and curves $B^1_{1,4}$, $B^2_{1,4}$, $B^1_{0,1}$, $B^2_{0,1}$ in $\mathbb{F}_4$ given by the equations

$$B^1_{1,4} : Y_0 + 2Y_2 = 0, \quad B^2_{1,4} : Y_1 + 2Y_2 = 0,$$

$$B^1_{0,1} : X_0 = 0, \quad B^2_{0,1} : X_1 = 0.$$

For (6.253), we obtain examples if we use a section C and curves $B^1_{1,3}$, $B^2_{1,3}$, $B^1_{0,1}$, $B^2_{0,1}$ in $\mathbb{F}_3$ given by the equations

$$B^1_{1,3} : Y_0 + 2Y_2 = 0, \quad B^2_{1,3} : Y_1 + 2Y_2 = 0,$$

$$B^1_{0,1} : X_0 = 0, \quad B^2_{0,1} : X_1 = 0.$$

For (6.211), we obtain examples if we use a section C and curves $B^1_{1,2}$, $B^2_{1,2}$, $B^1_{0,1}$, $B^2_{0,1}$ in $\mathbb{F}_2$ given by the equations

$$B^1_{1,2} : Y_0 + 2Y_2 = 0, \quad B^2_{1,2} : Y_1 + 2Y_2 = 0,$$

$$B^1_{0,1} : X_0 = 0, \quad B^2_{0,1} : X_1 = 0.$$

For (6.95), we obtain examples if we use a section C and curves $B^1_{1,1}$, $B^2_{1,1}$, $B^1_{0,1}$, $B^2_{0,1}$ in $\mathbb{F}_1$ given by the equations

$$B^1_{1,1} : Y_0 + 2Y_2 = 0, \quad B^2_{1,1} : Y_1 + 2Y_2 = 0,$$

$$B^1_{0,1} : X_0 = 0, \quad B^2_{0,1} : X_1 = 0.$$

For (6.254), we obtain an example if we use a section C and curves $B^1_{1,3}$, $B^2_{1,3}$, $B^1_{0,1}$, $B^2_{0,1}$, $B^3_{0,1}$ in $\mathbb{F}_3$ given by the equations

$$B^1_{1,3} : Y_0 + 2Y_2 = 0, \quad B^2_{1,3} : Y_1 + 2Y_2 = 0,$$

$$B^1_{0,1} : X_0 = 0, \quad B^2_{0,1} : X_1 = 0, \quad B^3_{0,1} : X_0 - X_1 = 0.$$

3.2 Complete proof of Theorem 1.5

In this section, we will show that there is no numerical class such that it has an Abelian $K3$ cover except the numerical classes which are mentioned in Subsection 3.1. Then by Subsection 3.1, we will get Theorem 1.5. From here, we use the notations that

(i) X is a $K3$ surface,

(ii) G is a finite Abelian subgroup of $\mathrm{Aut}(X)$ such that $X/G \cong \mathbb{F}_n$,

(iii) $p : X \to X/G$ is the quotient map, and

(iv) $B := \sum_{i=1}^{k} b_i B_i$ is the branch divisor of p.

Furthermore, we use the notation that $B^k_{i,j}$ (or simply $B_{i,j}$) is a smooth curve on $\mathbb{F}_n$ such that $B^k_{i,j} = iC + jF$ in $\mathrm{Pic}(\mathbb{F}_n)$ if $n \geq 0$ where $k \in \mathbb{N}$.

For the branch divisor $B = \sum_{i=1}^{m} \sum_{j=1}^{n(i)} b^i_j B^j_{s_i,t_i}$ where $m, n(i) \in \mathbb{N}$, we use the notation that

$$G^j_{s_i,t_i} := \{g \in G : g_{|p^{-1}(B^j_{s_i,t_i})} = \mathrm{id}_{p^{-1}(B^j_{s_i,t_i})}\}.$$

Recall that by Theorem 2.5, $G^j_{s_i,t_i}$ is a cyclic group of order b^i_j which is generated by a non-symplectic automorphism of order b^i_j. Since G is Abelian, the support of B and the support of p^*B are simple normal crossing.

Lemma 3.2 *We assume that $X/G \cong \mathbb{F}_n$ for $n \geq 1$. If $B = aC + \sum_{i=1}^{k} b_i(c_iC + nc_iF) + \sum_{j=1}^{l} d_jF_j$ in $\mathrm{Pic}(\mathbb{F}_n)$, where $a, b_i, d_j \geq 2$ and $c_i, l \geq 1$, then $3 \geq l \geq 2$ and $d_1 = \cdots = d_l$.*

Proof By Theorem 2.5, there are pairwise disjoint smooth curves $C_1, \cdots, C_m$ such that $p^*C = \sum_{i=1}^{m} aC_i$. Since $C_1, \cdots, C_m$ are pairwise disjoint, we get that $\big(\sum_{i=1}^{m} C_i \cdot \sum_{i=1}^{m} C_i\big) = \sum_{i=1}^{m}(C_i \cdot C_i) = m(C_i \cdot C_i)$ for $i = 1, \cdots, m$. Since $(C \cdot C) = -n < 0$, $(C_i \cdot C_i) < 0$ for $i = 1, \cdots, m$. Since X is a $K3$ surface, C_i is a smooth rational curve for $i = 1, \cdots, m$. Let $p_{|C_i} : C_i \to C$ be the finite map. Let B_{c_i,nc_i} be an irreducible curve on $\mathbb{F}_n$. Since $B_{c_i,nc_i} = c_iC + nc_iF$ in $\mathrm{Pic}(\mathbb{F}_n)$, we get that $C \cap B_{c_i,nc_i}$ is an empty set. Since the support of B is simple normal crossing , $p_{|C_i}$ is the Galois covering whose branch divisor is $\sum_{j=1}^{l} d_j(C \cap F_j)$. If $d_i \neq d_j$, then $p_{|C_i}$ must be non-trivial. Since G is an Abelian group, $p_{|C_i}$ is the Abelian cover, however by Theorem 2.3, this is a non-Abelian cover. This is a contradiction. Therefore, $d_1 = \cdots = d_l$.

By Lemma 3.2, the numerical class of B is not one of (6.128), (6.129), (6.132), (6.137), (6.143), (6.150), (6.151), (6.152), (6.154), (6.159), (6.160), (6.162), (6.170), (6.171), (6.172), (6.173), (6.174), (6.175), (6.179), (6.188), (6.193), (6.220), (6.227), (6.230), (6.235), (6.247), (6.248), (6.255), (6.256), (6.257), (6.264), (6.269), (6.271), (6.274), (6.276), (6.285), (6.288), (6.290), (6.295), (6.297), (6.301), (6.307), (6.310), (6.313), (6.315) of the list in Section 6.

Lemma 3.3 *We assume that $X/G \cong \mathbb{F}_n$ for $n \geq 1$. If $B = aC + \sum_{i=1}^{k} b_iB_i + \sum_{j=1}^{l} d_jB_{0,1}^j$ where $a, b_i, d_j \geq 2$, then $d_1 = \cdots = d_l$, $2 \leq \sum_{i=1}^{k}(C \cdot B_i) + \sum_{j=1}^{l}(C \cdot B_{0,1}^j) \leq 3$, and $b_i = d_1$ if $(C \cdot B_i) \neq 0$ for $i = 1, \cdots, k$.*

Proof In the same way of Lemma 3.2, we get that for $p^*C = \sum_{i=1}^{m} C_i$, the finite map $p_{|C_i} : C_i \to C$ is the Abelian cover between $\mathbb{P}^1$ whose branch divisor is $\sum_{j=1}^{l} d_j(C \cap F_j)$ and Galois group is $\{g \in G : | \ g(C_1) = C_1\}$. By Theorem 2.3, we get the claim.

By Lemma 3.3, the numerical class of B is not one of (6.127), (6.133), (6.134), (6.135), (6.145), (6.146), (6.156), (6.157), (6.158), (6.161), (6.163), (6.164), (6.165), (6.166), (6.167), (6.168), (6.169), (6.223), (6.224), (6.225), (6.236), (6.237), (6.238), (6.239), (6.240), (6.261), (6.262), (6.263), (6.268), (6.270), (6.272), (6.273), (6.275), (6.284), (6.289), (6.292), (6.296), (6.298), (6.299), (6.300), (6.306), (6.312), (6.314) of the list in Section 6.

Lemma 3.4 *If there are irreducible curves B_1 and B_2 and positive even integers $b_1, b_2 \geq 2$ such that $B = b_1B_1 + b_2B_2$ and $(B_1 \cdot B_2) \neq 0$, then $(B_1 \cdot B_2) = 8$.*

Proof By Theorem 2.5, $G = G_{B_1} \oplus G_{B_2}$ and $G_{B_i} \cong \mathbb{Z}/b_i\mathbb{Z}$ for $i = 1, 2$. Let $s_i \in G_{B_i}$ be a generator for $i = 1, 2$. Since G is Abelian, $s_1^{\frac{b_i}{2}} \circ s_2^{\frac{b_2}{2}}$ is a symplectic automorphism of order 2. Since X/G is smooth, $\mathrm{Fix}(s_1^{\frac{b_i}{2}} \circ s_2^{\frac{b_2}{2}}) = p^{-1}(B_1) \cap p^{-1}(B_2)$. Since the support of B is simple normal crossing and $|G| = b_1b_2$, we get that $|p^{-1}(B_1) \cap p^{-1}(B_2)| = (B_1 \cdot B_2)$. By the fact that the fixed locus of a symplectic automorphism of order 2 are 8 isolated points, we get that $(B_1 \cdot B_2) = 8$.

By Lemma 3.4, the numerical class of B is not one of (6.21), (6.25), (6.26), (6.28), (6.103), (6.112), (6.130), (6.176), (6.213), (6.216), (6.241) of the list in Section 6.

Lemma 3.5 *If there are irreducible curves B_1 and B_2 such that $B = 3B_1 + 3B_2$ and*

$(B_1 \cdot B_2) \neq 0$, then $(B_1 \cdot B_2) = 3$.

Proof By Theorem 2.5, $G = G_{B_1} \oplus G_{B_2}$ and $G_{B_i} \cong \mathbb{Z}/3\mathbb{Z}$ for $i = 1, 2$. Let $s_i \in G_{B_i}$ be a generator for $i = 1, 2$. Since G is Abelian, we may assume that $s_1 \circ s_2$ is a non-symplectic automorphism of order 3. By Theorem 2.5, $\mathrm{Fix}(s_1 \circ s_2)$ does not contain a curve. Then by [1, Theorem 2.8] or [14, Table 2], $\mathrm{Fix}(s_1 \circ s_2)$ is only three isolated points. Since X/G is smooth, $\mathrm{Fix}(s_1 \circ s_2) = p^{-1}(B_1) \cap p^{-1}(B_2)$. Since $B_1 + B_2$ is simple normal crossing and $G = G_{B_1} \oplus G_{B_2}$, we get that $|p^{-1}(B_1) \cap p^{-1}(B_2)| = (B_1 \cdot B_2)$. Therefore, we get $(B_1 \cdot B_2) = 3$.

By Lemma 3.5, the numerical class of B is not one of (6.22), (6.23), (6.212), (6.218) of the list in Section 6.

Lemma 3.6 *If there are irreducible curves B_i and positive integers $b_i \geq 2$ for $i = 1, \cdots, k$ such that $B = \sum_{i=1}^{k} b_i B_i$ and $G = G_{B_i}$ for some i, then $(B_i \cdot B_j) = 0$ for $j \neq i$.*

Proof Recall that by Theorem 2.5, G_{B_m} is generated by a non-symplectic automorphism of order b_m and $\mathrm{Fix}(G_{B_m}) \supset p^{-1}(B_m)$ for $m = 1, \cdots, k$. If $(B_i \cdot B_j) \neq 0$ for $j \neq 0$, then $p^{-1}(B_i) \cap p^{-1}(B_j)$ is not an empty set. By the fact that the fixed locus of an automorphism is a pairwise set of points and curves, this is a contradiction.

By Lemma 3.6, the numerical class of B is not one of (6.24), (6.131), (6.177), (6.219), (6.242) of the list in Section 6.

Lemma 3.7 *If there are irreducible curves B_1 and B_2 such that $B = 2B_1 + 2B_2$ and $(B_1 \cdot B_2) \neq 0$, then $\frac{B_i}{2} \in \mathrm{Pic}(\mathbb{F}_n)$ for $i = 1, 2$.*

Proof By Theorem 2.5, $G = G_{B_1} \oplus G_{B_2}$ and $G_{B_i} \cong \mathbb{Z}/2\mathbb{Z}$ for $i = 1, 2$. Since the fixed locus of a non-symplectic automorphism of order 2 is a set of pairwise set of smooth curves or empty set, X/G_{B_i} is smooth. Then there is a double cover $X/G_{B_i} \to X/G \cong \mathbb{F}_n$ whose branch divisor is $2B_j$ for $i, j = 1, 2$ and $i \neq j$. By Theorem 3.1, $\frac{B_i}{2} \in \mathrm{Pic}(\mathbb{F}_n)$ for $i = 1, 2$.

By Lemma 3.7, the numerical class of B is not one of (6.27), (6.113), (6.117) of the list in Section 6.

Lemma 3.8 *If there are irreducible curves B_1, B_2, B_3 such that $B = 2B_1 + 3B_2 + 6B_3$ and $(B_2 \cdot B_2) \geq 1$ and $(B_i \cdot B_j) \neq 0$ for $1 \leq i < j \leq 3$, then $(B_2 \cdot B_2) = 1$.*

Proof Theorem 2.5, $G_{B_1} \cong \mathbb{Z}/2\mathbb{Z}$, $G_{B_2} \cong \mathbb{Z}/3\mathbb{Z}$, $G_{B_3} \cong \mathbb{Z}/6\mathbb{Z}$. Since $(B_i \cdot B_j) \neq 0$ for $1 \leq i < j \leq 3$, we get $G_{B_1} \oplus G_{B_2} \cap G_{B_3} = \{\mathrm{id}_X\}$. Therefore, $G = G_{B_1} \oplus G_{B_2} \oplus G_{B_3}$. Since $(B_2 \cdot B_2) > 0$, we get that $p^*B_2 = 3C_2$ and the only curve of $\mathrm{Fix}(G_{B_2})$ is C_2.

We assume that $(B_2 \cdot B_2) \geq 2$. Since $|G| = 36$, $(C_{1,1}^2 \cdot C_{1,1}^2) \geq 8$, and hence the genus of $C_{1,1}^2$ is at least 5. By [1,14] and the only curve of $\mathrm{Fix}(G_{B_2})$ is C_2, this is a contradiction.

By Lemma 3.8, the numerical class of B is not one of (6.29), (6.214) of the list in Section 6.

Lemma 3.9 *If there are irreducible curves B_1, B_2, B_3 such that $B = 2B_1 + 4B_2 + 4B_3$ and $(B_i \cdot B_j) \neq 0$ for $1 \leq i < j \leq 3$, then $(B_1 \cdot B_2) = 1$.*

Proof Theorem 2.5, $G_{B_1} \cong \mathbb{Z}/2\mathbb{Z}$ and $G_{B_i} \cong \mathbb{Z}/4\mathbb{Z}$ for $i = 2, 3$. Since $(B_i \cdot B_j) \neq 0$ for $1 \leq i < j \leq 3$, we get $G_{B_1} \cap (G_{B_2} \oplus G_{B_3}) = \{\mathrm{id}_X\}$. Therefore, $G = G_{B_1} \oplus G_{B_2} \oplus G_{B_3}$. Let $s \in G_{B_1}$ and $t \in G_{B_2}$ be generators. Then $s \circ t$ is a non-symplectic automorphism of order 4 and $p^{-1}(B_1) \cap p^{-1}(B_2) \subset \mathrm{Fix}(s \circ t)$. By Theorem 2.5 and $G = G_{B_1} \oplus G_{B_2} \oplus G_{B_3}$, $\mathrm{Fix}(s \circ t)$ does not contain a curve. By [2, Proposition 1], the number of isolated points of $\mathrm{Fix}(s \circ t)$ is 4. If $(B_1 \cdot B_2) \geq 2$, then $|p^{-1}(B_1) \cap p^{-1}(B_2)| \geq 8$. This is a contradiction.

By Lemma 3.9, the numerical class of B is not (6.30), (6.109), (6.155), (6.215) of the list in Section 6.

Lemma 3.10 *We assume that $X/G \cong \mathbb{P}^1 \times \mathbb{P}^1$. Then $B \neq a(\{q\} \times \mathbb{P}^1) + bC_1 + cC_2$ where C_1 and C_2 are smooth curves on $\mathbb{P}^1 \times \mathbb{P}^1$, $C_1 \cap C_2 \neq \emptyset$, and a, b, c are even integers.*

Proof We assume that $B = a(\{q\} \times \mathbb{P}^1) + bC_1 + cC_2$ where C_1 and C_2 are smooth curves on $\mathbb{P}^1 \times \mathbb{P}^1$, $C_1 \cap C_2 \neq \emptyset$, and a, b, c are even integers. Since $C_1 \cap C_2 \neq \emptyset$, by Theorem 2.5, $G = G_{C_1} \oplus G_{C2}$. Since b, c are even integers, G_{C_1}, G_{C_2} are cyclic groups, and $G = G_{C_1} \oplus G_{C2}$, the number of non-symplectic involution of G is 2. Since $(B_{1,0} \cdot C_i) \neq 0$ and a is even, G must have at least 3 non-symplectic involutions. This is a contradiction.

By Lemma 3.10, the numerical class of B is not one of (6.31), (6.32), (6.33), (6.35), (6.36), (6.37), (6.38) of the list in Section 6.

Lemma 3.11 *If there are irreducible curves B_i and positive integers $b_i \geq 2$ for $i = 1, \cdots, k$ such that $B = \sum_{i=1}^{k} b_i B_i$, $G = G_{B_1} \oplus G_{B_2}$ and b_1 and b_2 are coprime, then for each $i = 1, 2$, $j = 3, \cdots, k$, we get that b_i and b_j are coprime if $(B_i \cdot B_j) \neq 0$.*

Proof Let $s \in G_{B_1}$ and $t \in G_{B_2}$ be generators. By Theorem 2.5, the order of s is b_1 and that of t is b_2. Since $G = G_{B_1} \oplus G_{B_2}$, there are integers u and v such that G_{B_j} is generated by $s^u \circ t^v$.

We assume that $(B_1 \cdot B_j) \neq 0$ and b_1 and b_j are not coprime. Since b_1 and b_2 are coprime, there is an integer l such that $(s^u \circ t^v)^l \neq \mathrm{id}_X$ and $(s^u \circ t^v)^l = s^m$ or t^m. Since b_1 and b_j are not coprime, we assume that $(s^u \circ t^v)^l = s^m$. Then $p^{-1}(B_1)$ and $p^{-1}(B_j)$ are contained in $\mathrm{Fix}(s^m)$. By the fact that the fixed locus of an automorphism is a pairwise set of points and curves, this is a contradiction.

By Theorem 2.5 and Lemma 3.11, the numerical class of B is not one of (6.34), (6.40), (6.265), (6.266), (6.293), (6.294), (6.308), (6.309) of the list in Section 6.

We assume that the numerical class of B is (6.39) of the list in Section 6. We denote B by $3B_{1,0} + 3B_{2,2} + 3B_{0,1}$. By Theorem 2.5, $G = G_{2,2}$. Since $G_{2,2} \cong \mathbb{Z}/3\mathbb{Z}$, G has 1 subgroups generated by a non-symplectic automorphism of order 3. Since $(B_{1,0} \cdot B_{2,2}) \neq 0$, G contains at least 2 such a subgroup from Theorem 2.5. This is a contradiction.

Lemma 3.12 *If there are irreducible curves B_1, B_2, B_3 such that $B = 2B_1 + 2B_2 + 2B_3$, and $(B_i \cdot B_j) \neq 0$ for $1 \leq i < j \leq 3$, then we get that $\frac{B_3}{2} \in \mathrm{Pic}(\mathbb{F}_n)$ if $(B_1 \cdot B_2) = 4$.*

Proof By Theorem 2.5, $G_{B_i} \cong \mathbb{Z}/2\mathbb{Z}$ for $i = 1, 2, 3$. Since $(B_i \cdot B_j) \neq 0$ for $1 \leq i < j \leq 3$, by Theorem 2.5, $G = G_{B_1} \oplus G_{B_2} \oplus G_{B_3}$.

We assume that $(B_1 \cdot B_2) = 4$. Then $p^{-1}(B_1) \cap p^{-1}(B_2)$ is a set of 8 points. Since the fixed locus of a symplectic automorphism of order 2 is a set of 8 isolated points, $X/G_{B_1} \oplus G_{B_2}$ is smooth. Then there is a double cover $X/G_{B_1} \oplus G_{B_2} \to X/G \cong \mathbb{F}_n$ whose branch divisor is $2B_3$. Thus, $\frac{B_3}{2} \in \mathrm{Pic}(\mathbb{F}_n)$ for $i = 1, 2$.

By Lemma 3.12, the numerical class of B is not one of (6.41), (6.119), (6.122), (6.217) of the list in Section 6.

Lemma 3.13 *If there are irreducible curves B_1, B_2, B_3 such that $B = 2B_1 + 2B_2 + 2B_3$, and $(B_i \cdot B_j) \neq 0$ for $1 \leq i < j \leq 3$, then $(B_i \cdot B_j) \leq 4$ for $1 \leq i < j \leq 3$.*

Proof By Theorem 2.5, $G_{B_i} \cong \mathbb{Z}/2\mathbb{Z}$ for $i = 1, 2, 3$ and $G = G_{B_1} \oplus G_{B_2} \oplus G_{B_3}$. Let $s, t \in G$ be generators of G_{B_i} and G_{B_j}, respectively, where $1 \leq i < j \leq 3$. Then $s \circ t$ is a symplectic automorphism of order 2 and $p^{-1}(B_i) \cap p^{-1}(B_j) \subset \mathrm{Fix}(s \circ t)$. Since $|G| = 8$, we get $2(B_i \cdot B_j) = |p^{-1}(B_i) \cap p^{-1}(B_j)|$. Thus, we have that $(B_i \cdot B_j) \leq 4$.

By Lemma 3.13, the numerical class of B is not one of (6.42), (6.120) of the list in Section 6.

Lemma 3.14 *We assume that $X/G \cong \mathbb{P}^1 \times \mathbb{P}^1$. Then $B \neq a_1(\{q_1\} \times \mathbb{P}^1) + a_2(\{q_2\} \times \mathbb{P}^1) + bC' + c(\mathbb{P}^1 \times \{q_3\})$, where C' is an irreducible curve, $C' = (nC + mF)$ in $\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)$, $n, m > 0$, and a_1a_2, b, c are even integers.*

Proof We assume that $B = a_1(\{q_1\} \times \mathbb{P}^1) + a_2(\{q_2\} \times \mathbb{P}^1) + bC' + c(\mathbb{P}^1 \times \{q_3\})$, where C' is an irreducible curve, $C' = (nC + mF)$, $n, m > 0$, and a_1a_2, b, c are even integers. By Theorem 2.5, $G = G^2_{1,0} \oplus G_{C'}$. By a_1a_2 and b are even integers, the number of non-symplectic involution of G is 2. Since $(B_{0,1} \cdot C') \neq 0$ and $(B_{0,1} \cdot B^i_{1,0}) \neq 0$ for $i = 1, 2$ and c is an even integer, G must have at least 3 non-symplectic involutions. This is a contradiction.

By Lemma 3.14, the numerical class of B is not one of (6.43), (6.44) of the list in Section 6.

Lemma 3.15 *We assume that $X/G \cong \mathbb{P}^1 \times \mathbb{P}^1$. Then $B \neq a_1(\{q_1\} \times \mathbb{P}^1) + b_1C_1 + b_2C_2 + a_2(\mathbb{P}^1 \times \{q_2\})$, where C_i is an irreducible curve, $C_i = (n_iC + m_iF)$ in $\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)$, $n_i, m_i > 0$ for $i = 1, 2$, and a_1, a_2, b_1b_2 are even integers.*

Proof We assume that $B = a_1(\{q_1\} \times \mathbb{P}^1) + b_1C_1 + b_2C_2 + a_2(\mathbb{P}^1 \times \{q_2\})$, where C_i is an irreducible curve, $C = (n_i, m_i)$ in $\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)$, $n_i, m_i > 0$ for $i = 1, 2$, and a_1, a_2, b_1b_2 are even integers. By Theorem 2.5, $G = G_{C_1} \oplus G_{C_2}$. By b_1b_2 is an even integer, the number of non-symplectic involutions of G is at most 2. Since $(B_{1,0} \cdot C_i) \neq 0$ and $(B_{0,1} \cdot C_i) \neq 0$ for $i = 1, 2$, and a_1 and a_2 are even integers, G must have at least 3 non-symplectic involutions. This is a contradiction.

By Lemma 3.15, the numerical class of B is not one of (6.47)–(6.52) of the list in Section 6.

We assume that the numerical class of B is (6.53) of the list in Section 6. We denote B by $3B_{1,0} + 2B^1_{1,1} + 6B^2_{1,1} + 3B_{0,1}$. By Theorem 2.5, $G^1_{1,1} \cong \mathbb{Z}/2\mathbb{Z}$ and $G^2_{1,1} \cong \mathbb{Z}/6\mathbb{Z}$ and $G = G^1_{1,1} \oplus G^2_{1,1}$. Then the number of subgroups of G which is generated by a non-symplectic automorphism of order 3 is 1. By Theorem 2.5 and $(B_{1,0} \cdot B^2_{1,1}) \neq 0$, G must have at least 2 such subgroups. This is a contradiction.

We assume that the numerical class of B is (6.54) of the list in Section 6. We denote B by $3B_{1,0} + 3B^1_{1,1} + 3B^2_{1,1} + 3B_{0,1}$. By Theorem 2.5, $G^i_{1,1} \cong \mathbb{Z}/3\mathbb{Z}$ for $i = 1, 2$, and $G = G^1_{1,1} \oplus G^2_{1,1}$. Then the number of subgroups of G which is generated by a non-symplectic automorphism of order 3 is 3. By Theorem 2.5, $(B_{1,0} \cdot B^i_{1,1}) \neq 0$ and $(B_{1,0} \cdot B_{0,1}) \neq 0$, G must have at least 4 such subgroups. This is a contradiction.

We assume that the numerical class of B is (6.56) of the list in Section 6. We denote B

by $2B^1_{1,0} + 6B^2_{1,0} + 3B_{1,2} + 3B_{0,1}$. By Theorem 2.5, $G^1_{1,0} \cong \mathbb{Z}/2\mathbb{Z}$ and $G_{1,2} \cong \mathbb{Z}/3\mathbb{Z}$, and $G = G^1_{1,0} \oplus G_{1,2}$. Then the number of subgroups of G which is generated by a non-symplectic automorphism of order 3 is 1. By Theorem 2.5 and $(B_{0,1} \cdot B_{1,2}) \neq 0$, G must have at least 2 such subgroups. This is a contradiction.

We assume that the numerical class of B is (6.58) of the list in Section 6. We denote B by $2B_{1,0} + 2B^1_{1,1} + 2B^2_{1,1} + 2B^3_{1,1} + 2B_{0,1}$. By Theorem 2.5, $G^i_{1,1} \cong \mathbb{Z}/2\mathbb{Z}$ for $i = 1, 2, 3$. Since $(B^i_{1,1} \cdot B^j_{1,1}) \neq 0$ for $1 \leq i < j \leq 3$ and $G^i_{1,1} \cong \mathbb{Z}/2\mathbb{Z}$ for $i = 1, 2, 3$, $G = G^1_{1,1} \oplus G^2_{1,1} \oplus G^3_{1,1}$. Then the number of non-symplectic involutions of G is 4. Since $(B_{1,0} \cdot B_{0,1}) \neq 0$, $(B_{0,1} \cdot C_i) \neq 0$ and $(B_{1,0} \cdot C_i) \neq 0$ for $i = 1, 2, 3$, G must have at least 5 non-symplectic involutions. This is a contradiction.

Lemma 3.16 *We assume that $X/G \cong \mathbb{P}^1 \times \mathbb{P}^1$. If $B = \sum_{i=1}^{2} a_i(\{p_i\} \times \mathbb{P}^1) + bC' + \sum_{j=1}^{2} c_j(\mathbb{P}^1 \times \{q_j\})$, where C' is an irreducible curve, $\{p_i\} \times \mathbb{P}^1 \cap C' \neq \emptyset$, $C \cap \mathbb{P}^1 \times \{q_i\} \neq \emptyset$, $a_i, c_1, c_2, b \in \mathbb{N}_{\geq 2}$, then $a_1 = a_2$ and $c_1 = c_2$.*

Proof Let C_{p_1} be one of irreducible components of $p^*(\{p_1\} \times \mathbb{P}^1)$. Since $(\{p_1\} \times \mathbb{P}^1 \cdot \{p_1\} \times \mathbb{P}^1) = 0$, C_{p_1} is an elliptic curve. Let $\pi : X \to Y := X/G_{C'}$ be the quotient map, and $G' := G/G_{C'}$ be a finite Abelian subgroup of $\mathrm{Aut}(Y)$. Since $\{p_i\} \times \mathbb{P}^1 \cap C \neq \emptyset$, the finite map $\pi_{|C_{p_1}} : C_{p_1} \to C'_{p_1} := \pi(C_{p_1})$ is a branched cover. Since C_{p_1} is an elliptic curve, C'_{p_1} is $\mathbb{P}^1$ Since the branch divisor of the quotient map $\pi' : Y \to Y/G' \cong \mathbb{P}^1 \times \mathbb{P}^1$ is $\sum_{i=1}^{2} a_i\{p_i\} \times \mathbb{P}^1 + \sum_{j=1}^{2} c_j \mathbb{P}^1 \times \{q_j\}$, the branch divisor of $\pi_{C'_{p_1}} : C'_{p_1} \to p_1 \times \mathbb{P}^1$ is $c_1 q_1 + c_2 q_2$. By Theorem 2.3, we get that $c_1 = c_2$. In the same way, we obtain that $a_1 = a_2$.

By Lemma 3.16, the numerical class of B is not one of (6.61)–(6.64) of the list in Section 6.

We assume that the numerical class of B is one of (6.69)–(6.78) of the list in Section 6. By Theorem 2.3, there are an Abelian surface and a finite group G such that $A/G = \mathbb{P}^1 \times \mathbb{P}^1$ and the branch divisor is B. By Theorem 2.2, there is a surjective morphism from a $K3$ surface to an Abelian surface. This is a contradiction.

Lemma 3.17 *If $X/G \cong \mathbb{F}_n$ where $n \geq 1$, then $B \neq aC + bB_{s,t} + cB_{u,v} + dB_{0,1}$, where $a, d \geq 0$ are even integers, $a = 0$ or $a \geq 2$, and $b, c > 0$ are even integers.*

Proof We assume that $B = aC + bB_{s,t} + cB_{u,v} + dB_{0,1}$ where $a, d \geq 0$ are even integers, $a = 0$ or $a \geq 2$, and $b, c > 0$ are even integers. By Theorem 2.5 and $(B_{s,t} \cdot B_{u,v}) \neq 0$, we get that $G = G_{s,t} \oplus G_{u,v}$. Then the number of non-symplectic involution of G is 2. Since $(B_{s,t} \cdot B_{0,1}) \neq 0$ and $(B_{u,v} \cdot B_{0,1}) \neq 0$, G must have at least 3 non-symplectic involutions. This is a contradiction.

By Lemma 3.17, the numerical class of B is not one of (6.104), (6.114), (6.115), (6.118), (6.141), (6.148), (6.184), (6.185), (6.187), (6.232), (6.234), (6.245), (6.246), (6.259), (6.291) of the list in Section 6.

Lemma 3.18 *For the branch divisor $B = \sum_{i=1}^{k} b_i B_i$, we get that $\frac{|G|}{b_i^2}(B_i \cdot B_i)$ is an even integer for $1 \leq i \leq k$.*

Proof For $i = 1, \cdots, k$, we put $p^*B_i = \sum_{j=1}^{l} b_i C_j$ where C_j is a smooth curve for $j = 1, \cdots, l$.

By Theorem 2.5, $C_1, \cdots, C_l$ are pairwise disjoint. Then we get that $\frac{|G|}{b_i^2}(B_i \cdot B_i) = \sum_{j=1}^{l}(C_j \cdot C_j)$. Since X is a $K3$ surface, $(C_j \cdot C_j)$ is an even integer, and hence $\frac{|G|}{b_i^2}(B_i \cdot B_i)$ is an even integer.

By Lemma 3.18, the numerical class of B is not one of (6.106), (6.107), (6.140), (6.147), (6.180), (6.183), (6.231), (6.233), (6.258) of the list in Section 6.

We assume that the numerical class of B is (6.110) of the list in Section 6. We denote B by $2B_{1,2} + 4B_{1,1}^1 + 4B_{1,1}^2 + 2B_{0,1}$. By Theorem 2.5, $G_{1,2} \cong \mathbb{Z}/2\mathbb{Z}$ and $G_{1,1}^i \cong \mathbb{Z}/4\mathbb{Z}$ for $i = 1, 2$ Since $(B_{1,2} \cdot B_{1,1}^i) \neq 0$ for $i = 1, 2$, by Theorem 2.5, $G = G_{1,2} \oplus G_{1,1}^1 \oplus G_{1,1}^2$, and hence $|G| = 36$. Let $s \in G_{1,2}$ and $t \in G_{1,1}^1$ be generators. Then $s \circ t$ is a non-symplectic automorphism of order 4 and $p^{-1}(B_{1,2}) \cap p^{-1}(B_{1,1}^1) \subset \mathrm{Fix}(s \circ t)$. Since $G = G_{1,2} \oplus G_{1,1}^1 \oplus G_{1,1}^2$ and $B = 2B_{1,2} + 4B_{1,1}^1 + 4B_{1,1}^2 + 2B_{0,1}$, by Theorem 2.5, $\mathrm{Fix}(s \circ t)$ does not contain a curve. By [2, Proposition 1], the number of isolated points of $\mathrm{Fix}(s \circ t)$ is 4. Since $(B_{1,2} \cdot B_{1,1}^1) = 2$, we get that $|p^{-1}(B_{1,2}) \cdot p^{-1}(B_{1,1}^1)| \geq 8$. This is a contradiction.

We assume that the numerical class of B is (6.121) of the list in Section 6. We denote B by $2B_{2,3} + 2B_{1,1}^1 + 2B_{1,1}^2 + 2B_{0,1}$. By Theorem 2.5, $G_{2,3} \cong G_{1,1}^1 \cong G_{1,1}^2 \cong \mathbb{Z}/2\mathbb{Z}$. Since an intersection of two of $B_{2,3}, B_{1,1}^1, B_{1,1}^2$ is not an empty set, by Theorem 2.5, $G = G_{2,3} \oplus G_{1,1}^1 \oplus G_{1,1}^2$, and hence $|G| = 8$. Let $s \in G_{2,3}$ and $t \in G_{0,1}$ be generators. Since s and t are non-symplectic involutions, $\mathrm{Fix}(s)$ and $\mathrm{Fix}(t)$ are sets of curves and $\mathrm{Fix}(s \circ t)$ is a set of 8 isolated points. Since $(B_{2,3} \cdot B_{0,1}) = 2$, $|p^{-1}(B_{2,3}) \cap p^{-1}(B_{1,1}^1)| = 4$. Since $\mathrm{Fix}(s \circ t) \supset p^{-1}(B_{2,3}) \cap p^{-1}(B_{0,1})$, $X/(G_{2,3} \oplus G_{0,1})$ has 2 singular points, however, since the branch divisor of the double cover $X/(G_{2,3} \oplus G_{0,1}) \to X/G$ is $2B_{1,1}^1 + 2B_{1,1}^2$ and $(B_{1,1}^1 \cdot B_{1,1}^2) = 1$, the number of singular points of $X/(G_{2,3} \oplus G_{0,1})$ must be 1. This is a contradiction.

As for the case of (6.121), the numerical class of B is not one of (6.191)–(6.192) of the list in Section 6.

We assume that the numerical class of B is (6.123) of the list in Section 6. We denote B by $2B_{2,2} + 2B_{1,2} + 2B_{1,1} + 2B_{0,1}$. By Theorem 2.5, $G_{2,2} \cong G_{1,2} \cong G_{1,1} \cong \mathbb{Z}/2\mathbb{Z}$. Since an intersection of two of $B_{2,2}, B_{1,2}, B_{1,1}$ is not an empty set, by Theorem 2.5, $G = G_{2,2} \oplus G_{1,2} \oplus G_{1,1}$. Since $(B_{2,2} \cdot B_{1,2}) = 4$, $X/(G_{2,2} \oplus G_{1,2})$ is smooth. Then there is a double cover $X/G_{2,2} \oplus G_{1,2} \to X/G \cong \mathbb{F}_1$ whose branch divisor is $2B_{1,1} + 2B_{0,1}$. Since $\frac{B_{1,1}+B_{0,1}}{2} \notin \mathrm{Pic}(\mathbb{F}_1)$, by Theorem 3.1, this is a contradiction.

We assume that the numerical class of B is (6.125) of the list in Section 6. We denote B by $2B_{1,2}^1 + 2B_{1,2}^2 + 2B_{1,1}^1 + 2B_{1,1}^2$. By Theorem 2.5, $G_{1,2}^i \cong G_{1,1}^i \cong \mathbb{Z}/2\mathbb{Z}$ for $i = 1, 2$. Since an intersection of two of $B_{1,2}^1, B_{1,2}^2, B_{1,1}^1, B_{1,1}^2$ is not an empty set, by Theorem 2.5, $G = G_{1,2}^1 \oplus G_{1,2}^2 \oplus G_{1,1}^1 \oplus G_{1,1}^2$ or $G = G_{1,2}^1 \oplus G_{1,2}^2 \oplus G_{1,1}^1$. We assume that $G = G_{1,2}^1 \oplus G_{1,2}^2 \oplus G_{1,1}^1 \oplus G_{1,1}^2$. Since $|G| = 16$ and $(B_{1,2}^1 \cdot B_{1,2}^2) = 3$, $|p^{-1}(B_{1,2}^1 \cap B_{1,2}^2)| \geq 12$. Since the number of isolated points of symplectic involution is 8, this is a contradiction. Therefore, $G = G_{1,2}^1 \oplus G_{1,2}^2 \oplus G_{1,1}^1$.

By Theorem 3.1, there are the Galois covers $p_1 : Y_1 \to \mathbb{F}_1$ and $p_2 : Y_2 \to \mathbb{F}_1$ such that the branch divisor of p_1 is $2B_{1,2}^1 + 2B_{1,2}^2$ and that of p_2 is $2B_{1,1}^1 + 2B_{1,1}^2$. Let $X' := Y_1 \times_{\mathbb{F}_1} Y_2$. Then there is the Galois cover $q : X' \to \mathbb{F}_1$ whose branch divisor is $2B_{1,2}^1 + 2B_{1,2}^2 + 2B_{1,1}^1 + 2B_{1,1}^2$ and Galois group is isomorphic to $\mathbb{Z}/2\mathbb{Z}^{\oplus 2}$ as a group. By Theorem 2.1, there is a symplectic automorphism of order 2, $s \in G$ such that $X' = X/\langle s \rangle$. Since s is symplectic, the minimal resolution $f : X'_m \to X'$ is a $K3$ surface. Let $e_1, \cdots, e_8$ be the exceptional divisors of f. We set $\{p_1, p_2, p_3\} := B_{1,2}^1 \cap B_{1,2}^2$ and $\{p_4\} := B_{1,1}^1 \cap B_{1,1}^2$. Let $\pi : \mathrm{Blow}_{\{p_1,p_2,p_3,p_4\}}\mathbb{F}_1 \to \mathbb{F}_1$ be the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at points p_1, p_2, p_3, p_4, and $E_i := \pi^{-1}(p_i)$ be an exceptional divisor of π for $i = 1, 2, 3, 4$. Since the support of B is simple normal crossing, in the same way of Proposition 3.6, there is a Galois cover $q : X'_m \to \mathrm{Blow}_{\{p_1,p_2,p_3,p_4\}}\mathbb{F}_1$ whose branch divisor is $2C^1_{1,2} + 2C^2_{1,2} + 2C^1_{1,1} + 2C^2_{1,1}$ and Galois group is isomorphic to $\mathbb{Z}/2\mathbb{Z}^{\oplus 2}$ as a group, where $C^1_{1,2}, C^2_{1,2}, C^1_{1,1}, C^2_{1,1}$ are proper transforms of $B^1_{1,2}, B^2_{1,2}, B^1_{1,1}, B^2_{1,1}$ by the birational map π^{-1} in order. Notice that $q^*\big(\sum_{i=1}^{4} E_i\big) = \sum_{j=1}^{8} e_j$ and there is the commutative diagram:

$$\begin{array}{ccc} X' & \longrightarrow & \mathbb{F}_1 \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle \pi} \\ X'_m & \xrightarrow[q]{} & \mathrm{Blow}_{\{p_1,p_2,p_3,p_4\}}\mathbb{F}_1. \end{array}$$

Furthermore, we put $\{x_1, \cdots, x_8\} := \mathrm{Fix}(s)$. Then $\mathrm{Blow}_{\{x_1,\cdots,x_8\}}X/\langle s\rangle = X'_m$, the branch divisor of the double cover $\mathrm{Blow}_{\{x_1,\cdots,x_8\}}X \to X'_m$ is $\sum_{j=1}^{8} e_j$, and there is the commutative diagram:

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \uparrow & & \uparrow \\ \mathrm{Blow}_{\{x_1,\cdots,x_8\}}X & \longrightarrow & X'_m. \end{array} .$$

In the same way of Proposition 3.6, we get that

$$\sum_{i=1}^{4} q^*E_i = 2(\pi \circ q)^*\Big(C + \frac{3}{2}F\Big) - 2C^1_{1,2} - 2C^1_{1,1} \quad \text{in } \mathrm{Pic}(X'_m).$$

Since $\mathrm{Blow}_{\{x_1,\cdots,x_8\}}X$ and X'_m are smooth, and $q^*\big(\sum_{i=1}^{4} E_i\big) = \sum_{j=1}^{8} e_j$, we get that $\frac{\sum_{i=1}^{4} q^*E_i}{2} \in \mathrm{Pix}(X'_m)$, and hence $\frac{F}{2} \in \mathrm{Pic}(X'_m)$.

Since $C^1_{1,2} \cap C^2_{1,2}$ is an empty set and $\frac{C^1_{1,2}+C^2_{1,2}}{2} \in \mathrm{Pic}(\mathrm{Blow}_{\{p_1,p_2,p_3,p_4\}}\mathbb{F}_1)$, by Theorem 3.1, there is the Galois cover $g : Z \to \mathrm{Blow}_{\{p_1,p_2,p_3,p_4\}}\mathbb{F}_1$ such that Z is smooth, the branch divisor is $2C^1_{1,2} + 2C^2_{1,2}$, and the Galois group is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ as a group. By Theorem 2.1, there is a non-symplectic automorphism of order 2ρ of X'_m such that $X'_m/\langle\rho\rangle = Z$. Let $h : X'_m \to Z$ be the quotient map. Then $q = g \circ h$, and hence $\frac{F}{2} \in \mathrm{Pic}(X'_m)^\rho$. Since the degree of g is 2 and $(C^1_{1,2} \cdot \frac{F}{2}) = \frac{1}{2}$ and $\frac{g^*C^1_{1,2}}{2} \in \mathrm{Pic}(Z)$, we get that $g^*\frac{F}{2} \notin \mathrm{Pic}(Z)$. Recall that $C^i_{1,1} = C + F - e_4$ in $\mathrm{Pic}(\mathrm{Blow}_{\{p_1,p_2,p_3,p_4\}}\mathbb{F}_1)$ for $i = 1, 2$. Since the branch divisor of h is $2g^*C^1_{1,1} + 2g^*C^2_{1,1}$, we get that $q^*\big(\frac{1}{2}C + \frac{1}{2}F - e_4\big) \in \mathrm{Pic}(X'_m)$. By [2], $\mathrm{Pic}(X'_m)^\rho$ is generated by $h^*\mathrm{Pic}(Z)$ and $q^*\big(\frac{1}{2}C + \frac{1}{2}F - e_4\big)$ over $\mathbb{Z}$. Since $g^*\frac{F}{2} \notin \mathrm{Pic}(Z)$ and $2q^*\big(\frac{1}{2}C + \frac{1}{2}F - e_4\big) \in h^*\mathrm{Pic}(Z)$, we may assume that there is $\alpha \in \mathrm{Pic}(Z)$ such that

$$q^*\frac{F}{2} = h^*\alpha + q^*\Big(\frac{1}{2}C + \frac{1}{2}F - e_4\Big).$$

Then $g^*\big(\frac{-1}{2}C + e_4\big) \in \mathrm{Pic}(Z)$. Since the degree of g is 2 and $(C^1_{1,2} \cdot \frac{-1}{2}C + e_4) = \frac{3}{2}$ and $\frac{g^*C^1_{1,2}}{2} \in \mathrm{Pic}(Z)$, we get that $\big(\frac{g^*C^1_{1,2}}{2} \cdot g^*\big(\frac{-1}{2}C + e_4\big)\big) = \frac{3}{2}$. By the assumption that $\frac{g^*C^1_{1,2}}{2} \in \mathrm{Pic}(Z)$ and $g^*\big(\frac{-1}{2}C + e_4\big) \in \mathrm{Pic}(Z)$, this is a contradiction. Therefore, the numerical class of B is not (6.125) of the list in Section 6.

We assume that the numerical class of B is (6.126) of the list in Section 6. We denote B by $2B_{1,2}+2B^1_{1,1}+2B^2_{1,1}+2B^3_{1,1}+2B_{0,1}$. By Theorem 2.5, $G_{1,2}\cong G^i_{1,1}\cong G_{0,1}\cong \mathbb{Z}/2\mathbb{Z}$ where $i=1,2,3$. Since an intersection of two of $B_{1,2},B^1_{1,1},B^2_{1,1},B^3_{1,1},B_{0,1}$ is not an empty set, by Theorem 2.5, $G=G_{1,2}\oplus G^1_{1,1}\oplus G^2_{1,1}\oplus G^3_{1,1}$. Let G_s be the subgroup of G which consists of symplectic automorphisms of G. Then $G_s\cong \mathbb{Z}/2\mathbb{Z}^{\oplus 3}$. By [16], the number of singular points of X/G_s is 14, however, since the branch divisor of the double cover $X/G_s\to X/G$ is $B=2B_{1,2}+2B^1_{1,1}+2B^2_{1,1}+2B^3_{1,1}+2B_{0,1}$ and the support of B is simple normal crossing, the number of singular points of X/G_s is 13. This is a contradiction. Therefore, the numerical class of B is not (6.126) of the list in Section 6.

Lemma 3.19 *If $X/G\cong \mathbb{F}_n$ where $n\geq 1$, then $B\neq aC+bB_{s,t}+cB_{u,v}$ where $a,b,c>0$ are even integers, and $(C\cdot B_{s,t})\neq 0$ and $(C\cdot B_{u,v})\neq 0$, i.e., $s\neq t$ or $u\neq v$.*

Proof We assume that $B=aC+bB_{s,t}+cB_{u,v}$ where $a,b,c>0$ are even integers, and $(C\cdot B_{s,t})\neq 0$ and $(C\cdot B_{u,v})\neq 0$ By Theorem 2.5 and $(B_{s,t}\cdot B_{u,v})\neq 0$, $G=G_{s,t}\oplus G_{u,v}$. Then the number of non-symplectic involutions of G is 2. Since $(C\cdot B_{s,t})\neq 0$ and $(C\cdot B_{u,v})\neq 0$, G must have at least 3 non-symplectic involutions. This is a contradiction.

By Lemma 3.19, the numerical class of B is not one of (6.139), (6.181), (6.182), (6.244) of the list in Section 6.

We assume that the numerical class of B is (6.189) of the list in Section 6. We denote B by $2B_{1,0}+2B_{1,4}+2B^1_{1,1}+2B^2_{1,1}$. By Theorem 2.5, $G_{1,0}\cong G_{1,4}\cong G^i_{1,1}\cong \mathbb{Z}/2\mathbb{Z}$ where $i=1,2$. Since $(B_{1,4}\cdot B^i_{1,1})\neq 0$ for $i=1,2$, by Theorem 2.5, $G=G_{1,4}\oplus G^1_{1,1}\oplus G^2_{1,1}$. Let $s\in G^1_{1,1}$ and $t\in G^2_{1,1}$ be generators. Since the number of non-symplectic automorphisms of order 2 of G is 4 and Theorem 2.5, we may assume that $\mathrm{Fix}(s)$ is the support of $p^*B^1_{1,1}$. Since the support of B is simple normal crossing and $(B_{1,4}\cdot B^1_{1,1})=4$, $X/(G_{1,4}\oplus G^1_{1,1})$ is smooth. Then there is the Galois cover $X/G_{1,4}\oplus G^1_{1,1}\to \mathbb{F}_1$ such that the branch divisor is $2B_{1,0}+2B^2_{1,1}$ and the Galois group is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ as a group. Since $\frac{B_{1,0}+B^2_{1,1}}{2}\notin \mathrm{Pic}(\mathbb{F}_1)$, this is a contradiction.

As for the case of (6.189), the numerical class of B does not (6.190) of the list in Section 6.

We assume that the numerical class of B is (6.228) of the list in Section 6. We denote B by $3B_{1,0}+3B_{1,2}+3B_{1,4}$. By Theorem 2.5 and $(B_{1,2}\cdot B_{1,4})\neq 0$, $G=G_{1,2}\oplus G_{1,4}$. Let $s\in G_{1,4}$ be a generator of $G_{1,4}$. Then the only curve of $\mathrm{Fix}(s)$ is $C_{1,4}$. Since $(B_{1,4}\cdot B_{1,4})=6$, the genus of $C_{1,4}$ is 4. By [1,14], $\mathrm{Fix}(s)$ does not have isolated points, and hence $X/G_{1,4}$ is smooth. Let $q:X/G_{1,4}\to X/G$ be the quotient map. Then the degree of q is 3, and the branch divisor of q is $3B_{1,0}+3B_{1,2}$. Since the degree of q is 3 and $X/G_{1,4}$ is smooth, $\frac{3}{3^2}(B_{1,0}\cdot B_{1,0})$ is an integer. Since $(B_{1,0}\cdot B_{1,0})=-2$, $\frac{3}{3^2}(B_{1,0}\cdot B_{1,0})=-\frac{2}{3}$. This is a contradiction.

We assume that the numerical class of B is (6.229) of the list in Section 6. We denote B by $3B_{1,0}+3B_{1,2}+3B_{1,3}+3B_{0,1}$. By Theorem 2.5, $G_{1,0}\cong G_{1,2}\cong G_{1,3}\cong G_{0,1}\cong \mathbb{Z}/3\mathbb{Z}$. Since $(B_{1,2}\cdot B_{1,3})\neq 0$, by Theorem 2.5, $G=G_{1,2}\oplus G_{1,3}$. Let $s,t\in G$ be generators of $G_{1,2}$ and $G_{1,3}$ respectively such that $s\circ t$ is a non-symplectic automorphism of order 3. Since $G=G_{1,2}\oplus G_{1,3}$, the number of subgroups of G which are generated by a non-symplectic automorphism of order 3 is 3. Since $(B_{1,2}\cdot B_{0,1})\neq 0$ and $(B_{1,3}\cdot B_{0,1})\neq 0$, we get that $p^{-1}(B_{0,1})$ is contained in $\mathrm{Fix}(s\circ t)$, and hence $p^{-1}B_{1,0}$ is contained in $\mathrm{Fix}(s)$. Since $|G|=9$, there is an elliptic curve $C_{0,1}$ on X such that $p^*B_{0,1}=3C_{0,1}$. By [1,14], the number of isolated points of $\mathrm{Fix}(s\circ t)$ is 3. Since $(B_{1,0}\cdot B_{1,3})=1$ and $(B_{1,2}\cdot B_{1,3})=3$, we have $|p^{-1}(B_{1,0}\cup B_{1,2})\cap p^{-1}(B_{1,3})|=4$. Since $p^{-1}(B_{1,0}\cup B_{1,2})\subset \mathrm{Fix}(s)$ and $p^{-1}(B_{1,3})\subset \mathrm{Fix}(t)$, we get that $p^{-1}(B_{1,0} \cup B_{1,2}) \cap p^{-1}(B_{1,3}) \subset \mathrm{Fix}(s \circ t)$. By the fact that the number of isolated points of $\mathrm{Fix}(s \circ t)$ is 3, this is a contradiction.

We assume that the numerical class of B is (6.243) of the list in Section 6. We denote B by $2B_{1,0} + 2B_{1,4} + 2B_{2,4}$. By Theorem 2.5, $G = G_{1,4} \oplus G_{2,4}$. Let $s \in G$ be a generator of $G_{1,4}$. Since $(B_{1,0} \cdot B_{1,4}) \neq 0$ and $(B_{1,4} \cdot B_{2,4}) \neq 0$, the only curve of $\mathrm{Fix}(s)$ is $C_{1,4}$. Since the fixed locus of a non-symplectic involution does not have isolated points, $X/G_{1,4}$ is smooth. Let $q : X/G_{1,4} \to X/G \cong \mathbb{F}_2$ be the quotient map. The degree of q is 2 and the branch divisor of q is $2B_{1,0} + 2B_{2,2}$. Since $\frac{B_{1,0}+B_{2,2}}{2} \notin \mathrm{Pic}(\mathbb{F}_2)$, by Theorem 3.1, this is a contradiction.

We assume that the numerical class of B is (6.249) of the list in Section 6. We denote B by $2B_{1,0} + 2B^1_{1,3} + 2B^2_{1,3} + 2B_{1,2}$. By Theorem 2.5, $G^i_{1,3} \cong G_{1,2} \cong \mathbb{Z}/2\mathbb{Z}$ where $i = 1, 2$. Since an intersection of two of $B^1_{1,3}, B^2_{1,3}, B_{1,2}$ is not an empty set, $G = G^1_{1,3} \oplus G^2_{1,3} \oplus G_{1,2}$. Since $|G| = 8$ and $(B^1_{1,3} \cdot B^2_{1,3}) = 4$, $Y := X/(G^1_{1,3} \oplus G^2_{1,3})$ is smooth. Then there is the Galois cover $q : Y \to X/G$ such that the branch divisor is $2B_{1,0} + 2B_{1,2}$, and the Galois group is $\mathbb{Z}/2\mathbb{Z}$ as a group. Since the fixed locus of a non-symplectic automorphism of order 2 does not have isolated points, $X/G^1_{1,3}$ is smooth, and there is the Galois cover $q'' : X/G^1_{1,3} \to Y$ such that the branch divisor of q'' is $2q^*B^1_{1,3}$ and the Galois group of q'' is $\mathbb{Z}/2\mathbb{Z}$ as a group. Since Y and $X/G^1_{1,3}$ are smooth, and the degree of q'' is two, we get that $\frac{q^*B^1_{1,3}}{2} \in \mathrm{Pic}(Y)$. Recall that the branch divisor of q is $2B_{1,0} + 2B_{1,2}$, and the degree of q is two. Since $\frac{q^*B_{1,2}}{2} \in \mathrm{Pic}(Y)$, we get that $\frac{q^*F}{2} = \frac{q^*B^1_{1,3}}{2} - \frac{q^*B_{1,2}}{2} \in \mathrm{Pic}(Y)$. Since $(B_{1,0} \cdot F) = 1$, we get that $\left(\frac{q^*B_{1,0}}{2} \cdot \frac{q^*F}{2}\right) = \frac{1}{2}$. Since $\frac{q^*B_{1,0}}{2} \in \mathrm{Pic}(Y)$ and $\frac{q^*F}{2} \in \mathrm{Pic}(Y)$, this is a contradiction. Therefore, the numerical class of B is not (6.249).

We assume that the numerical class of B is (6.250) of the list in Section 6. We denote B by $2B_{1,0} + 2B_{1,3} + 2B^1_{1,2} + 2B^2_{1,2} + 2B_{0,1}$. By Theorem 2.5, $G_{1,3} \cong G^i_{1,2} \cong \mathbb{Z}/2\mathbb{Z}$ where $i = 1, 2$. Since an intersection of two of $B_{1,3}, B^1_{1,2}, B^2_{1,2}$ is not an empty set, by Theorem 2.5, $G = G_{1,3} \oplus G^1_{1,2} \oplus G^2_{1,2}$. Let $s \in G^1_{1,2}$ be a generator. Since the number of non-symplectic automorphisms of order 2 of G is 4 and Theorem 2.5, we may assume that $p^{-1}(B^1_{1,3})$ and $p^{-1}(B_{1,0})$ are contained in $\mathrm{Fix}(s)$. Since the support of B is simple normal crossing and $(B_{1,3} \cdot B_{1,0} + B^1_{1,2}) = 4$, $X/(G_{1,3} \oplus G^1_{1,2})$ is smooth and there is the Galois cover $X/(G_{1,3} \oplus G^1_{1,2}) \to \mathbb{F}_2$ such that the branch divisor is $2B^2_{1,2} + 2B_{0,1}$ and the Galois group is $\mathbb{Z}/2\mathbb{Z}$ as a group. Since $\frac{B^2_{1,2}+B_{0,1}}{2} \notin \mathrm{Pic}(\mathbb{F}_2)$, this is a contradiction.

We assume that the numerical class of B is (6.286) of the list in Section 6. We denote B by $2B_{1,0} + 3B^1_{1,4} + 6B^2_{1,4}$. By Theorem 2.5, $G_{1,0} \cong \mathbb{Z}/2\mathbb{Z}$, $G^1_{1,4} \cong \mathbb{Z}/3\mathbb{Z}$, $G^2_{1,4} \cong \mathbb{Z}/6\mathbb{Z}$ and $G = G^1_{1,4} \oplus G^2_{1,4}$. Let s be a generator of $G^1_{1,4}$. Since $(B^1_{1,4} \cdot B^1_{1,4}) = 4$, the genus of $C^1_{1,4}$ is 5 where $p^*B^1_{1,4} = 3C^1_{1,4}$. Since $G_{1,0} \cong \mathbb{Z}/2\mathbb{Z}$ and $(B^1_{1,4} \cdot B^2_{1,4}) \neq 0$, the only curve of $\mathrm{Fix}(s)$ is $C^1_{1,4}$. By [1,14], this is a contradiction.

We assume that the numerical class of B is (6.287) of the list in Section 6. We denote B by $2B_{1,0} + 4B^1_{1,4} + 4B^2_{1,4}$. By Theorem 2.5, $G^i_{1,4} \cong \mathbb{Z}/4\mathbb{Z}$ for $i = 1, 2$. Since $(B^1_{1,4} \cdot B^2_{1,4}) \neq 0$, by Theorem 2.5, $G = G^1_{1,4} \oplus G^2_{1,4}$. Let $s \in G^1_{1,4}$ and $t \in G^2_{1,4}$ be generators. Then non-symplectic involutions of G are s^2 and t^2. By Theorem 2.5, we may assume that $\mathrm{Fix}(s^2) = p^{-1}(B_{1,0}) \cup p^{-1}(B^1_{1,4})$ and $\mathrm{Fix}(t^2) = p^{-1}(B^2_{1,4})$. For a symplectic involution $s^2 \circ t^2$, since X/G is smooth, $\mathrm{Fix}(s^2 \circ t^2) \subset \mathrm{Fix}(s^2) \cap \mathrm{Fix}(t^2)$. Since $(C \cdot B^i_{1,4}) = 0$ and $(B^1_{1,4} \cdot B^1_{1,4}) = 4$, we get that $p^{-1}(B_{1,0} \cup B^1_{1,4}) \cap p^{-1}(B^2_{1,4})$ are 4 points. By the fact that the fixed locus of a symplectic involution of a $K3$ surface are 8 isolated points, this is a contradiction.

We assume that the numerical class of B is (6.305) of the list in Section 6. We denote B by $3B_{1,0} + 2B^1_{1,6} + 6B^2_{1,6}$. By Theorem 2.5 and $(B^1_{1,6} \cdot B^2_{1,6}) \neq 0$, $G = G^1_{1,6} \oplus G^2_{1,6}$. Let $\rho_1, \rho_2 \in G$ be generators of $G_{B^1_{1,6}}$ and $G_{B^2_{1,6}}$, respectively. Then ρ_2^2 is a non-symplectic automorphism of order 3 and a generator of $G_{1,0}$. Since $(C \cdot C) = -6$ and $|G| = 12$, we get that $p^*C = \sum_{j=1}^{4} 3C_j$ where C_j is a smooth rational curve. Then $C_1, \cdots, C_4, C^2_{1,6} \subset \mathrm{Fix}(\rho_2^2)$ where $p^*B^2_{1,6} = 6C^2_{1,6}$. By [1,14], this is a contradiction.

We assume that the type of B is (6.45) of the list in Section 6. We denote B by $4B^1_{1,0}+4B^2_{1,0}+2B_{1,3} + 2B_{0,1}$. We take the Galois cover $q : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ whose branch divisor is $4B^1_{1,0} + 4B^2_{1,0}$. Since the support of B is simple normal crossing, $q^*(2B_{1,3} + 2B_{0,1}) = 2B_{4,3} + 2B_{0,1}$. By Theorem 2.2, there is the Galois morphism $g : X \to \mathbb{P}^1 \times \mathbb{P}^1$ such that the branch divisor is $2B_{4,3} + 2B_{0,1}$ and the Galois group is Abelian. Since the numerical class of $2B_{4,3} + 2B_{0,1}$ is (6.25), this is a contradiction.

As for the case of (6.45), the numerical class of B is not one of (6.46), (6.55), (6.57), (6.59), (6.60), (6.65), (6.66), (6.67), (6.68), (6.102), (6.105), (6.108), (6.111), (6.116), (6.124), (6.136), (6.138), (6.142), (6.144, (6.149), (6.153), (6.178), (6.186), (6.221), (6.222), (6.226), (6.260), (6.267) of the list in Section 6 by (6.25), (6.24), (6.27), (6.25), (6.37), (6.34), (6.40), (6.34), (6.34), (6.212), (6.213), (6.214), (6.215), (6.216), (6.217), (6.286), (6.286), (6.287), (6.287), (6.305), (6.228), (6.241), (6.243), (6.286), (6.287), (6.303), (6.305), (6.308) in order.

Therefore, we get Theorem 1.5.

4 Abelian Groups of *K*3 Surfaces with Smooth Quotient

In this section, first of all, we will show Theorems 1.1–1.2. Next, we will show Theorem 1.4. By Section 3, we had that if X/G is $\mathbb{P}^2$ or $\mathbb{F}_n$, then G is one of $\mathcal{A}G$ as a group.

Proposition 4.1 *Let X be a $K3$ surface and G be a finite subgroup of* $\mathrm{Aut}(X)$ *such that X/G is a smooth rational surface. For a birational morphism $f : X/G \to \mathbb{F}_n$, we get that $0 \leq n \leq 12$.*

Proof Let $f : X/G \to \mathbb{F}_n$ be a birational morphism, e_i be the exceptional divisors for $i = 1, \cdots, m$, and $B = \sum_{i=1}^{k} b_i B_i$ be the branch divisor. Since X/G and $\mathbb{F}_n$ are smooth and f is a birational morphism, we get $\mathrm{Pic}(X/G) = f^*\mathrm{Pic}(\mathbb{F}_n) \bigoplus_{i=1}^{m} \mathbb{Z}e_i$ and there are positive integers a_i for $i = 1, \cdots, m$ such that $K_{X/G} = f^*K_{\mathbb{F}_n} + \sum_{i=1}^{m} a_i e_i$. By Theorem 2.4,

$$0 = f^*K_{\mathbb{F}_n} + \sum_{i=1}^{m} a_i e_i + \sum_{i=1}^{k} \frac{b_i - 1}{b_i} B_i.$$

Since $\mathrm{Pic}(X/G) = f^*\mathrm{Pic}(\mathbb{F}_n) \bigoplus_{i=1}^{m} \mathbb{Z}e_i$, at least one of $B_1, \cdots, B_k$ is not an exceptional divisor of f. By rearranging if necessary, we assume that B_i is not an exceptional divisor of f for $1 \leq i \leq u$, and B_j is an exceptional divisor of f for $u+1 \leq j \leq k$. Then f_*B_i is an irreducible curve on $\mathbb{F}_n$ for $1 \leq i \leq u$. Therefore, for $1 \leq i \leq u$, there are non-negative integers c_i, d_i, g^i_j

such that

$$B_i = f^*(c_iC + d_iF) - \sum_{j=1}^{m} g_j^i e_j \quad \text{in } \mathrm{Pic}(X/G),$$

where $(c_i, d_i) = (1, 0)$, $(0, 1)$, or $d_i \geq c_i n > 0$. Since $K_{\mathbb{F}_n} = -2C - (n+2)F$ in $\mathrm{Pic}(\mathbb{F}_n)$, by Theorem 2.4, we get that $2 = \sum_i \frac{b_i - 1}{b_i} c_i$ and $n + 2 = \sum_i \frac{b_i - 1}{b_i} d_i$. In the same way as Proposition 3.1, we get this proposition.

Let X be a $K3$ surface, G be a finite subgroup of $\mathrm{Aut}(X)$ such that X/G is smooth, and $f : X/G \to \mathbb{F}_n$ be a birational morphism. By Proposition 4.1, we get $0 \leq n \leq 12$. By the proof of Proposition 4.1, the numerical class of f_*B is one of the list on Section 3. Let $B = \sum_{i=1}^{k} b_i B_i + \sum_{j=k+1}^{l} b_j B_j$, where B_i is not an exceptional divisor of f for $i = 1, \cdots, k$ and B_j is an exceptional divisor of f for $j = k+1, \cdots, l$. Since $(X/G) \backslash \bigcup_{j=k+1}^{l} B_j$ is isomorphic to $\mathbb{F}_n \backslash \bigcup_{j=k+1}^{l} f(B_j)$ and $f(B_j)$ is a point for $j = k+1, \cdots l$, $(X/G) \backslash \bigcup_{j=k+1}^{l} B_j$ is simply connected. By Theorem 2.5, G is generated by $G_1, \cdots, G_k$. Therefore, as for the case of Hirzebruch surface, we will guess G from the numerical class of f_*B. Recall that if G is Abelian, then G_i is a cyclic group, which is generated by a purely non-symplectic automorphism of order b_i. If $f_*B_1 = C$, or F, then G is generated by $G_2, \cdots, G_k$, and if $(f_*B_1, f_*B_2) = (C, F)$, then G is generated by $G_3, \cdots, G_k$.

Recall that since X/G is a smooth rational, X/G is given by blowups of $\mathbb{F}_n$. Next, we will investigate the relationship between a branch divisor and exceptional divisors of blow-ups.

Lemma 4.1 *Let X be a $K3$ surface, and $G \subset \mathrm{Aut}(X)$ a finite subgroup such that X/G is a smooth rational surface, and B be the branch divisor of the quotient map $p : X \to X/G$. For a birational morphism $h : X/G \to T$ where T is a smooth projective surface, let e_i be the exceptional divisor of h for $i = 1, \cdots, m$. Then for $i = 1, \cdots, m$ we have that $h(e_i) \in \mathrm{Supp}(h_*B)$.*

Proof Let $e_1, \cdots, e_m$ be the exceptional divisors of h. Since X/G and T are smooth and h is birational, $\mathrm{Pic}(X/G) = h^*\mathrm{Pic}(T) \bigoplus_{j=1}^{m} \mathbb{Z}e_j$ and there are positive integers a_i such that

$$K_{X/G} = h^*K_T + \sum_{i=1}^{m} a_i e_i.$$

We assume that $h(e_i) \notin \mathrm{Supp}(h_*B)$ for some $1 \leq i \leq m$. For simply, we assume that $i = 1$, i.e., $h(e_1) \notin \mathrm{Supp}(h_*B)$. Let $B_1, \cdots, B_k$ be irreducible components of B such that B_j is not an exceptional divisor of h for $j = 1, \cdots, k$. Since $h(e_1) \notin \mathrm{Supp}(h_*B)$, there are integers $c_{j,s}$ such that $B_j = h^*C_j + \sum_{s=2}^{m} c_{j,s} e_s$, where C_j is an irreducible curve in T. By Theorem 2.4, we get that

$$0 = \Big(h^*K_T + \sum_{i=1}^{m} a_i e_i\Big) + \sum_{j=1}^{k} \frac{b_j - 1}{b_j} \Big(h^*C_j + \sum_{s=2}^{m} c_{j,s} e_s\Big) + \sum_{j=1}^{m} l_j e_j \quad \text{in } \mathrm{Pic}(X/G),$$

where $l_j = 0$ or $l_j = \frac{d_j - 1}{d_j}$ for some an integer $d_j \geq 2$. Since $a_i \geq 1$, $c_{j,1} = 0$, $l_j \geq 0$ and $\mathrm{Pic}(X/G) = h^* \mathrm{Pic}(T) \bigoplus_{j=1}^{m} \mathbb{Z} e_j$, this is a contradiction.

Proposition 4.2 *Let X be a $K3$ surface, $G \subset \mathrm{Aut}(X)$ be a finite subgroup such that the quotient space X/G is smooth, and B be the branch divisor of the quotient morphism $p : X \to X/G$. Let $f : X/G \to T$ be a birational morphism where T is a smooth surface, $e_1, \cdots, e_m$ be the exceptional divisors of f, and $f_* B := \sum_{i=1}^{u} b_i \widetilde{B_i}$ where $\widetilde{B_i}$ is an irreducible curves on U for $i = 1, \cdots, u$. If $\widetilde{B_i}$ is smooth for each $1 \leq i \leq u$, then for $1 \leq j \leq m$ there are $1 \leq s < t \leq u$ such that $f(e_j) \in \widetilde{B_s} \cap \widetilde{B_t}$.*

Proof We set $B = \sum_{i=1}^{u} b_i B_i + \sum_{j=u+1}^{k} b_j B_j$, where B_i is not an exceptional divisor of f for $i = 1, \cdots, u$, and B_j is an exceptional divisor of f for $j = u+1, \cdots, k$. Then $f_* B = \sum_{i=1}^{u} b_i f_* B_i$. We assume that $f_* B_i$ is a smooth curve for $i = 1, \cdots, u$. By Lemma 4.1, $f(e_i) \in \mathrm{supp}(f_* B)$ for $i = 1, \cdots, m$.

Let $S := X/G$, $Z := \{f(e_1), \cdots, f(e_m)\} := \{z_1, \cdots, z_v\} \subset T$ where

$$v := |\{f(e_1), \cdots, f(e_m)\}|, \quad q : \mathrm{Blow}_Z T \to T$$

be the blow-up, and $E_i := q^{-1}(z_i)$ be the exceptional divisor of q for $1 \leq i \leq v$. Then there is a birational morphism $g : S \to \mathrm{Blow}_Z T$ such that $f = q \circ g$, i.e., the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Blow}_Z T & \xrightarrow{q} & T \\ {\scriptstyle g}\uparrow & \nearrow{\scriptstyle f} & \\ S. & & \end{array}$$

By changing the number if necessary, we assume that $g(e_i) = E_i$ for $1 \leq i \leq v$. Then the exceptional divisors of g are $e_{v+1}, \cdots, e_m$. Since $\mathrm{Pic}(\mathrm{Blow}_Z T) = q^* \mathrm{Pic}(T) \bigoplus_{j=1}^{v} \mathbb{Z} E_j$ and $f = q \circ g$,

$$\mathrm{Pic}(S) = g^* \mathrm{Pic}(\mathrm{Blow}_Z T) \bigoplus_{j=v+1}^{m} \mathbb{Z} e_j = \left(f^* \mathrm{Pic}(T) \bigoplus_{i=1}^{v} \mathbb{Z} g^* E_i \right) \bigoplus_{j=v+1}^{m} \mathbb{Z} e_j.$$

Since $K_{\mathrm{Blow}_Z T} = q^* K_T + \sum_{j=1}^{v} E_j$,

$$K_S = g^* K_{\mathrm{Blow}_Z T} + \sum_{i=v+1}^{m} a'_i e_i = \left(f^* K_T + \sum_{j=1}^{v} g^* E_i \right) + \sum_{i=v+1}^{m} a'_i e_i,$$

where a'_i is a positive integer for $i = v+1, \cdots, m$.

We assume that for some $1 \leq i \leq m$, $f(e_i) \notin f_* B_s \cap f_* B_t$ for each $1 \leq s < t \leq u$. Since $Z = \{f(e_1), \cdots, f(e_v)\}$, we assume that $1 \leq i \leq v$. For simplicity, we assume that $i = 1$. In addition, since $f(e_j) \in \mathrm{supp}(f_* B)$ for $j = 1, \cdots, m$, by changing the number if necessary, we assume that $f(e_1) \in \mathrm{supp}(f_* B_1)$, and $f(e_1) \notin \mathrm{supp}(f_* B_j)$ for $2 \leq j \leq u$. Recall that the exceptional divisors of q are $E_1, \cdots, E_v$, the exceptional divisors of g are $e_{v+1}, \cdots, e_m$, and

$g(e_i) = E_i$ for $1 \leq i \leq v$. Since $f = q \circ g$, for $j = 1, \cdots, u$ there are non-negative integers $c_{j,s}, c'_{j,t}$ such that

$$B_j = f^*f_*B_j - \sum_{s=1}^{v} c_{j,s} g^*E_s - \sum_{t=v+1}^{m} c'_{j,t} e_t \quad \text{in } \mathrm{Pic}(S).$$

Since $f(e_1) \notin f_*B_j$ for $2 \leq j \leq u$, we get that $c_{j,1} = 0$ for $2 \leq j \leq u$. Since f_*B_1 is smooth, $c_{1,1} = 1$. Since $K_S = f^*K_T + \sum_{j=1}^{v} g^*E_i + \sum_{i=v+1}^{m} a'_i e_i$ and $0 = K_S + \sum_{i=1}^{k} \frac{b_i-1}{b_i} B_i$ in $\mathrm{Pic}(S)$,

$$\begin{aligned} 0 = &\Big(f^*K_T + \sum_{j=1}^{v} g^*E_i + \sum_{i=v+1}^{m} a'_i e_i\Big) \\ &+ \sum_{i=1}^{u} \frac{b_i - 1}{b_i}\Big(f^*f_*B_j - \sum_{s=1}^{v} c_{j,s} g^*E_s - \sum_{t=v+1}^{m} c'_{j,t} e_t\Big) \\ &+ \sum_{j=u+1}^{k} \frac{b_j - 1}{b_j} B_j \quad \text{in } \mathrm{Pic}(S). \end{aligned}$$

From the coefficient of g^*E_1, we get that $1 = \frac{b_1-1}{b_1}$. Since $b_1 \geq 2$, this is a contradiction.

Let X be a $K3$ surface, G be a finite subgroup of $\mathrm{Aut}(X)$ such that X/G is a smooth rational surface, and B be the branch divisor of the quotient map $p : X \to X/G$. Let $h : X/G \to T$ be a birational morphism where T is a smooth projective surface, and $e_1, \cdots, e_m$ be the exceptional divisors of h. We set $h_*B := \sum_{j=1}^{l} b_j B'_j$. We write $B = \sum_{i=1}^{l} b_i B_i + \sum_{j=l+1}^{k} b_j B_j$ such that $h_*B_i = B'_i$ for $i = 1, \cdots, l$. Then B_j is one of the exceptional divisor of h for $j = l+1, \cdots, k$, and for $i = 1, \cdots, l$ there are non-negative integers $c_{i,1}, \cdots, c_{i,m}$ such that $B_i = h_*^{-1}B'_i - \sum_{t=1}^{m} c_{i,t} e_t$.

Remark 4.1 In the above situation, for e_u and e_v where $1 \leq u < v \leq m$ and $h(e_u) = h(e_v)$, we get that $c_{i,u} = 0$ if and only if $c_{i,v} = 0$.

Remark 4.2 In the situation of Proposition 4.2, we assume that $T = \mathbb{F}_n$. Then there are positive integers $a_1, \cdots, a_m$ such that $K_{X/G} = h^*K_{\mathbb{F}_n} + \sum_{i=1}^{m} a_i e_i$. By the proof of Proposition 4.2, we get that $a_1 = \cdots = a_u = 1$ and

$$1 + \frac{\beta_i - 1}{\beta_i} = \sum_{j=1}^{k} \frac{b_j - 1}{b_j} c_{i,j} \quad \text{for } i = 1, \cdots, u,$$

where $\beta_i = 1$ if e_i is not an irreducible component of B, and β_i is the ramification index at e_i if e_i is an irreducible component of B.

Furthermore, we assume that $X/G \neq \mathrm{Blow}_{\{h(e_1),\cdots,h(e_u)\}}\mathbb{F}_n$. For the birational morphism $g : X/G \to \mathrm{Blow}_{\{h(e_1),\cdots,h(e_u)\}}\mathbb{F}_n$ in the proof of Proposition 4.2, we rearrange the order so that $\{g(e_{u+1}), \cdots, g(e_{u+v})\} = \{g(e_{u+1}), \cdots, g(e_m)\}$, where $v := |\{g(e_{u+1}), \cdots, g(e_m)\}|$. Like the proof of Proposition 4.2, by considering the blow-up of $\mathrm{Blow}_{\{h(e_1),\cdots,h(e_u)\}}\mathbb{F}_n$ at $\{g(e_{u+1}), \cdots, g(e_{u+v})\}$, we get that $a_{u+1} = \cdots = a_{u+v} = 2$ and

$$2 + \frac{\beta_i - 1}{\beta_i} = \sum_{j=1}^{k} \frac{b_j - 1}{b_j} c_{i,j} \quad \text{for } i = u+1, \cdots, u+v,$$

where $\beta_i = 1$ if e_i is not an irreducible component of B, and β_i is the ramification index at e_i if e_i is an irreducible component of B.

Recall that by Theorem 2.5, G_{B_i} is generated by a non-symplectic automorphism of order b_i. As a corollary of Theorem 2.5 and Proposition 4.2, we get the following Theorem 4.1.

Theorem 4.1 *Let X be a $K3$ surface, G be a finite subgroup of $\mathrm{Aut}(X)$ such that X/G is smooth, and B be the branch divisor of the quotient map $p : X \to X/G$. Let $f : X \to S$ be the birational morphism where S is minimal rational surface. We put $f_*B := \sum_{i=1}^{k} b_i B_i$ where B_i is an irreducible curve for $i = 1, \cdots, k$. We denote by G_s the subgroup of G, which consists of symplectic automorphisms of G, and b the least common multiple of $b_1, \cdots, b_k$. Then there is a purely non-symplectic automorphism $g \in G$ of order b such that G is the semidirect product $G_s \rtimes \langle g \rangle$ of G_s and $\langle g \rangle$.*

Proof Since G_s is a normal subgroup of G and G/G_s is a cyclic group, in order to show Theorem 4.1, we only show that there is a purely non-symplectic automorphism $g \in G$ of order b.

First of all, we assume that $X/G \cong \mathbb{P}^2$. We put $B := \sum_{i=1}^{k} b_i B_i$ where B_i is an irreducible curve for $i = 1, \cdots, k$. By Theorem 2.4, $0 = \sum_{i=1}^{k} \frac{b_i - 1}{b_i} \deg B_i + \deg K_{\mathbb{P}^2}$, in which $K_{\mathbb{P}^2}$ is the canonical line bundle of $\mathbb{P}^2$. Since the degree of $K_{\mathbb{P}^2}$ is -3 and $\frac{1}{2} \le \frac{l-1}{l} < 1$ for any positive integer l, we get that $4 \le \sum_{i=1}^{k} \deg B_i \le 6$. If $\sum_{i=1}^{k} \deg B_i = 6$, then $b_1 = \cdots = b_k = 2$. By Theorem 2.5, in this case the statement of theorem is established. We assume that $\sum_{i=1}^{k} \deg B_i \le 5$. By [15, Theorem 2], $b = b_i$ for some $1 \le i \le k$ or $b = l.c.m(b_i, b_j)$ for $i < j$. By Theorem 2.5, in the former case, we get this theorem.

For the latter, i.e., if $b \ne b_i$ for $1 \le i \le k$, then B is one of (i) $3L_1 + 3L_2 + 3L_3 + 2L_4 + 2L_5$, where L_3 passes through the points $L_1 \cap L_2$ and $L_4 \cap L_5$ (see [15, pp. 408]), (ii) $3L_1 + 3L_2 + 3L_3 + 2Q$, where L_1, L_2 are the tangent to Q and L_3 is in general position with respect to $L_1 \cup L_2 \cup Q$ (see [15, pp. 408]), and (iii) $2L_1 + 2L_2 + 3L_3 + 3Q$, where L_1, L_2, L_3 are three distinct tangent lines to Q (see [15, pp. 410]). Here, L_i and Q are smooth curves on $\mathbb{P}^2$ with $\deg L_i = 1$ and $\deg Q = 2$ for $i = 1, \cdots, 5$. Then there are $1 \le i < j \le k$ such that $b = l.c.m(b_i, b_j)$, $B_i + B_j$ is simple normal crossing, and $(B_i \cap B_j) \setminus \bigcup_{s \ne i,j} B_s$ is not an empty set. For clarity, we may assume that $i = 1$, $j = 2$. We take one point $y \in (B_1 \cap B_2) \setminus \bigcup_{i=3}^{k} B_i$. Let $x \in p^{-1}(y)$. By the assumption for y and Theorem 2.1, there are open subset $V \subset \mathbb{P}^2$ and $U \subset X$ such that $y \in V$, $x \in U$, $p_{|U} : U \to V$ is isomorphic to $\{z \in \mathbb{C}^2 : \ |z| < 1\} \ni (z_1, z_2) \mapsto (z_1^{b_1}, z_2^{b_2}) \in \{z \in \mathbb{C}^2 : \ |z| < 1\}$, and hence $G_x := \{g \in G \,|\, g(x) = x\} \cong \mathbb{Z}/b_1\mathbb{Z} \oplus \mathbb{Z}/b_2\mathbb{Z}$. Since $b = l.c.m(b_1, b_2)$, there is a purely non-symplectic automorphism $g \in G$ with order b.

Next, we assume that $X/G \cong \mathbb{F}_n$. By the list of the numerical class of B in Section 6, if the numerical class of B is not one of (6.65), (6.70), (6.73), (6.77), (6.83), (6.92), (6.102), (6.127), (6.128), (6.132), (6.136), (6.143), (6.153), (6.154), (6.170), (6.235), (6.251), (6.252), (6.253), then $b = b_i$ for some $1 \le i \le k$. Therefore, by Theorem 2.5, we get this theorem. If the numerical class of B is one of (6.65), (6.70), (6.73), (6.77), (6.92), (6.127), (6.128), (6.132), (6.136), (6.143), (6.153), (6.154), (6.170), (6.235), (6.251), (6.252), (6.253), then there are $1 \leq i < j \leq k$ such that $b = l.c.m(b_i, b_j)$, $B_i + B_j$ is simple normal crossing, and $(B_i \cap B_j) \setminus \bigcup_{s \neq i,j} B_s$ is not an empty set. As for the case of $\mathbb{P}^2$, we get this theorem.

We assume that the numerical class of B is (6.83). We write $B = 3B_{3,3} + 2B^1_{0,1} + 2B^2_{0,1}$. Since $B^1_{0,1} \cap B^2_{0,1}$ is an empty set, if $B_{3,3} \cap B^1_{0,1}$ is not one point, then by $(B_{3,3} \cdot B^1_{0,1}) = 3$, there is a point $y \in B_{3,3} \cap B^1_{0,1}$ such that the support of B is simple normal crossing at y. Since $b = 6$, by Theorem 2.5, we get this theorem. Therefore, we assume that $B_{3,3} \cap B^1_{0,1}$ and $B_{3,3} \cap B^2_{0,1}$ are one point. Let $q : X/G_s \to X/G$ be the quotient map. Then the singular locus of X/G_s is $q^{-1}(B_{3,3} \cap B^1_{0,1}) \cup q^{-1}(B_{3,3} \cap B^2_{0,1})$. Since the Galois group of q is $G/G_s \cong \mathbb{Z}/6\mathbb{Z}$, the branch divisor of q is B, and $B_{3,3} \cap B^1_{0,1}$ and $B_{3,3} \cap B^2_{0,1}$ are one point, X/G_s has just two singular point. By [16, Theorem 3], this is a contradiction. Therefore, if the numerical class of B is (6.83), then we get this theorem. As for the case of (6.83), we get this theorem for (6.102).

Finally, we assume that X/G is not $\mathbb{P}^2$ or $\mathbb{F}_n$. We take a birational morphism $f : X/G \to \mathbb{F}^n$ where $0 \leq n$. Let $e_1, \cdots, e_m$ be the exceptional divisors of f. In the same way of the case where $X/G \cong \mathbb{P}^2$ or $\mathbb{F}_n$, we only consider the case that the numerical class of f_*B is one of (6.65), (6.70), (6.73), (6.77), (6.83), (6.92), (6.102), (6.127), (6.128), (6.132), (6.136), (6.143), (6.153), (6.154), (6.170), (6.235), (6.251), (6.252), (6.253).

We assume that the numerical class of f_*B is (6.65). By Remark 4.2, there are positive integers $a_1, \cdots, a_5, b$ such that

$$1 + \frac{b-1}{b} = \frac{2}{3}a_1 + \frac{5}{6}a_2 + \frac{1}{2}a_3 + \frac{3}{4}a_4 + \frac{3}{4}a_5.$$

Since the numerical class of f_*B is (6.65), we may assume that a_1 or a_2 is 0, and either a_4 or a_5 is 0. However, there are not such positive integers. Therefore, the numerical class of f_*B is not (6.65). As for the case of (6.65), the numerical class of B is not one of (6.73), (6.77), (6.128), (6.132), (6.170), (6.235), (6.251), (6.253).

We assume that the numerical class of f_*B is (6.70). By Remark 4.2, there are positive integers $a_1, \cdots, a_6, b$ such that

$$1 + \frac{b-1}{b} = \frac{1}{2}a_1 + \frac{2}{3}a_2 + \frac{5}{6}a_3 + \frac{1}{2}a_4 + \frac{3}{4}a_5 + \frac{3}{4}a_6.$$

Since the numerical class of f_*B is (6.70), we may assume that two of a_1, a_2 and a_3 are 0, and two of a_4, a_5 and a_6 are 0. The integers satisfying the above conditions is only $(a_1, \cdots, a_6, b) = (1, 0, 0, 1, 0, 0, 12)$. Therefore, for $B := \sum_{j=1}^{l} B_j B_j$, $b_i = 12$ for some $1 \leq i \leq l$. By Theorem 2.5, if the numerical class of f_*B is (6.65), then we get this theorem. As for the case of (6.70), if the numerical class of B is one of (6.136), (6.143), then we get this theorem.

We assume that the numerical class of B is (6.83). By Remark 4.2, there are positive integers $a_1, \cdots, a_6, b$ such that

$$1 + \frac{b-1}{b} = \frac{2}{3}a_1 + \frac{1}{2}a_2 + \frac{1}{2}a_3.$$

Since the numerical class of f_*B is (6.83), we may assume that either a_2 or a_3 is 0. The integers satisfying the above conditions is $(a_1, a_2, a_3, b) = (2, 1, 0, 6)$ or $(2, 0, 1, 6)$. Therefore, we get this of theorem. As for the case of (6.83), if the numerical class of B is one of (6.92), (6.102), (6.127), (6.153), (6.154), (6.252), then we get this theorem.

Theorem 4.2 *Let X be a $K3$ surface and G be a finite subgroup of $\mathrm{Aut}(X)$ such that X/G is smooth. For a birational morphism $f : X/G \to \mathbb{F}_n$ where $0 \leq n$, we get that n is not one of $5, 7, 9, 10, 11$.*

Proof Let $p : X \to X/G$ be the quotient map, and $B := \sum_{i=1}^{k} b_i B_i$ be the branch divisor of p. Let $f : X/G \to \mathbb{F}_n$ be a birational morphism where $0 \leq n$, and $e_1, \cdots, e_m$ be the exceptional divisors of f.

First we will show this theorem for the cases where f is an isomorphism, i.e., $X/G \cong \mathbb{F}_n$. By Theorem 2.4, $n = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ or 12. We assume that $n = 5, 7$ or 9. Then the numerical class of B is one of (6.296), (6.297), (6.298), (6.299), (6.300), (6.301), (6.310), (6.314), (6.315) of the list in Section 6.

We assume that the numerical class of B is (6.296). We denote B by $4B_{1,0} + 2B_{1,5} + 4B_{1,6}$. Let $p^*B_{1,0} = \sum_{i=1}^{m} 4C_i$ where C_i is a smooth curve for $i = 1, \cdots, m$. Since $(B_{1,0} \cdot B_{1,0}) < 0$, $(C_i \cdot C_i) < 0$. Since X is a $K3$ surface, and C_i is irreducible, we get that $(C_i \cdot C_i) = -2$. Since the degree of p is $|G|$ and $(B_{1,0} \cdot B_{1,0}) = -5$, we get that $\frac{-5|G|}{16} = -2m + 2 \sum_{1 \leq i < j \leq m} (C_i \cdot C_j)$, and hence $\frac{5|G|}{32} \leq m$. Let $p^*B_{1,6} = \sum_{j=1}^{l} 4C'_j$ where C'_j is a smooth curve for $j = 1, \cdots, l$. Since $(B_{1,0} \cdot B_{1,6}) = 1$, $\frac{|G|}{16} = m\big(C_1 \cdot \sum_{j=1}^{l} C'_j\big)$. Since $\big(C_1 \cdot \sum_{j=1}^{l} C'_j\big) \geq 1$, we get that $m \leq \frac{|G|}{16}$. By $\frac{5|G|}{32} \leq m$ and $m \leq \frac{|G|}{16}$, we get that the numerical class of B is not (6.296). As for the case of (6.296), the numerical class of f_*B is not one of (6.297), (6.298), (6.299), (6.300), (6.301), (6.310), (6.314), (6.315). Therefore, if $X/G \cong \mathbb{F}_n$, then $n \neq 5, 7, 9, 10, 11$.

Next, we assume that f is not an isomorphism, i.e., X/G is not a Hirzebruch surface $\mathbb{F}_n$. By the proof of Proposition 4.1, the numerical class of f_*B is one of the list in Section 6. As a result, $n = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ or 12. We assume that $n = 5, 7$ or 9. The numerical class of f_*B is one of (6.296), (6.297), (6.298), (6.299), (6.300), (6.301), (6.310), (6.314), (6.315).

We assume that the numerical class of f_*B is (6.296). Let $p^*B_{1,0} = 4\sum_{i=1}^{m} C_i$, where C_i is a smooth curve for $i = 1, \cdots, m$. Since the degree of p is $|G|$, by $(C \cdot F) = 1$, we get that $|G| = 4m(C_1 \cdot p^*f^*F)$, and hence $|G|$ is a multiple of $4m$. Since $f_*B_{1,0} = C$, $(B_{1,0}, B_{1,0}) \leq (C \cdot C) = -5$. By $\frac{|G|}{16}(B_{1,0} \cdot B_{1,0}) = -2m + 2 \sum_{1 \leq i < j \leq m} (C_i \cdot C_j)$, we get that $m = \frac{|G|}{4}$. Since the numerical class of f_*B is (6.296), there must be positive integers a_1, a_2, a_3, b such that

$$1 + \frac{b-1}{b} = \frac{3}{4}a_1 + \frac{1}{2}a_2 + \frac{3}{4}a_3,$$

and either a_1 or a_2 is 0. The integers satisfying the above conditions are only $(a_1, a_2, a_3, b) = (1, 0, 1, 2)$, and hence $f(e_i) \in f_*B_{1,5} \cap f_*B_{1,6}$ for each $i = 1, \cdots, l$. Since $(f_*B_{1,5} \cdot f_*B_{1,6}) = 1$, $f_*B_{1,5} \cap f_*B_{1,6}$ is one point. We put $x := f_*B_{1,5} \cap f_*B_{1,6}$. Let $q : \mathrm{Blow}_x\mathbb{F}_5 \to \mathbb{F}_5$ be the blow-up of $\mathbb{F}_5$ at x. Then there is a birational morphism $g : X/G \to \mathrm{Blow}_x\mathbb{F}_5$ such that $f = q \circ g$. Let $C' := g_*B_{1,0}$. Let E be the exceptional divisor of q. Since $f(e_i) = x$ for each $i = 1, \cdots, l$, $g(e_i) \in E$ for each $i = 1, \cdots, l$. Since $g_*B = 4C' + 2g_*B_{1,5} + 4g_*B_{1,6} + 2E$, if g is not an isomorphism, then there must be integers a_1, a_2, a_3, a_4, b such that

$$2 + \frac{b-1}{b} = \frac{3}{4}a_1 + \frac{1}{2}a_2 + \frac{3}{4}a_3 + \frac{1}{2}a_4,$$

and if a_1 is not 0, then either $a_2 = a_3 = 0$. However, there are not such positive integers. Therefore, g is an isomorphism, i.e., $X/G = \mathrm{Blow}_x\mathbb{F}_5$, and hence $B = 4B_{1,0}+2B_{1,5}+4B_{1,6}+2E$ and $(B_{1,0}\cdot E) = 1$. We put $p^*E = 2\sum_{j=1}^{u} C'_j$ where C'_j is a smooth curve for $j = 1,\cdots,u$. Since $m = \frac{|G|}{4}, \frac{|G|}{2} = |G|\big(C_1\cdot \sum_{j=1}^{u} C'_j\big)$. This is a contradiction. Therefore, the numerical class of B is not (6.296). As for the case of (6.296), the numerical class of B is not one of (6.310), (6.314).

We assume that the numerical class of f_*B is (6.297). Then there must be integers a_1, a_2, a_3, a_4, b such that

$$1+\frac{b-1}{b} = \frac{3}{4}a_1+\frac{1}{2}a_2+\frac{3}{4}a_3+\frac{3}{4}a_4,$$

and if a_1 is not zero, then $a_2 = a_3 = 0$. The integers satisfying the above condition is $(a_1,a_2,a_3,a_4,b) = (1,0,0,1,2)$ or $(0,0,1,1,2)$. Therefore, for each $i = 1,\cdots,l$, we get that $f(e_i)\in f_*B_{1,0}\cap f_*B_{0,1}$ or $f(e_i)\in f_*B^2_{1,5}\cap f_*B_{0,1}$. If $f(e_i)\in f_*B^2_{1,5}\cap f_*B_{0,1}$ for all $i = 1,\cdots,l$, then $(B_{1,0}\cdot B_{1,0}) = -5$ and $(B_{1,0}\cdot B_{0,1}) = 1$. However, as for the case of $X/G\cong\mathbb{F}_n$, we can see that such things can not happen. Therefore, $f(e_i)\in f_*B_{1,0}\cap f_*B_{0,1}$ for some $i = 1,\cdots,l$. By using the blow-up of $\mathbb{F}_5$ at $x := f_*B_{1,0}\cap f_*B_{0,1}$, as for the case of (6.296), this is a contradiction. Therefore, the numerical class of B is not (6.297). As for the case of (6.297), the numerical class of B is not (6.315).

We assume that the numerical class of f_*B is (6.298). Then there must be integers a_1, a_2, a_3, b such that

$$1+\frac{b-1}{b} = \frac{5}{6}a_1+\frac{1}{2}a_2+\frac{2}{3}a_3.$$

The integers satisfying the above condition are only $(a_1,a_2,a_3,b) = (1,0,1,2)$, and hence $f(e_i)\in f_*B_{1,5}\cap f_*B_{1,6}$ for each $i = 1,\cdots,l$. Since $(f_*B_{1,5}\cdot f_*B_{1,6}) = 1$, $f_*B_{1,5}\cap f_*B_{1,6}$ is one point. We put $x := f_*B_{1,5}\cap f_*B_{1,6}$. Let $q : \mathrm{Blow}_x\mathbb{F}_5\to\mathbb{F}_5$ be the blow-up of $\mathbb{F}_5$ at x. As for the case of (6.296), since there are no integers a_1,a_2,a_3,a_4,b such that

$$2+\frac{b-1}{b} = \frac{3}{4}a_1+\frac{1}{2}a_2+\frac{3}{4}a_3+\frac{1}{2}a_4,$$

we get that $X/G = \mathrm{Blow}_x\mathbb{F}_5$, and hence $B = 6B_{1,0}+2B_{1,6}+3B_{1,6}+2E$, and $(B_{1,0}\cdot E) = 1$. We put $p^*E = 2\sum_{j=1}^{u} C'_j$, where C'_j is a smooth curve for $j = 1,\cdots,u$. Since $(E\cdot E) = -1$, we get that $u = \frac{|G|}{4}+\sum_{1\le i<j\le u}(C'_i\cdot C'_j)$, and hence $u\ge\frac{|G|}{4}$. Since $(B_{1,0}\cdot E) = 1$, $\frac{|G|}{12}$ is a multiple of u. This is a contradiction. Therefore, the numerical class of B is not (6.298).

We assume that the numerical class of f_*B is (6.299). Then there must be positive integers a_1,a_2,a_3,a_4,b such that

$$1+\frac{b-1}{b} = \frac{5}{6}a_1+\frac{1}{2}a_2+\frac{2}{3}a_3+\frac{2}{3}a_4$$

and $a_1a_3 = 0$. The integers satisfying the above conditions are $(a_1,a_2,a_3,a_4,b) = (1,0,0,1,2)$ or $(0,1,1,1,6)$. Therefore, for each $i = 1,\cdots,l$, we get that $f(e_i)\in f_*B_{1,0}\cap f_*B_{0,1}$ or $f(e_i)\in f_*B_{1,6}\cap f_*B_{1,5}\cap f_*B_{0,1}$. If $f(e_i)\in f_*B_{1,6}\cap f_*B_{1,5}\cap f_*B_{0,1}$ for all $i = 1,\cdots,l$, then $(B_{1,0}\cdot B_{1,0}) = -5$ and $(B_{1,0}\cdot B_{0,1}) = 1$. We get that this is not established in the same way as in the case of $X/G\cong\mathbb{F}_n$. By using the blow-up of $\mathbb{F}_5$ at $x := f_*B_{1,0}\cap f_*B_{0,1}$, as for the case of (6.298), we get that there is no case where $f(e_i) \in f_*B_{1,0} \cap f_*B_{0,1}$ for some $i = 1, \cdots, l$. Therefore, the numerical class of B is not (6.299). As for the case of (6.299), the numerical class of B is not one of (6.300)–(6.301).

Corollary 4.1 *Let X be a $K3$ surface and G be a finite subgroup of* $\mathrm{Aut}(X)$ *such that X/G is smooth. If there is a birational morphism $f : X/G \to \mathbb{F}_n$ from the quotient space X/G to a Hirzebruch surface $\mathbb{F}_n$ where $n = 6, 8$ or 12, then f is an isomorphism, i.e., X/G is a Hirzebruch surface.*

Proof Let $n \geq 1$ and $C_{-n} \subset \mathbb{F}_n$ be the unique irreducible curve such that $(C_{-n} \cdot C_{-n}) = -n$. Since for $x \in \mathbb{F}_n$, if $x \in C_{-n}$, then $\mathrm{Blow}_x\mathbb{F}_n = \mathrm{Blow}_y\mathbb{F}_{n+1}$ where $y \in \mathbb{F}_{n+1} \backslash C_{-(n+1)}$, and if $x \notin C_{-n}$, then $\mathrm{Blow}_x\mathbb{F}_n = \mathrm{Blow}_y\mathbb{F}_{n-1}$ where $y \in C_{-(n-1)}$, by Theorem 4.2, we get this corollary.

Theorem 4.3 *Let X be a $K3$ surface and G be a finite Abelian subgroup of* $\mathrm{Aut}(X)$. *If X/G is smooth, then G is isomorphic to one of $\mathcal{AG}$ as groups.*

Proof Since X/G is smooth, the quotient space X/G is an Enriques surface or a rational surface. If X/G is Enriques, then $G \cong \mathbb{Z}/2\mathbb{Z}$ as a group and $\mathbb{Z}/2\mathbb{Z} \in \mathcal{AG}$. By Section 3, if $X/G \cong \mathbb{F}_n$, then G is isomorphic to one of $\mathcal{AG}$ as a group. By [15], if $X/G \cong \mathbb{P}^2$, then G is isomorphic to one of $\mathcal{AG}$ as a group. Therefore, we assume that X/G is rational, and $X/G \neq \mathbb{P}^2$ or $\mathbb{F}_n$.

Let $f : X/G \to \mathbb{F}_n$ be a birational morphism where $0 \leq n \leq 12$, and B be the branch divisor of G. By Theorem 4.2 and Corollary 4.1, we may assume that $0 \leq n \leq 4$. By the proof of Proposition 4.1, the numerical class of f_*B is one of the list in Section 6.

We assume that the numerical class of f_*B is one of (6.4), (6.5), (6.6), (6.10), (6.11), (6.12), (6.14), (6.15), (6.16), (6.19), (6.20), (6.25), (6.26), (6.27), (6.28), (6.32), (6.33), (6.36), (6.37), (6.38), (6.41), (6.42), (6.46), (6.51), (6.52), (6.57), (6.58), (6.59), (6.60), (6.79), (6.80), (6.81), (6.82), (6.85), (6.87), (6.88), (6.89), (6.91), (6.94), (6.96), (6.98), (6.112), (6.113), (6.114), (6.115), (6.116), (6.117), (6.118), (6.119), (6.120), (6.121), (6.122), (6.123), (6.124), (6.125), (6.126), (6.176), (6.177), (6.178), (6.180), (6.181), (6.182), (6.183), (6.184), (6.185), (6.186), (6.187), (6.189), (6.190), (6.191), (6.192), (6.195), (6.196), (6.197), (6.199), (6.200), (6.202), (6.203), (6.206), (6.216), (6.217), (6.241), (6.242), (6.243), (6.244), (6.245), (6.246), (6.249), (6.250), (6.270), (6.271), (6.272), (6.273), (6.274), (6.275), (6.276), (6.277), (6.279), (6.282) of the list in Section 6. By Theorem 2.5, G is generated by automorphisms $g_1, \cdots, g_m$, where $1 \leq m \leq 5$ and the order of g_i is two for $i = 1, \cdots, m$. Therefore, G is $\mathbb{Z}/2\mathbb{Z}^{\oplus a}$ where $1 \leq a \leq 5$ as a group.

We assume that the numerical class of f_*B is one of (6.1), (6.2), (6.3), (6.17), (6.18), (6.22), (6.23), (6.24), (6.39), (6.54), (6.55), (6.194), (6.198), (6.201), (6.204), (6.205), (6.212), (6.218), (6.219), (6.228), (6.229), (6.284), (6.285), (6.289), (6.290) of the list in Section 6. By Theorem 2.5, G is generated by automorphisms $g_1, \cdots, g_m$, where $1 \leq m \leq 3$ and the order of g_i is 3 for $i = 1, \cdots, m$. Therefore, G is $\mathbb{Z}/3\mathbb{Z}^{\oplus b}$ where $1 \leq b \leq 3$ as a group.

We assume that the numerical class of f_*B is one of (6.29), (6.34), (6.40), (6.44), (6.49), (6.50), (6.53), (6.56), (6.62), (6.63), (6.64), (6.66), (6.67), (6.68), (6.69), (6.71), (6.77), (6.83), (6.84), (6.92), (6.93), (6.102), (6.106), (6.107),(6.108), (6.127), (6.128), (6.133), (6.134), (6.135), (6.137), (6.138), (6.145), (6.146), (6.147), (6.148), (6.149), (6.151), (6.153), (6.154), (6.163), (6.164), (6.165), (6.166), (6.167), (6.168), (6.169), (6.174), (6.175), (6.179), (6.188), (6.193), (6.211), (6.214), (6.220), (6.221), (6.223), (6.224), (6.225), (6.226), (6.227), (6.230), (6.236), (6.237), (6.238), (6.239), (6.240), (6.248), (6.251), (6.252), (6.254), (6.256), (6.258), (6.259), (6.260), (6.265), (6.266), (6.267), (6.268), (6.269), (6.283), (6.286), (6.288), (6.292), (6.293), (6.294), (6.295) of the list in Section 6. By Theorem 2.5, G is generated by automorphisms $g_i, \cdots, g_m$, $h_1, \cdots h_n$, where $1 \leq m \leq 3$, $1 \leq n \leq 2$, the order of g_i is 2 for $i = 1, \cdots, m$, and the order of h_j is 3 for $j = 1, \cdots, n$. Therefore, G is $\mathbb{Z}/2\mathbb{Z}^{\oplus d} \oplus \mathbb{Z}/3\mathbb{Z}^{\oplus e}$, where $(d, e) = (1, 1)$, $(1, 2)$, $(1, 3)$, $(2, 1)$, $(2, 2)$, $(3, 1)$, $(3, 2)$ as a group.

We assume that the numerical class of f_*B is one of (6.7), (6.8), (6.9), (6.13), (6.21), (6.30), (6.31), (6.35), (6.43), (6.45), (6.47), (6.48), (6.61), (6.86), (6.90), (6.97), (6.99), (6.100), (6.103), (6.104), (6.105), (6.109), (6.110), (6.130), (6.131), (6.139), (6.140), (6.141), (6.142), (6.155), (6.156), (6.157), (6.158), (6.161), (6.162), (6.207), (6.208), (6.209), (6.210), (6.213), (6.215), (6.222), (6.231), (6.232), (6.233), (6.234), (6.255), (6.257), (6.261), (6.262), (6.263), (6.264), (6.278), (6.280), (6.281), (6.287), (6.291) of the list in Section 6. By Theorem 2.5, G is generated by automorphisms $g_i, \cdots, g_m$, $h_1, \cdots, h_n$, where the order of g_i is 2 for $i = 1, \cdots, m$, the order of h_j is 4 for $j = 1, \cdots, n$, and (n, m) is one of $(0, 1)$, $(0, 2)$, $(0, 3)$, $(1, 1)$, $(1, 2)$, $(2, 1)$, $(3, 1)$. Therefore, G is $\mathbb{Z}/2\mathbb{Z}^{\oplus f} \oplus \mathbb{Z}/4\mathbb{Z}^{\oplus g}$, where $(f, g) = (0, 1)$, $(0, 2)$, $(0, 3)$, $(1, 1)$, $(1, 2)$, $(2, 1)$, $(3, 1)$ as a group.

We assume that the numerical class of f_*B is (6.65) of the list in Section 6. We denote B by $3B^1_{1,0} + 6B^2_{1,0} + 2B_{1,1} + 4B^1_{0,1} + 4B^2_{0,1} + \sum_{j=1}^{l} b'_i B'_i$, where $f_*B^i_{1,0} = (1, 0)$, $f_*B^i_{0,1} = (0, 1)$ in $\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)$, and B'_j is an exceptional divisor of f for $j = 1, \cdots, l$. By Theorem 2.5, $G \cong \mathbb{Z}/2\mathbb{Z}^{\oplus i} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ where i =0 or 1. If $G \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$, then G is one of $\mathcal{AG}$ as a group. We assume that $G \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$. By Remark 4.2, there are integers $\beta, a_j \geq 0$ such that

$$1 + \frac{\beta - 1}{\beta} = \frac{5}{6}a_1 + \frac{1}{2}a_2 + \frac{2}{3}a_3 + \frac{11}{12}a_4 + \frac{11}{12}a_5.$$

Since $G \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$, β=1, 2, 3, 4, 6 or 12. Since $f_*B = 3(1, 0) + 6(1, 0) + 2(1, 1) + 4(0, 1) + 4(0, 1)$, the support of f_*B is simple normal crossing. Since each irreducible component of f_*B is smooth, $a_j = 0$ or 1 for each $1 \leq j \leq 5$. Since $f_*B = 3(1, 0) + 6(1, 0) + 2(1, 1) + 4(0, 1) + 4(0, 1)$, the non-zero element of $\{a_1, a_2\}$ is just one, and the non-zero element of $\{a_4, a_5\}$ is just one. The integers which satisfy the above condition are $(\beta, a_1, a_2, a_3) = (12, 1, 0, 1)$ and $(a_4, a_5) = (1, 0)$ or (0,1). Therefore, $f(e_i) \notin f_*B^2_{1,0}$ for $i = 1, \cdots, l$. By the fact that $f_*B^2_{1,0} = (1, 0)$ and $f_*B_{1,1} = (1, 1)$ in $\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)$ and the fact that $f(e_i) \notin f_*B^2_{1,0}$ for $i = 1, \cdots, l$, we get that $B^2_{1,0} \cap B_{1,1}$ is not an empty set, and hence $p^{-1}(B^2_{1,0}) \cap p^{-1}(B_{1,1})$ is an empty set. Since $G \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$, the number of subgroup of G which is generated by a non-symplectic automorphism of order 2 is one. Since each ramification index of $B^2_{1,0}$ and $B_{1,1}$ is divided by 2, by Theorem 2.5, there is a non-symplectic automorphism g of order 2 such that $\mathrm{Fix}(g) \supset f^{-1}B^2_{1,0}$ and $\mathrm{Fix}(g) \supset f^{-1}B_{1,1}$. Since $p^{-1}(B^2_{1,0}) \cap p^{-1}(B_{1,1}) \neq \emptyset$, this is a contradiction. Therefore, if the numerical class of f_*B is (6.65), then G is one of $\mathcal{AG}$ as a group.

As for the case of (6.65), if the numerical class of f_*B is one of (6.95), (6.136), (6.150), (6.159), (6.235), (6.247), (6.253) of the list in Section 6, then G is one of $\mathcal{AG}$ as a group.

We assume that the numerical class of f_*B is (6.70) of the list in Section 6. We denote B by $2B^1_{1,0} + 3B^2_{1,0} + 6B^3_{1,0} + 2B^1_{0,1} + 4B^2_{0,1} + 4B^3_{0,1} + \sum_{j=1}^{l} b'_i B'_i$, where $f_*B^i_{1,0} = (1, 0)$, $f_*B^i_{0,1} = (0, 1)$, and

B'_j is an exceptional divisor of f for $j = 1, \cdots, l$. By Theorem 2.5, $G \cong \mathbb{Z}/2\mathbb{Z}^{\oplus i} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ where $i = 0, 1$ or 2. There are some integers β, a_j such that

$$1 + \frac{\beta - 1}{\beta} = \frac{1}{2}a_1 + \frac{2}{3}a_2 + \frac{5}{6}a_3 + \frac{1}{2}a_4 + \frac{3}{4}a_5 + \frac{3}{4}a_6.$$

Since $G \cong \mathbb{Z}/2\mathbb{Z}^{\oplus i} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ where $i = 0, 1$ or 2, we get β=1, 2, 3, 4, 6 or 12. Since $f_*B = 2(1,0) + 3(1,0) + 6(1,0) + 2(0,1) + 4(0,1) + 4(0,1)$, the support of f_*B is simple normal crossing. Since each irreducible component of f_*B is smooth, $a_j = 0$ or 1 for each $1 \le j \le 6$, and by Proposition 4.2 the non-zero element of $\{a_1, a_2, a_3\}$ is just one, and the non-zero element of $\{a_4, a_5, a_6\}$ is just one. From the above, $(\beta, a_1, a_2, a_3, b_1, b_2, b_3) = (1, 1, 0, 0, 1, 0, 0)$. Therefore, $f(e_j) \in f_*(B^1_{1,0}) \cap f_*(B^1_{0,1})$ for $j = 1, \cdots, l$. Since $((1,0) \cdot (1,0)) = 0$, we get that $(p^*B^i_{1,0} \cdot p^*B^i_{1,0}) = 0$ for $i = 2, 3$. Since X is a $K3$ surface, the support of $p^*B^i_{1,0}$ is a union of elliptic curves for $i = 2, 3$. Since $G \cong \mathbb{Z}/2\mathbb{Z}^{\oplus i} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ where $i = 0, 1$ or 2, the number of subgroups of G which are generated by a non-symplectic automorphism of order 3 is one, and hence there is a non-symplectic automorphism g of order 3 such that $\mathrm{Fix}(g)$ has at least two elliptic curves. By [1,14], this is a contradiction. Therefore, the numerical class of f_*B is not (6.70).

As for the case of (6.70), the numerical class of f_*B is not one of (6.75), (6.143) of the list in Section 6.

If the numerical class of f_*B is (6.72) of the list in Section 6, then by Theorem 2.5, $G \cong \mathbb{Z}/2\mathbb{Z}^{\oplus i} \oplus \mathbb{Z}/4\mathbb{Z}^{\oplus j}$ where (i, j) is one of (0,1), (0,2), (1,1), (1,2), (2,1), (2,2), (3,1). We assume that $G \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 2} \oplus \mathbb{Z}/4\mathbb{Z}^{\oplus 2}$. Since G is generated by non-symplectic automorphism of order 2 and 4, $G_s := \{g \in G : \ g \text{ is symplectic}\} \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 2} \oplus \mathbb{Z}/4\mathbb{Z}$. By the classification of finite symplectic groups (see [13, 10, 16]), we see that there is no G_s where $G_s \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 2} \oplus \mathbb{Z}/4\mathbb{Z}$. Therefore, $G \cong \mathbb{Z}/2\mathbb{Z}^{\oplus i} \oplus \mathbb{Z}/4\mathbb{Z}^{\oplus j}$ where (i, j) is one of (0,1), (0,2), (1,1), (1,2), (2,1), (3,1), and if the numerical class of f_*B is (6.72), then G is one of $\mathcal{A}G$ as a group.

As for the case of (6.72), if the numerical class of f_*B one of (6.74), (6.78), (6.111), (6.144) of the list in Section 6, then G is one of $\mathcal{A}G$ as a group.

We assume that the numerical class of f_*B is (6.73) of the list in Section 6. We denote B by $2B^1_{1,0} + 4B^2_{1,0} + 4B^3_{1,0} + 3B^1_{0,1} + 3B^2_{0,1} + 3B^3_{0,1} + \sum_{j=1}^{l} b'_i B'_i$ By Theorem 2.5, $G \cong \mathbb{Z}/2\mathbb{Z}^{\oplus i} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ where $i = 0, 1$ or 2. As for the case of (6.68), there are integers β, a_j such that

$$1 + \frac{\beta - 1}{\beta} = \frac{1}{2}a_1 + \frac{3}{4}a_2 + \frac{3}{4}a_3 + \frac{2}{3}a_4 + \frac{2}{3}a_5 + \frac{2}{3}a_6,$$

and $a_j = 0$ or 1 for each $1 \le j \le 6$, $\beta = 1, 2, 3, 4, 6$ or 12, the non-zero element of $\{a_1, a_2, a_3\}$ is only one, and the non-zero element of $\{a_4, a_5, a_6\}$ is only one, however, integers which satisfy the above condition do not exist. Therefore, the numerical class of f_*B is not (6.73).

As for the case of (6.73), the numerical class of f_*B is not one of (6.101), (6.129), (6.132), (6.152), (6.160), (6.170), (6.171), (6.172), (6.173) of the list in Section 6.

We assume that the numerical class of f_*B is (6.76) of the list in Section 6. We denote B by $2B^1_{1,0} + 4B^2_{1,0} + 4B^3_{1,0} + 2B^1_{0,1} + 2B^2_{0,1} + 2B^3_{0,1} + 2B^4_{0,1} + \sum_{i=1}^{n} b'_i B'_i$, where $f_*B^i_{1,0} = (1,0)$, $f_*B^i_{0,1} = (0,1)$ in $\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)$ and $f_*B'_i = 0$. By Theorem 2.5, $G \cong \mathbb{Z}/2\mathbb{Z}^{\oplus i} \oplus \mathbb{Z}/4\mathbb{Z}$, where

$i = 0, 1, 2, 3$ or 4. We assume that $G \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 4} \oplus \mathbb{Z}/4\mathbb{Z}$. By Theorem 2.5, $G = G^1_{1,0} \oplus G^2_{1,0} \oplus G^1_{0,1} \oplus G^2_{0,1} \oplus G^3_{0,1}$. As for the case of (6.70), we get that $f(e_i) \in B^1_{1,0} \cap B^j_{0,1}$ for each $i = 1, \cdots, m$ where $j = 1, 2, 3, 4$. Therefore, we get $(B^2_{1,0} \cdot B^j_{0,1}) = 1$. Let $s \in G^2_{1,0}$ be a generator. Since $G = G^1_{1,0} \oplus G^2_{1,0} \oplus G^1_{0,1} \oplus G^2_{0,1} \oplus G^3_{0,1}$, by Theorem 2.5, there is a non-symplectic automorphism $t \in G^j_{0,1}$ for some $j = 1, 2, 3$ such that $\mathrm{Fix}(t \circ s)$ does not have a curve. Since $(B^2_{1,0} \cdot B^j_{0,1}) = 1$ and $|G| = 2^3 4$, we get that $|p^{-1}(B^2_{1,0}) \cap p^{-1}(B^j_{0,1})| = 8$. By [2, Proposition 1], the number of isolated points of $\mathrm{Fix}(t \circ s)$ is 4. This is a contradiction. Therefore, if the numerical class of f_*B is (6.76), then $G \cong \mathbb{Z}/2\mathbb{Z}^{\oplus i} \oplus \mathbb{Z}/4\mathbb{Z}$ where $i = 0, 1, 2$ or 3, and hence G is one of $\mathcal{A}G$ as a group.

We assume that the numerical class of f_*B is (6.150) of the list in Section 6. We denote B by $3B_{1,0} + 2B^1_{1,1} + 6B^2_{1,1} + 4B^1_{0,1} + 12B^2_{0,1} + \sum_{j=1}^{l} b'_i B'_i$ where $f_*B^i_{s,t} = sC + tF$ in $\mathrm{Pic}(\mathbb{F}_1)$, and B'_j is an exceptional divisor of f for $j = 1, \cdots, l$. By Theorem 2.5, $G \cong \mathbb{Z}/2\mathbb{Z}^{\oplus i} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ where $i = 0$ or 1. Then the number of subgroup of G which is generated by a non-symplectic automorphism of order 3 is one. By the above, for e_i, there are integers $\beta, a_j \geq 0$ such that

$$1 + \frac{\beta - 1}{\beta} = \frac{2}{3}a_1 + \frac{1}{2}a_2 + \frac{5}{6}a_3 + \frac{3}{4}a_4 + \frac{11}{12}a_5.$$

Since $G \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$, β=1, 2, 3, 4, 6 or 12. Since $f_*B = 3(1,0) + 6(1,0) + 2(1,1) + 4(0,1) + 4(0,1)$, the support of f_*B is simple normal crossing. Since each irreducible component of f_*B is smooth, $a_j = 0$ or 1 for each $1 \leq j \leq 5$. The integers which satisfy the above condition are $(\beta, a_1, a_2, a_3, a_4, a_5) = (4, 0, 0, 1, 0, 1)$. Therefore, $f(e_i) \notin f_*B_{1,0} \cap f_*B^2_{0,1}$ for $i = 1, \cdots, l$ and hence $p^{-1}(B_{1,0}) \cap p^{-1}(B^2_{0,1})$ is not an empty set. Since $G_{1,0} \cong \mathbb{Z}/3\mathbb{Z}$, $G^2_{0,1} \cong \mathbb{Z}/12\mathbb{Z}$, and $p^{-1}(B_{1,0}) \cap p^{-1}(B^2_{0,1})$ is not an empty set, we get that the number of subgroup of G which is generated by a non-symplectic automorphism of order 3 is at least two. This is a contradiction. Therefore, the numerical class of f_*B is not (6.150).

As for the case of (6.150), the numerical class of f_*B is not (6.159) of the list in Section 6.

5 Abelian Groups of Enriques Surfaces with Smooth Quotient

Let E be an Enriques surface and H be a finite Abelian subgroup of $\mathrm{Aut}(E)$ such that E/H is smooth. Let X be the $K3$-cover of E, and $G := \{s \in \mathrm{Aut}(X) : s$ is a lift of some $h \in H\}$. Then G is a finite Abelian group, G has a non-symplectic involution whose fixed locus is empty, $X/G = E/H$, and the branch divisor of G is that of H.

Theorem 5.1 *Let E be an Enriques surface and H be a finite subgroup of* $\mathrm{Aut}(E)$. *We assume that the quotient space E/H is smooth and there is a birational morphism from E/H to a Hirzebruch surface $\mathbb{F}_n$, where $0 \leq n$. Then $0 \leq n \leq 4$.*

Proof Let $f : E/H \to \mathbb{F}_n$ be a birational morphism, and $B := \sum_{i=1}^{k} b_i B_i$ be the branch divisor of the quotient map $E \to E/H$. Since the canonical line bundle of an Enriques surface is numerically trivial, by Theorem 2.4, the numerical class of f_*B is one of Section 3. By [11, Proposition 4.5], G does not have a non-symplectic automorphism whose order is odd. Therefore, b_i is even number for each $i = 1, \cdots, k$ by Theorem 2.5. By the list of the numerical class of Section 3, we get the claim.

Theorem 5.2 *For each numerical classes* (6.6), (6.8), (6.9), (6.11), (6.12), (6.13), (6.16), (6.89), (6.90), (6.91), (6.94), (6.96), (6.97), (6.98), (6.101), (6.203), (6.206), (6.209), (6.210), (6.281) *of the list in Section* 6*, there is an Enriques surface E and a finite Abelian subgroup H of* $\mathrm{Aut}(E)$ *such that E/H is a Hirzebruch surface $\mathbb{F}_n$, and the numerical class of the branch divisor B of the quotient map $E \to E/H$ is* (6.6), (6.8), (6.9), (6.11), (6.12), (6.13), (6.16), (6.89), (6.90), (6.91), (6.94), (6.96), (6.97), (6.98), (6.101), (6.203), (6.206), (6.209), (6.210), (6.281)*.*

Furthermore, for a pair (E, H) of an Enriques surface E and a finite Abelian subgroup H of $\mathrm{Aut}(E)$*, if $E/H \cong \mathbb{F}_n$ and the numerical class of the branch divisor B of the quotient map $E \to E/H$ is* (6.6), (6.8), (6.9), (6.11), (6.12), (6.13), (6.16), (6.89), (6.90), (6.91), (6.94), (6.96), (6.97), (6.98), (6.101), (6.203, (6.206), (6.209), (6.210), (6.281)*, then H is $\mathbb{Z}/2\mathbb{Z}^{\oplus 2}$, $\mathbb{Z}/4\mathbb{Z}^{\oplus 2}$, $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}^{\oplus 4}$, $\mathbb{Z}/2\mathbb{Z}^{\oplus 3}$, $\mathbb{Z}/2\mathbb{Z}^{\oplus 2} \oplus \mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}^{\oplus 3}$, $\mathbb{Z}/2\mathbb{Z}^{\oplus 2}$, $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}^{\oplus 3}$, $\mathbb{Z}/2\mathbb{Z}^{\oplus 3}$, $\mathbb{Z}/2\mathbb{Z}^{\oplus 3}$, $\mathbb{Z}/2\mathbb{Z}^{\oplus 2} \oplus \mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}^{\oplus 4}$, $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}^{\oplus 2}$, $\mathbb{Z}/2\mathbb{Z}^{\oplus 3}$, $\mathbb{Z}/4\mathbb{Z}^{\oplus 2}$, $\mathbb{Z}/2\mathbb{Z}^{\oplus 2} \oplus \mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$, in order, as a group.*

Proof Let X be the $K3$-cover of E, $G := \{s \in \mathrm{Aut}(X) : s \text{ is a lift of some } h \in H\}$, and $p : X \to X/G$ be the quotient map. Then G is a finite Abelian group, $X/G \cong \mathbb{F}_n$, and the branch divisor of p is B. Since b_i is power of two for each $i = 1, \cdots, k$, $G \cong \mathbb{Z}/2\mathbb{Z}^{\oplus s} \oplus \mathbb{Z}/4\mathbb{Z}^{\oplus t} \oplus \mathbb{Z}/8\mathbb{Z}^{\oplus u}$ where $s, t, u \geq 0$. By Theorem 2.5, and the assumption that G has a non-symplectic automorphism of order 2 such that whose fixed locus is an empty set, we get $s + t + u \geq 3$, and hence the numerical class of B is one of (6.6), (6.8), (6.9), (6.10), (6.11), (6.12), (6.13), (6.15), (6.16), (6.19), (6.20), (6.81), (6.82), (6.87), (6.88), (6.89), (6.90), (6.91), (6.94), (6.96), (6.97), (6.98), (6.100), (6.101), (6.199), (6.200), (6.203), (6.206), (6.208), (6.209), (6.210), (6.281), (6.282) of the list in Section 6.

We assume that the numerical class of B is (6.6). We denote B by $2B_{1,0}^1 + 2B_{1,0}^2 + 2B_{2,2} + 2B_{0,1}^1 + 2B_{0,1}^2$. By Proposition 3.3, $G = G_{1,0}^1 \oplus G_{2,2} \oplus G_{0,1}^1 \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 3}$. Let $s, t, u, \in G$ be generators of $G_{1,0}^1$, $G_{0,1}^1$ and $G_{2,2}$, respectively. Then the non-symplectic automorphisms of G are s, t, u, and $s \circ t \circ u$.

From here, we will show that $\mathrm{Fix}(s \circ t \circ u)$ is an empty set. We assume that the curves of $\mathrm{Fix}(s)$ are only $p^{-1}(B_{1,0}^1)$. Since s is a non-symplectic automorphism of order 2, the quotient space $X/\langle s \rangle$ is a smooth rational surface. The quotient map $q : X/\langle s \rangle \to X/G \cong \mathbb{P}^1 \times \mathbb{P}^1$ is the Galois cover such that the branch divisor is $2B_{0,1}^2 + 2B_{2,2} + 2B_{0,1}^1 + 2B_{0,1}^2$, and the Galois group is isomorphic to $\mathbb{Z}/2\mathbb{Z}^{\oplus 2}$ as a group. By Theorem 3.1, there is the Galois cover $g : Y \to X/G$ whose branch divisor is $2B_{2,2} + 2B_{0,1}^1 + 2B_{0,1}^2$ and Galois group is isomorphic to $\mathbb{Z}/2\mathbb{Z}^{\oplus 2}$ as a group. Since $\mathrm{Fix}(s)$ is not an empty set and the order of s is 2, $X/\langle s \rangle$ is a smooth rational surface. By Theorem 2.2, there is the Galois cover $h : X/\langle s \rangle \to Y$ such that $q = g \circ h$. Since the degree of q is 4 and that of g is 4, h is an isomorphism. Since the branch divisor of q is not that of g, this is a contradiction. Therefore, $\mathrm{Fix}(s)$ is $p^{-1}(B_{1,0}^1) \cup p^{-1}(B_{1,0}^2)$. In the same way, $\mathrm{Fix}(t)$ is $p^{-1}(B_{0,1}^1) \cup p^{-1}(B_{0,1}^2)$. Therefore, by Theorem 2.5, $\mathrm{Fix}(s \circ t \circ u)$ is an empty set, and hence $E := X/\langle s \circ t \circ u \rangle$ is an Enriques surface. Let $H := G/\langle s \circ t \circ u \rangle$. Then $E/H \cong \mathbb{P}^1 \times \mathbb{P}^1$, $H \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 2}$, and the branch divisor of H is B. It is easy to show that for an Enriques surface E and a finite Abelian subgroup H of $\mathrm{Aut}(E)$ such that $E/H \cong \mathbb{P}^1 \times \mathbb{P}^1$ if the numerical class of H is (6.6), then $H \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 2}$.

As for the case of (6.6), the claim is established for (6.89).

We assume that the numerical class of B is (6.8). We denote B by $4B_{1,0}^1 + 4B_{1,0}^2 + 2B_{1,1} +$

$4B_{0,1}^1 + 4B_{0,1}^2$. By Proposition 3.3, $G = G_{1,0}^1 \oplus G_{1,1} \oplus G_{0,1}^1 \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}^{\oplus 2}$. Let $s, t, u \in G$ be generators of $G_{1,0}^1$, $G_{0,1}^1$ and $G_{1,1}$, respectively. By Theorem 2.5, s and t are non-symplectic automorphism of order 4 and u is a non-symplectic automorphism of order 2. By Theorem 2.5, $G_{1,0}^2$ is generated by $s \circ t^{2x} \circ u^y$ where $x, y = 0$ or 2. Since $(s \circ t^{2x} \circ u^y)^2 = s^2$ for $x, y = 0$ or 2, we get that $\mathrm{Fix}(s^2)$ is $p^{-1}(B_{1,0}^1) \cup p^{-1}(B_{1,0}^2)$. As for the case of (6.6), we get the claim for (6.8).

As for the case of (6.8), the claim is established for (6.101).

We assume that the numerical class of B is (6.9). We denote B by $4B_{1,0}^1 + 4B_{1,0}^2 + 2B_{1,2} + 2B_{0,1}^1 + 2B_{0,1}^2$. By Proposition 3.3, $G = G_{1,0}^1 \oplus G_{1,2} \oplus G_{0,1}^1 \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 2} \oplus \mathbb{Z}/4\mathbb{Z}$. Let $s, t, u \in G$ be generators of $G_{1,0}^1$, $G_{0,1}^1$ and $G_{1,2}$, respectively. As for the case of (6.6), $\mathrm{Fix}(t)$ is $p^{-1}(B_{0,1}^1) \cup p^{-1}(B_{0,1}^2)$. As for the case of (6.8), $\mathrm{Fix}(s)$ is the support of $p^{-1}(B_{1,0}^1) \cup p^{-1}(B_{1,0}^2)$. As for the case of (6.6), we get the claim for (6.101).

We assume that the numerical class of B is (6.10). We denote B by $2B_{1,0}^1 + 2B_{1,0}^2 + 2B_{1,0}^3 + 2B_{1,4}$. Let $s_1, s_2, t \in G$ be generators of $G_{1,0}^1$, $G_{1,0}^2$ and $G_{1,4}$, respectively. By Proposition 3.3, $G = G_{1,0}^1 \oplus G_{1,0}^2 \oplus G_{1,4} \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 3}$. Then the non-symplectic involutions of G are s_1, s_2, t and $s_1 \circ s_2 \circ t$.

We assume that $\mathrm{Fix}(s_1)$ is $p^{-1}(B_{1,0}^1) \cup p^{-1}(B_{1,0}^3)$. Then $X/\langle s_1 \rangle$ is a smooth rational surface, and the quotient map $q : X/\langle s_1 \rangle \to X/G \cong \mathbb{P}^1 \times \mathbb{P}^1$ is the Galois cover such that the branch divisor is $2B_{0,1}^2 + 2B_{1,4}$, and the Galois group is isomorphic to $\mathbb{Z}/2\mathbb{Z}^{\oplus 2}$ as a group. Since $\mathbb{P}^1 \times \mathbb{P}^1 \backslash B_{1,0}^2$ is simply connected, in the same way of the proof of Theorem 2.5, this is a contradiction. Therefore, $\mathrm{Fix}(s_i)$ is $p^{-1}(B_{1,0}^i)$ for $i = 1, 2$, and hence $\mathrm{Fix}(s_1 \circ s_2 \circ t)$ is $p^{-1}(B_{1,0}^3)$. There is not an Enriques surface E and a finite Abelian subgroup H of $\mathrm{Aut}(E)$ such that $E/H \cong \mathbb{P}^1 \times \mathbb{P}^1$ and the numerical class of the branch divisor of H is (6.10).

As for the case of (6.10), we get the claim for (6.87), (6.100).

We assume that the numerical class of B is (6.11). We denote B by $2B_{1,0}^1 + 2B_{1,0}^2 + 2B_{1,0}^3 + 2B_{1,1} + 2B_{0,1}^1 + 2B_{0,1}^2 + 2B_{0,1}^3$. By Proposition 3.3, $G = \oplus_{i=1}^2 G_{1,0}^i \oplus G_{1,1} \oplus_{i=1}^2 G_{0,1}^i$, and hence the number of non-symplectic automorphisms of order 2 of G is 16. By Theorem 2.5, G has a non-symplectic automorphism of order 2 whose fixed locus is an empty set. Furthermore, it is easy to show that for an Enriques surface E and a finite Abelian subgroup H of $\mathrm{Aut}(E)$ such that $E/H \cong \mathbb{P}^1 \times \mathbb{P}^1$ if the numerical class of H is (6.11), then $H \cong \mathbb{Z}/2\mathbb{Z}^{\oplus 4}$.

As for the case of (6.11), the claim is established for (6.12), (6.13), (6.16), (6.91), (6.94), (6.96), (6.97), (6.98), (6.206), 6.210).

We assume that the numerical class of B is (6.15). We denote B by $2B_{1,0}^1 + 2B_{1,0}^2 + 2B_{1,2}^1 + 2B_{1,2}^2$. By Proposition 3.4, $G = G_{1,0}^1 \oplus G_{1,2}^1 \oplus G_{1,2}^2$. Let $s, t, u \in G$ be generators of $G_{1,0}^1$, $G_{1,2}^1$ and $G_{1,2}^2$, respectively. Then the non-symplectic automorphisms of order 2 of G are s, t, u and $s \circ t \circ u$. We assume that $\mathrm{Fix}(s \circ t \circ u)$ is an empty set. Since $(B_{1,0}^i \cdot B_{1,2}^j) \neq 0$ for $i, j = 1, 2$, $\mathrm{Fix}(s)$ is $p^{-1}(B_{1,0}^1) \cup p^{-1}(B_{1,0}^2)$. Since $(B_{1,0}^1 + B_{1,0}^2 \cdot B_{1,2}^1) = 4$, $X/(G_{1,0}^1 \oplus G_{1,2}^1)$ is smooth. Since $G = G_{1,0}^1 \oplus G_{1,2}^1 \oplus G_{1,2}^2$, the branch divisor of the quotient map $X/(G_{1,0}^1 \oplus G_{1,2}^1) \to X/G \cong \mathbb{F}_2$ is $2B_{1,0}^2$ and its degree is 2. Since $\frac{B_{1,0}^2}{2} \notin \mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)$ and $X/(G_{1,0}^1 \oplus G_{1,2}^1)$ is smooth, this is a contradiction. Therefore, there is not an Enriques surface E and a finite Abelian subgroup H of $\mathrm{Aut}(E)$ such that $E/H \cong \mathbb{F}_1$ and the numerical class of branch divisor of H is (6.15).

As for the case of (6.15), we get that there is not an Enriques surface E and a finite Abelian subgroup H of $\mathrm{Aut}(E)$ such that $E/H \cong \mathbb{F}_n$ and the numerical class of the branch divisor of H is (6.88).

We assume that the numerical class of B is (6.19). We denote B by $2B^1_{1,1}+2B^2_{1,1}+2B^3_{1,1}+2B^4_{1,1}$. By Proposition 3.6, $G=G^1_{1,1}\oplus G^2_{1,1}\oplus G^3_{1,1}$. Let $s_i\in G^i_{1,1}$ be a generator of $G^i_{1,1}$ for $i=1,2,3,4$. By Theorem 2.5, $\mathrm{Fix}(s_i)$ is not an empty set for $i=1,2,3,4$. Since $G\cong\mathbb{Z}/2\mathbb{Z}^{\oplus 3}$, $s_4=s_1\circ s_2\circ s_3$, and hence G does not have a non-symplectic automorphism of order 2 whose fixed locus is an empty set. Therefore, there is not an Enriques surface E and a finite Abelian subgroup H of $\mathrm{Aut}(E)$ such that $E/H\cong\mathbb{F}_1$ and the numerical class of the branch divisor of H is (6.19).

As for the case of (6.19), we get that there is not an Enriques surface E and a finite Abelian subgroup H of $\mathrm{Aut}(E)$ such that $E/H\cong\mathbb{F}_n$ and the numerical class of the branch divisor of H is (6.19), (6.20), (6.81), (6.82), (6.200).

We assume that the numerical class of B is (6.90). We denote B by $2B_{1,0}+2B_{1,1}+2B_{2,2}+4B^1_{0,1}+4B^2_{0,1}$. By Corollary 3.3, $G=G_{1,1}\oplus G_{2,2}\oplus G^1_{0,1}$. Let $q:X/\langle G_{1,0},G_{1,1},G_{2,2}\rangle\to X/G\cong\mathbb{F}_1$ be the quotient map. Then the branch divisor of q is $4B^1_{0,1}+4B^2_{0,1}$. By Theorem 2.2, $X/\langle G_{1,0},G_{1,1},G_{2,2}\rangle\cong\mathbb{F}_4$, and the branch divisor of $\langle G_{1,0},G_{1,1},G_{2,2}\rangle$ is $2B_{1,0}+2q^*B_{1,1}+2q^*B_{2,2}$. Let $s,t,u\in G$ be generators of $G_{1,1}$, $G_{2,2}$ and $G^1_{0,1}$, respectively. Then $\mathrm{Fix}(s)$ is the support of $p^*B_{1,0}$ and that of $p^*B_{1,1}$. Then as for the case of (6.6), we get the claim.

As for the case of (6.90), the claim is established for (6.203), (6.209), (6.281).

We assume that the numerical class of B is (6.199). We denote B by $2B_{1,0}+2B_{1,4}+2B^1_{1,2}+2B^2_{1,2}$. By Corollary 3.5, $G=G_{1,4}\oplus G^1_{1,2}\oplus G^2_{1,2}$. Let $s,t,u\in G$ be generators of $G_{1,4}$, $G^1_{1,2}$ and $G^2_{1,2}$, respectively. Then the non-symplectic automorphisms of G are s, t, u and $s\circ t\circ u$. Since each fixed locus of s, t and u is not an empty set, by Theorem 2.5, if G has a non-symplectic automorphism of order 2 whose fixed locus is an empty set, then that is $s\circ t\circ u$. We assume that $\mathrm{Fix}(s\circ t\circ u)$ is an empty set. Then we may assume that $\mathrm{Fix}(t)$ is $p^{-1}(B_{1,0})\cup p^{-1}(B^1_{1,2})$. Since $(B_{1,0}+B^1_{1,2}\cdot B_{1,4})=6$, we get $|p^{-1}(B_{1,0}\cup B^1_{1,2})\cap p^{-1}(B_{1,4})|=12$. Since $s\circ t$ is a symplectic automorphism of order 2 and $p^{-1}(B_{1,0}\cup B^1_{1,2})\cap p^{-1}(B_{1,4})$ is contained in $\mathrm{Fix}(s\circ t)$, this is a contradiction. Therefore, there is not an Enriques surface E and a finite Abelian subgroup H of $\mathrm{Aut}(E)$ such that $E/H\cong\mathbb{F}_1$ and the numerical class of the branch divisor of H is (6.199).

We assume that the numerical class of B is (6.208). We denote B by $4B_{1,0}+2B_{1,3}+4B_{1,2}+2B^1_{0,1}+2B^2_{0,1}$. By Proposition 3.8, $G=G_{1,3}\oplus G_{1,2}\oplus G^1_{0,1}$. Let $s,t,u\in G$ be generators of $G_{1,3}$, $t\in G_{1,2}$ and $u\in G^1_{0,1}$, respectively. Then the non-symplectic automorphisms of G are s, t^2, u and $s\circ t^2\circ u$. Since each fixed locus of s, t^2 and u is not an empty set by Theorem 2.5, if G has a non-symplectic automorphism of order 2 whose fixed locus is an empty set, then that is $s\circ t^2\circ u$.

We assume that $\mathrm{Fix}(s\circ t^2\circ u)$ is an empty set. Then $\mathrm{Fix}(t^2)$ is $p^{-1}(B_{1,0})\cup p^{-1}(B_{1,2})$ and $\mathrm{Fix}(u)$ is $p^{-1}(B^1_{1,0})\cup p^{-1}(B^2_{1,0})$. Since $(B_{1,3}\cdot B^1_{0,1}+B^2_{0,1})=4$, we get that $X/(G_{1,3}\oplus G^1_{0,1})$ is smooth, and the branch divisor of the quotient map $f:X/(G_{1,3}\oplus G^1_{0,1})\to X/G\cong\mathbb{F}_2$ is $4B_{1,0}+4B_{1,2}$, and the Galois group is $\mathbb{Z}/4\mathbb{Z}$, which is induced by t. Furthermore, since $(B_{1,3}\cdot B_{1,0}+B_{1,2})=4$ and $(B_{1,3}\cdot B^1_{0,1}+B^2_{0,1})=4$, $G/\langle s,t^2,u\rangle$ is smooth, and the branch divisor of the quotient map $g:X/\langle s,t^2,u\rangle\to X/G\cong\mathbb{F}_2$ is $2B_{1,0}+2B_{1,2}$, and the Galois group is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ as a group. Let $E_{1,0}$ and $E_{1,2}$ be the support of $g^*B_{1,0}$ and $g^*B_{1,2}$, respectively. Then $g^*B_{1,0}=2E_{1,0}$ and $g^*B_{1,2}=2E_{1,2}$. Moreover, by Theorem 3.1, there is the double cover $h:X/(G_{1,3}\oplus G^1_{0,1})\to X/\langle s,t^2,u\rangle$ such that $f=g\circ h$ and the branch divisor is $2E_{1,0}+2E_{1,2}$. Since $X/(G_{1,3}\oplus G^1_{0,1})$ and $X/\langle s,t^2,u\rangle$ are smooth, we get

$\frac{E_{1,0}+E_{1,2}}{2} \in \mathrm{Pic}(X/\langle s,t^2,u\rangle)$. Since $g^*B_{1,2} = g^*B_{1,0} + 2g^*F$ in $\mathrm{Pic}(X/\langle s,t^2,u\rangle)$,

$$2E_{1,2} = 2E_{1,0} + 2g^*F \quad \text{in } \mathrm{Pic}(X/\langle s,t^2,u\rangle).$$

Since $X/\langle s,t^2,u\rangle$ is a smooth rational surface, $\mathrm{Pic}(X/(G_{1,3} \oplus G^1_{0,1}))$ is torsion free. Therefore, we get

$$E_{1,2} = E_{1,0} + g^*F \quad \text{in } \mathrm{Pic}(X/\langle s,t^2,u\rangle),$$

and hence

$$E_{1,2} + E_{1,0} = 2E_{1,0} + g^*F \quad \text{in } \mathrm{Pic}(X/\langle s,t^2,u\rangle).$$

Since $\frac{E_{1,2}+E_{1,0}}{2} \in \mathrm{Pic}(X/\langle s,t^2,u\rangle)$, we get

$$\frac{g^*F}{2} \in \mathrm{Pic}(X/\langle s,t^2,u\rangle).$$

Since $(B_{1,0} \cdot F) = 1$, the degree of g is two, $\frac{g^*B_{1,0}}{2}$ and $\frac{g^*F}{2}$ are elements of $\mathrm{Pic}(X/\langle s,t^2,u\rangle)$, this is a contradiction. Therefore, there is not an Enriques surface E and a finite Abelian subgroup H of $\mathrm{Aut}(E)$ such that $E/H \cong \mathbb{F}_1$ and the numerical class of the branch divisor of H is (6.208).

We assume that the numerical class of B is (6.282). We denote B by $2B_{1,0}+2B^1_{1,4}+2B^2_{1,4}+2B^3_{1,4}$. By Corollary 3.5, $G = \oplus_{i=1}^3 G^i_{1,4}$. Let $s_i \in G^i_{1,4}$ be a generator for $i = 1,2,3$. Then the non-symplectic automorphisms of G are s_i and $s_1 \circ s_2 \circ s_3$ where $i = 1,2,3$. Since each fixed locus of s_i is not an empty set for each $i = 1,2,3$ by Theorem 2.5, if G has a non-symplectic automorphism of order 2 whose fixed locus is an empty set, then that is $s_1 \circ s_2 \circ s_3$. We assume that $\mathrm{Fix}(s_1 \circ s_2 \circ s_3)$ is an empty set. Then we may assume that $\mathrm{Fix}(s_1)$ is $p^{-1}(B_{1,0}) \cup p^{-1}(B^1_{1,4})$. Since $(B_{1,0} + B^1_{1,4} \cdot B_{1,4}) = 4$, we get that $X/(G^1_{1,4} \oplus G^2_{1,4})$ is smooth, and the branch divisor of the quotient map $X/(G^1_{1,4} \oplus G^1_{1,4}) \to X/G \cong \mathbb{F}_4$ is $2B^3_{1,4}$. This is a contradiction as the degree of the quotient map is 2. Therefore, there is not an Enriques surface E and a finite Abelian subgroup H of $\mathrm{Aut}(E)$ such that $E/H \cong \mathbb{F}_4$ and the numerical class of the branch divisor of H is (6.282).

By Theorem 5.2, we get Theorem 1.7.

Theorem 5.3 *Let E be an Enriques surface and H be a finite Abelian subgroup of* $\mathrm{Aut}(E)$. *If E/H is smooth, then H is isomorphic to one of $\mathcal{AG}(E)$ as a group.*

Proof Let X be the $K3$-cover of E, $G := \{s \in \mathrm{Aut}(X) : s \text{ is a lift of some } h \in H\}$, and $p : X \to X/G$ be the quotient map. Then G is a finite Abelian group, $X/G = E/H$, and the branch divisor of p is B. We classified H for the case of $E/H \cong \mathbb{F}_n$ in Theorem 5.2. From here, we assume that E/H is smooth and $E/H \ncong \mathbb{F}_n$ or $\mathbb{P}^2$. Since G does not have a non-symplectic automorphism whose order is odd (see [11]), by Theorems 2.5 and 1.4, $G \cong \mathbb{Z}/2\mathbb{Z}^{\oplus s} \oplus \mathbb{Z}/4\mathbb{Z}^{\oplus t} \oplus \mathbb{Z}/8\mathbb{Z}^{\oplus u}$ where $s,t,u \geq 0$. By the assumption that G has a non-symplectic automorphism of order 2 such that whose fixed locus is an empty set, and the fact that G is generated by non-symplectic automorphisms whose fixed locus have a curve, we get $s+t+u \geq 3$. Therefore, G is one of the following as a group:

$$\{\mathbb{Z}/2\mathbb{Z}^{\oplus a},\ \mathbb{Z}/4\mathbb{Z}^{\oplus 3},\ \ \mathbb{Z}/2\mathbb{Z}^{\oplus f} \oplus \mathbb{Z}/4\mathbb{Z}^{\oplus g},\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z} :$$

$$3 \leq a \leq 5,\ (f,g) = (1,2),(2,1),(3,1)\}.$$

If G is one of

$$\{\mathbb{Z}/2\mathbb{Z}^{\oplus a},\ \ \mathbb{Z}/2\mathbb{Z}^{\oplus f} \oplus \mathbb{Z}/4\mathbb{Z}^{\oplus g} : 3 \leq a \leq 5,\ (f,g) = (1,2),(2,1),(3,1)\}$$

as a group, then quotient group G/K of G by a subgroup $K \cong \mathbb{Z}/2\mathbb{Z}$ is one of

$$\{\mathbb{Z}/2\mathbb{Z}^{\oplus a},\ \mathbb{Z}/4\mathbb{Z}^{\oplus 2},\ \mathbb{Z}/2\mathbb{Z}^{\oplus f} \oplus \mathbb{Z}/4\mathbb{Z} : a = 2,3,4\ f = 1,2\} \subset \mathcal{A}G(E)$$

as a group. Let $f : X/G \to \mathbb{F}_n$ be the birational morphism. We assume that $G \cong \mathbb{Z}/4\mathbb{Z}^{\oplus 3}$. By the assumption that $G \cong \mathbb{Z}/4\mathbb{Z}^{\oplus 3}$ and Theorem 2.5, the numerical class of f_*B is only (6.142). We denote B by $2B_{1,0} + 4B^1_{1,4} + 4B^2_{1,4} + 4B^1_{0,1} + 4B^2_{0,1} + \sum_{i=1}^{n} b'_i B'_i$, where $f_*B_{1,0} = C$, $f_*B^i_{1,4} = C+4F$, $f_*B^i_{0,1} = F$ and $f_*B'_i = 0$ in $\mathrm{Pic}(\mathbb{F}_4)$. Since $G \cong \mathbb{Z}/4\mathbb{Z}^{\oplus 3}$, by Theorem 2.5, we get that $G = G^1_{1,4} \oplus G^2_{1,4} \oplus G^1_{0,1}$. Let $s \in G^1_{1,4}$, $t \in G^2_{1,4}$ and $u \in G^1_{0,1}$ be generators respectively. The non-symplectic involutions of G are s^2, t^2, u^2 and $s^2 \circ t^2 \circ u^2$. Since each fixed locus of s^2, t^2 and u^2 is not an empty set, if G has a non-symplectic automorphism of order 2 whose fixed locus is an empty set, then that is $s^2 \circ t^2 \circ u^2$. If the fixed locus of $s^2 \circ t^2 \circ u^2$ is an empty set, then the fixed locus of $s \circ t \circ u$ is an empty set. By [2], this is a contradiction. Therefore, G is not $\mathbb{Z}/4\mathbb{Z}^{\oplus 3}$ as a group.

We assume that $G \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$. By Theorem 2.5, the numerical class of f_*B is only (6.101). By the proof of Theorem 4.3, f is an isomorphism, i.e., $X/G \cong \mathbb{F}_1$. By Theorem 5.2, we get the claim.

By Theorems 5.2–5.3, we get Theorem 1.8.

6 The List of a Numerical Class

Here, we will give the list of a numerical class of an effective divisor $B = \sum_{i=1}^{k} b_i B_i$ on $\mathbb{F}_n$ such that B_i is a smooth curve for each $i = 1, \cdots, k$ and $K_{\mathbb{F}_n} + \sum_{i=1}^{k} \frac{b_i - 1}{b_i} B_i = 0$ in $\mathrm{Pic}(\mathbb{F}_n)$.

If there is a $K3$ surface X and a finite subgroup G of $\mathrm{Aut}(X)$ such that $X/G = \mathbb{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$, then by Theorem 2.4 the numerical class of B is one of the following:

$$3(3C+3F)\ \ \mathbb{Z}/3\mathbb{Z} \tag{6.1}$$
$$3C+3C+3(C+3F)\ \ \mathbb{Z}/3\mathbb{Z}^{\oplus 2} \tag{6.2}$$
$$3C+3C+3(C+F)+3F+3F\ \ \mathbb{Z}/3\mathbb{Z}^{\oplus 3} \tag{6.3}$$
$$2(4C+4F)\ \ \mathbb{Z}/2\mathbb{Z} \tag{6.4}$$
$$2C+2C+2(2C+4F)\ \ \mathbb{Z}/2\mathbb{Z}^{\oplus 2} \tag{6.5}$$
$$2C+2C+2(2C+2F)+2F+2F\ \ \mathbb{Z}/2\mathbb{Z}^{\oplus 3} \tag{6.6}$$
$$4C+4C+2(C+4F)\ \ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \tag{6.7}$$
$$4C+4C+2(C+F)+4F+4F\ \ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}^{\oplus 2} \tag{6.8}$$
$$4C+4C+2(C+2F)+2F+2F\ \ \mathbb{Z}/2\mathbb{Z}^{\oplus 2} \oplus \mathbb{Z}/4\mathbb{Z} \tag{6.9}$$
$$2C+2C+2C+2(C+4F)\ \ \mathbb{Z}/2\mathbb{Z}^{\oplus 3} \tag{6.10}$$
$$2C+2C+2C+2(C+F)+2F+2F+2F\ \ \mathbb{Z}/2\mathbb{Z}^{\oplus 5} \tag{6.11}$$
$$2C+2C+2C+2(C+2F)+2F+2F\ \ \mathbb{Z}/2\mathbb{Z}^{\oplus 4} \tag{6.12}$$

$$2C + 2C + 2C + 2(C+F) + 4F + 4F \quad \mathbb{Z}/2\mathbb{Z}^{\oplus 3} \oplus \mathbb{Z}/4\mathbb{Z} \tag{6.13}$$
$$2(2C+2F) + 2(2C+2F) \quad \mathbb{Z}/2\mathbb{Z}^{\oplus 2} \tag{6.14}$$
$$2C + 2C + 2(C+2F) + 2(C+2F) \quad \mathbb{Z}/2\mathbb{Z}^{\oplus 3} \tag{6.15}$$
$$2C + 2C + 2(C+F) + 2(C+F) + 2F + 2F \quad \mathbb{Z}/2\mathbb{Z}^{\oplus 4} \tag{6.16}$$
$$3(C+F) + 3(C+F) + 3(C+F) \quad \mathbb{Z}/3\mathbb{Z}^{\oplus 2} \tag{6.17}$$
$$3C + 3(C+F) + 3(C+2F) \quad \mathbb{Z}/3\mathbb{Z}^{\oplus 2} \tag{6.18}$$
$$2(C+F) + 2(C+F) + 2(C+F) + 2(C+F) \quad \mathbb{Z}/2\mathbb{Z}^{\oplus 3} \tag{6.19}$$
$$2C + 2(C+F) + 2(C+F) + 2(C+2F) \quad \mathbb{Z}/2\mathbb{Z}^{\oplus 3} \tag{6.20}$$
$$2(C+F) + 4(2C+2F) \tag{6.21}$$
$$3(C+F) + 3(2C+2F) \tag{6.22}$$
$$3(C+2F) + 3(2C+F) \tag{6.23}$$
$$3C + 3(2C+3F) \tag{6.24}$$
$$2C + 2(3C+4F) \tag{6.25}$$
$$2(C+F) + 2(3C+3F) \tag{6.26}$$
$$2(C+2F) + 2(3C+2F) \tag{6.27}$$
$$2(C+3F) + 2(3C+F) \tag{6.28}$$
$$2(C+F) + 3(C+F) + 6(C+F) \tag{6.29}$$
$$2(C+F) + 4(C+F) + 4(C+F) \tag{6.30}$$
$$2C + 4(2C+2F) + 2F \tag{6.31}$$
$$4C + 2(2C+2F) + 4F \tag{6.32}$$
$$2C + 2(3C+3F) + 2F \tag{6.33}$$
$$3C + 6C + 2(C+4F) \tag{6.34}$$
$$4C + 2(C+F) + 4(C+2F) \tag{6.35}$$
$$2C + 2(C+F) + 2(2C+3F) \tag{6.36}$$
$$2C + 2(C+2F) + 2(2C+2F) \tag{6.37}$$
$$2C + 2(C+3F) + 2(2C+F) \tag{6.38}$$
$$3C + 3(2C+2F) + 3F \tag{6.39}$$
$$2C + 6C + 3(C+3F) \tag{6.40}$$
$$2(C+F) + 2(C+F) + 2(2C+2F) \tag{6.41}$$
$$2(C+2F) + 2(C+F) + 2(2C+F) \tag{6.42}$$
$$2C + 4C + 4(C+2F) + 2F \tag{6.43}$$
$$3C + 6C + 2(C+3F) + 2F \tag{6.44}$$
$$4C + 4C + 2(C+3F) + 2F \tag{6.45}$$
$$2C + 2C + 2(2C+3F) + 2F \tag{6.46}$$
$$2C + 4(C+F) + 4(C+F) + 2F \tag{6.47}$$
$$4C + 2(C+F) + 4(C+F) + 4F \tag{6.48}$$

$$2C + 3(C+F) + 6(C+F) + 2F \tag{6.49}$$
$$6C + 2(C+F) + 3(C+F) + 6F \tag{6.50}$$
$$2C + 2(C+F) + 2(2C+2F) + 2F \tag{6.51}$$
$$2C + 2(C+2F) + 2(2C+F) + 2F \tag{6.52}$$
$$3C + 2(C+F) + 6(C+F) + 3F \tag{6.53}$$
$$3C + 3(C+F) + 3(C+F) + 3F \tag{6.54}$$
$$3C + 3C + 3(C+2F) + 3F \tag{6.55}$$
$$2C + 6C + 3(C+2F) + 3F \tag{6.56}$$
$$2C + 2C + 2(C+F) + 2(C+3F) \tag{6.57}$$
$$2C + 2(C+F) + 2(C+F) + 2(C+F) + 2F \tag{6.58}$$
$$2C + 2C + 2C + 2(C+3F) + 2F \tag{6.59}$$
$$2C + 2C + 2(C+F) + 2(C+2F) + 2F \tag{6.60}$$
$$2C + 4C + 4(C+F) + 2F + 4F \tag{6.61}$$
$$2C + 3C + 6(C+F) + 2F + 3F \tag{6.62}$$
$$2C + 6C + 3(C+F) + 2F + 6F \tag{6.63}$$
$$3C + 6C + 2(C+F) + 3F + 6F \tag{6.64}$$
$$3C + 6C + 2(C+F) + 4F + 4F \tag{6.65}$$
$$2C + 6C + 3(C+F) + 3F + 3F \tag{6.66}$$
$$3C + 6C + 2(C+2F) + 2F + 2F \tag{6.67}$$
$$3C + 6C + 2(C+F) + 2F + 2F + 2F \tag{6.68}$$
$$2C + 3C + 6C + 2F + 3F + 6F \tag{6.69}$$
$$2C + 3C + 6C + 2F + 4F + 4F \tag{6.70}$$
$$2C + 3C + 6C + 3F + 3F + 3F \tag{6.71}$$
$$2C + 4C + 4C + 2F + 4F + 4F \tag{6.72}$$
$$2C + 4C + 4C + 3F + 3F + 3F \tag{6.73}$$
$$3C + 3C + 3C + 3F + 3F + 3F \tag{6.74}$$
$$2C + 3C + 6C + 2F + 2F + 2F + 2F \tag{6.75}$$
$$2C + 4C + 4C + 2F + 2F + 2F + 2F \tag{6.76}$$
$$3C + 3C + 3C + 2F + 2F + 2F + 2F \tag{6.77}$$
$$2C + 2C + 2C + 2C + 2F + 2F + 2F + 2F. \tag{6.78}$$

If there is a $K3$ surface X and a finite subgroup G of $\mathrm{Aut}(X)$ such that $X/G \cong \mathbb{F}_1$, then by Theorem 2.4 the numerical class of B is one of the following:

$$2(4C+6F) \quad \mathbb{Z}/2\mathbb{Z} \tag{6.79}$$
$$2(2C+4F) + 2(2C+2F) \quad \mathbb{Z}/2\mathbb{Z}^{\oplus 2} \tag{6.80}$$
$$2C + 2(C+2F) + 2(C+2F) + 2(C+2F) \quad \mathbb{Z}/2\mathbb{Z}^{\oplus 3} \tag{6.81}$$
$$2(C+3F) + 2(C+F) + 2(C+F) + 2(C+F) \quad \mathbb{Z}/2\mathbb{Z}^{\oplus 3} \tag{6.82}$$

$$3(3C+3F)+2F+2F \quad \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/3\mathbb{Z} \tag{6.83}$$

$$3C+3(2C+2F)+6F+6F \quad \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/3\mathbb{Z}^{\oplus 2} \tag{6.84}$$

$$2(4C+4F)+2F+2F \quad \mathbb{Z}/2\mathbb{Z}^{\oplus 2} \tag{6.85}$$

$$2C+2(3C+3F)+4F+4F \quad \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/4\mathbb{Z} \tag{6.86}$$

$$2C+2(3C+3F)+2F+2F+2F \quad \mathbb{Z}/2\mathbb{Z}^{\oplus 3} \tag{6.87}$$

$$2C+2(C+F)+2(2C+3F)+2F+2F \quad \mathbb{Z}/2\mathbb{Z}^{\oplus 3} \tag{6.88}$$

$$2(2C+2F)+2(2C+2F)+2F+2F \quad \mathbb{Z}/2\mathbb{Z}^{\oplus 3} \tag{6.89}$$

$$2C+2(C+F)+2(2C+2F)+4F+4F \quad \mathbb{Z}/2\mathbb{Z}^{\oplus 2}\oplus\mathbb{Z}/4\mathbb{Z} \tag{6.90}$$

$$2C+2(C+F)+2(2C+2F)+2F+2F+2F \quad \mathbb{Z}/2\mathbb{Z}^{\oplus 4} \tag{6.91}$$

$$3(C+F)+3(C+F)+3(C+F)+2F+2F \quad \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/3\mathbb{Z}^{\oplus 2} \tag{6.92}$$

$$3C+3(C+F)+3(C+F)+6F+6F \quad \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}3\mathbb{Z}^{\oplus 3} \tag{6.93}$$

$$2C+2(C+2F)+2(C+F)+2(C+F)+2F+2F \quad \mathbb{Z}/2\mathbb{Z}^{\oplus 4} \tag{6.94}$$

$$6C+2(C+F)+3(C+F)+12F+12F \quad \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/3\mathbb{Z}^{\oplus 2}\oplus\mathbb{Z}/4\mathbb{Z} \tag{6.95}$$

$$2(C+F)+2(C+F)+2(C+F)+2(C+F)+2F+2F \quad \mathbb{Z}/2\mathbb{Z}^{\oplus 4} \tag{6.96}$$

$$2C+2(C+F)+2(C+F)+2(C+F)+4F+4F \quad \mathbb{Z}/2\mathbb{Z}^{\oplus 3}\oplus\mathbb{Z}/4\mathbb{Z} \tag{6.97}$$

$$2C+2(C+F)+2(C+F)+2(C+F)+2F+2F+2F \quad \mathbb{Z}/2\mathbb{Z}^{\oplus 5} \tag{6.98}$$

$$2C+4(2C+2F)+4F+4F \quad \mathbb{Z}/4\mathbb{Z}^{\oplus 2} \tag{6.99}$$

$$2C+4(2C+2F)+2F+2F+2F \quad \mathbb{Z}/2\mathbb{Z}^{\oplus 2}\oplus\mathbb{Z}/4\mathbb{Z} \tag{6.100}$$

$$4C+2(C+F)+4(C+F)+8F+8F \quad \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/4\mathbb{Z}\oplus\mathbb{Z}/8\mathbb{Z} \tag{6.101}$$

$$3(2C+2F)+3(C+F)+2F+2F \tag{6.102}$$

$$4(2C+2F)+2(C+3F) \tag{6.103}$$

$$4(2C+2F)+2(C+2F)+2F \tag{6.104}$$

$$4(2C+2F)+2(C+F)+2F+2F \tag{6.105}$$

$$2(C+3F)+3(C+F)+6(C+F) \tag{6.106}$$

$$2(C+2F)+3(C+F)+6(C+F)+2F \tag{6.107}$$

$$2(C+F)+3(C+F)+6(C+F)+2F+2F \tag{6.108}$$

$$2(C+3F)+4(C+F)+4(C+F) \tag{6.109}$$

$$2(C+2F)+4(C+F)+4(C+F)+2F \tag{6.110}$$

$$2(C+F)+4(C+F)+4(C+F)+2F+2F \tag{6.111}$$

$$2(4C+5F)+2F \tag{6.112}$$

$$2(3C+aF)+2(C+(6-a)F), a\geq 3 \tag{6.113}$$

$$2(3C+4F)+2(C+F)+2F \tag{6.114}$$

$$2(3C+3F)+2(C+2F)+2F \tag{6.115}$$

$$2(3C+3F)+2(C+F)+2F+2F \tag{6.116}$$

$$2(2C+3F)+2(2C+3F) \tag{6.117}$$

$$2(2C+3F)+2(2C+2F)+2F \tag{6.118}$$

$$2(2C+4F)+2(C+F)+2(C+F) \tag{6.119}$$

$$2(2C+3F)+2(C+2F)+2(C+F) \tag{6.120}$$

$$2(2C+3F)+2(C+F)+2(C+F)+2F \tag{6.121}$$

$$2(2C+2F)+2(C+2F)+2(C+2F) \tag{6.122}$$

$$2(2C+2F)+2(C+2F)+2(C+F)+2F \tag{6.123}$$

$$2(2C+2F)+2(C+F)+2(C+F)+2F+2F \tag{6.124}$$

$$2(C+2F)+2(C+2F)+2(C+F)+2(C+F) \tag{6.125}$$

$$2(C+2F)+2(C+F)+2(C+F)+2(C+F)+2F \tag{6.126}$$

$$3C+3(2C+3F)+2F+2F \tag{6.127}$$

$$3C+3(2C+2F)+2F+2F+3F \tag{6.128}$$

$$3C+3(2C+2F)+4F+12F \tag{6.129}$$

$$2C+4(2C+4F) \tag{6.130}$$

$$2C+4(2C+3F)+4F \tag{6.131}$$

$$2C+4(2C+2F)+3F+6F \tag{6.132}$$

$$2C+3(C+2F)+6(C+2F) \tag{6.133}$$

$$2C+3(C+2F)+6(C+F)+6F \tag{6.134}$$

$$2C+3(C+F)+6(C+2F)+3F \tag{6.135}$$

$$2C+3(C+F)+6(C+F)+4F+4F \tag{6.136}$$

$$2C+3(C+F)+6(C+F)+3F+6F \tag{6.137}$$

$$2C+3(C+F)+6(C+F)+2F+2F+2F \tag{6.138}$$

$$2C+4(C+2F)+4(C+2F) \tag{6.139}$$

$$2C+4(C+F)+4(C+3F) \tag{6.140}$$

$$2C+4(C+F)+4(C+2F)+4F \tag{6.141}$$

$$2C+4(C+F)+4(C+F)+4F+4F \tag{6.142}$$

$$2C+4(C+F)+4(C+F)+3F+6F \tag{6.143}$$

$$2C+4(C+F)+4(C+F)+2F+2F+2F \tag{6.144}$$

$$3C+2(C+3F)+6(C+F)+3F \tag{6.145}$$

$$3C+2(C+2F)+6(C+F)+2F+3F \tag{6.146}$$

$$3C+2(C+F)+6(C+3F) \tag{6.147}$$

$$3C+2(C+F)+6(C+2F)+6F \tag{6.148}$$

$$3C+2(C+F)+6(C+F)+6F+6F \tag{6.149}$$

$$3C+2(C+F)+6(C+F)+4F+12F \tag{6.150}$$

$$3C+2(C+F)+6(C+F)+2F+2F+3F \tag{6.151}$$

$$3C+3(C+F)+3(C+F)+4F+12F \tag{6.152}$$

$$3C+3(C+2F)+3(C+F)+2F+2F \tag{6.153}$$

$$3C+3(C+F)+3(C+F)+2F+2F+3F \tag{6.154}$$

$$4C + 2(C + 3F) + 4(C + 2F) \tag{6.155}$$

$$4C + 2(C + 3F) + 4(C + F) + 4F \tag{6.156}$$

$$4C + 2(C + 2F) + 4(C + 2F) + 2F \tag{6.157}$$

$$4C + 2(C + 2F) + 4(C + F) + 2F + 4F \tag{6.158}$$

$$4C + 2(C + F) + 4(C + F) + 6F + 12F \tag{6.159}$$

$$4C + 2(C + F) + 4(C + F) + 5F + 20F \tag{6.160}$$

$$4C + 2(C + F) + 4(C + 2F) + 2F + 2F \tag{6.161}$$

$$4C + 2(C + F) + 4(C + F) + 2F + 2F + 4F \tag{6.162}$$

$$6C + 2(C + 3F) + 3(C + F) + 6F \tag{6.163}$$

$$6C + 2(C + 2F) + 3(C + 3F) \tag{6.164}$$

$$6C + 2(C + 2F) + 3(C + 2F) + 3F \tag{6.165}$$

$$6C + 2(C + 2F) + 3(C + F) + 3F + 3F \tag{6.166}$$

$$6C + 2(C + 2F) + 3(C + F) + 2F + 6F \tag{6.167}$$

$$6C + 2(C + F) + 3(C + 3F) + 2F \tag{6.168}$$

$$6C + 2(C + F) + 3(C + 2F) + 2F + 3F \tag{6.169}$$

$$6C + 2(C + F) + 3(C + F) + 10F + 15F \tag{6.170}$$

$$6C + 2(C + F) + 3(C + F) + 9F + 18F \tag{6.171}$$

$$6C + 2(C + F) + 3(C + F) + 8F + 24F \tag{6.172}$$

$$6C + 2(C + F) + 3(C + F) + 7F + 42F \tag{6.173}$$

$$6C + 2(C + F) + 3(C + F) + 2F + 3F + 3F \tag{6.174}$$

$$6C + 2(C + F) + 3(C + F) + 2F + 2F + 6F \tag{6.175}$$

$$2C + 2(3C + 6F) \tag{6.176}$$

$$2C + 2(3C + 5F) + 2F \tag{6.177}$$

$$2C + 2(3C + 4F) + 2F + 2F \tag{6.178}$$

$$2C + 2(3C + 3F) + 3F + 6F \tag{6.179}$$

$$2C + 2(C + 4F) + 2(2C + 2F) \tag{6.180}$$

$$2C + 2(C + 3F) + 2(2C + 3F) \tag{6.181}$$

$$2C + 2(C + 2F) + 2(2C + 4F) \tag{6.182}$$

$$2C + 2(C + F) + 2(2C + 5F) \tag{6.183}$$

$$2C + 2(C + 3F) + 2(2C + 2F) + 2F \tag{6.184}$$

$$2C + 2(C + 2F) + 2(2C + 3F) + 2F \tag{6.185}$$

$$2C + 2(C + 2F) + 2(2C + 2F) + 2F + 2F \tag{6.186}$$

$$2C + 2(C + F) + 2(2C + 4F) + 2F \tag{6.187}$$

$$2C + 2(C + F) + 2(2C + 2F) + 3F + 6F \tag{6.188}$$

$$2C + 2(C + 4F) + 2(C + F) + 2(C + F) \tag{6.189}$$

$$2C + 2(C + 3F) + 2(C + 2F) + 2(C + F) \tag{6.190}$$

$$2C + 2(C + 3F) + 2(C + F) + 2(C + F) + 2F \tag{6.191}$$

$$2C + 2(C + 2F) + 2(C + 2F) + 2(C + F) + 2F \tag{6.192}$$

$$2C + 2(C + F) + 2(C + F) + 2(C + F) + 3F + 6F. \tag{6.193}$$

If there is a $K3$ surface X and a finite subgroup G of $\mathrm{Aut}(X)$ such that $X/G \cong \mathbb{F}_2$, then by Theorem 2.4 the numerical class of B is one of the following:

$$3(3C + 6F) \quad \mathbb{Z}/3\mathbb{Z} \tag{6.194}$$

$$2(4C + 8F) \quad \mathbb{Z}/2\mathbb{Z} \tag{6.195}$$

$$2(2C + 4F) + 2(2C + 4F) \quad \mathbb{Z}/2\mathbb{Z}^{\oplus 2} \tag{6.196}$$

$$2C + 2(C + 2F) + 2(2C + 6F) \quad \mathbb{Z}/2\mathbb{Z}^{\oplus 2} \tag{6.197}$$

$$3(C + 2F) + 3(C + 2F) + 3(C + 2F) \quad \mathbb{Z}/3\mathbb{Z}^{\oplus 2} \tag{6.198}$$

$$2C + 2(C + 4F) + 2(C + 2F) + 2(C + 2F) \quad \mathbb{Z}/2\mathbb{Z}^{\oplus 3} \tag{6.199}$$

$$2(C + 2F) + 2(C + 2F) + 2(C + 2F) + 2(C + 2F) \quad \mathbb{Z}/2\mathbb{Z}^{\oplus 3} \tag{6.200}$$

$$3C + 3(2C + 4F) + 3F + 3F \quad \mathbb{Z}/3\mathbb{Z}^{\oplus 2} \tag{6.201}$$

$$2C + 2(3C + 6F) + 2F + 2F \quad \mathbb{Z}/2\mathbb{Z}^{\oplus 2} \tag{6.202}$$

$$2C + 2(C + 2F) + 2(2C + 4F) + 2F + 2F \quad \mathbb{Z}/2\mathbb{Z}^{\oplus 3} \tag{6.203}$$

$$3C + 3(C + 3F) + 3(C + 3F) \quad \mathbb{Z}/3\mathbb{Z}^{\oplus 2} \tag{6.204}$$

$$3C + 3(C + 2F) + 3(C + 2F) + 3F + 3F \quad \mathbb{Z}/3\mathbb{Z}^{\oplus 3} \tag{6.205}$$

$$2C + 2(C + 2F) + 2(C + 2F) + 2(C + 2F) + 2F + 2F \quad \mathbb{Z}/2\mathbb{Z}^{\oplus 4} \tag{6.206}$$

$$2C + 4(2C + 4F) + 2F + 2F \quad \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \tag{6.207}$$

$$4C + 2(C + 3F) + 4(C + 2F) + 2F + 2F \quad \mathbb{Z}/2\mathbb{Z}^{\oplus 2} \oplus \mathbb{Z}/4\mathbb{Z} \tag{6.208}$$

$$4C + 2(C + 2F) + 4(C + 2F) + 4F + 4F \quad \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}^{\oplus 2} \tag{6.209}$$

$$4C + 2(C + 2F) + 4(C + 2F) + 2F + 2F + 2F \quad \mathbb{Z}/2\mathbb{Z}^{\oplus 3} \oplus \mathbb{Z}/4\mathbb{Z} \tag{6.210}$$

$$6C + 2(C + 2F) + 3(C + 2F) + 6F + 6F \quad \mathbb{Z}/2\mathbb{Z}^{\oplus 2} \oplus \mathbb{Z}/3\mathbb{Z}^{\oplus 2} \tag{6.211}$$

$$3(C + 2F) + 3(2C + 4F) \tag{6.212}$$

$$2(C + 2F) + 4(2C + 4F) \tag{6.213}$$

$$2(C + 2F) + 3(C + 2F) + 6(C + 2F) \tag{6.214}$$

$$2(C + 2F) + 4(C + 2F) + 4(C + 2F) \tag{6.215}$$

$$2(3C + 6F) + 2(C + 2F) \tag{6.216}$$

$$2(2C + 4F) + 2(C + 2F) + 2(C + 2F) \tag{6.217}$$

$$3C + 3(2C + 6F) \tag{6.218}$$

$$3C + 3(2C + 5F) + 3F \tag{6.219}$$

$$3C + 3(2C + 4F) + 2F + 6F \tag{6.220}$$

$$2C + 3(C + 2F) + 6(C + 2F) + 2F + 2F \tag{6.221}$$

$$2C + 4(C + 2F) + 4(C + 2F) + 2F + 2F \tag{6.222}$$

$$3C + 2(C + 3F) + 6(C + 3F) \tag{6.223}$$

$$3C + 2(C + 3F) + 6(C + 2F) + 6F \tag{6.224}$$

$$3C + 2(C + 2F) + 6(C + 3F) + 2F \tag{6.225}$$
$$3C + 2(C + 2F) + 6(C + 2F) + 3F + 3F \tag{6.226}$$
$$3C + 2(C + 2F) + 6(C + 2F) + 2F + 6F \tag{6.227}$$
$$3C + 3(C + 2F) + 3(C + 4F) \tag{6.228}$$
$$3C + 3(C + 2F) + 3(C + 3F) + 3F \tag{6.229}$$
$$3C + 3(C + 2F) + 3(C + 2F) + 2F + 6F \tag{6.230}$$
$$4C + 2(C + 5F) + 4(C + 2F) \tag{6.231}$$
$$4C + 2(C + 4F) + 4(C + 2F) + 2F \tag{6.232}$$
$$4C + 2(C + 2F) + 4(C + 4F) \tag{6.233}$$
$$4C + 2(C + 2F) + 4(C + 3F) + 4F \tag{6.234}$$
$$4C + 2(C + 2F) + 4(C + 2F) + 3F + 6F \tag{6.235}$$
$$6C + 2(C + 4F) + 3(C + 3F) \tag{6.236}$$
$$6C + 2(C + 4F) + 3(C + 2F) + 3F \tag{6.237}$$
$$6C + 2(C + 3F) + 3(C + 3F) + 2F \tag{6.238}$$
$$6C + 2(C + 3F) + 3(C + 2F) + 2F + 3F \tag{6.239}$$
$$6C + 2(C + 2F) + 3(C + 3F) + 2F + 2F \tag{6.240}$$
$$2C + 2(3C + 8F) \tag{6.241}$$
$$2C + 2(3C + 7F) + 2F \tag{6.242}$$
$$2C + 2(C + 4F) + 2(2C + 4F) \tag{6.243}$$
$$2C + 2(C + 3F) + 2(2C + 5F) \tag{6.244}$$
$$2C + 2(C + 3F) + 2(2C + 4F) + 2F \tag{6.245}$$
$$2C + 2(C + 2F) + 2(2C + 5F) + 2F \tag{6.246}$$
$$6C + 2(C + 2F) + 3(C + 2F) + 4F + 12F \tag{6.247}$$
$$6C + 2(C + 2F) + 3(C + 2F) + 2F + 2F + 3F \tag{6.248}$$
$$2C + 2(C + 3F) + 2(C + 3F) + 2(C + 2F) \tag{6.249}$$
$$2C + 2(C + 3F) + 2(C + 2F) + 2(C + 2F) + 2F. \tag{6.250}$$

If there is a $K3$ surface X and a finite subgroup G of $\mathrm{Aut}(X)$ such that $X/G \cong \mathbb{F}_3$, then by Theorem 2.4 the numerical class of B is one of the following:

$$3C + 3(2C + 6F) + 2F + 2F \quad \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \tag{6.251}$$
$$3C + 3(C + 3F) + 3(C + 3F) + 2F + 2F \quad \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}^{\oplus 2} \tag{6.252}$$
$$6C + 2(C + 3F) + 3(C + 3F) + 4F + 4F \quad \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \tag{6.253}$$
$$6C + 2(C + 3F) + 3(C + 3F) + 2F + 2F + 2F \quad \mathbb{Z}/2\mathbb{Z}^{\oplus 3} \oplus \mathbb{Z}/3\mathbb{Z} \tag{6.254}$$
$$2C + 4(2C + 6F) + 2F \tag{6.255}$$
$$2C + 3(C + 3F) + 6(C + 3F) + 2F \tag{6.256}$$
$$2C + 4(C + 3F) + 4(C + 3F) + 2F \tag{6.257}$$
$$3C + 2(C + 5F) + 6(C + 3F) \tag{6.258}$$

$$3C+2(C+4F)+6(C+3F)+2F \tag{6.259}$$

$$3C+2(C+3F)+6(C+3F)+2F+2F \tag{6.260}$$

$$4C+2(C+4F)+4(C+4F) \tag{6.261}$$

$$4C+2(C+4F)+4(C+3F)+4F \tag{6.262}$$

$$4C+2(C+3F)+4(C+4F)+2F \tag{6.263}$$

$$4C+2(C+3F)+4(C+3F)+2F+4F \tag{6.264}$$

$$6C+2(C+6F)+3(C+3F) \tag{6.265}$$

$$6C+2(C+5F)+3(C+3F)+2F \tag{6.266}$$

$$6C+2(C+4F)+3(C+3F)+2F+2F \tag{6.267}$$

$$6C+2(C+3F)+3(C+4F)+6F \tag{6.268}$$

$$6C+2(C+3F)+3(C+3F)+3F+6F \tag{6.269}$$

$$2C+2(3C+10F) \tag{6.270}$$

$$2C+2(3C+9F)+2F \tag{6.271}$$

$$2C+2(C+4F)+2(2C+6F) \tag{6.272}$$

$$2C+2(C+3F)+2(2C+7F) \tag{6.273}$$

$$2C+2(C+3F)+2(2C+6F)+2F \tag{6.274}$$

$$2C+2(C+4F)+2(C+3F)+2(C+3F) \tag{6.275}$$

$$2C+2(C+3F)+2(C+3F)+2(C+3F)+2F. \tag{6.276}$$

If there is a $K3$ surface X and a finite subgroup G of $\mathrm{Aut}(X)$ such that $X/G \cong \mathbb{F}_4$, then by Theorem 2.4 the numerical class of B is one of the following:

$$2C+2(3C+12F)\quad \mathbb{Z}/2\mathbb{Z} \tag{6.277}$$

$$2C+4(2C+8F)\quad \mathbb{Z}/4\mathbb{Z} \tag{6.278}$$

$$2C+2(C+4F)+2(2C+8F)\quad \mathbb{Z}/2\mathbb{Z}^{\oplus 2} \tag{6.279}$$

$$4C+2(C+6F)+4(C+4F)\quad \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/4\mathbb{Z} \tag{6.280}$$

$$4C+2(C+4F)+4(C+4F)+2F+2F\quad \mathbb{Z}/2\mathbb{Z}^{\oplus 2}\oplus\mathbb{Z}/4\mathbb{Z} \tag{6.281}$$

$$2C+2(C+4F)+2(C+4F)+2(C+4F)\quad \mathbb{Z}/2\mathbb{Z}^{\oplus 3} \tag{6.282}$$

$$6C+2(C+4F)+3(C+4F)+3F+3F\quad \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/3\mathbb{Z}^{\oplus 2} \tag{6.283}$$

$$3C+3(2C+9F) \tag{6.284}$$

$$3C+3(2C+8F)+3F \tag{6.285}$$

$$2C+3(C+4F)+6(C+4F) \tag{6.286}$$

$$2C+4(C+4F)+4(C+4F) \tag{6.287}$$

$$3C+2(C+4F)+6(C+4F)+3F \tag{6.288}$$

$$3C+3(C+4F)+3(C+5) \tag{6.289}$$

$$3C+3(C+4F)+3(C+4F)+3F \tag{6.290}$$

$$4C+2(C+5F)+4(C+4F)+2F \tag{6.291}$$

$$6C+2(C+5F)+3(C+4F)+6F \tag{6.292}$$

$$6C + 2(C + 4F) + 3(C + 6F) \tag{6.293}$$

$$6C + 2(C + 4F) + 3(C + 5F) + 3F \tag{6.294}$$

$$6C + 2(C + 4F) + 3(C + 4F) + 2F + 6F. \tag{6.295}$$

If there is a $K3$ surface X and a finite subgroup G of $\mathrm{Aut}(X)$ such that $X/G \cong \mathbb{F}_5$, then by Theorem 2.4 the numerical class of B is one of the following:

$$4C + 2(C + 5F) + 4(C + 6F) \tag{6.296}$$

$$4C + 2(C + 5F) + 4(C + 5F) + 4F \tag{6.297}$$

$$6C + 2(C + 6F) + 3(C + 6F) \tag{6.298}$$

$$6C + 2(C + 6F) + 3(C + 5F) + 3F \tag{6.299}$$

$$6C + 2(C + 5F) + 3(C + 6F) + 2F \tag{6.300}$$

$$6C + 2(C + 5F) + 3(C + 5F) + 2F + 3F. \tag{6.301}$$

If there is a $K3$ surface X and a finite subgroup G of $\mathrm{Aut}(X)$ such that $X/G \cong \mathbb{F}_6$, then by Theorem 2.4 the numerical class of B is one of the following:

$$3C + 3(2C + 12F) \quad \mathbb{Z}/3\mathbb{Z} \tag{6.302}$$

$$3C + 3(C + 6F) + 3(C + 6F) \quad \mathbb{Z}/3\mathbb{Z}^{\oplus 2} \tag{6.303}$$

$$6C + 2(C + 6F) + 3(C + 6F) + 2F + 2F \quad \mathbb{Z}/2\mathbb{Z}^{\oplus 2} \oplus \mathbb{Z}/3\mathbb{Z} \tag{6.304}$$

$$3C + 2(C + 6F) + 6(C + 6F) \tag{6.305}$$

$$4C + 2(C + 7F) + 4(C + 6F) \tag{6.306}$$

$$4C + 2(C + 6F) + 4(C + 6F) + 2F \tag{6.307}$$

$$6C + 2(C + 8F) + 3(C + 6F) \tag{6.308}$$

$$6C + 2(C + 7F) + 3(C + 6F) + 2F. \tag{6.309}$$

If there is a $K3$ surface X and a finite subgroup G of $\mathrm{Aut}(X)$ such that $X/G \cong \mathbb{F}_7$, then by Theorem 2.4 the numerical class of B is one of the following:

$$6C + 2(C + 7F) + 3(C + 7F) + 6F. \tag{6.310}$$

If there is a $K3$ surface X and a finite subgroup G of $\mathrm{Aut}(X)$ such that $X/G \cong \mathbb{F}_8$, then by Theorem 2.4 the numerical class of B is one of the following:

$$4C + 2(C + 8F) + 4(C + 8F) \quad \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \tag{6.311}$$

$$6C + 2(C + 8F) + 3(C + 9F) \tag{6.312}$$

$$6C + 2(C + 8F) + 3(C + 8F) + 3F. \tag{6.313}$$

If there is a $K3$ surface X and a finite subgroup G of $\mathrm{Aut}(X)$ such that $X/G \cong \mathbb{F}_9$, then by Theorem 2.4 the numerical class of B is one of the following:

$$6C + 2(C + 10F) + 3(C + 9F) \tag{6.314}$$

$$6C + 2(C + 9F) + 3(C + 9F) + 2F. \tag{6.315}$$

By Theorem 2.4 there is not a $K3$ surface X and a finite subgroup G of $\mathrm{Aut}(X)$ such that $X/G \cong \mathbb{F}_l$ for $l = 10, 11$.

If there is a $K3$ surface X and a finite subgroup G of $\mathrm{Aut}(X)$ such that $X/G \cong \mathbb{F}_{12}$, then by Theorem 2.4 the numerical class of B is the following:

$$6C + 2(C + 12F) + 3(C + 12F) \quad \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}. \tag{6.316}$$

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