Chin. Ann. Math. Ser. B
44(2), 2023, 165–192
DOI: 10.1007/s11401-023-0009-x

Chinese Annals of Mathematics, Series B
© The Editorial Office of CAM and Springer-Verlag Berlin Heidelberg 2023

A Theory of Orbit Braids*

Fengling LI[1] Hao LI[2] Zhi LÜ[3]

Abstract In this paper, the authors systematically discuss orbit braids in $M \times I$ with regards to orbit configuration space $F_G(M, n)$, where M is a connected topological manifold of dimension at least 2 with an effective action of a finite group G. These orbit braids form a group, named orbit braid group, which enriches the theory of ordinary braids.

The authors analyze the substantial relations among various braid groups associated to those configuration spaces $F_G(M, n), F(M/G, n)$ and $F(M, n)$. They also consider the presentations of orbit braid groups in terms of orbit braids as generators by choosing $M = \mathbb{C}$ with typical actions of $\mathbb{Z}_p$ and $(\mathbb{Z}_2)^2$.

Keywords Orbit braid, Orbit configuration space
2000 MR Subject Classification 20F36, 55Q05

1 Introduction

Braid groups are fundamental objects in mathematics, which were first defined rigorously and studied by Artin in 1925 (see [3–4]), although they already implicitly appeared in the works of Hurwitz [18] in 1891 and Fricke-Klein [16] in 1897. The subject has continued to further develop and flourish by extending ideas of braid groups or combining with various ideas and theories from other research areas since then. For example, Fox and Neuwrith [14] gave an alternative description of the classical braid groups by using the fundamental group of (unordered) configuration spaces. Brieskorn [9] extended the notion to Artin groups or the generalized braid groups by associating to all finite Coxeter groups.

Compatible with various points of view, the notion of braid groups was uniformly defined by Vershinin [23] in a general way as follows: Choose a connected topological manifold $\mathbb{M}$ admitting an action of a finite group $\mathbb{G}$. Let $Y_{\mathbb{G}}$ be the subspace of $\mathbb{M}$ formed by all points of free orbit type. So the action of $\mathbb{G}$ restricted to $Y_{\mathbb{G}}$ is free. Assume that $Y_{\mathbb{G}}$ is connected. Then there is a fibration $Y_{\mathbb{G}} \to X_{\mathbb{G}}$ with fiber $\mathbb{G}$, which gives a short exact sequence:

$$1 \to \pi_1(Y_{\mathbb{G}}) \to \pi_1(X_{\mathbb{G}}) \to \mathbb{G} \to 1.$$

The fundamental group $\pi_1(X_{\mathbb{G}})$ is called the braid group of the action of $\mathbb{G}$ on $\mathbb{M}$, denoted by

Manuscript received March 8, 2022. Revised October 26, 2022.
[1]School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, China.
E-mail: fenglingli@dlut.edu.cn
[2]Corresponding author. School of Mathematical Sciences, Fudan University, Shanghai 200433, China.
E-mail: 14110840001@fudan.edu.cn
[3]School of Mathematical Sciences, Fudan University, Shanghai 200433, China.
E-mail: zlu@fudan.edu.cn
*This work was supported by the National Natural Science Foundation of China (No. 11971112).

$Br(\mathbb{M},\mathbb{G})$, and the fundamental group $\pi_1(Y_{\mathbb{G}})$ is called the pure braid group of the action of $\mathbb{G}$ on $\mathbb{M}$, denoted by $P(\mathbb{M},\mathbb{G})$.

As an example of the notion above, for a connected topological manifold M of dimension greater than one, take $\mathbb{M}=M^{\times n}$ (Cartesian product of n copies of M). Then there is a natural action of the symmetric group $\mathbb{G}=\Sigma_n$ on $\mathbb{M}$, defined by $\sigma(x_1,\cdots,x_n)=(x_{\sigma^{-1}(1)},\cdots,x_{\sigma^{-1}(n)})$, $\sigma\in\Sigma_n$. So Y_{Σ_n} will be the ordered configuration space

$$F(M,n)=\{(x_1,\cdots,x_n)\in M^{\times n}\mid x_i\neq x_j \text{ for } i\neq j\}$$

(introduced by Fadell and Neuwrith [15]) and X_{Σ_n} will be the unordered configuration space $F(M,n)/\Sigma_n$. Thus, the braid group $Br(\mathbb{M},\Sigma_n)$ is the fundamental group $\pi_1(F(M,n)/\Sigma_n)$, also simply denoted by $B_n(M)$, and the pure braid group $P(\mathbb{M},\Sigma_n)$ is the fundamental group $\pi_1(F(M,n))$, also simply denoted by $P_n(M)$.

Theory of braids considered as above (called ordinary or classical braids here) obviously possesses the following basic theoretical features:

(F_1) (Geometric feature) Each braid corresponds to a collection of strings in $M\times I$, and the equivalence relation between braids is isotopy equivalence;

(F_2) (Homotopic feature) Each braid group is realized as the fundamental group of the orbit space.

Such two theoretical features are based upon the restriction of free actions. Recently Allcock and Basak in their series papers [1–2] studied the braid-like groups, regarded as the orbifold fundamental groups of the spaces with non-free action of a discrete group, given by removing a locally finite arrangement of complex hyperplanes in complex Euclidean space $\mathbb{C}^n$, or complex hyperbolic space $\mathbb{CH}^n$, or the Hermitian symmetric space for an orthogonal group $O(2,n)$, they gave homotopic generators of braid-like groups. This work implies that although above two features are not applicable for non free actions case, the study of braids can also be carried out.

The objective of this paper is to extend braids from configuration spaces to orbit braids from orbit configuration spaces in both geometric and homotopic views. In particular, fundamental group can be replaced by orbifold fundamental group in homotopic description and the geometric orbit braids are quite different from ordinary ones—the equivalence of orbit braids are no longer regarded as isotopy classes, because they contain singular points. Our strategy is to combine the original idea of Artin and the theory of transformation groups together.

Compared with the homotopic fundamental group of transformation groups in [1–2, 19], we give geometric definition of orbit braids and prove that they are isomorphism. Furthermore, we discuss not only the geometric generators of orbit braid group, but also their relations.

Let M be a connected topological manifold of dimension at least two with an effective action of a finite group G (the action of G on M is not assumed to be free), and $F_G(M,n)$ be the orbit configuration space where $n\geq 2$.

We use the paths in $F_G(M,n)$ to describe the braids in $M\times I$. In the sense of Artin, an orbit braid will be defined as the orbit of a braid in $M\times I$ under the action of G (see Definition 2.1), but generally it may not be the disjoint union of some ordinary strings.

However, equivariant isotopy classes of orbit braids will not work very well. Based upon the nature of orbit braids, our approach is to detect whether there exist two isotopic ordinary braids

compatible with the G-action in two orbit braids (see Definition 2.3). This equivalence relation among orbit braids can also be described in terms of the homotopy of paths in $F_G(M,n)$ (see Proposition 2.2). In this way, the key difficulty is overcome.

Moreover, we conclude that the set of the equivalence classes of all orbit braids at a fixed orbit base point forms a group, called the orbit braid group, denoted by $\mathcal{B}_n^{\rm orb}(M,G)$.

On the other hand, we obtain a homotopy description of the orbit braid group $\mathcal{B}_n^{\rm orb}(M,G)$.

Theorem 1.1 (see Theorem 2.1) *There is the following isomorphism*:

$$\mathcal{B}_n^{\rm orb}(M,G) \cong \pi_1^E(F_G(M,n),\mathbf{x},\mathbf{x}^{\rm orb}),$$

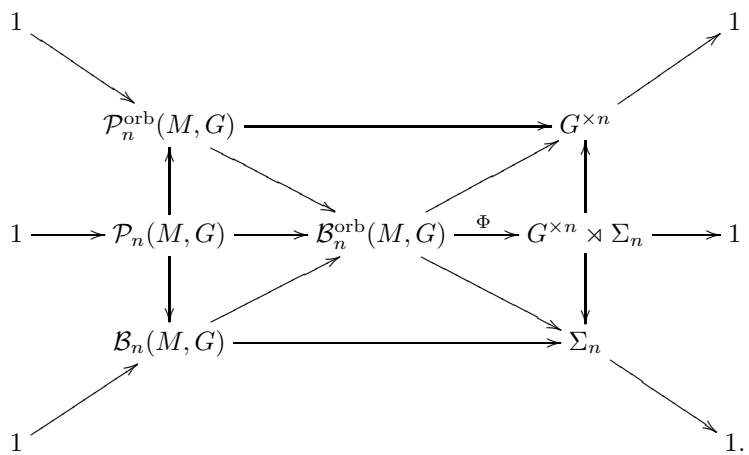
where $\pi_1^E(F_G(M,n),\mathbf{x},\mathbf{x}^{\rm orb})$ *is called the extended fundamental group*[1].

The orbit braid group $\mathcal{B}_n^{\rm orb}(M,G)$ is large enough to contain some interesting subgroups $\mathcal{P}_n^{\rm orb}(M,G)$, $\mathcal{B}_n(M,G)$ and $\mathcal{P}_n(M,G)$ (see Definition 2.4). Each class of $\mathcal{B}_n^{\rm orb}(M,G)$ determines a unique pair $(g,\sigma)\in G^{\times n}\times\Sigma_n$. This leads us to obtain an epimorphism

$$\Phi : \mathcal{B}_n^{\rm orb}(M,G) \to G^{\times n}\rtimes\Sigma_n,$$

we can further analyze the relations among $\mathcal{B}_n^{\rm orb}(M,G)$ and its subgroups. Our result is stated as follows.

Theorem 1.2 (see Theorem 2.2) *There are five short exact sequences around* $\mathcal{B}_n^{\rm orb}(M,G)$, *which form the following commutative diagram*:



Remark 1.1 If $\dim M > 2$, then it is easy to see that $\mathcal{P}_n(M,G) \cong \pi_1(M^{\times n},\mathbf{x})$ and $\mathcal{B}_n^{\rm orb}(M,G)\cong\pi_1^E(M,\mathbf{x},G(\mathbf{x}))^{\times n}\rtimes\Sigma_n$. Thus $\mathcal{B}_n^{\rm orb}(M,G)$ makes sense only when $\dim M = 2$.

Next we consider the geometric presentation of orbit braid groups. As is known to all, Artin first studied braids on $\mathbb{R}^2$. Thus we will carry out our work from the cases of $\mathbb{C}\approx\mathbb{R}^2$ with the following two typical actions.

The first one is $\mathbb{Z}_p \curvearrowright^{\phi_1} \mathbb{C}$ defined by $(\mathrm{e}^{\frac{2k\pi\mathbf{i}}{p}},z)\mapsto \mathrm{e}^{\frac{2k\pi\mathbf{i}}{p}}z$, which is non-free and fixes only the origin of $\mathbb{C}$, where p is a prime, and $\mathbb{Z}_p$ is regarded as the subgroup $\{\mathrm{e}^{\frac{2k\pi\mathbf{i}}{p}} \mid 0\le k<p\}$. If

[1]Golasiński Marek told us that the extended fundamental group is actually the fundamental group of a transformation group in the sense of Rhodes in [21] and Looijenga in [19].

the action ϕ_1 is restricted to $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$, then the action $\mathbb{Z}_p \curvearrowright^{\phi_1} \mathbb{C}^\times$ is free. The other one is $(\mathbb{Z}_2)^2 \curvearrowright^{\phi_2} \mathbb{C}$ defined by

$$\begin{cases} z \mapsto \overline{z}, \\ z \mapsto -\overline{z}, \end{cases}$$

which is the standard representation of $(\mathbb{Z}_2)^2$ on $\mathbb{C} \approx \mathbb{R}^2$, and this action is non-free.

We obtain the presentations of orbit braid groups $\mathcal{B}_n^{\mathrm{orb}}(\mathbb{C}, \mathbb{Z}_p)$, $\mathcal{B}_n^{\mathrm{orb}}(\mathbb{C}^\times, \mathbb{Z}_p)$ and part information of $\mathcal{B}_n^{\mathrm{orb}}(\mathbb{C}, (\mathbb{Z}_2)^2)$.

Proposition 1.1 (see Proposition 3.2) *$\mathcal{B}_n^{\mathrm{orb}}(\mathbb{C}, \mathbb{Z}_p)$ has a presentation, generated by $\mathbf{b}_k$ $(1 \le k \le n-1)$ and $\mathbf{b}$, with relations*:

(1) $\mathbf{b}^p = e$,
(2) $(\mathbf{b}\mathbf{b}_1)^2 = (\mathbf{b}_1\mathbf{b})^2$,
(3) $\mathbf{b}_k\mathbf{b} = \mathbf{b}\mathbf{b}_k$ $(k > 1)$,
(4) $\mathbf{b}_k\mathbf{b}_{k+1}\mathbf{b}_k = \mathbf{b}_{k+1}\mathbf{b}_k\mathbf{b}_{k+1}$,
(5) $\mathbf{b}_k\mathbf{b}_l = \mathbf{b}_l\mathbf{b}_k$ $(|k-l| > 1)$.

Proposition 1.2 (see Proposition 3.3) *$\mathcal{B}_n^{\mathrm{orb}}(\mathbb{C}^\times, \mathbb{Z}_p)$ has a presentation, generated by $\mathbf{b_k}$ $(1 \le k \le n-1)$ and $\mathbf{b}'$, with relations*:

(1) $(\mathbf{b}'\mathbf{b}_1)^2 = (\mathbf{b}_1\mathbf{b}')^2$,
(2) $\mathbf{b}_k\mathbf{b}' = \mathbf{b}'\mathbf{b}_k$ $(k > 1)$,
(3) $\mathbf{b}_k\mathbf{b}_{k+1}\mathbf{b}_k = \mathbf{b}_{k+1}\mathbf{b}_k\mathbf{b}_{k+1}$,
(4) $\mathbf{b}_k\mathbf{b}_l = \mathbf{b}_l\mathbf{b}_k$ $(|k-l| > 1)$.

Lemma 1.1 (see Lemma 3.2) *$\mathcal{B}_n^{\mathrm{orb}}(\mathbb{C}, (\mathbb{Z}_2)^2)$ is generated by $\mathbf{b}_k$ $(1 \le k \le n-1)$, $\mathbf{b}^x$ and $\mathbf{b}^y$, satisfying the relations*

(1) $(\mathbf{b}^x)^2 = (\mathbf{b}^y)^2 = e$,
(2) $\mathbf{b}^x\mathbf{b}^y = \mathbf{b}^y\mathbf{b}^x$,
(3) $\mathbf{b}^x\mathbf{b}_1\mathbf{b}^x\mathbf{b}_1^{-1} = \mathbf{b}_1\mathbf{b}^x\mathbf{b}_1^{-1}\mathbf{b}^x$, $\mathbf{b}^y\mathbf{b}_1\mathbf{b}^y\mathbf{b}_1^{-1} = \mathbf{b}_1\mathbf{b}^y\mathbf{b}_1^{-1}\mathbf{b}^y$,
(4) $\mathbf{b}_k\mathbf{b}^x = \mathbf{b}^x\mathbf{b}_k$, $\mathbf{b}_k\mathbf{b}^y = \mathbf{b}^y\mathbf{b}_k$ $(k > 1)$,
(5) $\mathbf{b}_k\mathbf{b}_{k+1}\mathbf{b}_k = \mathbf{b}_{k+1}\mathbf{b}_k\mathbf{b}_{k+1}$,
(6) $\mathbf{b}_k\mathbf{b}_l = \mathbf{b}_l\mathbf{b}_k$ $(|k-l| > 1)$.

For $p = 2$, since $(\mathbb{Z}_2)^n \rtimes \Sigma_n$ is isomorphic to the finite Coxeter group B_n and the action $\mathbb{Z}_2 \curvearrowright^{\phi_1} \mathbb{C}^\times$ is free, $\mathcal{B}_n^{\mathrm{orb}}(\mathbb{C}^\times, \mathbb{Z}_2)$ is exactly isomorphic to the generalized braid group $Br(B_n)$ defined by Brieskorn. In addition, we will see that the generalized braid group $Br(D_n)$ is isomorphic to a subgroup of $\mathcal{B}_n^{\mathrm{orb}}(\mathbb{C}, \mathbb{Z}_2)$.

It should be pointed out that although the group G is assumed to be finite, many aspects of our work do not need this restriction.

The paper is organized as follows. Section 2 is the main part of this paper, where we will discuss how to establish the theoretical framework of orbit braids. We give the definitions of the orbit braid group, and show that such group can be described in terms of homotopy (i.e., Theorem 1.1). Furthermore, we introduce some subgroups of orbit braid group and study various possible relations among orbit braid group and its subgroups. In Section 3, we present the orbit braid groups of two typical actions on $\mathbb{C}$, from which we see that the generalized

braid group $Br(B_n)$ actually agrees with the orbit braid group $\mathcal{B}_n^{\rm orb}(\mathbb{C}^\times, \mathbb{Z}_2)$ and $Br(D_n)$ is a subgroup of the orbit braid group $\mathcal{B}_n^{\rm orb}(\mathbb{C}, \mathbb{Z}_2)$. In Section 4, we give the notion of extended fundamental groups and state some properties in a general way in the category of topology, which were essentially due to Rhodes in [21].

2 Orbit Braids from Orbit Configuration Space

Given a topological group G and a topological space X. Assume that X admits an effective G-action. Then the orbit configuration space of the G-space X is defined by

$$F_G(X,n) = \{(x_1, \cdots, x_n) \in X^{\times n} \mid G(x_i) \cap G(x_j) = \emptyset \text{ for } i \neq j\}$$

with subspace topology, where $n \geq 2$ and $G(x)$ denotes the orbit of x. In the case where G acts trivially on X, the space $F_G(X,n)$ is the classical configuration space $F(X,n)$.

The action of G on X induces a natural action of $G^{\times n}$ on $F_G(X,n)$. In addition, $F_G(X,n)$ also admits a canonical free action of the symmetric group Σ_n. However, generally these two actions do not commutative. There is a natural homomorphism $\varphi : \Sigma_n \to {\rm Aut}(G^{\times n})$ defined by $\varphi(\sigma)(g) = g_\sigma$ for $\sigma \in \Sigma_n$ and $g = (g_1, \cdots, g_n) \in G^{\times n}$, where $g_\sigma = (g_{\sigma(1)}, \cdots, g_{\sigma(n)})$. Then we may obtain a semidirect product $G^{\times n} \rtimes_\varphi \Sigma_n$ via φ, on which the operation " $\cdot$ " is given by

$$(g,\sigma)\cdot(h,\tau) = (gh_\sigma, \sigma\tau)$$

for $(g,\sigma),(h,\tau) \in G^{\times n} \rtimes_\varphi \Sigma_n$. Now we see that $F_G(X,n)$ admits the action of $G^{\times n} \rtimes_\varphi \Sigma_n$, given by

$$((g,\sigma),x) \mapsto gx_\sigma,$$

where $gx_\sigma = (g_1 x_{\sigma(1)}, \cdots, g_n x_{\sigma(n)})$ for $x = (x_1, \cdots, x_n) \in F_G(X,n)$.

Remark 2.1 The notion of orbit configuration space was introduced by Xicoténcatl in the thesis [24] of his Ph.D. Since then, the study of the algebraic topology (especially cohomology) and relevant topics of orbit configuration spaces has been further developed.

This equivariant case is quite different from the classical case. In particular, if the action of G on X is non-free, then the singular points (i.e., points of non-free orbit type) in X will bring difficulty to our study. An effective approach to deal with this difficulty is to throw out all singular points from X (see [5]). Another approach is to choose nice behaved equivariant manifolds. For example, in [10], for two kinds of equivariant manifolds with non-free actions introduced by Davis and Januszkiewicz [12], the combinatorial structures of the orbit spaces of the equivariant manifolds can determine all singular points, so that an explicit formula of Euler characteristic for orbit configuration spaces can be obtained in terms of combinatorics.

In the following, we shall pay more attention to the case in which X is a connected topological manifold M of dimension greater than one, and G is a finite group. In this case $F_G(M,n)$ is connected. Here we shall focus on the relation between orbit braids in $M \times I$ and paths in $F_G(M,n)$. Actually, no matter the paths are closed or unclosed, by endowing an operation, one can always form various kinds of groups. This extends the notion of ordinary fundamental groups to the equivariant ones, as seen in the work of Rhodes in [21] (also see Looijenga's paper [19]).

2.1 Notions and properties of orbit braids

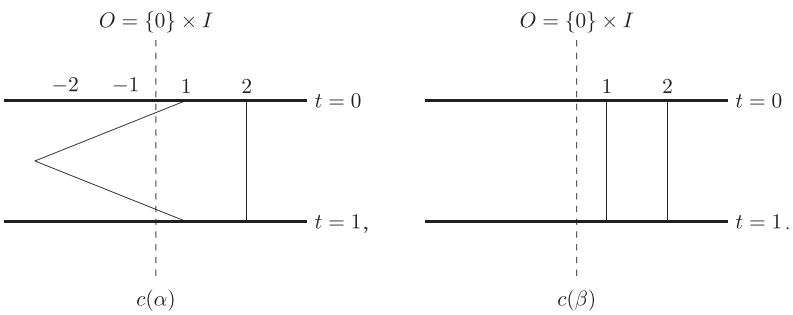
A path

$$\alpha = (\alpha_1, \cdots, \alpha_n) : I \to F_G(M, n)$$

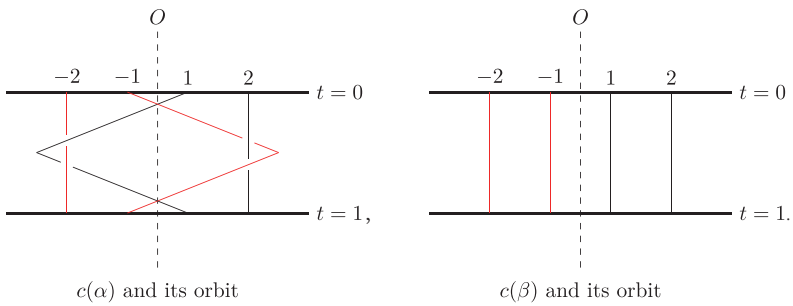
uniquely determines a configuration $c(\alpha) = \{c(\alpha_1), \cdots, c(\alpha_n)\}$ of n strings in $M \times I$, where $I = [0, 1]$ admits a trivial action of G and each string $c(\alpha_i) = \{(\alpha_i(s), s) \mid s \in I\}$ is homeomorphic to I. For each $s \in I$, since $\alpha(s) = (\alpha_1(s), \cdots, \alpha_n(s)) \in F_G(M, n)$, it follows that the intersection of any two different $c(\alpha_i)$ and $c(\alpha_j)$ is empty, so we may write $c(\alpha) = \coprod_{i=1}^{n} c(\alpha_i)$, which is naturally an unordered disjoint union of n intervals. Furthermore, it is easy to see that $c(\alpha)$ can determine $n!$ paths $\alpha_\sigma = (\alpha_{\sigma(1)}, \cdots, \alpha_{\sigma(n)}), \sigma \in \Sigma_n$ in $F_G(M, n)$ such that $c(\alpha_\sigma) = c(\alpha)$.

For the path α satisfying that $\alpha(0) = (x_1, \cdots, x_n)$ and $\alpha(1) = (x_{\sigma(1)}, \cdots, x_{\sigma(n)})$ for some $\sigma \in \Sigma_n$. if we consider the action of G on M, $c(\alpha)$ would be different from the classical Artin braid, see the following examples.

Example 2.1 Consider the orbit configuration space $F_{\mathbb{Z}_2}(\mathbb{C}, n)$ where the action of $\mathbb{Z}_2$ on $\mathbb{C}$ is given by $z \mapsto -z$, so this action is non-free and fixes only the origin of $\mathbb{C}$. In the case of $n = 2$, let us see two closed paths $\alpha, \beta : I \to F_{\mathbb{Z}_2}(\mathbb{C}, 2)$ at the point $\mathbf{x} = (1, 2)$ such that their corresponding configurations $c(\alpha)$ and $c(\beta)$ are as shown below:



If we forget the action of $\mathbb{Z}_2$ on $\mathbb{C}$, then clearly $c(\alpha)$ and $c(\beta)$ are isotopic relative to endpoints in $\mathbb{C} \times I$. However, under the condition that $\mathbb{C}$ admits the action of $\mathbb{Z}_2$, $c(\alpha)$ and $c(\beta)$ are even not homotopic relative to endpoints in $\mathbb{C} \times I$, since the first string of $c(\alpha)$ cannot go through the orbit of the second string of $c(\alpha)$, as we can see from the following picture:



For a path $\alpha = (\alpha_1, \cdots, \alpha_n) : I \to F_G(M, n)$, since M admits an action of G, we may define the orbit of α as follows:

$$G^{\times n}(\alpha) := \{g\alpha = (g_1\alpha_1, \cdots, g_n\alpha_n) \mid g = (g_1, \cdots, g_n) \in G^{\times n}\},$$

a collection of $|G|^n$ paths in $F_G(M, n)$. Then the corresponding configuration $c(\alpha) = \{c(\alpha_1), \cdots, c(\alpha_n)\}$ in $M \times I$ with trivial action of G on I gives its orbit configuration

$$\widetilde{c(\alpha)} = \{\widetilde{c(\alpha_1)}, \cdots, \widetilde{c(\alpha_n)}\},$$

where each orbit string $\widetilde{c(\alpha_i)} = \{hc(\alpha_i) \mid h \in G\}$ is the orbit of the string $c(\alpha_i)$ under the action of G, consisting of $|G|$ strings. We shall note that the $|G|$ strings in each orbit string $\widetilde{c(\alpha_i)}$ may intersect with each other, but the intersection of any two different orbit strings $\widetilde{c(\alpha_i)}$ and $\widetilde{c(\alpha_j)}$ must be empty since $g\alpha(s) = (g_1\alpha_1(s), \cdots, g_n\alpha_n(s)) \in F_G(M, n)$ for any $g = (g_1, \cdots, g_n) \in G^{\times n}$. On the other hand, for two paths α and α', if $\widetilde{c(\alpha)} = \widetilde{c(\alpha')}$, then there must be $g \in G^{\times n}$ and $\sigma \in \Sigma_n$ such that $\alpha = g\alpha'_\sigma$.

Now we are going to give the definition of orbit braids. Choose a point $\mathbf{x} = (x_1, \cdots, x_n)$ in $F_G(M, n)$ such that for each $1 \le i \le n$, the orbit $G(x_i)$ is of free type. Here and hereafter, $\mathbf{x}$ stands for the base point. Given $\sigma \in \Sigma_n$, $g = (g_1, \cdots, g_n) \in G^{\times n}$, we denote $(g_1 x_{\sigma(1)}, \cdots, g_n x_{\sigma(n)})$ by $g\mathbf{x}_\sigma$.

Definition 2.1 *Let $\alpha = (\alpha_1, \cdots, \alpha_n) : I \to F_G(M, n)$ be a path such that $\alpha(0) = \mathbf{x}$ and $\alpha(1) = g\mathbf{x}_\sigma$ for some $(g, \sigma) \in G^{\times n} \times \Sigma_n$. Then $\widetilde{c(\alpha)}$ is called an orbit braid in $M \times I$.*

Remark 2.2 Without loss of generality we may assume that $\alpha(0) = \mathbf{x}$. In fact, if $\alpha(0) \neq \mathbf{x}$, we may write $\alpha(0) = h\mathbf{x}_\tau$ where $h \in G^{\times n}$ and $\tau \in \Sigma_n$. We can construct path $\alpha' = h^{-1}_{\tau^{-1}}\alpha_{\tau^{-1}}$ such that $\alpha'(0) = \mathbf{x}$ and $\widetilde{c(\alpha')} = \widetilde{c(\alpha)}$.

Obviously, each orbit braid $\widetilde{c(\alpha)}$ has the property that $\widetilde{c(\alpha)}|_{s=0}$ and $\widetilde{c(\alpha)}|_{s=1}$ are homeomorphic to an unordered collection of the orbits of n coordinates of $\mathbf{x}$ under the action of G,

$$\widetilde{c(\mathbf{x})} = \{G(x_1), \cdots, G(x_n)\}.$$

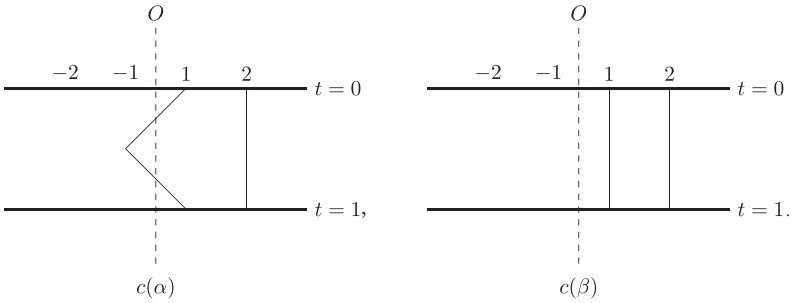
Namely, two endpoints of each orbit braid $\widetilde{c(\alpha)}$ are the same. Here we also call $\widetilde{c(\mathbf{x})}$ the (unordered) orbit base point.

Remark 2.3 In the theory of classical braids (see [4, 6]), it is easy to see that for two paths $\alpha, \beta : I \to F(M, n)$ with the same endpoints, α and β are homotopic relative to ∂I (also write $\alpha \simeq \beta$ rel ∂I) if and only if $c(\alpha)$ and $c(\beta)$ are isotopic relative to endpoints in $M \times I$[2].

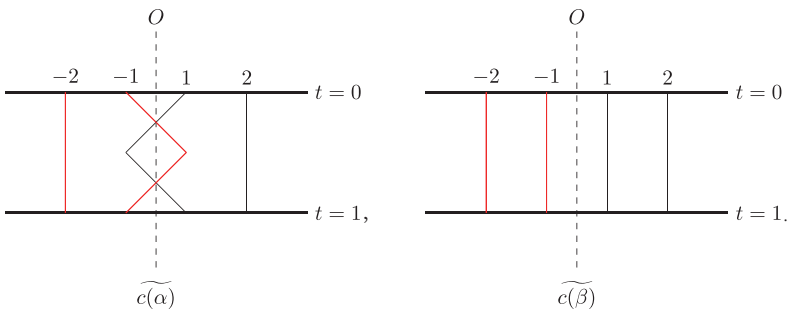
Since we are working on the case of M with an effective G-action, naturally we wonder whether the equivalence of homotopy and isotopy in Remark 2.3 still holds in equivariant case. The answer is no, see the following example.

Example 2.2 Let the action of $\mathbb{Z}_2$ on $\mathbb{C}$ be the same as that in Example 2.1. Consider the orbit configuration space $F_{\mathbb{Z}_2}(\mathbb{C}, n)$. In the case of $n = 2$, take two closed paths $\alpha(s) = (\mathrm{e}^{2s\pi\mathbf{i}}, 2)$ and $\beta(s) = (1, 2)$, their corresponding ordinary braids $c(\alpha)$ and $c(\beta)$ are shown as follows:

[2]Here the equivalence of $c(\alpha)$ and $c(\beta)$ up to isotopy is compatible with the Definition 3 of Artin's paper [4] since $c(\alpha)$ and $c(\beta)$ are given by two paths in $F(M, n)$.



Clearly, $\alpha \simeq \beta$ in $F_{\mathbb{Z}_2}(\mathbb{C}, 2)$. However, $\widetilde{c(\alpha)}$ and $\widetilde{c(\beta)}$ are not equivariant isotopic.



Definition 2.2 *Let $\alpha, \beta : I \to F_G(M, n)$ be two paths with the same endpoints. We say that $c(\alpha)$ and $c(\beta)$ are isotopic with respect to the G-action relative to endpoints in $M \times I$, denoted by $c(\alpha) \sim^G_{\rm iso} c(\beta)$, if there exist n homotopy maps $F_i : I \times I \to M$, $i = 1, \cdots, n$, which induce $\widehat{F}_i : I \times I \to M \times I$ given by $\widehat{F}_i(s,t) = (F_i(s,t), s)$, such that*

(1) $\coprod_{i=1}^{n} \widehat{F}_i(s,0) = c(\alpha)$ *and* $\coprod_{i=1}^{n} \widehat{F}_i(s,1) = c(\beta)$.

(2) $\coprod_{i=1}^{n} \widehat{F}_i(0,t) = c(\alpha)|_{s=0} = c(\beta)|_{s=0}$ *and* $\coprod_{i=1}^{n} \widehat{F}_i(1,t) = c(\alpha)|_{s=1} = c(\beta)|_{s=1}$.

(3) *For any $(s,t) \in I \times I$, if $i \neq j$ then $G(F_i(s,t)) \cap G(F_j(s,t)) = \emptyset$.*

With this understanding, we have the following result.

Proposition 2.1 *Let $\alpha, \beta : I \to F_G(M, n)$ be two paths with the same endpoints. Then $\alpha \simeq \beta \operatorname{rel} \partial I$ if and only if $c(\alpha) \sim^G_{\rm iso} c(\beta)$.*

Proof Assume that $F = (F_1, \cdots, F_n) : I \times I \to F_G(M, n)$ is a homotopy relative to ∂I from α to β. Then we can use F to define n homotopy maps

$$\widehat{F}_i : I \times I \to M \times I$$

by $\widehat{F}_i(s,t) = (F_i(s,t), s)$, $i = 1, \cdots, n$, satisfying (1)–(3) of Definition 2.2. Thus, $c(\alpha) \sim^G_{\rm iso} c(\beta)$.

Conversely, suppose that $c(\alpha) \sim^G_{\rm iso} c(\beta)$. Then there are n homotopy maps

$$\widehat{F}_i : I \times I \to M \times I$$

given by $\widehat{F}_i(s,t) = (F_i(s,t), s)$, $i = 1, \cdots, n$, satisfying (1)–(3) of Definition 2.2. These F_i give a map $F = (F_1, \cdots, F_n) : I \times I \to F_G(M,n)$, which is exactly the homotopy relative to ∂I from α to β.

Based upon this observation, we define the following equivalence relation among orbit braids.

Definition 2.3 *Let α and β be two paths in $F_G(M,n)$ such that $\alpha(0) = \beta(0) = \mathbf{x}$ and $\alpha(1) = g\mathbf{x}_\sigma$, $\beta(1) = h\mathbf{x}_\tau$ for some $(g,\sigma), (h,\tau) \in G^{\times n} \times \Sigma_n$. We say that orbit braids $\widetilde{c(\alpha)}$ and $\widetilde{c(\beta)}$ are equivalent, denoted by $\widetilde{c(\alpha)} \sim \widetilde{c(\beta)}$, if $c(\alpha) \sim^G_{\rm iso} c(\beta)$.*

Remark 2.4 It should be pointed out that if the action of G on M is free, then $\widetilde{c(\alpha)} \sim \widetilde{c(\beta)}$ if and only if $\widetilde{c(\alpha)}$ and $\widetilde{c(\beta)}$ are equivariantly isotopic relative to endpoints. However, if the action of G on M is not free, then generally $\widetilde{c(\alpha)}$ and $\widetilde{c(\beta)}$ are not equivariantly isotopic even if $\widetilde{c(\alpha)} \sim \widetilde{c(\beta)}$, as seen in Example 2.2.

Proposition 2.2 *$\widetilde{c(\alpha)} \sim \widetilde{c(\beta)}$ if and only if there are two paths $\alpha' = g\alpha_\sigma$, $\beta' = h\beta_\tau$, such that $\alpha'(0) = \beta'(0) = \mathbf{x}$ and α' is homotopic to β' relative to ∂I.*

Proof This is a consequence of Proposition 2.1 and Definition 2.3.

Using the equivalence relation in Definition 2.3, we define $\mathcal{B}^{\rm orb}_n(M,G)$ as the set consisting of the equivalence classes of all orbit braids at the orbit base point $\widetilde{c(\mathbf{x})}$ in $M \times I$.

Lemma 2.1 *Each class $[\widetilde{c(\alpha)}]$ in $\mathcal{B}^{\rm orb}_n(M,G)$ determines a unique pair $(g,\sigma) \in G^{\times n} \times \Sigma_n$.*

Proof By Remark 2.2, we may write $\alpha(0) = \mathbf{x} = (x_1, \cdots, x_n)$. Next let us look at the ending point $\alpha(1)$ of α. There must be $g \in G^{\times n}$ and permutation $\sigma \in \Sigma_n$ such that $\alpha(1) = g\mathbf{x}_\sigma$. Consider a path α' such that $\widetilde{c(\alpha')} \sim \widetilde{c(\alpha)}$. Then there exists a pair $(h,\tau) \in G^{\times n} \times \Sigma_n$ such that α' is homotopic to $h\alpha_\tau$ relative to ∂I. So $\alpha'(0) = h\alpha_\tau(0) = h\mathbf{x}_\tau$ and $\alpha'(1) = h\alpha_\tau(1) = h(g\mathbf{x}_\sigma)_\tau$. We can use h and τ to change the endpoints of α' such that

$$h^{-1}_{\tau^{-1}}\alpha'_{\tau^{-1}}(0) = h^{-1}_{\tau^{-1}}(h\mathbf{x}_\tau)_{\tau^{-1}} = \mathbf{x}\ h^{-1}_{\tau^{-1}}\alpha'_{\tau^{-1}}(1) = g\mathbf{x}_\sigma.$$

Since $\widetilde{c(\alpha')} = c(\widetilde{h^{-1}_{\tau^{-1}}\alpha'_{\tau^{-1}}})$, we obtain that $\widetilde{c(\alpha')}$ also determines the pair (g,σ), this implies that (g,σ) does not depend upon the choice of representatives of $[\widetilde{c(\alpha)}]$.

Let $\pi^E_1(F_G(M,n), \mathbf{x}, (G^{\times n} \rtimes \Sigma_n)(\mathbf{x}))$ denote the set consisting of the homotopy classes relative to ∂I of all paths $\alpha : I \to F_G(M,n)$ with $\alpha(0) = \mathbf{x}$ and $\alpha(1) \in \mathbf{x}^{\rm orb}$, where $\mathbf{x}^{\rm orb} = \{g\mathbf{x}_\sigma \mid g \in G^{\times n}, \sigma \in \Sigma_n\}$——the orbit at $\mathbf{x}$ under action of $G^{\times n} \rtimes \Sigma_n$. From Proposition 2.2, we have

Corollary 2.1 *$\mathcal{B}^{\rm orb}_n(M,G)$ bijectively corresponds to $\pi^E_1(F_G(M,n), \mathbf{x}, (G^{\times n} \rtimes \Sigma_n)(\mathbf{x}))$ as sets.*

Remark 2.5 Given $\sigma \in \Sigma_n$ and $g \in G^{\times n}$, we see easily that $\pi^E_1(F_G(M,n), \mathbf{x}, (G^{\times n} \rtimes \Sigma_n)(\mathbf{x}))$ bijectively corresponds to $\pi^E_1(F_G(M,n), g\mathbf{x}_\sigma, (G^{\times n} \rtimes \Sigma_n)(\mathbf{x}))$ by mapping $[\alpha]$ to $[g\alpha_\sigma]$, so $\pi^E_1(F_G(M,n), g\mathbf{x}_\sigma, (G^{\times n} \rtimes \Sigma_n)(\mathbf{x}))$ also bijectively corresponds to $\mathcal{B}^{\rm orb}_n(M,G)$.

2.2 Groups of orbit braids and their homotopy descriptions

Let $[\widetilde{c(\alpha)}]$ and $[\widetilde{c(\beta)}]$ be two classes in $\mathcal{B}^{\rm orb}_n(M,G)$.

First let us consider the operation between $\widetilde{c(\alpha)}$ and $\widetilde{c(\beta)}$ in an intuitive way. Since two orbit braids have the same endpoints, intuitively we can obtain a new orbit braid $\widetilde{c(\alpha)} \circ \widetilde{c(\beta)}$ by gluing the starting points of $\widetilde{c(\beta)}$ to the ending points of $\widetilde{c(\alpha)}$. More precisely,

$$\widetilde{c(\alpha)} \circ \widetilde{c(\beta)}(s) = \begin{cases} \widetilde{c(\alpha)}(2s), & \text{if } s \in \left[0, \frac{1}{2}\right], \\ \widetilde{c(\beta)}(2s-1), & \text{if } s \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Clearly this operation $\circ$ is well-defined, but not associative. By Corollary 2.1, this new orbit braid $\widetilde{c(\alpha)} \circ \widetilde{c(\beta)}$ should be determined by a path $\gamma : I \to F_G(M,n)$ with $\gamma(0) = \mathbf{x}$ and $\gamma(1) \in \mathbf{x}^{\mathrm{orb}}$. Such a path γ can be constructed as follows:

By Lemma 2.1, there exists a unique pair $(g,\sigma),(h,\tau) \in G^{\times n} \rtimes \Sigma_n$ such that $\alpha(1) = g\mathbf{x}_\sigma$, $\beta(1) = h\mathbf{x}_\tau$. Consider $\widehat{\beta} = g\beta_\sigma$, since $\beta(0) = \mathbf{x}$, we have that $\widehat{\beta}(0) = \alpha(1) = g\mathbf{x}_\sigma$, and we know from previous discussion $\widetilde{c(\beta)} = \widetilde{c(\widehat{\beta})}$. Then we can construct a new path

$$\gamma(s) = \alpha \circ \widehat{\beta}(s) = \begin{cases} \alpha(2s), & \text{if } s \in \left[0, \frac{1}{2}\right], \\ \widehat{\beta}(2s-1), & \text{if } s \in \left[\frac{1}{2}, 1\right] \end{cases}$$

with $\gamma(0) = \mathbf{x}$ and $\gamma(1) = g\beta_\sigma(1) = g(h\mathbf{x}_\tau)_\sigma = gh_\sigma \mathbf{x}_{\sigma\tau}$, as desired.

Remark 2.6 In the above construction of γ, we see that two pairs (g,σ) and (h,τ) actually produce a new pair $(gh_\sigma, \sigma\tau) = (g,\sigma)\cdot(h,\tau)$, which is compatible with $[\widetilde{c(\alpha)} \circ \widetilde{c(\beta)}] = [\widetilde{c(\gamma)}]$.

Now we define an operation $*$ on $\mathcal{B}_n^{\mathrm{orb}}(M,G)$ by

$$[\widetilde{c(\alpha)}] * [\widetilde{c(\beta)}] = [\widetilde{c(\alpha)} \circ \widetilde{c(\beta)}].$$

We claim that the operation $*$ is well-defined and associative. It suffices to show that for any $\alpha' \in [\alpha]$ and any $\beta' \in [\beta]$,

$$[\widetilde{c(\alpha')}] * [\widetilde{c(\beta')}] = [\widetilde{c(\alpha')} \circ \widetilde{c(\beta')}] = [\widetilde{c(\alpha)} \circ \widetilde{c(\beta)}] = [\widetilde{c(\alpha)}] * [\widetilde{c(\beta)}].$$

Since $\alpha(i) = \alpha'(i)$ and $\beta(i) = \beta'(i)$ for $i = 0, 1$, we have that $g\beta'_\sigma(0) = \alpha(1)$ and $\widetilde{c(\beta')} = \widetilde{c(g\beta'_\sigma)}$. In a similar way to the construction of γ as above, we may define $\gamma' = \alpha' \circ (g\beta'_\sigma)$. Furthermore, homotopy theory (see [22]) tells us that $\gamma' = \alpha' \circ (g\beta'_\sigma)$ is homotopic to $\gamma = \alpha \circ (g\beta_\sigma)$ relative to ∂I, By Corollary 2.1, this implies the operation $*$ is well-defined. Since the operation $*$ is essentially reduced to the operation on the homotopy classes of paths, it is also associative.

Proposition 2.3 *$\mathcal{B}_n^{\mathrm{orb}}(M,G)$ forms a group under the operation $*$, called the orbit braid group of the G-manifold M.*

Proof Obviously, the class $[\widetilde{c(c_{\mathbf{x}})}]$ is just the unit element, where $c_{\mathbf{x}}$ is the constant path with $c_{\mathbf{x}}(s) = \mathbf{x}, s \in I$.

Let $[\widetilde{c(\alpha)}]$ be an element in $\mathcal{B}_n^{\mathrm{orb}}(M,G)$. Consider the inverse path $\overline{\alpha}$ of α, i.e., $\overline{\alpha}(s) = \alpha(1-s)$. It is well-known in homotopy theory that $\alpha \circ \overline{\alpha}$ is homotopic to $c_{\mathbf{x}}$. Thus,

$$[\widetilde{c(\alpha)}] * [\widetilde{c(\overline{\alpha})}] = [\widetilde{c(\alpha)} \circ \widetilde{c(\overline{\alpha})}] = [\widetilde{c(c_{\mathbf{x}})}]$$

gives that $[\widetilde{c(\alpha)}]^{-1} = [\widetilde{c(\overline{\alpha})}]$.

When the action of G on M is trivial, $\mathcal{B}_n^{\rm orb}(M,G)$ will degenerate into the ordinary braid group $B_n(M)$. Thus the notion of orbit braid group is a generalization for ordinary braid groups.

Putting some restrictions on endpoints of orbit braids, we may define some subgroups of $\mathcal{B}_n^{\rm orb}(M,G)$ as follows.

Definition 2.4 (Subgroups of $\mathcal{B}_n^{\rm orb}(M,G)$)

(1) *Those classes* $[\widetilde{c(\alpha)}]$ *with* $\alpha(1) \in G^{\times n}(\mathbf{x})$ *of* $\mathcal{B}_n^{\rm orb}(M,G)$ *form a subgroup of* $\mathcal{B}_n^{\rm orb}(M,G)$, *which is called the pure orbit braid group, denoted by* $\mathcal{P}_n^{\rm orb}(M,G)$.

(2) *Those classes* $[\widetilde{c(\alpha)}]$ *with* $\alpha(1) \in \Sigma_n(\mathbf{x}) = \{\mathbf{x}_\sigma \mid \sigma \in \Sigma_n\}$ *of* $\mathcal{B}_n^{\rm orb}(M,G)$ *form a subgroup of* $\mathcal{B}_n^{\rm orb}(M,G)$, *which is called the braid group, denoted by* $\mathcal{B}_n(M,G)$.

(3) *Those classes* $[\widetilde{c(\alpha)}]$ *with* $\alpha(1) = \mathbf{x}$ *of* $\mathcal{B}_n^{\rm orb}(M,G)$ *form a subgroup of* $\mathcal{B}_n^{\rm orb}(M,G)$, *which is called the pure braid group, denoted by* $\mathcal{P}_n(M,G)$.

The group structure on $\mathcal{B}_n^{\rm orb}(M,G)$ gives us an insight to

$$\pi_1^E(F_G(M,n),\mathbf{x},(G^{\times n} \rtimes \Sigma_n)(\mathbf{x})),$$

on which we can also endow an operation $\bullet$ defined by

$$[\alpha] \bullet [\beta] = [\alpha \circ (g\beta_\sigma)], \tag{2.1}$$

where $(g,\sigma) \in G^{\times n} \times \Sigma_n$ is the unique pair determined by $[\widetilde{c(\alpha)}]$. Then it is easy to see that $\pi_1^E(F_G(M,n),\mathbf{x},(G^{\times n} \rtimes \Sigma_n)(\mathbf{x}))$ becomes a group under this operation. Indeed, this group exactly agrees with the fundamental group of transformation group in the sense of Rhodes in [21].

Now from Corollary 2.1, we have the following result.

Theorem 2.1 *The map*

$$\Lambda : \pi_1^E(F_G(M,n),\mathbf{x},(G^{\times n} \rtimes \Sigma_n)(\mathbf{x})) \to \mathcal{B}_n^{\rm orb}(M,G)$$

given by $[\alpha] \mapsto [\widetilde{c(\alpha)}]$ *is an isomorphism.*

Similarly, those subgroups defined above of $\mathcal{B}_n^{\rm orb}(M,G)$ can also be described in terms of the homotopy classes of paths in $F_G(M,n)$.

Corollary 2.2 *Homotopy descriptions of subgroups* $\mathcal{P}_n^{\rm orb}(M,G)$, $\mathcal{B}_n(M,G)$ *and* $\mathcal{P}_n(M,G)$:

(1) $\mathcal{P}_n^{\rm orb}(M,G) \cong \pi_1^E(F_G(M,n),\mathbf{x},G^{\times n}(\mathbf{x}))$;

(2) $\mathcal{B}_n(M,G) \cong \pi_1^E(F_G(M,n),\mathbf{x},\Sigma_n(\mathbf{x}))$;

(3) $\mathcal{P}_n(M,G) \cong \pi_1(F_G(M,n),\mathbf{x})$.

Remark 2.7 The above viewpoint can also be used in the theory of ordinary braids. Consider the case in which $G = \{e\}$. Then $\mathcal{B}_n^{\rm orb}(M,G)$ degenerates into the ordinary braid group $B_n(M)$, which is isomorphic to the group $\pi_1^E(F(M,n),\mathbf{x},\Sigma_n(\mathbf{x}))$. In this case, there is the following short exact sequence

$$1 \to \pi_1(F(M,n),\mathbf{x}) \to \pi_1^E(F(M,n),\mathbf{x},\Sigma_n(\mathbf{x})) \to \Sigma_n \to 1,$$

from which we see that $\pi_1^E(F(M,n),\mathbf{x},\Sigma_n(\mathbf{x}))$ is actually the fundamental group of the unordered configuration space $F(M,n)/\Sigma_n$.

Corollary 2.2 tells us that, as in the theory of ordinary braids, the pure braid group $\mathcal{P}_n(M,G)$ can be realized as the fundamental group $\pi_1(F_G(M,n),\mathbf{x})$. Later on, we will show that $\mathcal{B}_n(M,G)$ can be realized as the fundamental group of $F_G(M,n)/\Sigma_n$, and we shall see much more information on $\mathcal{B}_n^{\rm orb}(M,G)$ and $\mathcal{P}_n^{\rm orb}(M,G)$.

2.3 Short exact sequences

Consider the semidirect product $G^{\times n}\rtimes_\varphi\Sigma_n$ defined at the beginning of Section 2. Then by Lemma 2.1 and Remark 2.6, we can define a homomorphism

$$\Phi:\mathcal{B}_n^{\rm orb}(M,G)\to G^{\times n}\rtimes_\varphi\Sigma_n$$

by $\Phi([\widetilde{c(\alpha)}])=(g,\sigma)$, where (g,σ) is the unique pair determined by $[\widetilde{c(\alpha)}]$.

Lemma 2.2 *The homomorphism*

$$\Phi:\mathcal{B}_n^{\rm orb}(M,G)\to G^{\times n}\rtimes_\varphi\Sigma_n$$

is an epimorphism.

Proof Given a pair (g,σ) in $G^{\times n}\rtimes_\varphi\Sigma_n$, since $F_G(M,n)$ is path-connected, there must be a path $\alpha:I\to F_G(M,n)$ such that $\alpha(0)=\mathbf{x}$ and $\alpha(1)=g\mathbf{x}_\sigma$, which gives $\Phi([\widetilde{c(\alpha)}])=(g,\sigma)$. Thus Φ is an epimorphism.

Based upon the Definition 2.4, when Φ is restricted to $\mathcal{P}_n^{\rm orb}(M,G)$, each class $[\widetilde{c(\alpha)}]$ will uniquely determine the pair (g,e_{Σ_n}), where e_{Σ_n} is the unit element of Σ_n. Thus, Φ induces a homomorphism

$$\Phi_G:\mathcal{P}_n^{\rm orb}(M,G)\to G^{\times n}$$

given by $\Phi_G([\widetilde{c(\alpha)}])=g$, which is an epimorphism.

When Φ is restricted to $\mathcal{B}_n(M,G)$, each class $[\widetilde{c(\alpha)}]$ will uniquely determine the pair $(e_{G^{\times n}},\sigma)$, where $e_{G^{\times n}}$ is the unit element of $G^{\times n}$. So Φ induces a homomorphism

$$\Phi_\Sigma:\mathcal{B}_n(M,G)\to\Sigma_n,$$

which is also an epimorphism.

We can observe that Ker Φ, Ker Φ_G and Ker Φ_Σ are exactly the pure braid group $\mathcal{P}_n(M,G)$.

On the other hand, there are two natural projections $p|_\Sigma:G^{\times n}\rtimes_\varphi\Sigma_n\to\Sigma_n$ and $p|_G:G^{\times n}\rtimes_\varphi\Sigma_n\to G^{\times n}$, which give two maps

$$p|_\Sigma\circ\Phi:\mathcal{B}_n^{\rm orb}(M,G)\to\Sigma_n$$

and

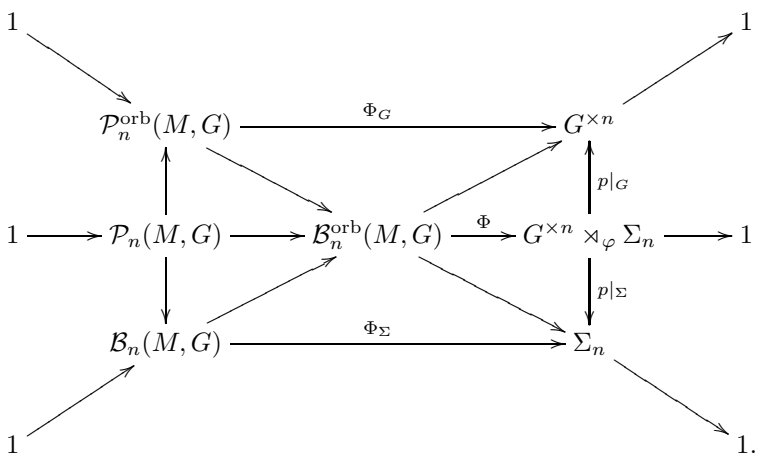
$$p|_G\circ\Phi:\mathcal{B}_n^{\rm orb}(M,G)\to G^{\times n}.$$

We see by Lemma 2.2 that such two maps are still surjective. In addition, it is easy to see that $\mathrm{Ker}(p|_\Sigma\circ\Phi)=\mathcal{P}_n^{\rm orb}(M,G)$ and $\mathrm{Ker}(p|_G\circ\Phi)=\mathcal{B}_n(M,G)$. However, we note that $p|_G\circ\Phi$ is not

a group homomorphism since $p|_G$ is not a group homomorphism, and $p|_\Sigma \circ \Phi$ is still a group homomorphism.

Together with all arguments above, we have the following result.

Theorem 2.2 *The following diagram commutes and contains five short exact sequences*:



Remark 2.8 We note that because the map $p|_G \circ \Phi : \mathcal{B}_n^{\rm orb}(M,G) \to G^{\times n}$ is not a group homomorphism,

$$1 \to \mathcal{B}_n(M,G) \to \mathcal{B}_n^{\rm orb}(M,G) \to G^{\times n} \to 1$$

is not an exact sequence in the sense that all maps must be group homomorphisms. However, it can still be regarded as an exact sequence in the sense of Switzer [22] for topological spaces.

Lemma 2.3 *The braid group $\mathcal{B}_n(M,G)$ is isomorphic to $\pi_1(F_G(M,n)/\Sigma_n, p_\Sigma(\mathbf{x}))$.*

Proof We see from Corollary 2.2 that $\mathcal{B}_n(M,G) \cong \pi_1^E(F_G(M,n), \mathbf{x}, \Sigma_n(\mathbf{x}))$, and the action of Σ_n on $F_G(M,n)$ is free, then the required result follows.

Remark 2.9 Corollary 2.2 and Lemma 2.3 tell us that the short exact sequence in Theorem 2.2,

$$1 \to \mathcal{P}_n(M,G) \to \mathcal{B}_n(M,G) \to \Sigma_n \to 1$$

geometrically corresponds to the short exact sequence

$$1 \to \pi_1(F_G(M,n), \mathbf{x}) \to \pi_1(F_G(M,n)/\Sigma_n, p_\Sigma(\mathbf{x})) \to \Sigma_n \to 1$$

given by the fibration $F_G(M,n) \to F_G(M,n)/\Sigma_n$ with fiber Σ_n.

2.4 Liftings of paths

For the projection $p^G : F_G(M,n) \to F(M/G,n)$, write $p^G = (p_1^G, \cdots, p_n^G)$ and $\overline{\mathbf{x}} = p^G(\mathbf{x})$. Consider the path $\overline{\alpha} = (\overline{\alpha}_1, \cdots, \overline{\alpha}_n) : I \to F(M/G,n)$ from $\overline{\mathbf{x}}$ to $\overline{\mathbf{x}}_\sigma = p^G(\mathbf{x}_\sigma), \sigma \in \Sigma_n$, and we have

$$(p^G)^{-1}(\overline{\alpha}(I)) = ((p_1^G)^{-1}(\overline{\alpha}_1(I)), \cdots, (p_n^G)^{-1}(\overline{\alpha}_n(I))).$$

Since G is finite, we see easily that there must be at least $|G|^n$ path liftings $\alpha : I \to F_G(M,n)$

$$\begin{array}{ccc} & & F_G(M,n) \\ & \overset{\alpha}{\nearrow} & \big\downarrow p^G \\ I & \xrightarrow{\overline{\alpha}} & F(M/G,n) \end{array}$$

such that

$$G^{\times n}(\alpha(I)) = (p^G)^{-1}(\overline{\alpha}(I)) = ((p_1^G)^{-1}(\overline{\alpha}_1(I)), \cdots, (p_n^G)^{-1}(\overline{\alpha}_n(I))).$$

In particular, there must be at least one path lifting α with $\alpha(0) = \mathbf{x}$.

Then we have the following result.

Lemma 2.4 *The projection $p^G : F_G(M,n) \to F(M/G,n)$ induces an epimorphism*

$$p_*^G : \pi_1^E(F_G(M,n), \mathbf{x}, (G^{\times n} \rtimes \Sigma_n)(\mathbf{x})) \to \pi_1^E(F(M/G,n), \overline{\mathbf{x}}, \Sigma_n(\overline{\mathbf{x}}))$$

by $p_*^G([\alpha]) = [p^G(\alpha)]$.

Remark 2.10 p_*^G may not be injective in general. In fact, because of the existence of non-free orbit point, it is possible that there exist two path liftings α, α' of $\overline{\alpha}$ such that $\alpha(0) = \alpha'(0)$ but $\alpha \not\simeq \alpha'$ rel ∂I, so $[\widetilde{c(\alpha)}] \neq [\widetilde{c(\alpha')}]$.

In a similar way as Lemma 2.3, we have that

$$\pi_1^E(F(M/G,n), \overline{\mathbf{x}}, \Sigma_n(\overline{\mathbf{x}})) \cong \pi_1(F(M/G,n)/\Sigma_n, \widehat{\overline{\mathbf{x}}}) = B_n(M/G),$$

where $\widehat{\overline{\mathbf{x}}}$ is the image of $\overline{\mathbf{x}}$ under the projection $F(M/G,n) \to F(M/G,n)/\Sigma_n$. Furthermore, together with Corollary 2.2, Lemma 2.4 and Theorem 2.2, we conclude that

Proposition 2.4 *There is an epimorphism between two short exact sequences*:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{P}_n^{\mathrm{orb}}(M,G) & \longrightarrow & \mathcal{B}_n^{\mathrm{orb}}(M,G) & \longrightarrow & \Sigma_n & \longrightarrow & 1 \\ & & \big\downarrow & & \big\downarrow & & =\big\downarrow & & \\ 1 & \longrightarrow & P_n(M/G) & \longrightarrow & B_n(M/G) & \longrightarrow & \Sigma_n & \longrightarrow & 1. \end{array}$$

Now let us consider the case in which the action of G on M is free. In this case, the projection $p^G : F_G(M,n) \to F(M/G,n)$ becomes a fibration with fiber $G^{\times n}$.

Lemma 2.5 *The following statements are equivalent.*

(1) *The action of G on M is free.*

(2) *For any path $\overline{\alpha} : I \to F(M/G,n)$, there are exactly $|G|^n$ path liftings of $\overline{\alpha}$.*

(3) *For any path $\overline{\alpha} : I \to F(M/G,n)$ and any two path liftings α' and α'' of $\overline{\alpha}$, $\widetilde{c(\alpha')} = \widetilde{c(\alpha'')}$.*

Proof The equivalence of (1) and (2) is obvious. Assume that there are exactly $|G|^n$ path liftings of $\overline{\alpha}$. Then we see that for each $1 \leq i \leq n$, $(p^G)^{-1}\overline{\alpha}_i$ consists of $|G|$ path liftings of $\overline{\alpha}_i$, all of which do not intersect to each other. Furthermore, we have that for any two path liftings α' and α'' of $\overline{\alpha}$, there is some $g \in G^{\times n}$ such that $\alpha' = g\alpha''$, so $\widetilde{c(\alpha')} = \widetilde{c(\alpha'')}$.

Conversely, let α' and α'' be two path liftings of $\overline{\alpha}$, and assume that $\widetilde{c(\alpha')} = \widetilde{c(\alpha'')}$. Then there must be $g \in G^{\times n}$ and $\sigma \in \Sigma_n$ such that $\alpha' = g\alpha''_\sigma$. Since $p^G(\alpha') = p^G(\alpha'')$, we know that

σ must be the unit. This implies that $\{g\alpha' \mid g \in G^{\times n}\}$ gives all different path liftings of $\overline{\alpha}$, which consist of exactly $|G|^n$ path liftings.

Remark 2.11 Lemma 2.5 tells us that if the action of G on M is non-free, then there must be some path $\overline{\alpha} : I \to F(M/G, n)$ and two different path liftings α' and α'' of $\overline{\alpha}$ such that $\widetilde{c(\alpha')} \neq \widetilde{c(\alpha'')}$. In fact, since the action of G on M is non-free, we may assume that there is some $s \in I$ such that $(p^G)^{-1}(\overline{\alpha}(s))$ is not the free orbit in $F_G(M, n)$. Furthermore, there would be more than $|G|^n$ path liftings of $\overline{\alpha}$ since we have more choices of path liftings via those points of non-free orbit in $(p^G)^{-1}(\overline{\alpha}(s))$.

Corollary 2.3 *If the action of G on M is free, then*
(1) $\mathcal{P}_n^{\rm orb}(M, G) \cong P_n(M/G)$, *so* $\mathcal{P}_n^{\rm orb}(M, G)$ *is realizable as* $\pi_1(F(M/G, n), \overline{\mathbf{x}})$.
(2) $\mathcal{B}_n^{\rm orb}(M, G) \cong B_n(M/G)$, *so* $\mathcal{B}_n^{\rm orb}(M, G)$ *is realizable as* $\pi_1(F(M/G, n)/\Sigma_n, \widehat{\overline{\mathbf{x}}})$.

Proof It is a consequence of Lemma 2.5 and Proposition 2.4.

Remark 2.12 In the viewpoint of the theory of covering spaces, generally $F_G(M, n)$ is not the covering space of $F(M/G, n)$. However, paths and the homotopies between two paths in $F(M/G, n)$ can still be lifted to $F_G(M, n)$ but liftings with the same starting point may not be unique. Thus, if the action of G on M is non-free, then the homomorphism

$$p_*^G : \mathcal{P}_n(M, G) \to P_n(M/G)$$

induced by the projection $p^G : F_G(M, n) \to F(M/G, n)$ is no longer injective. Actually, p_*^G is the composition of a monomorphism and an epimorphism

$$\mathcal{P}_n(M, G) \rightarrowtail \mathcal{P}_n^{\rm orb}(M, G) \twoheadrightarrow P_n(M/G).$$

2.5 Relation between orbit configuraiton space and ordinary configuraiton space

There is a natural embedding $i : F_G(M, n) \hookrightarrow F(M, n)$ from orbit configuration space to its corresponding ordinary configuration space.

Lemma 2.6 *The induced homomorphism $i_* : \pi_1(F_G(M, n), \mathbf{x}) \to \pi_1(F(M, n), \mathbf{x})$ is an epimorphism.*

Proof Take an element $[\alpha]$ in $\pi_1(F(M, n), \mathbf{x})$. If for any $s \in I$, $\alpha(s) \in F_G(M, n)$, then $[\alpha]$ is also an element of $\pi_1(F_G(M, n), \mathbf{x})$.

Now assume that there is some $s \in I$ (possibly s can be any point of the whole $(0, 1)$) such that $\alpha(s) \notin F_G(M, n)$. This means that there are at least two i, j with $i \neq j$ such that $G(\alpha_i(s)) = G(\alpha_j(s))$, where $\alpha = (\alpha_1, \cdots, \alpha_n)$. Clearly, $\alpha_i(s)$ or $\alpha_j(s)$ is not a G-fixed point since $\alpha_i(s) \neq \alpha_j(s)$. So there exists some $g \neq e$ in G such that $\alpha_i(s) = g\alpha_j(s)$. Since G is finite, there exists a G-invariant open neighborhood N of $\alpha_i(s)$ which is a disjoint union of some connected open subsets in $F(M, n)$ such that for a small enough connected open neighborhood $N_s \subset I$ of s, $\alpha_i(N_s) \cap N$ and $\alpha_j(N_s) \cap N$ lie in two different components of N. Then we can always do a slight homotopy deformation on α_i in N, changing α_i into α_i', such that $\alpha_i'(N_s) \cap N$

never meets with the orbits of other $\alpha_k(N_s)$, $k \neq i$. This gives a change on a small open arc of the path α_i up to homotopy.

As long as there are also finitely or infinitely many points s in I such that $\alpha_i(s)$ meets some orbit of $\alpha_k(s), k \neq i$, since $\alpha_i(I)$ is compact, we can perform the above approach finite times, so that α_i can be finally changed into a new path α_i' such that $\alpha_i \simeq \alpha_i'$ rel ∂I in $F(M,n)$, and for any $s \in I$ and any $k \neq i$, $G(\alpha_i'(s)) \cap G(\alpha_k(s)) = \emptyset$. This procedure only does change the component path α_i, so α is changed into $(\alpha_1, \cdots, \alpha_{i-1}, \alpha_i', \alpha_{i+1}, \cdots, \alpha_n)$, denoted by α'. Clearly, $\alpha \simeq \alpha'$ rel ∂I in $F(M,n)$.

If α' is not a path in $F_G(M,n)$ yet, then we will perform the above procedure on other component paths $\alpha_k, k \neq i$. Since $c(\alpha)$ only contains n strings, we can end our procedure until we obtain a path β such that for any $s \in I$, $\beta(s) \in F_G(M,n)$ and $\alpha \simeq \beta$ rel ∂I in $F(M,n)$. Thus, $[\alpha]$ is in the image of i_*. This completes the proof.

In a similar way as above, we can show that the following homomorphism induced by the embedding $i : F_G(M,n) \hookrightarrow F(M,n)$:

$$\pi_1^E(F_G(M,n), \mathbf{x}, \Sigma_n(\mathbf{x})) \to \pi_1^E(F(M,n), \Sigma_n(\mathbf{x}))$$

is also an epimorphism. Therefore we have

Proposition 2.5 *There is an epimorphism between two short exact sequences*:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{P}_n(M,G) & \longrightarrow & \mathcal{B}_n(M,G) & \longrightarrow & \Sigma_n & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & = \downarrow & & \\ 1 & \longrightarrow & P_n(M) & \longrightarrow & B_n(M) & \longrightarrow & \Sigma_n & \longrightarrow & 1. \end{array}$$

3 Presentations of Orbit Braid Groups of Two Typical Actions on $\mathbb{C}$

The geometric presentation of classical braid group $B_n(\mathbb{R}^2)$ in $\mathbb{R}^2 \times I$ (see [4, 6]) motivates us to present orbit braid groups. We begin with our work from the case of $\mathbb{C} \approx \mathbb{R}^2$ with the following two typical actions:

(I) The action $\mathbb{Z}_p \curvearrowright^{\phi_1} \mathbb{C}$ defined by $(\mathrm{e}^{\frac{2k\pi\mathbf{i}}{p}}, z) \mapsto \mathrm{e}^{\frac{2k\pi\mathbf{i}}{p}} z$, which is non-free and fixes only the origin of $\mathbb{C}$, where p is a prime, and $\mathbb{Z}_p$ is regarded as the group $\{\mathrm{e}^{\frac{2k\pi\mathbf{i}}{p}} \mid 0 \leq k < p\}$. If the action ϕ_1 is restricted to $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$, then the action $\mathbb{Z}_p \curvearrowright^{\phi_1} \mathbb{C}^\times$ is free.

(II) Non-free action $(\mathbb{Z}_2)^2 \curvearrowright^{\phi_2} \mathbb{C}$ defined by

$$\begin{cases} z \mapsto \overline{z}, \\ z \mapsto -\overline{z}. \end{cases}$$

This action is just the standard representation of $(\mathbb{Z}_2)^2$ on $\mathbb{C} \approx \mathbb{R}^2$.

Throughout this section, fix

$$\mathbf{z} = (1 + \mathbf{i}, 2 + 2\mathbf{i}, \cdots, n + n\mathbf{i})$$

as the base point in $F_{\mathbb{Z}_p}(\mathbb{C}, n)$, $F_{\mathbb{Z}_p}(\mathbb{C}^\times, n)$ and $F_{(\mathbb{Z}_2)^2}(\mathbb{C}, n)$, where $\mathbf{i} = \sqrt{-1}$. Clearly, each coordinate of $\mathbf{z}$ is free orbit point in above actions. For convenience, we denote $l + l\mathbf{i}$ by $\widehat{l}$, so $\mathbf{z} = (1 + \mathbf{i}, 2 + 2\mathbf{i}, \cdots, n + n\mathbf{i}) = (\widehat{1}, \widehat{2}, \cdots, \widehat{n})$.

3.1 Orbit braid group $\mathcal{B}_n^{\rm orb}(\mathbb{C},\mathbb{Z}_p)$

For a path $\alpha=(\alpha_1,\cdots,\alpha_n)$ in $F_{\mathbb{Z}_p}(\mathbb{C},n)$ with $\alpha(0)=\mathbf{z}$ and $\alpha(1)=g\mathbf{z}_\sigma$ where $(g,\sigma)\in(\mathbb{Z}_p)^n\rtimes\Sigma_n$, it is easy to see that the corresponding orbit braid $\widetilde{c(\alpha)}=\coprod_{i=1}^n\widetilde{c(\alpha_i)}$,

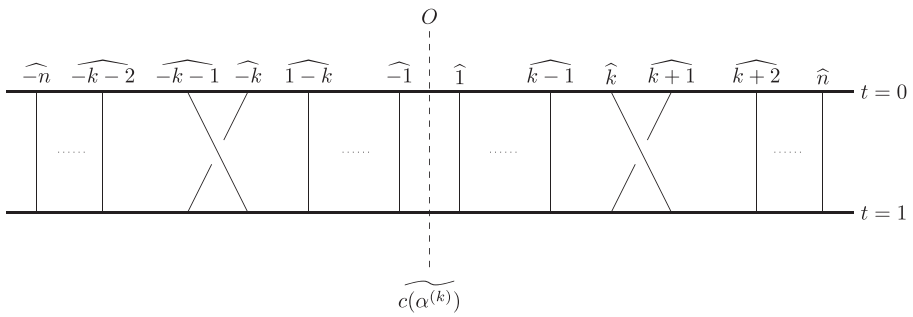
$$\widetilde{c(\alpha_i)}=\Big\{(\alpha_i(s),s),\Big(\mathrm{e}^{\frac{2\pi\mathbf{i}}{p}}\alpha_i(s),s\Big),\cdots,\Big(\mathrm{e}^{\frac{2(p-1)\pi\mathbf{i}}{p}}\alpha_i(s),s\Big)\mid s\in I\Big\}.$$

is symmetric with respect to the line $O=\{0\}\times I$ in $\mathbb{C}\times I$.

First let us consider the case $p=2$. To describe $\mathcal{B}_n^{\rm orb}(\mathbb{C},\mathbb{Z}_2)$, we construct a family of basic "bricks" $\mathbf{b}_k,k=1,\cdots,n-1$, and $\mathbf{b}$, where each orbit braid $\mathbf{b}_k$ is chosen as the class $[\widetilde{c(\alpha^{(k)})}]$ given by the path

$$\alpha^{(k)}(s)=\Big(1+\mathbf{i},\cdots,k+(k+1)\mathbf{i}+\mathrm{e}^{-\frac{\pi}{2}\mathbf{i}(1-s)},(k+1)+k\mathbf{i}+\mathbf{i}\mathrm{e}^{\frac{\pi}{2}\mathbf{i}s},\cdots,n+n\mathbf{i}\Big)\tag{3.1}$$

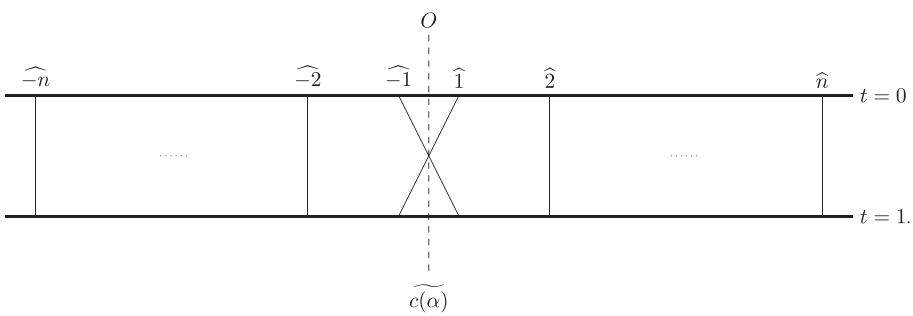
as shown in the following picture



and $\mathbf{b}$ is chosen as the class $[\widetilde{c(\alpha)}]$ given by the path

$$\alpha(s)=((1+\mathbf{i})(1-2s),2+2\mathbf{i},\cdots,n+n\mathbf{i})\tag{3.2}$$

as shown in the following picture



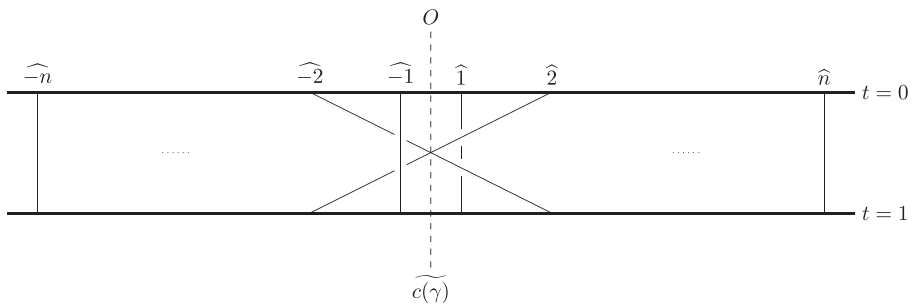
Remark 3.1 In the above picture for $\widetilde{c(\alpha)}$, we see that the first string and its orbit can exactly intersect at O. This can happen because the origin in $\mathbb{C}$ is just a fixed point of $\mathbb{Z}_2$-action.

Furthermore, even if the first string and its orbit do not intersect, then the corresponding orbit braid can still be equivalent to $\widetilde{c(\alpha)}$. In fact, we can choose the following path

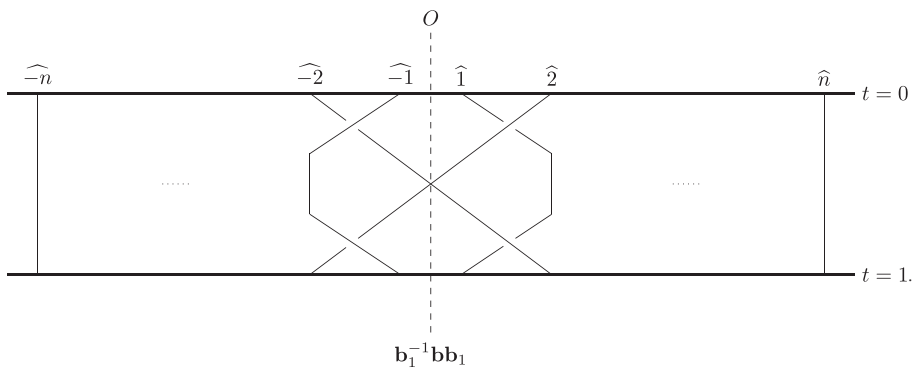
$$\beta(s) = ((1+\mathbf{i})\mathrm{e}^{\pi\mathbf{i}s}, 2+2\mathbf{i}, \cdots, n+n\mathbf{i}), \tag{3.3}$$

which never goes through the origin, but $[\widetilde{c(\alpha)}] = [\widetilde{c(\beta)}]$. Thus, $[\widetilde{c(\beta)}]$ can also be chosen as $\mathbf{b}$.

In addition, we note that there are also other typical orbit braids $\widetilde{c(\gamma)}$ in which the i-th string $c(\gamma_i)$ connects $\widehat{i}$ and $\widehat{-i}$, other strings remain constant, where $i = 2, \cdots, n$. However, these orbit braids are not basic "bricks". In fact, we see easily that for each i, $[\widetilde{c(\gamma)}]$ can be represented as a composition $\mathbf{b}_{i-1}^{-1} \cdots \mathbf{b}_1^{-1}\mathbf{b}\mathbf{b}_1 \cdots \mathbf{b}_{i-1}$. The following two pictures illustrate the case of $i = 2$.



and



Lemma 3.1 *$\mathcal{B}_n^{\mathrm{orb}}(\mathbb{C}, \mathbb{Z}_2)$ has a presentation, generated by $\mathbf{b}_k$ $(1 \le k \le n-1)$ and $\mathbf{b}$, satisfying the following relations*:

$R_1 : \mathbf{b}^2 = e$,

$R_2 : \mathbf{b}\mathbf{b}_1\mathbf{b}\mathbf{b}_1 = \mathbf{b}_1\mathbf{b}\mathbf{b}_1\mathbf{b}$,

$R_3 : \mathbf{b}_k\mathbf{b} = \mathbf{b}\mathbf{b}_k$ $(k > 1)$,

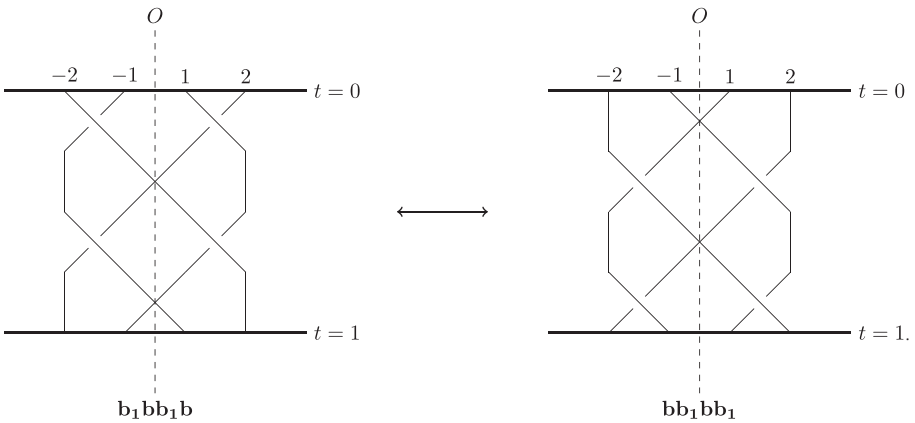
$R_4 : \mathbf{b}_k\mathbf{b}_{k+1}\mathbf{b}_k = \mathbf{b}_{k+1}\mathbf{b}_k\mathbf{b}_{k+1}$,

$R_5 : \mathbf{b}_k\mathbf{b}_l = \mathbf{b}_l\mathbf{b}_k$ $(|k-l| > 1)$.

Proof First, every class in $\mathcal{B}_n^{\rm orb}(\mathbb{C},\mathbb{Z}_2)$ can be reduced into the composition of $\mathbf{b}_k$ and $\mathbf{b}$ because each crossing of two adjacent orbit strings just decide a basic "brick".

Each $\mathbf{b}_k$ has a symmetric structure with respect to O and its half part is used as a generator of classical braid group $B_n(\mathbb{R}^2)$. Thus, the relations R_4 and R_5 follow from the theory of classical braid groups (see [6]).

We can construct homotopy deformation maps that connect both sides of the equations in relations R_1, R_2 and R_3. Actually, we can see this intuitively. Let us look at the pictures of orbit braids in both sides of relation R_2, as shown below.



Since we can always do a slight homotopy deformation on $\mathbf{b}$ near O such that the first string and its orbit do not intersect at O as stated in Remark 3.1, the intersection of second string and its orbit is equivalent to the orbit of second string over second string in the above picture. This illustrates the equivalence of two orbit braids.

We can also use a similar way to prove relations R_1 and R_3. We would like to leave them as exercises to the reader.

Proposition 3.1 $\mathcal{B}_n^{\rm orb}(\mathbb{C},\mathbb{Z}_2) \cong \langle \mathbf{b}_1,\cdots,\mathbf{b}_{n-1},\mathbf{b} \mid R_1,R_2,R_3,R_4,R_5\rangle$, *where* R_1–R_5 *are the relations stated in Lemma* 3.1.

Proof It suffices to prove the completeness of relations R_1–R_5, i.e., there doesn't exist any extra relation in $\mathcal{B}_n^{\rm orb}(\mathbb{C},\mathbb{Z}_2)$.

Assume that $\widetilde{\mathcal{B}}_n^{\rm orb}(\mathbb{C},\mathbb{Z}_2)$ is the pure algebraic presentation group as follows:

$$\widetilde{\mathcal{B}}_n^{\rm orb}(\mathbb{C},\mathbb{Z}_2) = \left\langle \mathbf{a}_1,\cdots,\mathbf{a}_{n-1},\mathbf{a} \,\middle|\, \begin{matrix} (1)\ \mathbf{a}^2 = e;\quad (2)\ \mathbf{a}\mathbf{a}_1\mathbf{a}\mathbf{a}_1 = \mathbf{a}_1\mathbf{a}\mathbf{a}_1\mathbf{a};\quad (3)\ \mathbf{a}_k\mathbf{a} = \mathbf{a}\mathbf{a}_k\ (k>1); \\ (4)\ \mathbf{a}_k\mathbf{a}_{k+1}\mathbf{a}_k = \mathbf{a}_{k+1}\mathbf{a}_k\mathbf{a}_{k+1};\quad (5)\ \mathbf{a}_k\mathbf{a}_l = \mathbf{a}_l\mathbf{a}_k\ (|k-l|>1) \end{matrix} \right\rangle.$$

Then there is an epimorphism

$$\phi_n : \widetilde{\mathcal{B}}_n^{\rm orb}(\mathbb{C},\mathbb{Z}_2) \to \mathcal{B}_n^{\rm orb}(\mathbb{C},\mathbb{Z}_2)$$

defined by $\phi_n(\mathbf{a}_k) = \mathbf{b}_k$ $(1 \le k \le n-1)$ and $\phi_n(\mathbf{a}) = \mathbf{b}$.

Consider the homomorphism $\widetilde{\Phi} : \widetilde{\mathcal{B}}_n^{\rm orb}(\mathbb{C},\mathbb{Z}_2) \to (\mathbb{Z}_2)^n \rtimes_\varphi \Sigma_n$ as a composition of ϕ_n and $\Phi : \mathcal{B}_n^{\rm orb}(\mathbb{C},\mathbb{Z}_2) \to (\mathbb{Z}_2)^n \rtimes_\varphi \Sigma_n$, where $(\mathbb{Z}_2)^n \rtimes_\varphi \Sigma_n$ is regarded as the symmetric group of permutations of $2n$ numbers $\pm 1, \cdots, \pm n$ preserving the action of $\mathbb{Z}_2$. Follow the idea of Chow in [11], let $\widetilde{\mathcal{B}}_n^{\rm o,orb}(\mathbb{C},\mathbb{Z}_2)$ be the subgroup consisting of those elements in $\widetilde{\mathcal{B}}_n^{\rm orb}(\mathbb{C},\mathbb{Z}_2)$ whose images as permutations under $\widetilde{\Phi}$ all leave the numbers $\pm n$ invariant. The index of $\widetilde{\mathcal{B}}_n^{\rm o,orb}(\mathbb{C},\mathbb{Z}_2)$ in $\widetilde{\mathcal{B}}_n^{\rm orb}(\mathbb{C},\mathbb{Z}_2)$ is obviously $2n$, and the representatives of $2n$ right cosets can be chosen as

$$M_0 = e, \quad M_i = \mathbf{a}_{n-1}\cdots\mathbf{a}_{i+1}\mathbf{a}_i, \quad i = 1, \cdots, n-1$$

and

$$M_{-1} = \mathbf{a}_{n-1}\cdots\mathbf{a}_1\mathbf{a}, \quad M_{-i} = \mathbf{a}_{n-1}\cdots\mathbf{a}_1\mathbf{a}\mathbf{a}_1^{-1}\cdots\mathbf{a}_{i-2}^{-1}\mathbf{a}_{i-1}^{-1}, \quad i = 2, \cdots, n.$$

The application of the well-known Reidemeister-Schreier method to $\widetilde{\mathcal{B}}_n^{\rm o,orb}(\mathbb{C},\mathbb{Z}_2)$ gives its $(3n-3)$ generators:

$$\mathbf{a}_1, \cdots, \mathbf{a}_{n-2}, \mathbf{a}, \tau_i = M_i\mathbf{a}_i^2 M_i^{-1}, \quad \tau_{-i} = M_{-i}\mathbf{a}_i^2 M_{-i}^{-1}, \quad i = 1, \cdots, n-1,$$

where $\tau_i\tau_{-i} = \tau_{-i}\tau_i$. Then we can define a homomorphism $\widetilde{p}_n : \widetilde{\mathcal{B}}_n^{\rm o,orb}(\mathbb{C},\mathbb{Z}_2) \to \widetilde{\mathcal{B}}_{n-1}^{\rm orb}(\mathbb{C},\mathbb{Z}_2)$ by assigning to each $\tau_{\pm i}$ the identity. Obviously, $\widetilde{G}_n = \ker \widetilde{p}_n$ is generated by $\tau_{\pm i}, i = 1, \cdots, n-1$.

Geometrically, it is easy to see that $\mathcal{B}_n^{\rm o,orb}(\mathbb{C},\mathbb{Z}_2) = \{\alpha \in \mathcal{B}_n^{\rm orb}(\mathbb{C},\mathbb{Z}_2) \mid \alpha_n(0) = \alpha_n(1)\}$ is exactly the image $\phi_n(\widetilde{\mathcal{B}}_n^{\rm o,orb}(\mathbb{C},\mathbb{Z}_2))$. Let G_n denote the kernel of the homomorphism p_n: $\mathcal{B}_n^{\rm o,orb}(\mathbb{C},\mathbb{Z}_2) \to \mathcal{B}_{n-1}^{\rm orb}(\mathbb{C},\mathbb{Z}_2)$ defined by removing the n-th string. Then we have the following commutative diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \widetilde{G}_n & \longrightarrow & \widetilde{\mathcal{B}}_n^{\rm o,orb}(\mathbb{C},\mathbb{Z}_2) & \xrightarrow{\widetilde{p}_n} & \widetilde{\mathcal{B}}_{n-1}^{\rm orb}(\mathbb{C},\mathbb{Z}_2) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \phi_n} & & \downarrow{\scriptstyle \phi_n} & & \downarrow{\scriptstyle \phi_{n-1}} & & \\
1 & \longrightarrow & G_n & \longrightarrow & \mathcal{B}_n^{\rm o,orb}(\mathbb{C},\mathbb{Z}_2) & \xrightarrow{p_n} & \mathcal{B}_{n-1}^{\rm orb}(\mathbb{C},\mathbb{Z}_2) & \longrightarrow & 1.
\end{array}$$

Now, to complete the proof, it needs to show that $\phi_n : \widetilde{\mathcal{B}}_n^{\rm orb}(\mathbb{C},\mathbb{Z}_2) \to \mathcal{B}_n^{\rm orb}(\mathbb{C},\mathbb{Z}_2)$ is an isomorphism. We perform an induction on n.

When $n = 1$, this is trivial.

When $n < k$, assume inductively that $\phi_n : \widetilde{\mathcal{B}}_n^{\rm orb}(\mathbb{C},\mathbb{Z}_2) \to \mathcal{B}_n^{\rm orb}(\mathbb{C},\mathbb{Z}_2)$ is an isomorphism.

When $n = k$, we see easily from the above commutative diagram that $\phi_k : \widetilde{G}_k \to G_k$ is an epimorphism, so G_k is generated by $\phi_k(\tau_i), \phi_k(\tau_{-i}), i = 1, \cdots, k-1$. Of course, $\phi_k(\tau_{-i})\phi_k(\tau_i) = \phi_k(\tau_i)\phi_k(\tau_{-i})$. Take a word $\omega \in \widetilde{\mathcal{B}}_k^{\rm orb}(\mathbb{C},\mathbb{Z}_2)$ such that $\phi_k(\omega) = e \in \mathcal{B}_k^{\rm orb}(\mathbb{C},\mathbb{Z}_2)$. Then $\omega \in \widetilde{\mathcal{B}}_k^{\rm o,\ orb}(\mathbb{C},\mathbb{Z}_2)$. Since ϕ_{k-1} is an isomorphism by induction assumption, $\widetilde{p}_k(\omega) = e$ so $\omega \in \widetilde{G}_k$ and $\phi_k(\omega) \in G_k$. Write $\omega = \tau_{j_1}^{\epsilon_1}\cdots\tau_{j_l}^{\epsilon_l}$ where for each $1 \le u \le l$, $\epsilon_u = 0$ or 1 and $j_u \in \{-k+1, \cdots, -1, 1, \cdots, k-1\}$. Then $\phi_k(\omega) = \phi_k(\tau_{j_1})^{\epsilon_1}\cdots\phi_k(\tau_{j_l})^{\epsilon_l} = e$ in G_k.

On the other hand, we see easily that G_k is also the kernel of the homomorphism p'_k: $\mathcal{P}_k(\mathbb{C},\mathbb{Z}_2) \to \mathcal{P}_{k-1}(\mathbb{C},\mathbb{Z}_2)$ by removing the k-th string. Thus $\phi_n(\omega) = e$ gives a relation in $\mathcal{P}_k(\mathbb{C},\mathbb{Z}_2) = \pi_1(F_{\mathbb{Z}_2}(\mathbb{C},k))$. Now we see that $F_{\mathbb{Z}_2}(\mathbb{C},k)$ is actually the complement space of $n(n+1)$ hyperplanes $H_{ij}^{\pm}, 1 \le i < j \le k$ in $\mathbb{C}^k$ as follows:

$$F_{\mathbb{Z}_2}(\mathbb{C},k) = \mathbb{C}^k \setminus \bigcup_{i<j} H_{ij}^{\pm},$$

where $H_{ij}^{\pm} = \{(z_1, \cdots, z_n) \in \mathbb{C}^k \mid z_i = \pm z_j\}$. Randell [20] gave a nice description of the fundamental group of the complement space of the complexication of real arrangements. Thus we can apply this description of Randell to read out $\pi_1(F_{\mathbb{Z}_2}(\mathbb{C}, n))$ whose generators exactly correspond to those hyperplanes $H_{ij}^{\pm}$ and whose all relations correspond to codim-2 intersections of those hyperplanes $H_{ij}^{\pm}$. It is easy to see that all codim-2 intersections are of the following three types:

$$\{z_i = \pm z_j; z_u = \pm z_v\}, \quad \{z_i = z_j = 0\}, \quad \{z_i = \pm z_j = \pm z_u\}.$$

Each codim-2 intersection of the first and second types is just the intersection of some two hyperplanes, so the corresponding relation must be of the form $ab = ba$, where a, b denote the conjugate classes of the generators corresponding to those two hyperplanes. Each codim-2 intersection of the third type is the intersection of some three hyperplanes, so the corresponding relation must be of the form $abc = bca = cab$, where a, b, c denote the conjugate classes of the generators corresponding to those three hyperplanes.

A careful check shows that for $1 \leq i \leq k-1$, $\phi_k(\tau_i)$ corresponds to hyperplane H_{ik}^{+}, and $\phi_k(\tau_{-i})$ corresponds to hyperplane H_{ik}^{-}, so $\phi_k(\omega)$ is the word formed by those generators corresponding to all $H_{ik}^{\pm}$. Thus, $\phi_k(\omega) = e$ means that the possible relations for generators $\phi_k(\tau_{\pm i})$ must be of $\phi_k(\tau_{-i})\phi_k(\tau_i) = \phi_k(\tau_i)\phi_k(\tau_{-i})$ which corresponds to codim-2 intersections $\{z_i = z_k = 0\}$. Therefore, $\phi_k(\omega) = e \Leftrightarrow \omega$ is a composition of those $\tau_{-i}\tau_i\tau_{-i}^{-1}\tau_i^{-1}$ and their conjugate classes. This implies that $\omega = e$ in $\widetilde{\mathcal{B}}_k^{\rm orb}(\mathbb{C}, \mathbb{Z}_2)$ so ϕ_k is an isomorphism.

For the general prime p, we first need to modify the path α in (3.2) or β in (3.3) into the general form

$$\alpha(s) = \Big((1+\mathbf{i})\Big(1 + \Big(\mathrm{e}^{\frac{2\pi\mathbf{i}}{p}} - 1\Big)s\Big), 2 + 2\mathbf{i}, \cdots, n + n\mathbf{i}\Big) \tag{3.4}$$

or

$$\beta(s) = \Big((1+\mathbf{i})\mathrm{e}^{\frac{2\pi\mathbf{i}s}{p}}, 2 + 2\mathbf{i}, \cdots, n + n\mathbf{i}\Big). \tag{3.5}$$

Then we can use the paths $\alpha^{(k)}$ in (3.1) and α in (3.4) (or β in (3.5)) to construct the required basic "bricks" $\mathbf{b}_k = [\widetilde{c(\alpha^{(k)})}]$ for $1 \leq k \leq n-1$ and $\mathbf{b} = [\widetilde{c(\alpha)}]$ or $[\widetilde{c(\beta)}]$, each of which would consist of p symmetric parts with respect to the line O.

It is not difficult to see that each class in $\mathcal{B}_n^{\rm orb}(\mathbb{C}, \mathbb{Z}_p)$ is also a composition of $\mathbf{b}_k$ $(1 \leq k \leq n-1)$ and $\mathbf{b}$.

To get the presentation of $\mathcal{B}_n^{\rm orb}(\mathbb{C}, \mathbb{Z}_p)$, an easy observation shows that we merely need to change the relations R_1 in Proposition 3.1 into $\mathbf{b}^p = e$. Any tangle of the first orbit string surround the line O can be untangled since the origin of $\mathbb{C}$ is a fixed point of the action, this illustrates $\mathbf{b}^p = e$. As for relation R_2 in Proposition 3.1, both sides represent the orbit braid of path $\gamma(s) = ((1+\mathbf{i})(1 + (\mathrm{e}^{\frac{2\pi\mathbf{i}}{p}} - 1)s), (2+2\mathbf{i})(1 + (\mathrm{e}^{\frac{2\pi\mathbf{i}}{p}} - 1)s), \cdots, n + n\mathbf{i})$. Thus we have the following proposition.

Proposition 3.2 *$\mathcal{B}_n^{\rm orb}(\mathbb{C}, \mathbb{Z}_p)$ has a presentation, generated by $\mathbf{b}_k$ $(1 \leq k \leq n-1)$ and $\mathbf{b}$, with relations*

(1) $\mathbf{b}^p = e$,

(2) $(\mathbf{b}\mathbf{b}_1)^2 = (\mathbf{b}_1\mathbf{b})^2$,
(3) $\mathbf{b}_k\mathbf{b} = \mathbf{b}\mathbf{b}_k\,(k > 1)$,
(4) $\mathbf{b}_k\mathbf{b}_{k+1}\mathbf{b}_k = \mathbf{b}_{k+1}\mathbf{b}_k\mathbf{b}_{k+1}$,
(5) $\mathbf{b}_k\mathbf{b}_l = \mathbf{b}_l\mathbf{b}_k\,(|k-l| > 1)$.

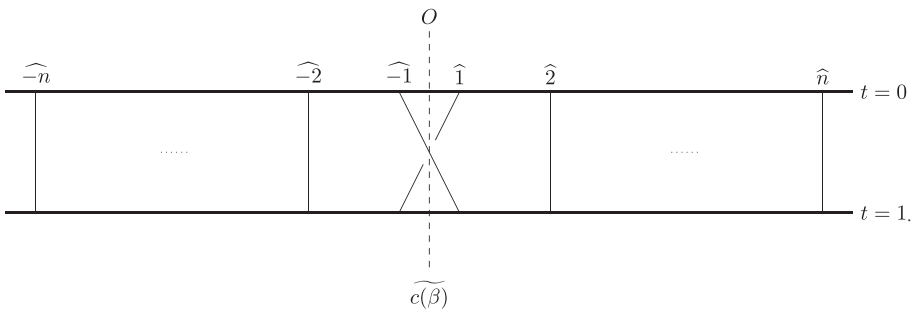
The completeness of relations in $\mathcal{B}_n^{\mathrm{orb}}(\mathbb{C}, \mathbb{Z}_p)$ can be proved in the similar way as in Proposition 3.1. We just point out main differences. First, $(\mathbb{Z}_p)^n \rtimes_\varphi \Sigma_n$ can be regarded as the symmetric group formed by those permutations of pn symmetric numbers $\mathrm{e}^{\frac{2k\pi\mathbf{i}}{p}}1, \cdots, \mathrm{e}^{\frac{2k\pi\mathbf{i}}{p}}n, k = 0, \cdots, p-1$. So the corresponding $\widetilde{G}_n$ is generated by $p(n-1)$ generators $\tau_1^k, \cdots, \tau_{n-1}^k, k = 0, \cdots, p-1$; second, $F_{\mathbb{Z}_p}(\mathbb{C}, n)$ can still be regarded as the complement space of complex hyperplane arrangement in $\mathbb{C}^n$ as follows:

$$F_{\mathbb{Z}_p}(\mathbb{C}, n) = \mathbb{C}^n \setminus \bigcup_{i<j} H_{ij}^k,$$

where $H_{ij}^k = \{(z_1, \cdots, z_n) \in \mathbb{C}^n \mid z_i = \mathrm{e}^{\frac{2k\pi\mathbf{i}}{p}} z_j\}$. In this case, each codim-2 intersection $\{z_i = z_n = 0\}$ will be the intersection of p hyperplanes H_{in}^k, and the corresponding relation will be of the form $a_1 \cdots a_p = a_2 \cdots a_p a_1 = a_p a_1 \cdots a_{p-1}$, where $a_1, \cdots, a_p$ denote the conjugate classes of the generators corresponding to those p hyperplanes H_{in}^k.

3.2 Orbit braid group $\mathcal{B}_n^{\mathrm{orb}}(\mathbb{C}^\times, \mathbb{Z}_p)$

In the similar way to the case of $\mathcal{B}_n^{\mathrm{orb}}(\mathbb{C}, \mathbb{Z}_p)$, we can describe $\mathcal{B}_n^{\mathrm{orb}}(\mathbb{C}^\times, \mathbb{Z}_p)$. In this case, a family of basic "bricks" named after $\mathbf{b}_k$ and $\mathbf{b}'$ can also be constructed by the paths $\alpha^{(k)}$ in (3.1) and β in (3.5). Here we make sure that p ordinary strings of the first orbit string in $\widetilde{c(\beta)}$ do not intersect because $\mathbb{Z}_p$ acts freely on $\mathbb{C}^\times$, as shown in the following picture for the case of $p = 2$:



Since we are working on the case of $\mathbb{C}^\times$ with a free $\mathbb{Z}_p$-action, this means that we cannot untangle any tangle surround the line O of the first orbit string of $\mathbf{b}'$, so $\mathbf{b}'$ in $\mathcal{B}_n^{\mathrm{orb}}(\mathbb{C}^\times, \mathbb{Z}_p)$ is an element of infinite order. Actually, the only difference between $\mathcal{B}_n^{\mathrm{orb}}(\mathbb{C}^\times, \mathbb{Z}_p)$ and $\mathcal{B}_n^{\mathrm{orb}}(\mathbb{C}, \mathbb{Z}_p)$ is the order of $\mathbf{b}'$. Indeed, on $\widetilde{\mathbf{b}_k}$, we see that there is not any direct twine among p symmetric parts with respect to O in $\widetilde{c(\alpha^{(k)})}$, and only thing that happens is that two strings within each symmetric part do an exchange of ending points. So $\mathbf{b}_k$ in $\mathcal{B}_n^{\mathrm{orb}}(\mathbb{C}, \mathbb{Z}_p)$ has the same property as in $\mathcal{B}_n^{\mathrm{orb}}(\mathbb{C}^\times, \mathbb{Z}_p)$. Thus we have that

Proposition 3.3 $\mathcal{B}_n^{\rm orb}(\mathbb{C}^\times, \mathbb{Z}_p)$ *has a presentation, generated by* $\mathbf{b_k}\,(1 \le k \le n-1)$ *and* $\mathbf{b}'$, *with relations*:

(1) $(\mathbf{b}'\mathbf{b}_1)^2 = (\mathbf{b}_1\mathbf{b}')^2$,

(2) $\mathbf{b}_k\mathbf{b}' = \mathbf{b}'\mathbf{b}_k\ (k > 1)$,

(3) $\mathbf{b}_k\mathbf{b}_{k+1}\mathbf{b}_k = \mathbf{b}_{k+1}\mathbf{b}_k\mathbf{b}_{k+1}$,

(4) $\mathbf{b}_k\mathbf{b}_l = \mathbf{b}_l\mathbf{b}_k\ (|k-l| > 1)$.

Proof Since $\mathbb{Z}_p$ acts on $\mathbb{C}^\times$ freely, the completeness of relations in $\mathcal{B}_n^{\rm orb}(\mathbb{C}^\times, \mathbb{Z}_p)$ can be verified in several ways. Here we give the simplest one. Actually, by Corollary 2.3, $\mathcal{B}_n^{\rm orb}(\mathbb{C}^\times, \mathbb{Z}_p) \cong B_n(\mathbb{C}^\times) \cong \mathcal{B}_n^{\rm orb}(\mathbb{C}^\times, \mathbb{Z}_2)$. As we will see in Section 3.4, $\mathcal{B}_n^{\rm orb}(\mathbb{C}^\times, \mathbb{Z}_2) \cong Br(B_n)$, where $Br(B_n)$ denotes the generalized braid group corresponding to the finite Coxeter group B_n (see [23]), and in particular, $Br(B_n)$ consists of the same generators with four type of relations exactly listed as in Proposition 3.3.

3.3 Orbit braid group $\mathcal{B}_n^{\rm orb}(\mathbb{C}, (\mathbb{Z}_2)^2)$

Identify $\mathbb{C}$ with $\mathbb{R}^2$, the action $(\mathbb{Z}_2)^2 \curvearrowright^{\phi_2} (\mathbb{Z}_2)^2$ is just the standard $(\mathbb{Z}_2)^2$-representation, where $(\mathbb{Z}_2)^2$ is generated by two reflections g^x and g^y with respect to x-axis and y-axis. For a path $\alpha = (\alpha_1, \cdots, \alpha_n)$ in $F_{(\mathbb{Z}_2)^2}(\mathbb{C}, n)$ with $\alpha(0) = \mathbf{z}$ and $\alpha(1) = g\mathbf{z}_\sigma$, the correspoding orbit braid $\widetilde{c(\alpha)} = \coprod_{i=1}^{n} \widetilde{c(\alpha_i)}$ is symmetric with respect to the line O, real axis$\times I$ and imaginary axis$\times I$ in $\mathbb{C} \times I$, where

$$\widetilde{c(\alpha_i)} = \{(\alpha_i(s), s), (-\alpha_i(s), s), (\overline{\alpha_i(s)}, s), (-\overline{\alpha_i(s)}, s) \mid s \in I\}$$

and $\overline{\alpha_i(s)}$ means the conjugacy of $\alpha_i(s)$.

Based upon the symmetries of the orbit braids in $\mathcal{B}_n^{\rm orb}(\mathbb{C}, (\mathbb{Z}_2)^2)$, we construct a family of basic "bricks" named after $\mathbf{b}_k$, $\mathbf{b}^x$ and $\mathbf{b}^y$ as follows:

(1) $\mathbf{b}_k$ is chosen as $[\widetilde{c(\alpha^{(k)})}]$ where $\alpha^{(k)}$ is the path in (3.1),

(2) $\mathbf{b}^x$ is chosen as $[\widetilde{c(\alpha^x)}]$ where α^x is the path given by

$$\alpha^x(s) = (1 + (1-2s)\mathbf{i}, 2 + 2\mathbf{i}, \cdots, n + n\mathbf{i})$$

such that α_1^x and $\overline{\alpha_1^x}$ intersect at real axis$\times I$,

(3) $\mathbf{b}^y$ is chosen as $[\widetilde{c(\alpha^y)}]$ where α^y is the path given by

$$\alpha^y(s) = ((1-2s) + \mathbf{i}, 2 + 2\mathbf{i}, \cdots, n + n\mathbf{i})$$

such that α_1^y and $-\overline{\alpha_1^y}$ intersect at imaginary axis$\times I$.

We can read six classes of relations among these basic "bricks" in the same way as previous two examples.

Lemma 3.2 $\mathcal{B}_n^{\rm orb}(\mathbb{C}, (\mathbb{Z}_2)^2)$ *is generated by* $\mathbf{b}_k\,(1 \le k \le n-1)$, $\mathbf{b}^x$ *and* $\mathbf{b}^y$, *satisfying the relations*

(1) $(\mathbf{b}^x)^2 = (\mathbf{b}^y)^2 = e$,

(2) $\mathbf{b}^x\mathbf{b}^y = \mathbf{b}^y\mathbf{b}^x$,

(3) $\mathbf{b}^x\mathbf{b}_1\mathbf{b}^x\mathbf{b}_1^{-1} = \mathbf{b}_1\mathbf{b}^x\mathbf{b}_1^{-1}\mathbf{b}^x$, $\mathbf{b}^y\mathbf{b}_1\mathbf{b}^y\mathbf{b}_1^{-1} = \mathbf{b}_1\mathbf{b}^y\mathbf{b}_1^{-1}\mathbf{b}^y$,

(4) $\mathbf{b}_k\mathbf{b}^x = \mathbf{b}^x\mathbf{b}_k$, $\mathbf{b}_k\mathbf{b}^y = \mathbf{b}^y\mathbf{b}_k\ (k > 1)$,

(5) $\mathbf{b}_k\mathbf{b}_{k+1}\mathbf{b}_k = \mathbf{b}_{k+1}\mathbf{b}_k\mathbf{b}_{k+1}$,
(6) $\mathbf{b}_k\mathbf{b}_l = \mathbf{b}_l\mathbf{b}_k$ $(|k-l|>1)$.

We shall emphasis here that we failed to find a way to prove the completeness of six relations in $\mathcal{B}_n^{\mathrm{orb}}(\mathbb{C},\mathbb{Z}_2^2)$ although we believe it is correct. The obstacles lie in two points: First the action of $(\mathbb{Z}_2)^2$ on $\mathbb{C}$ is non free, so that we cannot use known techniques about ordinary braid groups; second $F_{(\mathbb{Z}_2)^2}(\mathbb{C},n)$ is a real arrangement. As far as we know, it seems that there is not an effective approach to deal with the fundamental group of the complement space of real arrangements yet.

3.4 Compare with generalized braid groups (Artin groups)

Recall homomorphism $\Phi : \mathcal{B}_n^{\mathrm{orb}}(M,G) \to G^{\times n} \rtimes_\varphi \Sigma_n$, here we consider two kinds of group actions—$\mathbb{Z}_2 \curvearrowright^{\phi_1} \mathbb{C}$ and $\mathbb{Z}_2 \curvearrowright^{\phi_1} \mathbb{C}^\times$. An easy argument shows that $\mathbb{Z}_2^n \rtimes_\varphi \Sigma_n$ is exactly isomorphic to finite Coxeter group B_n. Then we have the following short exact sequences from Theorem 2.2:

$$1 \to \mathcal{P}_n(\mathbb{C},\mathbb{Z}_2) \to \mathcal{B}_n^{\mathrm{orb}}(\mathbb{C},\mathbb{Z}_2) \to B_n \to 1 \tag{3.6}$$

and

$$1 \to \mathcal{P}_n(\mathbb{C}^\times,\mathbb{Z}_2) \to \mathcal{B}_n^{\mathrm{orb}}(\mathbb{C}^\times,\mathbb{Z}_2) \to B_n \to 1. \tag{3.7}$$

Here we pay our attention to the relations between the orbit braid groups $\mathcal{B}_n^{\mathrm{orb}}(\mathbb{C},\mathbb{Z}_2)$, $\mathcal{B}_n^{\mathrm{orb}}(\mathbb{C}^\times,\mathbb{Z}_2)$ and the generalised braid groups $Br(D_n)$, $Br(B_n)$ (for the concept of generalized braid group or Artin group, see Appendix A)

It was known in [17] that two orbit configuration spaces $F_{\mathbb{Z}_2}(\mathbb{C},n)$ and $F_{\mathbb{Z}_2}(\mathbb{C}^\times,n)$ are classifying space of two generalized pure braid groups $P(D_n)$ and $P(B_n)$. Moreover the actions of D_n and B_n on $F_{\mathbb{Z}_2}(\mathbb{C},n)$ and $F_{\mathbb{Z}_2}(\mathbb{C}^\times,n)$ respectively are free. From this viewpoint, we have the following two short exact sequences:

$$1 \to P(D_n) \to Br(D_n) \to D_n \to 1 \tag{3.8}$$

and

$$1 \to P(B_n) \to Br(B_n) \to B_n \to 1. \tag{3.9}$$

In addition, we also have that

$$\mathcal{P}_n(\mathbb{C},\mathbb{Z}_2) \cong P(D_n) \cong \pi_1(F_{\mathbb{Z}_2}(\mathbb{C},n),\mathbf{z})$$

and

$$\mathcal{P}_n(\mathbb{C}^\times,\mathbb{Z}_2) \cong P(B_n) \cong \pi_1(F_{\mathbb{Z}_2}(\mathbb{C}^\times,n),\mathbf{z}).$$

First let us look at the case of $F_{\mathbb{Z}_2}(\mathbb{C}^\times,n)$. It can be seen from Corollary 2.3 that two short exact sequences (3.7) and (3.9) are essentially the same—both come from fibration from $F_{\mathbb{Z}_2}(\mathbb{C}^\times,n)$ to $F_{\mathbb{Z}_2}(\mathbb{C}^\times,n)/B_n$. Thus we have that

Proposition 3.4 *$\mathcal{B}_n^{\mathrm{orb}}(\mathbb{C}^\times,\mathbb{Z}_2)$ is isomorphic to the generalized braid group $Br(B_n)$.*

As for the case of $F_{\mathbb{Z}_2}(\mathbb{C}, n)$, compare two short exact sequences (3.6) and (3.8), we see that $\mathcal{P}_n(\mathbb{C}, \mathbb{Z}_2) \cong P(D_n)$ but $B_n \ncong D_n$, so $\mathcal{B}_n^{\rm orb}(\mathbb{C}, \mathbb{Z}_2)$ and $Br(D_n)$ are not isomorphic. Next let us analyze the connection between $\mathcal{B}_n^{\rm orb}(\mathbb{C}, \mathbb{Z}_2)$ and $Br(D_n)$.

It is well-known that the finite Coxeter group D_n is generated by $s, \sigma_1, \cdots, \sigma_{n-1}$ with relations:

(1) $s^2 = \sigma_i^2 = e$,
(2) $(\sigma_i\sigma_{i+1})^3 = e$,
(3) $(s\sigma_2)^3 = e$,
(4) $(s\sigma_i)^2 = e\,(i \neq 2)$,
(5) $(\sigma_i\sigma_j)^2 = e\,(|i-j| > 1)$.

And the corresponding generalized braid group $Br(D_n)$ is generated by $s, \sigma_1, \cdots, \sigma_{n-1}$ with relations:

(1) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$,
(2) $s\sigma_2 s = \sigma_2 s\sigma_2$,
(3) $s\sigma_i = \sigma_i s\,(i \neq 2)$,
(4) $\sigma_i\sigma_j = \sigma_j\sigma_i\,(|i-j| > 1)$.

Remark 3.2 We can observe how D_n acts freely on $F_{\mathbb{Z}_2}(\mathbb{C}, n)$ in terms of σ_i and s. For each $(z_1, \cdots, z_n) \in F_{\mathbb{Z}_2}(\mathbb{C}, n)$,

$$\sigma_i(z_1, \cdots, z_{i-1}, z_i, z_{i+1}, z_{i+2}, \cdots, z_n) = (z_1, \cdots, z_{i-1}, z_{i+1}, z_i, z_{i+2}, \cdots, z_n)$$

(i.e., σ_i only permutes i-th and $(i+1)$-th coordinates of z) and

$$s(z_1, z_2, z_3, \cdots, z_n) = (-z_2, -z_1, z_3, \cdots, z_n)$$

(i.e., s just transfers z_1 to $-z_2$ and z_2 to $-z_1$), we can verify easily that these transformations exactly satisfy the relations (1)–(5) in D_n.

Now by Remark 3.2, the generator s in $Br(D_n)$ can be regarded as the class $\mathbf{bb}_1\mathbf{b}$ in $\mathcal{B}_n^{\rm orb}(\mathbb{C}, \mathbb{Z}_2)$, and each σ_i can be regarded as the class $\mathbf{b}_i$ in $\mathcal{B}_n^{\rm orb}(\mathbb{C}, \mathbb{Z}_2)$. Thus, we can define a homomorphism

$$f : Br(D_n) \to \mathcal{B}_n^{\rm orb}(\mathbb{C}, \mathbb{Z}_2)$$

by $f(s) = \mathbf{bb}_1\mathbf{b}$ and $f(\sigma_i) = \mathbf{b}_i$. A direct check shows that f is a monomorphism.

Proposition 3.5 *$Br(D_n)$ is isomorphic to a subgroup of $\mathcal{B}_n^{\rm orb}(\mathbb{C}, \mathbb{Z}_2)$.*

4 Extended Fundamental Groups of Topological Spaces

As noted before, the extended fundamental group used as the homotopy description of orbit braid group is actually the fundamental group of a transformation group in the sense of Rhodes in [21]. In this section, we give the notion of extended fundamental groups and state some properties in a general way in the category of topology, which are essentially due to Rhodes in [21].

Let X be a path-connected topological space and let $\mathrm{Home}(X)$ be the group given by all homeomorphisms from X to itself. Here we write $\mathrm{Home}(X)$ as $\mathbb{G}(X)$ for a convenience. When

X is a connected smooth manifold, we may consider the group given by all diffeomorphisms from X to itself.

Let H be a subgroup of $\mathbb{G}(X)$. Fix a point x_0 in X as base point. Let $\Omega(X, x_0, H)$ consist of all pairs (α, h_α) with $h_\alpha \in H$, where $\alpha : I \to X$ is a path in X such that $\alpha(0) = x_0$ and $\alpha(1) = h_\alpha x_0$ are in the orbit $H(x_0)$ at x_0.

There is a natural operation $\circ : \Omega(X, x_0, H) \times \Omega(X, x_0, H) \to \Omega(X, x_0, H)$ defined by mapping $((\alpha, h_\alpha), (\beta, h_\beta))$ to $(\alpha \circ (h_\alpha\beta), h_\alpha h_\beta)$, where

$$\alpha \circ (h_\alpha\beta)(s) = \begin{cases} \alpha(2s), & \text{if } s \in \left[0, \frac{1}{2}\right], \\ h_\alpha\beta(2s-1), & \text{if } s \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

In addition, there is also an equivalence relation $\sim$ on $\Omega(X, x_0, H)$ as follows: For two pairs (α, h_α) and (β, h_β),

$$(\alpha, h_\alpha) \sim (\beta, h_\beta) \Leftrightarrow \begin{cases} h_\alpha = h_\beta, \\ \alpha \simeq \beta \text{ rel } \partial I. \end{cases}$$

Furthermore, there is an induced operation $\bullet$ on the quotient set $\Omega(X, x_0, H))/\sim$ given by

$$[\alpha, h_\alpha] \bullet [\beta, h_\beta] = [\alpha \circ (h_\alpha\beta), h_\alpha h_\beta].$$

Rhodes showed in [21] that $\Omega(X, x_0, H(x_0))/\sim$ forms a group under the operation $\bullet$. Here we call this group the extended fundamental group or the equivariant fundamental group, denoted by $\pi_1^E(X, x_0, H)$ or $\pi_1^H(X, x_0)$.

Remark 4.1 Generally, two classes $[\alpha, h_\alpha]$ and $[\alpha, h'_\alpha]$ with $h_\alpha \neq h'_\alpha$ are distinct even if $h_\alpha x_0 = h'_\alpha x_0$. Indeed, choose a point x in the orbit $H(x_0)$, we know from [7] that there exists a coset $hH_{x_0} \in H/H_{x_0}$ such that for any $h' \in hH_{x_0}$, $x = h'x_0$, where H_{x_0} is the isotropy subgroup at x_0. If x_0 is of free orbit type, then $H_{x_0} = \{e\}$, so in this case, for two $h \neq h'$ in H, $hx_0 \neq h'x_0$ and $\pi_1^E(X, x_0, H)$ may also be written as $\pi_1^E(X, x_0, H(x_0))$, as used in Section 2.

When H is exactly the trivial group $\{e\}$, $\pi_1^E(X, x_0, H)$ degenerates into the ordinary fundamental group $\pi_1(X, x_0)$.

The following results are due to Rhodes [21].

Theorem 4.1 *Let X be a path-connected topological space and let H be a subgroup of $\mathbb{G}(X)$. Then*

(1) (see [21, Theorems 1–2]) *up to isomorphism, $\pi_1^H(X, x_0)$ does not depend upon the choice of the base point x_0,*

(2) (see [21, Theorem 3]) *$\pi_1^H(X, x_0)$ is homotopy invariant of X with H-action,*

(3) (see [21, §8, p644]) *there is the following short exact sequence*

$$1 \to \pi_1(X, x_0) \to \pi_1^H(X, x_0) \to H \to 1.$$

The map $\Delta : \pi_1^H(X, x_0) \to H/H_{x_0}$ defined by $[\alpha, h_\alpha] \mapsto h_\alpha H_{x_0}$ is surjective, and the preimage of H_{x_0} is exactly the group $\pi_1^{H_{x_0}}(X, x_0)$. Thus we have that

Corollary 4.1 *There is the following short exact sequence*

$$1 \to \pi_1^{H_{x_0}}(X, x_0) \to \pi_1^H(X, x_0) \to H/H_{x_0} \to 1.$$

Finally, based upon the arguments in Section 2, we end this section with the following properties:

(A) If H is finite, then the projection $p : X \to X/H$ induces an epimorphism

$$p_* : \pi_1^H(X, x_0) \to \pi_1(X/H, p(x_0)).$$

(B) If H is finite and the action of H on X is free, then the projection $p : X \to X/H$ induces an isomorphism

$$\pi_1^H(X, x_0) \cong \pi_1(X/H, p(x_0)).$$

(C) If x_0 is of free orbit type under the action of $\mathbb{G}(X)$, then X gives a direct system

$$\{\pi_1^H(X, x_0) \mid H \leq \mathbb{G}(X)\}$$

such that the limit of this direct system is exactly $\pi_1^{\mathbb{G}(X)}(X, x_0)$.

A Generalized Braid Group

Generalized braid groups, with respect to all finite Coxeter groups, were introduced by Brieskorn [9] in the 1970's. They are also Artin groups.

Following the terminology and notation of the paper by Vershinin [23], let V be an n-dimensional real vector space and let W be a finite subgroup of $GL(V)$ generated by reflections. Let $\mathcal{M}$ be the set of hyperplanes such that W is generated by the orthogonal reflections in the $M \in \mathcal{M}$. For any $w \in W$ and any $M \in \mathcal{M}$ we assume that $w(M)$ belongs to $\mathcal{M}$. Consider the complexification $V_{\mathbb{C}}$ of the space V and the complexification $M_{\mathbb{C}}$ of $M \in \mathcal{M}$. Set

$$Y_W = V_{\mathbb{C}} - \bigcup_{M \in \mathcal{M}} M_{\mathbb{C}}.$$

Then W acts freely on Y_W, and the orbit space of this action is denoted by $X_W = Y_W/W$. Then the fundamental group $\pi_1(X_W)$ is called the braid group of action of W on V, denoted by $Br(V, W)$. The fundamental group $\pi_1(Y_W)$ is called the pure braid group of action of W on V, denoted by $P(V, W)$.

For a finite Coxeter group

$$W = \langle w_1, \cdots, w_k \mid w_i^2 = e, (w_i w_j)^{m_{i,j}} = e, m_{i,j} = m_{j,i} \rangle,$$

the generalized braid group $Br(W)$ of W is defined as the group with generators w_i and relations

$$\mathrm{prod}(m_{i,j}; w_i, w_j) = \mathrm{prod}(m_{j,i}; w_j, w_i),$$

where the symbol $\mathrm{prod}(m; x, y)$ stands for the product $xyxy\cdots$ with m factors. By adding the relation $w_i^2 = \mathrm{e}$ to the above presentation we obtain a presentation of W. The following theorem is due to Brieskorn [8] and Deligne [13].

Theorem A.1 (see [8, 13])

(1) *The fundamental group $\pi_1(X_W)$ is isomorphic to the generalized braid group $Br(W)$.*

(2) *The universal covering of X_W is contractible, and hence X_W is a space of $K(\pi; 1)$.*

This theorem means that X_W is the classifying space of the generalized braid group $Br(W)$. In addition, it is easy to see that Y_W is also a space of $K(\pi; 1)$, so Y_W is the classifying space of the generalized pure braid group $P(W)$ of W.

References

[1] Allcock, D. and Basak, T., Geometric generators for braid-like groups, *Geom. Topol.*, **20**, 2016, 747–778.

[2] Allcock, D. and Basak, T., Generators for a complex hyperbolic braid group, *Geom. Topol.*, **22**(6), 2018, 3435–3500.

[3] Artin, E., Theorie der Zöpfe, *Ahb. Math. Sem. Univ. Hamburg*, **4**, 1925, 47–72.

[4] Artin, E., Theory of braids, *Ann. of Math.*, **48**, 1947, 101–126.

[5] Bibby, C. and Gadish, N., Combinatorics of orbit configuration spaces, 2018, arXiv:1804.06863.

[6] Birman, J. S., Braids, Links, and Mapping Class Groups, Princeton Univ. Press, Princeton, NJ, 1974.

[7] Bredon, G. E., Intrduction to compact transformation groups, *Pure and Applied Mathematics*, **46**, Academic Press, New York, London, 1972.

[8] Brieskorn, E., Die Fundamentalgruppe des Raumes der regulären Orbits einer endlichen komplexen Spiegelungsgruppe, *Invent. Math.*, **12**, 1971, 57–61.

[9] Brieskorn, E., Sur les groupes de tresses (d'après V.I. Arnol'd), Séminaire Bourbaki, 24ème année (1971/1972), *Lecture Notes in Math.*, **317**, 1973, 21–44.

[10] Chen, J. D., Lü, Z. and Wu, J., Orbit configuration spaces of small covers and quasi-toric manifolds, *Science China Mathematics.*, **64**(1), 2021, 167–196.

[11] Chow, W. L., On the algebraical braid group, *Ann. of Math.*, **49**, 1948, 654–658.

[12] Davis, M. and Januszkiewicz, T., Convex polytopes, Coxeter orbifolds and torus actions, *Duke Math. J.*, **61**, 1991, 417–451.

[13] Deligne, P., Les immeubles des groupes de tresses généralisés, *Invent. Math.*, **17**, 1972, 273–302.

[14] Fadell, E. and Neuwirth, L., Configuration spaces, *Math. Scand.*, **10**, 1962, 111–118.

[15] Fox, R. and Neuwirth, L., The braid groups, *Math. Scand.*, **10**, 1962, 119–126.

[16] Fricke, R. and Klein, F., Vorlesungen über die Theorie der automorphen Funktionen, Bd. I. Gruppentheoretischen Grundlagen, Teubner, Leipzig, 1897, Johnson, New York, 1965.

[17] Goryunov, V. V., The cohomology of braid groups of series C and D, *Trudy Moskov. Mat. Obshch.*, **42**, 1981, 234–242.

[18] Hurwitz, A., Über Riemannsche Flächen mit gegebenen Verzweigungspunkten, *Math. Ann.*, **39**(1), 1891, 1–60.

[19] Looijenga, E., Artin groups and the fundamental groups of some moduli spaces, *J. Topol.*, **1**(1), 2008, 187–216.

[20] Randell, R., The fundamental group of the complement of a union of complex hyperplanes: Correction, *Invent. math.*, **80**, 1985, 467–468.

[21] Rhodes, F., On the fundamental group of a transformation group, *Proceedings of the London Mathematical Society*, **3**(1), 1966, 635–650.

[22] Switzer, R. M., Algebraic Topology–Homotopy and Homology, Springer-Verlag, New York, Heidelberg, 1975.

[23] Vershinin, V. V., Braid groups and loop spaces, *Russian Mathematical Surveys*, **54**, 1999, 273–350.

[24] Xicoténcatl, M. A., m Orbit configuration spaces, infinitesimal braid relations in homology and equivariant loop spaces, Thesis (Ph.D.), University of Rochester, Rochester, 1997.