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3-Bihom-ρ-Lie Algebras, 3-Pre-Bihom-ρ-Lie Algebras

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Abstract The purpose is to introduce the notions of 3-Bihom-ρ-Lie algebras and 3-pre-Bihom-ρ-Lie algebras. The authors describe their constructions and express the related lemmas and theorems. Also, they define the 3-Bihom-ρ-Leibniz algebras and show that a 3-Bihom-ρ-Lie algebra is a 3-Bihom-ρ-Leibniz algebra with the ρ-Bihom-skew symmetry property. Moreover, a combination of a 3-Bihom-ρ-Lie algebra bracket and a Rota-Baxer operator gives a 3-pre-Bihom-ρ-Lie algebra structure.

Keywords 3-Bihom-ρ-Lie algebra, 3-Pre-Bihom-ρ-Lie algebra, Rota-Baxer operator
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1 Introduction

The origin of Hom-structures can be found in the physics literature around 1900, appearing in the study of quasi deformations of Lie algebras of vector fields. Hom-Lie algebras first appeared in [10, 19]. This class of Lie algebras was introduced by Hartwig, Larsson and Silvestrov. In 2012 a new notion was introduced by Yuan [23], as an extension of Hom-Lie superalgebras to G-graded algebras, in the name of Hom-Lie color algebras (see [2] for more details). As an extension of Hom-Lie algebras, Graziani, Makhlouf, Menini and Panaite introduced the notion of Bihom-Lie algebras in [8]. Bihom-Lie algebras are Lie algebras with a bracket and two homomorphisms α, β. In special case if $\alpha = \beta$, we have a Hom-Lie algebra and if $\alpha = \beta = Id$, a Bihom-Lie algebra will be returned to a Lie algebra. This kind of Lie algebras has got more attention and the notions like Bihom-Lie superalgebras, Bihom-Lie colour algebras, Bihom-associative algebras appeared in [1, 12, 14, 22].

In 1985 the concept of n-Lie algebras was introduced by Filippov [6]. The other names of this Lie algebras are Filippov algebra, Nambu-Lie algebra and Lie n-algebras. If $n = 3$, we have the special case of this Lie algebra which has close relationships with many important fields in mathematics and mathematical physics (see [3, 9, 17, 21, 24]). For example, the metric 3-Lie algebras are used to describe a world volume of multiple M2-branes (see [3, 9]). Recently, 3-Lie algebras have been generalized to concepts such as 3-Hom-Lie algebras, 3-Lie color algebras, 3-Hom-Lie color algebras and 3-Bihom-Lie algebras (see [3, 5, 11, 13, 15, 19–20, 24]).

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Pre-Lie algebras are the other kind of Lie algebras which were given by Gerstenhaber in [7]. We can find this Lie algebra with various literature such as left-symmetric algebra and right-symmetric algebra. This idea was a good case to be generalized on the other algebras, and new notions like Hom-pre-Lie algebras, 3-pre-Lie algebra, 3-pre-(Hom)-ρ-Lie algebras and Bihom-pre-Lie algebra were developed [4, 16, 18, 20].

In this paper, we intend to introduce the notions of 3-Bihom-ρ-Lie algebras and 3-pre-Bihom-ρ-Lie algebras as extensions of 3-Bihom-Lie algebras.

This paper is arranged as follows: Section 2 contains a brief summary. In Section 3, we describe the class of Bihom-ρ-Lie algebras which is called 3-Bihom-ρ-Lie algebras. This Bihom-Lie algebra includes a multiplication $[\cdot,\cdot,\cdot]_{\mathcal{B}}$, a two-cycle and two even linear maps α,β, the combination of these structures give the spacial conditions. This section also contains the notion of 3-Bihom-ρ-Leibniz algebras and establishes the relation between 3-Bihom-ρ-Lie algebras and 3-Bihom-ρ-Leibniz algebras. In this section, the combination of a 3-Bihom-ρ-Lie algebra structure and a Rota-Baxer operator of weight λ gives us a new 3-Bihom-ρ-Lie algebra structure, and by combining a 3-Bihom-ρ-Leibniz algebra structure and a Rota-Baxer operator of weight λ, we create a new 3-Bihom-ρ-Leibniz algebra structure. Section 4 contains the construction of 3-pre-Bihom-ρ-Lie algebras. At first we introduce the pre-Bihom-Lie algebras and the reader will get some results. At the end, by combining the 3-pre-ρ-Lie algebra structure $\{\cdot,\cdot,\cdot\}$, two even linear maps α,β, a 3-Bihom-ρ-Lie algebra structure $[\cdot,\cdot,\cdot]_\beta$ and a Rota-Baxer operator P of weight 0, we create two new 3-pre-Bihom-ρ-Lie algebras.

2 3-Bihom-ρ-Lie Algebras

Let us consider the following assumptions:

(i) $(G,+)$ as an Abelian group over a field $\mathbb{K}$ ($\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$),

(ii) a map $\rho : G\times G \to \mathbb{K}^\star$.

If ρ satisfies the following conditions

$$\rho(a,b)=\rho(b,a)^{-1}, \quad a,b\in G, \tag{2.1}$$

$$\rho(a+b,c)=\rho(a,c)\rho(b,c), \quad a,b,c\in G, \tag{2.2}$$

then ρ is called a two-cycle. The above two conditions give us

$$\rho(a,b)\neq 0, \quad \rho(0,b)=1,$$
$$\rho(c,c)=\pm 1, \quad \forall a,b,c\in G,\ c\neq 0.$$

Note that if $\mathcal{B}$ is a G-graded vector space, an element $f\in\mathcal{B}_a$ is called homogeneous of G-degree a. We denote by $Hg(\mathcal{B})$ the set of homogeneous elements in $\mathcal{B}$ and by $|f|$ the G-degree of $f\in Hg(\mathcal{B})$. In the next, for simplicity we use $\rho(f,g)$ instead of $\rho(|f|,|g|)$.

Definition 2.1 *A Bihom-ρ-Lie algebra is a G-graded vector space $\mathcal{B}$ together with a two-cycle ρ, an even bilinear map $[\cdot,\cdot] : \mathcal{B}\times\mathcal{B}\to\mathcal{B}$ and two even linear maps $\alpha,\beta : \mathcal{B}\to\mathcal{B}$*

satisfying

$$\alpha \circ \beta = \beta \circ \alpha,$$
$$[\beta(f), \alpha(g)] = -\rho(f,g)[\beta(g), \alpha(f)],$$
$$\circlearrowleft_{f,g,h} \rho(h,f)[\beta^2(f), [\beta(g), \alpha(h)]] = 0 \quad (\rho\text{-Bihom-Jacobi identity}).$$

Here, we describe the class of Bihom-ρ-Lie algebras, which includes the special multiplication $[\cdot,\cdot,\cdot]_{\mathcal{B}}$, a two-cycle and two even linear maps α, β. This Lie algebras is called 3-Bihom-ρ-Lie algebras.

Definition 2.2 *Multiple* $(\mathcal{B}, [\cdot,\cdot,\cdot]_{\mathcal{B}}, \rho, \alpha, \beta)$ *is called a* 3*-Bihom-*ρ*-Lie algebra if*

- $|[f_1, f_2, f_3]_{\mathcal{B}}| = |f_1| + |f_2| + |f_3|,$
- $\alpha \circ \beta = \beta \circ \alpha,$
- $\alpha[f_1, f_2, f_3]_{\mathcal{B}} = [\alpha(f_1), \alpha(f_2), \alpha(f_3)]_{\mathcal{B}}, \quad \beta[f_1, f_2, f_3]_{\mathcal{B}} = [\beta(f_1), \beta(f_2), \beta(f_3)]_{\mathcal{B}},$
- $[\beta^2(f_1), \beta^2(f_2), [\beta(g_1), \beta(g_2), \alpha(g_3)]_{\mathcal{B}}]_{\mathcal{B}}$

$$\begin{aligned} &= \rho(f_1+f_2+g_1, g_2+g_3)[\beta^2(g_2), \beta^2(g_3), [\beta(f_1), \beta(f_2), \alpha(g_1)]_{\mathcal{B}}]_{\mathcal{B}} \\ &\quad - \rho(f_1+f_2, g_1+g_3)\rho(g_2,g_3)[\beta^2(g_1), \beta^2(g_3), [\beta(f_1), \beta(f_2), \alpha(g_2)]_{\mathcal{B}}]_{\mathcal{B}} \\ &\quad + \rho(f_1+f_2, g_1+g_2)[\beta^2(g_1), \beta^2(g_2), [\beta(f_1), \beta(f_2), \alpha(g_3)]] \ (\rho\text{-Bihom-Jacobi identity}) \end{aligned} \tag{2.3}$$

- $[\beta(f), \beta(g), \alpha(h)]_{\mathcal{B}} = -\rho(f,g)[\beta(g), \beta(f), \alpha(h)]_{\mathcal{B}} = -\rho(g,h)[\beta(f), \beta(h), \alpha(g)]_{\mathcal{B}}$

for any $f_1, f_2, f_3, g_1, g_2 \in Hg(\mathcal{B})$*.* $(\mathcal{B}, [\cdot,\cdot,\cdot]_{\mathcal{B}}, \rho, \alpha, \beta)$ *consists of a* G*-graded vector space* $\mathcal{B}$*, two even linear maps* $\alpha, \beta : \mathcal{B} \to \mathcal{B}$*, a two-cycle* ρ *and a* 3*-linear map* $[\cdot,\cdot,\cdot]_{\mathcal{B}}$ *on* $\mathcal{B}$*.*

Definition 2.3 *A linear map* $\delta : \mathcal{A} \to \mathcal{B}$ *is called a morphism of two* 3*-Bihom-*ρ*-Lie algebras* $(\mathcal{A}, [\cdot,\cdot,\cdot]_{\mathcal{A}}, \rho, \alpha, \beta)$ *and* $(\mathcal{B}, [\cdot,\cdot,\cdot]_{\mathcal{B}}, \rho, \alpha', \beta')$ *if*

$$\delta[f_1, f_2, f_3]_{\mathcal{A}} = [\delta(f_1), \delta(f_2), \delta(f_3)]_{\mathcal{B}},$$
$$\delta \circ \alpha = \alpha' \circ \delta, \quad \delta \circ \beta = \beta' \circ \delta.$$

Example 2.1 Let us consider $\mathcal{B}$ as a $G = \mathbb{Z}_2^3$-graded vector space with the basis $\{l_1, l_2, l_3, l_4\}$. Then $\mathcal{B} = \mathcal{B}_{(1,0,0)} \oplus \mathcal{B}_{(0,1,0)} \oplus \mathcal{B}_{(0,0,1)} \oplus \mathcal{B}_{(1,1,1)}$ with

$$l_1 \in \mathcal{B}_{(1,0,0)}, \quad l_2 \in \mathcal{B}_{(0,1,0)}, \quad l_3 \in \mathcal{B}_{(0,0,1)}, \quad l_4 \in \mathcal{B}_{(1,1,1)}.$$

We define $\rho : G \times G \to \mathbb{C}^*$ by the matrix

$$[\rho(a,b)] = \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

where $(a,b) \in \{(1,0,0), (0,1,0), (0,0,1), (1,1,1)\} \times \{(1,0,0), (0,1,0), (0,0,1), (1,1,1)\}$. We consider the 3-$\rho$-Lie algebra bracket associated to the basis $\{l_1, l_2, l_3, l_4\}$ by

$$[l_1, l_2, l_3]_{\mathcal{B}} = l_4, \quad [l_1, l_2, l_4]_{\mathcal{B}} = l_3,$$

$$[l_2, l_3, l_4]_{\mathcal{B}} = l_1, \quad [l_1, l_3, l_4]_{\mathcal{B}} = l_2.$$

If we define two even linear maps $\alpha, \beta : \mathcal{B} \to \mathcal{B}$ by

$$\alpha = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \beta = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

So it is easy to check that $(\mathcal{B}, [\cdot,\cdot,\cdot]_{\mathcal{B}}, \rho, \alpha, \beta)$ is a 3-Bihom-ρ-Lie algebra.

Lemma 2.1 *Consider the following assumptions*:
(i) $([\cdot,\cdot,\cdot]_{\mathcal{B}}, \rho, \alpha, \beta)$ *is a* 3*-Bihom-*ρ*-Lie algebra,*
(ii) f *is an even element of* $\mathcal{B}$ $(|f| = 0)$ *such that* $\alpha(f) = \beta(f) = f$.
If we define a multiplication $[\cdot,\cdot]$ *on* $\mathcal{B}$ *by*

$$[g, h] = [f, g, h]_{\mathcal{B}}, \tag{2.4}$$

then $(\mathcal{B}, [\cdot,\cdot], \rho, \alpha, \beta)$ *is a Bihom-*ρ*-Lie algebra.*

Proof First, we show that $[\beta(g), \alpha(h)] = -\rho(g, h)[\beta(h), \alpha(g)]$. For this we have

$$\begin{aligned} [\beta(g), \alpha(h)] &= [f, \beta(g), \alpha(h)]_{\mathcal{B}} \\ &= [\beta(f), \beta(g), \alpha(h)]_{\mathcal{B}} \\ &= -\rho(g, h)[\beta(f), \beta(h), \alpha(g)]_{\mathcal{B}} \\ &= -\rho(g, h)[f, \beta(h), \alpha(g)]_{\mathcal{B}} \\ &= -\rho(g, h)[\beta(h), \alpha(g)]. \end{aligned}$$

Now, we show that the multiplication $[\cdot,\cdot]$ satisfies the ρ-Bihom-Jacobi identity. We have

$$\begin{aligned} &\rho(e, g)[\beta^2(g), [\beta(h), \alpha(e)]] + \rho(g, h)[\beta^2(h), [\beta(e), \alpha(g)]] + \rho(h, e)[\beta^2(e), [\beta(g), \alpha(h)]] \\ &= \rho(e, g)[f, \beta^2(g), [f, \beta(h), \alpha(e)]_{\mathcal{B}}]_{\mathcal{B}} + \rho(g, h)[f, \beta^2(h), [f, \beta(e), \alpha(g)]_{\mathcal{B}}]_{\mathcal{B}} \\ &\quad + \rho(h, e)[f, \beta^2(e), [f, \beta(g), \alpha(h)]_{\mathcal{B}}]_{\mathcal{B}} \\ &= \rho(e, g)[\beta^2(f), \beta^2(g), [\beta(f), \beta(h), \alpha(e)]_{\mathcal{B}}]_{\mathcal{B}} \\ &\quad + \rho(g, h)[\beta^2(f), \beta^2(h), [\beta(f), \beta(e), \alpha(g)]_{\mathcal{B}}]_{\mathcal{B}} \\ &\quad + \rho(h, e)[\beta^2(f), \beta^2(e), [\beta(f), \beta(g), \alpha(h)]_{\mathcal{B}}]_{\mathcal{B}}. \end{aligned}$$

Thus

$$\begin{aligned} &\rho(e, g)[\beta^2(g), [\beta(h), \alpha(e)]] + \rho(g, h)[\beta^2(h), [\beta(e), \alpha(g)]] + \rho(h, e)[\beta^2(e), [\beta(g), \alpha(h)]] \\ &= \rho(e, g)[\beta^2(f), \beta^2(g), [\beta(f), \beta(h), \alpha(e)]_{\mathcal{B}}]_{\mathcal{B}} + \rho(g, h)[\beta^2(f), \beta^2(h), [\beta(f), \beta(e), \alpha(g)]_{\mathcal{B}}]_{\mathcal{B}} \\ &\quad + \rho(h, e)[\beta^2(f), \beta^2(e), [\beta(f), \beta(g), \alpha(h)]_{\mathcal{B}}]_{\mathcal{B}}. \end{aligned}$$

Since $([\cdot,\cdot,\cdot]_{\mathcal{B}}, \rho, \alpha, \beta)$ is a 3-Bihom-ρ-Lie algebra, on the right side of the above relation, the first term can be changed by the ρ-Bihom-Jacobi identity. Therefore

$$\rho(e, g)[\beta^2(g), [\beta(h), \alpha(e)]] + \rho(g, h)[\beta^2(h), [\beta(e), \alpha(g)]] + \rho(h, e)[\beta^2(e), [\beta(g), \alpha(h)]]$$

$$
\begin{aligned}
&= \rho(e,g)\rho(g,h+e)[\beta^2(h),\beta^2(e),[\beta(f),\beta(g),\alpha(f)]_{\mathcal{B}}]_{\mathcal{B}} \\
&\quad - \rho(h,e)[\beta^2(f),\beta^2(e),[\beta(f),\beta(g),\alpha(h)]_{\mathcal{B}}]_{\mathcal{B}} \\
&\quad + \rho(e,g)\rho(g,h)[\beta^2(f),\beta^2(h),[\beta(f),\beta(g),\alpha(e)]_{\mathcal{B}}]_{\mathcal{B}} \\
&\quad + \rho(g,h)[\beta^2(f),\beta^2(h),[\beta(f),\beta(e),\alpha(g)]_{\mathcal{B}}]_{\mathcal{B}} \\
&\quad + \rho(h,e)[\beta^2(f),\beta^2(e),[\beta(f),\beta(g),\alpha(h)]_{\mathcal{B}}]_{\mathcal{B}} \\
&= 0.
\end{aligned}
$$

Proposition 2.1 *The following actions acquire a 3-Bihom-ρ-Lie algebra structure*

$$
\begin{aligned}
&[a_1+b_1,a_2+b_2,a_3+b_3]_{\mathcal{A}\oplus\mathcal{B}} = [a_1,a_2,a_3]_{\mathcal{A}} + [b_1,b_2,b_3]_{\mathcal{B}}, \\
&(\alpha\oplus\alpha')(a+b) = \alpha(a)+\alpha'(b), \\
&(\beta\oplus\beta')(a+b) = \beta(a)+\beta'(b),
\end{aligned}
$$

when $(\mathcal{A},[\cdot,\cdot,\cdot]_{\mathcal{A}},\rho,\alpha,\beta)$ and $(\mathcal{B},[\cdot,\cdot,\cdot]_{\mathcal{B}},\rho,\alpha',\beta')$ are two 3-Bihom-ρ-Lie algebras.

Proof The proof of this proposition by the definition of the 3-Bihom-ρ-Lie algebra is straightforward.

Definition 2.4 *Let $\mathcal{B}$ be a G-graded vector space, ρ be a two-cycle and $[\cdot,\cdot,\cdot]_{\mathcal{B}}$ be a 3-linear map on $\mathcal{B}$ such that*

$$
\begin{aligned}
&|[f_1,f_2,f_3]_{\mathcal{B}}| = |f_1|+|f_2|+|f_3|, \\
&[f_1,f_2,[g_1,g_2,g_3]_{\mathcal{B}}]_{\mathcal{B}} \\
&= [[f_1,f_2,g_1]_{\mathcal{B}},g_2,g_3]_{\mathcal{B}} + \rho(f_1+f_2,g_1)[g_1,[f_1,f_2,g_2]_{\mathcal{B}},g_3]_{\mathcal{B}} \\
&\quad + \rho(f_1+f_2,g_1+g_2)[g_1,g_2,[f_1,f_2,g_3]_{\mathcal{B}}]_{\mathcal{B}},
\end{aligned} \tag{2.5}
$$

then the triple $(\mathcal{B},[\cdot,\cdot,\cdot]_{\mathcal{B}},\rho)$ is called a 3-ρ-Leibniz algebra.

Definition 2.5 *Let $\mathcal{B}$ be a G-graded vector space, ρ be a two-cycle and $[\cdot,\cdot,\cdot]_{\mathcal{B}}$ be a 3-linear map on $\mathcal{B}$. If $\alpha,\beta:\mathcal{B}\to\mathcal{B}$ are two even linear maps satisfying $\alpha\circ\beta=\beta\circ\alpha$, $\alpha[f_1,f_2,f_3]_{\mathcal{B}} = [\alpha(f_1),\alpha(f_2),\alpha(f_3)]_{\mathcal{B}}$, $\beta[f_1,f_2,f_3]_{\mathcal{B}} = [\beta(f_1),\beta(f_2),\beta(f_3)]_{\mathcal{B}}$, and*

$$
\begin{aligned}
&[\alpha\beta(f_1),\alpha\beta(f_2),[g_1,g_2,g_3]_{\mathcal{B}}]_{\mathcal{B}} \\
&= [[\beta(f_1),\beta(f_2),g_1]_{\mathcal{B}},\beta(g_2),\beta(g_3),]_{\mathcal{B}} \\
&\quad + \rho(f_1+f_2,g_1)[\beta(g_1),[\beta(f_1),\beta(f_2),g_2]_{\mathcal{B}},\beta(g_3),]_{\mathcal{B}} \\
&\quad + \rho(f_1+f_2,g_1+g_2)[\beta(g_1),\beta(g_2),[\alpha(f_1),\alpha(f_2),g_3]_{\mathcal{B}}]_{\mathcal{B}},
\end{aligned} \tag{2.6}
$$

then $(\mathcal{B},[\cdot,\cdot,\cdot]_{\mathcal{B}},\rho,\alpha,\beta)$ is called a 3-Bihom-ρ-Leibniz algebra.

Definition 2.6 *A morphism of 3-ρ-Leibniz algebras $(\mathcal{A},[\cdot,\cdot,\cdot]_{\mathcal{A}},\rho)$ and $(\mathcal{B},[\cdot,\cdot,\cdot]_{\mathcal{B}},\rho)$ is a linear map $\delta:\mathcal{A}\to\mathcal{B}$ such that $\delta[f,g,h]_{\mathcal{A}} = [\delta(f),\delta(g),\delta(h)]_{\mathcal{B}}$ for all $f,g,h\in\mathcal{A}$. When $(\mathcal{A},[\cdot,\cdot,\cdot]_{\mathcal{A}},\rho,\alpha,\beta)$ and $(\mathcal{B},[\cdot,\cdot,\cdot]_{\mathcal{B}},\rho,\alpha',\beta')$ are 3-Bihom-ρ-Leibniz algebras, $\delta:\mathcal{A}\to\mathcal{B}$ is a* *morphism of* 3*-Bihom-ρ-Leibniz algebras if*

$$\delta[f,g,h]_{\mathcal{A}} = [\delta(f),\delta(g),\delta(h)]_{\mathcal{B}},$$
$$\alpha'\circ\delta = \delta\circ\alpha,\quad \beta'\circ\delta = \delta\circ\beta.$$

Proposition 2.2 *Consider the following assumptions*:
(i) $(\mathcal{B},[\cdot,\cdot,\cdot]_{\mathcal{B}},\rho)$ *is a* 3*-ρ-Leibniz algebra,*
(ii) $\alpha,\beta:\mathcal{B}\to\mathcal{B}$ *are two even commuting morphisms of* 3*-ρ-Leibniz algebras.*
Then $(\mathcal{B},\langle\cdot,\cdot,\cdot\rangle_{\mathcal{B}},\rho,\alpha,\beta)$ *is a* 3*-ρ-Bihom-Leibniz algebra, where*

$$\langle f_1,f_2,f_3\rangle_{\mathcal{B}} = [\alpha(f_1),\alpha(f_2),\beta(f_3)]_{\mathcal{B}}. \tag{2.7}$$

Proof Using (2.5)–(2.7), we have

$$\begin{aligned}
&\langle\alpha\beta(f_1),\alpha\beta(f_2),\langle g_1,g_2,g_3\rangle_{\mathcal{B}}\rangle_{\mathcal{B}}\\
&=[\alpha^2\beta(f_1),\alpha^2\beta(f_2),[\alpha\beta(g_1),\alpha\beta(g_2),\beta^2(g_3)]_{\mathcal{B}}]_{\mathcal{B}}\\
&=[[\alpha^2\beta(f_1),\alpha^2\beta(f_2),\alpha\beta(g_1)]_{\mathcal{B}},\alpha\beta(g_2),\beta^2(g_3)]_{\mathcal{B}}\\
&\quad+\rho(f_1+f_2,g_1)[\alpha\beta(g_1),[\alpha^2\beta(f_1),\alpha^2\beta(f_2),\alpha\beta(g_2)]_{\mathcal{B}},\beta^2(g_3)]_{\mathcal{B}}\\
&\quad+\rho(f_1+f_2,g_1+g_2)[\alpha\beta(g_1),\alpha\beta(g_2),[\alpha^2\beta(f_1),\alpha^2\beta(f_2),\beta^2(g_3)]_{\mathcal{B}}]_{\mathcal{B}}.
\end{aligned}$$

Now, again by (2.7), the above relation is rewritten as follows

$$\begin{aligned}
&\langle\alpha\beta(f_1),\alpha\beta(f_2),\langle g_1,g_2,g_3\rangle_{\mathcal{B}}\rangle_{\mathcal{B}}\\
&=\langle\langle\beta(f_1),\beta(f_2),g_1\rangle_{\mathcal{B}},\beta(g_2),\beta(g_3)\rangle_{\mathcal{B}}\\
&\quad+\rho(f_1+f_2,g_1)\langle\beta(g_1),\langle\beta(f_1),\beta(f_2),g_2\rangle_{\mathcal{B}},\beta(g_3)\rangle_{\mathcal{B}}\\
&\quad+\rho(f_1+f_2,g_1+g_2)\langle\beta(g_1),\beta(g_2),\langle\alpha(f_1),\alpha(f_2),g_3\rangle_{\mathcal{B}}\rangle_{\mathcal{B}}
\end{aligned}$$

for the elements $\alpha^2\beta(f),\alpha\beta(g),\beta^2(h)$ of $\mathcal{B}$.

Proposition 2.3 *A* 3*-Bihom-ρ-Leibniz algebra* $(\mathcal{B},[\cdot,\cdot,\cdot]_{\mathcal{B}},\rho,\alpha,\beta)$ *satisfying the ρ-Bihom-skew-symmetry condition*

$$[\beta(f_1),\beta(f_2),\alpha(f_3)]_{\mathcal{B}} = -\rho(f_1,f_2)[\beta(f_2),\beta(f_1),\alpha(f_3)]_{\mathcal{B}} = -\rho(f_2,f_3)[\beta(f_1),\beta(f_3),\alpha(f_1)]_{\mathcal{B}},$$

is a 3*-Bihom-ρ-Lie algebra.*

Proof It is enough to show that (2.3) is equivalent to (2.6). At first (2.3) is equivalent to

$$\begin{aligned}
&[\beta^2(f_1),\beta^2(f_2),\beta[g_1,g_2,\alpha\beta^{-1}(g_3)]_{\mathcal{B}}]_{\mathcal{B}}\\
&=\rho(f_1+f_2+g_1,g_2+g_3)[\beta^2(g_2),\beta^2(g_3),\beta[f_1,f_2,\alpha\beta^{-1}(g_1)]_{\mathcal{B}}]_{\mathcal{B}}\\
&\quad-\rho(f_1+f_2,g_1+g_3)\rho(g_2,g_3)[\beta^2(g_1),\beta^2(g_3),\beta[f_1,f_2,\alpha\beta^{-1}(g_2)]_{\mathcal{B}}]_{\mathcal{B}}\\
&\quad+\rho(f_1+f_2,g_1+g_2)[\beta^2(g_1),\beta^2(g_2),\beta[f_1,f_2,\alpha\beta^{-1}(g_3)]_{\mathcal{B}}]_{\mathcal{B}}.
\end{aligned}$$

Thus

$$\begin{aligned}
&[\beta(f_1),\beta(f_2),[g_1,g_2,\alpha\beta^{-1}(g_3)]_{\mathcal{B}}]_{\mathcal{B}}\\
&=\rho(f_1+f_2+g_1,g_2+g_3)[\beta(g_2),\beta(g_3),[f_1,f_2,\alpha\beta^{-1}(g_1)]_{\mathcal{B}}]_{\mathcal{B}}\\
&\quad-\rho(f_1+f_2,g_1+g_3)\rho(g_2,g_3)[\beta(g_1),\beta(g_3),[f_1,f_2,\alpha\beta^{-1}(g_2)]_{\mathcal{B}}]_{\mathcal{B}}\\
&\quad+\rho(f_1+f_2,g_1+g_2)[\beta(g_1),\beta(g_2),[f_1,f_2,\alpha\beta^{-1}(g_3)]_{\mathcal{B}}]_{\mathcal{B}},
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&[\beta(f_1),\beta(f_2),[g_1,g_2,\alpha\beta^{-1}(g_3)]_{\mathcal{B}}]_{\mathcal{B}}\\
&=\rho(f_1+f_2+g_1,g_2+g_3)[\beta(g_2),\beta(g_3),\alpha[\alpha^{-1}(f_1),\alpha^{-1}(f_2),\beta^{-1}(g_1)]_{\mathcal{B}}]_{\mathcal{B}}\\
&\quad-\rho(f_1+f_2,g_1+g_3)\rho(g_2,g_3)[\beta(g_1),\beta(g_3),\alpha[\alpha^{-1}(f_1),\alpha^{-1}(f_2),\beta^{-1}(g_2)]_{\mathcal{B}}]_{\mathcal{B}}\\
&\quad+\rho(f_1+f_2,g_1+g_2)[\beta(g_1),\beta(g_2),[f_1,f_2,\alpha\beta^{-1}(g_3)]_{\mathcal{B}}]_{\mathcal{B}}.
\end{aligned}$$

The ρ-Bihom-skew symmetry condition can change the second and third terms of the above relation as follows

$$\begin{aligned}
&[\beta(f_1),\beta(f_2),[g_1,g_2,\alpha\beta^{-1}(g_3)]_{\mathcal{B}}]_{\mathcal{B}}\\
&=\rho(f_1+f_2+g_1,g_2+g_3)\rho(g_2+g_3,f_1+f_2+g_1)\\
&\quad\cdot[\beta[\alpha^{-1}(f_1),\alpha^{-1}(f_2),\beta^{-1}(g_1)]_{\mathcal{B}},\beta(g_2),\beta(g_3)]_{\mathcal{B}}\\
&\quad+\rho(f_1+f_2,g_1+g_3)\rho(g_2,g_3)\rho(g_3,f_1+f_2+g_2)\\
&\quad\cdot[\beta(g_1),\beta[\alpha^{-1}(f_1),\alpha^{-1}(f_2),\beta^{-1}(g_2)]_{\mathcal{B}},\alpha(g_3)]_{\mathcal{B}}\\
&\quad+\rho(f_1+f_2,g_1+g_2)[\beta(g_1),\beta(g_2),[f_1,f_2,\alpha\beta^{-1}(g_3)]_{\mathcal{B}}]_{\mathcal{B}}.
\end{aligned}$$

In the last relation, it is enough to replace f_1, f_2 with $\alpha(f_1)$, $\alpha(f_2)$ and g_3 with $\alpha^{-1}\beta(g_3)$, then

$$\begin{aligned}
&[\alpha\beta(f_1),\alpha\beta(f_2),[g_1,g_2,g_3]_{\mathcal{B}}]_{\mathcal{B}}\\
&=[[\beta(f_1),\beta(f_2),g_1]_{\mathcal{B}},\beta(g_2),\beta(g_3)]_{\mathcal{B}}\\
&\quad+\rho(f_1+f_2,g_1)[\beta(g_1),[\beta(f_1),\beta(f_2),g_2]_{\mathcal{B}},\beta(g_3)]_{\mathcal{B}}\\
&\quad+\rho(f_1+f_2,g_1+g_2)[\beta(g_1),\beta(g_2),[\alpha(f_1),\alpha(f_2),g_3]_{\mathcal{B}}]_{\mathcal{B}}.
\end{aligned}$$

Definition 2.7 *Let $\mathcal{B}$ be a G-graded vector space, $[\cdot,\cdot,\cdot]_{\mathcal{B}}:\otimes^3\mathcal{B}\to\mathcal{B}$ be a 3-linear map and $\lambda\in\mathbb{K}$. An even linear map $P:\mathcal{B}\to\mathcal{B}$ is called a Rota-Baxer operator of weight λ for $(\mathcal{B},[\cdot,\cdot,\cdot]_{\mathcal{B}})$ if it satisfies the following condition*

$$\begin{aligned}
&[P(f_1),P(f_2),P(f_3)]_{\mathcal{B}}\\
&=P([P(f_1),P(f_2),f_3]_{\mathcal{B}}+[P(f_1),f_2,P(f_3)]_{\mathcal{B}}+[f_1,P(f_2),P(f_3)]_{\mathcal{B}}\\
&\quad+\lambda[P(f_1),f_2,f_3]_{\mathcal{B}}+\lambda[f_1,P(f_2),f_3]_{\mathcal{B}}+\lambda[f_1,f_2,P(f_3)]_{\mathcal{B}}+\lambda^2[f_1,f_2,f_3]_{\mathcal{B}}). \qquad (2.8)
\end{aligned}$$

Remark 2.1 If P is a Rota-Baxer operator of weight λ for $(\mathcal{B},[\cdot,\cdot,\cdot]_{\mathcal{B}})$ and we define a new multiplication $\langle\cdot,\cdot,\cdot\rangle_{\mathcal{B}}$ on $\mathcal{B}$ by

$$\langle f_1,f_2,f_3\rangle_{\mathcal{B}}=[P(f_1),P(f_2),f_3]_{\mathcal{B}}+[P(f_1),f_2,P(f_3)]_{\mathcal{B}}+[f_1,P(f_2),P(f_3)]_{\mathcal{B}}$$

$$+\lambda[P(f_1),f_2,f_3]_{\mathcal{B}}+\lambda[f_1,P(f_2),f_3]_{\mathcal{B}}$$
$$+\lambda[f_1,f_2,P(f_3)]_{\mathcal{B}}+\lambda^2[f_1,f_2,f_3]_{\mathcal{B}},$$

then P is a Rota-Baxer operator of weight λ for $(\mathcal{B},\langle\cdot,\cdot,\cdot\rangle_{\mathcal{B}})$.

Proposition 2.4 *Consider the following assumptions*:
(i) $\mathcal{B}$ *is a* G*-graded vector space,*
(ii) $\alpha,\beta:\mathcal{B}\to\mathcal{B}$ *are even linear maps,*
(iii) $[\cdot,\cdot,\cdot]_{\mathcal{B}}$ *is a* 3*-linear map on* $\mathcal{B}$*,*
(iv) $P:\mathcal{B}\to\mathcal{B}$ *is an even linear map such that* $P\circ\alpha=\alpha\circ P$ *and* $P\circ\beta=\beta\circ P$*,*
(v) $\lambda\in\mathbb{K}$ *is a fixed scalar.*
We define a multiplication $\{\cdot,\cdot,\cdot\}^{\alpha,\beta}$ *on* $\mathcal{B}$ *by*

$$\begin{aligned}\{f_1,f_2,f_3\}^{\alpha,\beta}=&[P(f_1),P(f_2),f_3]_{\mathcal{B}}+[P(f_1),f_2,P(f_3)]_{\mathcal{B}}+[f_1,P(f_2),P(f_3)]_{\mathcal{B}}\\&+\lambda[P(f_1),f_2,f_3]_{\mathcal{B}}+\lambda[f_1,P(f_2),f_3]_{\mathcal{B}}\\&+\lambda[f_1,f_2,P(f_3)]_{\mathcal{B}}+\lambda^2[f_1,f_2,f_3]_{\mathcal{B}},\quad\forall f_1,f_2,f_3\in Hg(\mathcal{B}).\end{aligned}\tag{2.9}$$

Then
(i) *If* $\alpha[f_1,f_2,f_3]_{\mathcal{B}}=[\alpha(f_1),\alpha(f_2),\alpha(f_3)]_{\mathcal{B}}$, $\beta[f_1,f_2,f_3]_{\mathcal{B}}=[\beta(f_1),\beta(f_2),\beta(f_3)]_{\mathcal{B}}$*, then*

$$\alpha\{f_1,f_2,f_3\}^{\alpha,\beta}=\{\alpha(f_1),\alpha(f_2),\alpha(f_3)\}^{\alpha,\beta},$$
$$\beta\{f_1,f_2,f_3\}^{\alpha,\beta}=\{\beta(f_1),\beta(f_2),\beta(f_3)\}^{\alpha,\beta}.$$

(ii) *If*

$$[\beta(f_1),\beta(f_2),\alpha(f_3)]_{\mathcal{B}}=-\rho(f_1,f_2)[\beta(f_2),\beta(f_1),\alpha(f_3)]_{\mathcal{B}},$$
$$[\beta(f_1),\beta(f_2),\alpha(f_3)]_{\mathcal{B}}=-\rho(f_2,f_3)[\beta(f_1),\beta(f_3),\alpha(f_2)]_{\mathcal{B}},$$

then

$$\{\beta(f_1),\beta(f_2),\alpha(f_3)\}^{\alpha,\beta}=-\rho(f_1,f_2)\{\beta(f_2),\beta(f_1),\alpha(f_3)\}^{\alpha,\beta},$$
$$\{\beta(f_1),\beta(f_2),\alpha(f_3)\}^{\alpha,\beta}=-\rho(f_2,f_3)\{\beta(f_1),\beta(f_3),\alpha(f_2)\}^{\alpha,\beta}.$$

(iii) *If*

$$\begin{aligned}&[P(f_1),P(f_2),P(f_3)]_{\mathcal{B}}\\&=P([P(f_1),P(f_2),f_3]_{\mathcal{B}}+[P(f_1),f_2,P(f_3)]_{\mathcal{B}}+[f_1,P(f_2),P(f_3)]_{\mathcal{B}}\\&\quad+\lambda[P(f_1),f_2,f_3]_{\mathcal{B}}+\lambda[f_1,P(f_2),f_3]_{\mathcal{B}}+\lambda[f_1,f_2,P(f_3)]_{\mathcal{B}}+\lambda^2[f_1,f_2,f_3]_{\mathcal{B}}),\end{aligned}$$

then $P(\{f_1,f_2,f_3\}^{\alpha,\beta})=[P(f_1),P(f_2),P(f_3)]_{\mathcal{B}}$.

Proof (i) By (2.9), we have

$$\{\alpha(f_1),\alpha(f_2),\alpha(f_3)\}^{\alpha,\beta}=[P\alpha(f_1),P\alpha(f_2),\alpha(f_3)]_{\mathcal{B}}+[P\alpha(f_1),\alpha(f_2),P\alpha(f_3)]_{\mathcal{B}}$$

$$+ [\alpha(f_1), P\alpha(f_2), P\alpha(f_3)]_{\mathcal{B}} + \lambda[P\alpha(f_1), \alpha(f_2), \alpha(f_3)]_{\mathcal{B}}$$
$$+ \lambda[\alpha(f_1), P\alpha(f_2), \alpha(f_3)]_{\mathcal{B}} + \lambda[\alpha(f_1), \alpha(f_2), P\alpha(f_3)]_{\mathcal{B}}$$
$$+ \lambda^2[\alpha(f_1), \alpha(f_2), \alpha(f_3)]_{\mathcal{B}}.$$

Since $P \circ \alpha = \alpha \circ P$ and $\alpha[f_1, f_2, f_3]_{\mathcal{B}} = [\alpha(f_1), \alpha(f_2), \alpha(f_3)]_{\mathcal{B}}$, we have

$$\begin{aligned}\{\alpha(f_1), \alpha(f_2), \alpha(f_3)\}^{\alpha,\beta} &= \alpha[P(f_1), P(f_2), (f_3)]_{\mathcal{B}} + \alpha[P(f_1), (f_2), P(f_3)]_{\mathcal{B}} \\ &+ \alpha[(f_1), P(f_2), P(f_3)]_{\mathcal{B}} + \alpha\lambda[P(f_1), (f_2), (f_3)]_{\mathcal{B}} \\ &+ \alpha\lambda[(f_1), P(f_2), (f_3)]_{\mathcal{B}} + \alpha\lambda[(f_1), (f_2), P(f_3)]_{\mathcal{B}} \\ &+ \alpha\lambda^2[(f_1), (f_2), (f_3)]_{\mathcal{B}} = \alpha\{f_1, f_2, f_3\}^{\alpha,\beta}.\end{aligned}$$

In a similar way, we can show that $\{\beta(f_1), \beta(f_2), \beta(f_3)\}^{\alpha,\beta} = \beta\{f_1, f_2, f_3\}^{\alpha,\beta}$.

(ii) By (2.9), we obtain

$$\begin{aligned}\{\beta(f_1), \beta(f_2), \alpha(f_3)\}^{\alpha,\beta} &= [P\beta(f_1), P\beta(f_2), \alpha(f_3)]_{\mathcal{B}} + [P\beta(f_1), \beta(f_2), P\alpha(f_3)]_{\mathcal{B}} \\ &+ [\beta(f_1), P\beta(f_2), P\alpha(f_3)]_{\mathcal{B}} + \lambda[P\beta(f_1), \beta(f_2), \alpha(f_3)]_{\mathcal{B}} \\ &+ \lambda[\beta(f_1), P\beta(f_2), \alpha(f_3)]_{\mathcal{B}} + \lambda[\beta(f_1), \beta(f_2), P\alpha(f_3)]_{\mathcal{B}} \\ &+ \lambda^2[\beta(f_1), \beta(f_2), \alpha(f_3)]_{\mathcal{B}}.\end{aligned}$$

Since $\beta \circ P = P \circ \beta, \alpha \circ P = P \circ \alpha$, $[\beta(f_1), \beta(f_2), \alpha(f_3)]_{\mathcal{B}} = -\rho(f_1, f_2)[\beta(f_2), \beta(f_1), \alpha(f_3)]_{\mathcal{B}}$ and $[\beta(f_1), \beta(f_2), \alpha(f_3)]_{\mathcal{B}} = -\rho(f_2, f_3)[\beta(f_1), \beta(f_3), \alpha(f_2)]_{\mathcal{B}}$, we get

$$\{\beta(f_1), \beta(f_2), \alpha(f_3)\}^{\alpha,\beta} = -\rho(f_1, f_2)\{\beta(f_2), \beta(f_1), \alpha(f_3)\}^{\alpha,\beta}$$

and

$$\{\beta(f_1), \beta(f_2), \alpha(f_3)\}^{\alpha,\beta} = -\rho(f_2, f_3)\{\beta(f_1), \beta(f_3), \alpha(f_2)\}^{\alpha,\beta}.$$

(iii) By (2.9), the proof is clear.

Proposition 2.5 *Consider the following assumptions*:
(i) $(\mathcal{B}, [\cdot,\cdot,\cdot]_{\mathcal{B}}, \rho, \alpha, \beta)$ *is a* 3*-Bihom-ρ-Lie algebra,*
(ii) $\lambda \in \mathbb{K}$ *is a fixed scalar,*
(iii) $P : \mathcal{B} \to \mathcal{B}$ *is a Rota-Baxer operator of weight* λ *such that* $P\circ\alpha = \alpha\circ P$ *and* $P\circ\beta = \beta\circ P$.
We define a multiplication $\{\cdot,\cdot,\cdot\}^{\alpha,\beta}$ *on* $\mathcal{B}$ *by*

$$\begin{aligned}\{f_1, f_2, f_3\}^{\alpha,\beta} &= [P(f_1), P(f_2), f_3]_{\mathcal{B}} + [P(f_1), f_2, P(f_3)]_{\mathcal{B}} + [f_1, P(f_2), P(f_3)]_{\mathcal{B}} \\ &+ \lambda[P(f_1), f_2, f_3]_{\mathcal{B}} + \lambda[f_1, P(f_2), f_3]_{\mathcal{B}} \\ &+ \lambda[f_1, f_2, P(f_3)]_{\mathcal{B}} + \lambda^2[f_1, f_2, f_3]_{\mathcal{B}}, \quad \forall f_1, f_2, f_3 \in Hg(\mathcal{B}).\end{aligned} \tag{2.10}$$

Then $(\mathcal{B}, \{f_1, f_2, f_3\}^{\alpha,\beta}, \rho, \alpha, \beta)$ *is a* 3*-Bihom-ρ-Lie algebra.*

Proof The proof is completed by long and straightforward computations.

Proposition 2.6 *Consider the following assumptions*:

(i) $(\mathcal{B},[\cdot,\cdot,\cdot]_{\mathcal{B}},\rho,\alpha,\beta)$ *is a* 3-*Bihom-*ρ*-Leibniz algebra,*

(ii) $\lambda\in\mathbb{K}$ *is a fixed scalar,*

(iii) $P:\mathcal{B}\to\mathcal{B}$ *is a Rota-Baxer operator of weight* λ *such that* $P\circ\alpha=\alpha\circ P$ *and* $P\circ\beta=\beta\circ P$. *Defining a multiplication* $\{\cdot,\cdot,\cdot\}^{\alpha,\beta}$ *on* $\mathcal{B}$ *by*

$$\begin{aligned}\{f_1,f_2,f_3\}^{\alpha,\beta}=&[P(f_1),P(f_2),f_3]_{\mathcal{B}}+[P(f_1),f_2,P(f_3)]_{\mathcal{B}}+[f_1,P(f_2),P(f_3)]_{\mathcal{B}}\\&+\lambda[P(f_1),f_2,f_3]_{\mathcal{B}}+\lambda[f_1,P(f_2),f_3]_{\mathcal{B}}\\&+\lambda[f_1,f_2,P(f_3)]_{\mathcal{B}}+\lambda^2[f_1,f_2,f_3]_{\mathcal{B}},\quad\forall f_1,f_2,f_3\in Hg(\mathcal{B}),\end{aligned}\tag{2.11}$$

then $(\mathcal{B},\{f_1,f_2,f_3\}^{\alpha,\beta},\rho,\alpha,\beta)$ *is a* 3*-Bihom-*ρ*-Leibniz algebra.*

Proof The proof is completed by long and straightforward computations.

3 3-Pre-Bihom-ρ-Lie Algebras

In this part, we intend to study 3-pre-Bihom-ρ-Lie algebras and their properties. Also, the Rota-Baxer operators in this section help us to make the 3-pre-Bihom-ρ-Lie algebra structure.

Definition 3.1 *Triplet* $(\mathcal{B},\{\cdot,\cdot,\cdot\},\rho)$ *consisting of a G-graded vector space* $\mathcal{B}$*, a two-cycle* ρ *and a* 3*-linear map* $\{\cdot,\cdot,\cdot\}:\otimes^3\mathcal{B}\to\mathcal{B}$ *satisfying the following relations*

$$\begin{aligned}&|\{f_1,f_2,f_3\}|=|f_1|+|f_2|+|f_3|,\\&\{f_1,f_2,f_3\}=-\rho(f_1,f_2)\{f_2,f_1,f_3\},\\&\{f_1,f_2,\{g_1,g_2,g_3\}\}=\{[f_1,f_2,g_1]_c,g_2,g_3\}+\rho(f_1+f_2,g_1)\{g_1,[f_1,f_2,g_2]_c,g_3\}\\&\qquad+\rho(f_1+f_2,g_1+g_2)\{g_1,g_2,\{f_1,f_2,g_3\}\},\\&\{[f_1,f_2,f_3]_c,g_1,g_2\}=\{f_1,f_2,\{f_3,g_1,g_2\}\}+\rho(f_1,f_2+f_3)\{f_2,f_3,\{f_1,g_1,g_2\}\}\\&\qquad+\rho(f_1+f_2,f_3)\{f_3,f_1,\{f_2,g_1,g_2\}\},\end{aligned}$$

where

$$[f_1,f_2,f_3]_c=\{f_1,f_2,f_3\}+\rho(f_1,f_2+f_3)\{f_2,f_3,f_1\}+\rho(f_1+f_2,f_3)\{f_3,f_1,f_2\}$$

is called a 3*-pre-*ρ*-Lie algebra.*

Definition 3.2 *Multiple* $(\mathcal{B},\{\cdot,\cdot,\cdot\}^{\alpha,\beta},\rho,\alpha,\beta)$ *is called a* 3*-pre-Bihom-*ρ*-Lie algebra if*

(i) $\mathcal{B}$ *is a G-graded vector space,*

(ii) $\alpha,\beta:\mathcal{B}\to\mathcal{B}$ *are even linear maps,*

(iii) $\{\cdot,\cdot,\cdot\}^{\alpha,\beta}:\otimes^3\mathcal{B}\to\mathcal{B}$ *is a* 3*-linear map satisfying the following relations*

$$\begin{aligned}|\{f_1,f_2,f_3\}^{\alpha,\beta}|&=|f_1|+|f_2|+|f_3|,\\\{\beta(f_1),\beta(f_2),\alpha(f_3)\}^{\alpha,\beta}&=-\rho(f_1,f_2)\{\beta(f_2),\beta(f_1),\alpha(f_3)\}^{\alpha,\beta}\\&=-\rho(f_2,f_3)\{\beta(f_1),\beta(f_3),\alpha(f_2)\}^{\alpha,\beta},\end{aligned}$$

$$\alpha\{f_1,f_2,f_3\}^{\alpha,\beta}=\{\alpha(f_1),\alpha(f_2),\alpha(f_3)\}^{\alpha,\beta},$$
$$\beta\{f_1,f_2,f_3\}^{\alpha,\beta}=\{\beta(f_1),\beta(f_2),\beta(f_3)\}^{\alpha,\beta},$$
$$\begin{aligned}&\{\alpha\beta(f_1),\alpha\beta(f_2),\{\alpha(g_1),\alpha(g_2),g_3\}^{\alpha,\beta}\}^{\alpha,\beta}\\=&\{[\beta(f_1),\beta(f_2),\alpha(g_1)]_c^{\alpha,\beta},\alpha\beta(g_2),\beta(g_3)\}^{\alpha,\beta}\\&+\rho(f_1+f_2,g_1)\{\alpha\beta(g_1),[\beta(f_1),\beta(f_2),\alpha(g_2)]_c^{\alpha,\beta},\beta(g_3)\}^{\alpha,\beta}\\&+\rho(f_1+f_2,g_1+g_2)\{\alpha\beta(g_1),\alpha\beta(g_2),\{\alpha(f_1),\alpha(f_2),g_3\}^{\alpha,\beta}\}^{\alpha,\beta},\end{aligned}\tag{3.1}$$
$$\begin{aligned}&\{[g_1,g_2,g_3]_c^{\alpha,\beta},\alpha\beta(f_1),\beta(f_2)\}^{\alpha,\beta}\\=&\rho(g_1,g_2+g_3)\{\alpha\beta(g_2),\alpha\beta(g_3),\{\alpha(g_1),\alpha(f_1),f_2\}^{\alpha,\beta}\}^{\alpha,\beta}\\&-\rho(g_2,g_3)\{\alpha\beta(g_1),\alpha\beta(g_3),\{\alpha(g_2),\alpha(f_1),f_2\}^{\alpha,\beta}\}^{\alpha,\beta}\\&+\{\alpha\beta(g_1),\alpha\beta(g_2),\{\alpha(g_3),\alpha(f_1),f_2\}^{\alpha,\beta}\}^{\alpha,\beta},\end{aligned}\tag{3.2}$$
$$\begin{aligned}&\{\alpha\beta(f_2),[\beta(g_1),\beta(g_2),\alpha(g_3)]_c^{\alpha,\beta},\beta(f_1)\}^{\alpha,\beta}\\=&\rho(f_2,g_1+g_2)\{\alpha\beta(g_1),\alpha\beta(g_2),\{\alpha(f_2),\alpha(g_3),f_1\}^{\alpha,\beta}\}^{\alpha,\beta}\\&+\rho(f_2+g_1,g_2+g_3)\{\alpha\beta(g_2),\alpha\beta(g_3),\{\alpha(f_2),\alpha(g_1),f_1\}^{\alpha,\beta}\}^{\alpha,\beta}\\&-\rho(f_2,g_1+g_3)\rho(g_2,g_3)\{\alpha\beta(g_1),\alpha\beta(g_3),\{\alpha(f_2),\alpha(g_2),f_1\}^{\alpha,\beta}\}^{\alpha,\beta},\end{aligned}\tag{3.3}$$

where

$$\begin{aligned}[f_1,f_2,f_3]_c^{\alpha,\beta}=&\{f_1,f_2,f_3\}^{\alpha,\beta}+\rho(f_1,f_2+f_3)\{f_2,\alpha^{-1}\beta(f_3),\alpha\beta^{-1}(f_1)\}^{\alpha,\beta}\\&+\rho(f_1+f_2,f_3)\{\alpha^{-1}\beta(f_3),f_1,\alpha\beta^{-1}(f_2)\}^{\alpha,\beta}.\end{aligned}$$

Proposition 3.1 *Let $(\mathcal{B},\{\cdot,\cdot,\cdot\}^{\alpha,\beta},\rho,\alpha,\beta)$ be a 3-pre-Bihom-ρ-Lie algebra. Then $(\mathcal{B}, [\cdot,\cdot,\cdot]_c^{\alpha,\beta},\rho,\alpha,\beta)$ is a 3-Bihom-ρ-Lie algebra, which is called the sub-adjacent 3-Bihom-ρ-Lie algebra of the 3-pre-Bihom-ρ-Lie algebra $(\mathcal{B},\{\cdot,\cdot,\cdot\}^{\alpha,\beta},\rho,\alpha,\beta)$.*

Proof By the definition of 3-Bihom-ρ-Lie algebra, we have

$$\begin{aligned}&[\beta^2(f_1),\beta^2(f_2),[\beta(g_1),\beta(g_2),\alpha(g_3)]_c^{\alpha,\beta}]_c^{\alpha,\beta}\\=&\{\beta^2(f_1),\beta^2(f_2),\{\beta(g_1),\beta(g_2),\alpha(g_3)\}^{\alpha,\beta}\}^{\alpha,\beta}\\&+\rho(f_1,f_2+g_1+g_2+g_3)\{\beta^2(f_2),\{\alpha^{-1}\beta^2(g_1),\alpha^{-1}\beta^2(g_2),\beta(g_3)\}^{\alpha,\beta},\alpha\beta(f_1)\}^{\alpha,\beta}\\&+\rho(f_1+f_2,g_1+g_2+g_3)\{\{\alpha^{-1}\beta^2(g_1),\alpha^{-1}\beta^2(g_2),\beta(g_3)\}^{\alpha,\beta},\beta^2(f_1),\alpha\beta(f_2)\}^{\alpha,\beta}\\&+\rho(g_1,g_2+g_3)\{\beta^2(f_1),\beta^2(f_2),\{\beta(g_2),\beta(g_3),\alpha(g_1)\}^{\alpha,\beta}\}^{\alpha,\beta}\\&+\rho(g_1,g_2+g_3)\rho(f_1,f_2+g_1+g_2+g_3)\{\beta^2(f_2),\{\alpha^{-1}\beta^2(g_2),\alpha^{-1}\beta^2(g_3),\beta(g_1)\}^{\alpha,\beta},\beta^2(f_1)\}^{\alpha,\beta}\\&+\rho(g_1+g_2,g_3)\{\beta^2(f_1),\beta^2(f_2),\{\beta(g_3),\beta(g_1),\alpha(g_2)\}^{\alpha,\beta}\}^{\alpha,\beta}\\&+\rho(g_1+g_2,g_3)\rho(f_1,f_2+g_1+g_2+g_3)\{\beta^2(f_2),\{\alpha^{-1}\beta^2(g_3),\alpha^{-1}\beta^2(g_1),\beta(g_2)\}^{\alpha,\beta},\alpha\beta(f_1)\}^{\alpha,\beta}\\&+\rho(g_1+g_2,g_3)\rho(f_1+f_2,g_1+g_2+g_3)\{\{\alpha^{-1}\beta^2(g_3),\alpha^{-1}\beta^2(g_1),\beta(g_2)\}^{\alpha,\beta},\beta^2(f_1),\alpha\beta(f_2)\}^{\alpha,\beta}\\&+\rho(g_1,g_2+g_3)\rho(f_1+f_2,g_1+g_2+g_3)\{\{\alpha^{-1}\beta^2(g_2),\alpha^{-1}\beta^2(g_3),\beta(g_1)\}^{\alpha,\beta},\beta^2(f_1),\alpha\beta(f_2)\}^{\alpha,\beta}.\end{aligned}$$

Also

$$\rho(f_1+f_2+g_1,g_2+g_3)[\beta^2(g_2),\beta^2(g_3),[\beta(f_1),\beta(f_2),\alpha(g_1)]_c^{\alpha,\beta}]_c^{\alpha,\beta}$$

$$
\begin{aligned}
&= \rho(f_1+f_2+g_1, g_2+g_3)\{\beta^2(g_2), \beta^2(g_3), \{\beta(f_1), \beta(f_2), \alpha(g_1)\}^{\alpha,\beta}\}^{\alpha,\beta}\\
&\quad + \rho(f_1+f_2+g_1, g_3)\rho(g_2,g_3)\{\beta^2(g_3), \{\alpha^{-1}\beta^2(f_1), \alpha^{-1}\beta^2(f_2), \beta(g_1)\}^{\alpha,\beta}, \alpha\beta(g_2)\}^{\alpha,\beta}\\
&\quad + \{\{\alpha^{-1}\beta^2(f_1), \alpha^{-1}\beta^2(f_2), \beta(g_1)\}^{\alpha,\beta}, \beta^2(g_2), \alpha\beta(g_3)\}^{\alpha,\beta}\\
&\quad + \rho(f_1+f_2+g_1, g_2+g_3)\rho(f_1, f_2+g_1)\{\beta^2(g_2), \beta^2(g_3), \{\beta(f_2), \beta(g_1), \alpha(f_1)\}^{\alpha,\beta}\}^{\alpha,\beta}\\
&\quad + \rho(f_1+f_2+g_1, g_3)\rho(f_1, f_2+g_1)\rho(g_2,g_3)\\
&\quad \cdot \{\beta^2(g_3), \{\alpha^{-1}\beta^2(f_2), \alpha^{-1}\beta^2(g_1), \beta(f_1)\}^{\alpha,\beta}, \beta^2(g_2)\}^{\alpha,\beta}\\
&\quad + \rho(f_1, f_2+g_1)\{\{\alpha^{-1}\beta^2(f_2), \alpha^{-1}\beta^2(g_1), \beta(f_1)\}^{\alpha,\beta}, \beta^2(g_2), \alpha\beta(g_3)\}^{\alpha,\beta}\\
&\quad + \rho(f_1+f_2+g_1, g_2+g_3)\rho(f_1+f_2, g_1)\{\beta^2(g_2), \beta^2(g_3), \{\beta(g_1), \beta(f_1), \alpha(f_2)\}^{\alpha,\beta}\}^{\alpha,\beta}\\
&\quad + \rho(f_1+f_2+g_1, g_3)\rho(f_1+f_2, g_1)\rho(g_2,g_3)\{\beta^2(g_3),\\
&\quad \cdot \{\alpha^{-1}\beta^2(g_1), \alpha^{-1}\beta^2(f_1), \beta(f_2)\}^{\alpha,\beta}, \alpha\beta(g_2)\}^{\alpha,\beta}\\
&\quad + \rho(f_1+f_2, g_1)\{\{\alpha^{-1}\beta^2(g_1), \alpha^{-1}\beta^2(f_1), \beta(f_2)\}^{\alpha,\beta}, \beta^2(g_2), \alpha\beta(g_3)\}^{\alpha,\beta},
\end{aligned}
$$

and

$$
\begin{aligned}
&\rho(f_1+f_2, g_1+g_3)\rho(g_2,g_3)[\beta^2(g_1), \beta^2(g_3), [\beta(f_1), \beta(f_2), \alpha(g_2)]_c^{\alpha,\beta}]_c^{\alpha,\beta}\\
&= \rho(f_1+f_2, g_1+g_3)\rho(g_2,g_3)\{\beta^2(g_1), \beta^2(g_3), \{\beta(f_1), \beta(f_2), \alpha(g_2)\}^{\alpha,\beta}\}^{\alpha,\beta}\\
&\quad + \rho(f_1+f_2+g_2, g_3)\rho(g_1, g_2+g_3)\{\beta^2(g_3), \{\alpha^{-1}\beta^2(f_1), \alpha^{-1}\beta^2(f_2), \beta(g_2)\}^{\alpha,\beta}, \alpha\beta(g_1)\}^{\alpha,\beta}\\
&\quad + \rho(g_1,g_2)\{\{\alpha^{-1}\beta^2(f_1), \alpha^{-1}\beta^2(f_2), \beta(g_2)\}^{\alpha,\beta}, \beta^2(g_1), \alpha\beta(g_3)\}^{\alpha,\beta}\\
&\quad + \rho(f_1+f_2, g_1+g_3)\rho(g_2,g_3)\rho(f_1, f_2+g_2)\{\beta^2(g_1), \beta^2(g_3), \{\beta(f_2), \beta(g_2), \alpha(f_1)\}^{\alpha,\beta}\}^{\alpha,\beta}\\
&\quad + \rho(f_1+f_2+g_2, g_3)\rho(f_1, f_2+g_2)\rho(g_1, g_2+g_3)\\
&\quad \cdot \{\beta^2(g_3), \{\alpha^{-1}\beta^2(f_2), \alpha^{-1}\beta^2(g_2), \beta(f_1)\}^{\alpha,\beta}, \beta^2(g_1)\}^{\alpha,\beta}\\
&\quad + \rho(f_1, f_2+g_2)\rho(g_1,g_2)\{\{\alpha^{-1}\beta^2(f_2), \alpha^{-1}\beta^2(g_2), \beta(f_1)\}^{\alpha,\beta}, \beta^2(g_1), \alpha\beta(g_3)\}^{\alpha,\beta}\\
&\quad + \rho(f_1+f_2, g_1+g_3)\rho(g_2,g_3)\rho(f_1+f_2, g_2)\{\beta^2(g_1), \beta^2(g_3), \{\beta(g_2), \beta(f_1), \alpha(f_2)\}^{\alpha,\beta}\}^{\alpha,\beta}\\
&\quad + \rho(f_1+f_2+g_2, g_3)\rho(g_1, g_2+g_3)\rho(f_1+f_2, g_2)\\
&\quad \cdot \{\beta^2(g_3), \{\alpha^{-1}\beta^2(g_2), \alpha^{-1}\beta^2(f_1), \beta(f_2)\}^{\alpha,\beta}, \alpha\beta(g_1)\}^{\alpha,\beta}\\
&\quad + \rho(g_1,g_2)\rho(f_1+f_2, g_2)\{\{\alpha^{-1}\beta^2(g_2), \alpha^{-1}\beta^2(f_1), \beta(f_2)\}^{\alpha,\beta}, \beta^2(g_1), \alpha\beta(g_3)\}^{\alpha,\beta}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&\rho(f_1+f_2, g_1+g_2)[\beta^2(g_1), \beta^2(g_2), [\beta(f_1), \beta(f_2), \alpha(g_3)]_c^{\alpha,\beta}]_c^{\alpha,\beta}\\
&= \rho(f_1+f_2, g_1+g_2)\{\beta^2(g_1), \beta^2(g_2), \{\beta(f_1), \beta(f_2), \alpha(g_3)\}^{\alpha,\beta}\}^{\alpha,\beta}\\
&\quad + \rho(f_1+f_2, g_2)\rho(g_1, g_2+g_3)\{\beta^2(g_2), \{\alpha^{-1}\beta^2(f_1), \alpha^{-1}\beta^2(f_2), \beta(g_3)\}^{\alpha,\beta}, \alpha\beta(g_1)\}^{\alpha,\beta}\\
&\quad + \rho(g_1+g_2, g_3)\{\{\alpha^{-1}\beta^2(f_1), \alpha^{-1}\beta^2(f_2), \beta(g_3)\}^{\alpha,\beta}, \beta^2(g_1), \alpha\beta(g_2)\}^{\alpha,\beta}\\
&\quad + \rho(f_1+f_2, g_1+g_2)\rho(f_1, f_2+g_3)\{\beta^2(g_1), \beta^2(g_2), \{\beta(f_2), \beta(g_3), \alpha(f_1)\}^{\alpha,\beta}\}^{\alpha,\beta}\\
&\quad + \rho(f_1+f_2, g_2)\rho(f_1, f_2+g_3)\rho(g_1, g_2+g_3)\\
&\quad \cdot \{\beta^2(g_2), \{\alpha^{-1}\beta^2(f_2), \alpha^{-1}\beta^2(g_3), \beta(f_1)\}^{\alpha,\beta}, \beta^2(g_1)\}^{\alpha,\beta}
\end{aligned}
$$

$$
\begin{aligned}
&+\rho(f_1,f_2+g_3)\rho(g_1+g_2,g_3)\{\{\alpha^{-1}\beta^2(f_2),\alpha^{-1}\beta^2(g_3),\beta(f_1)\}^{\alpha,\beta},\beta^2(g_1),\alpha\beta(g_2)\}^{\alpha,\beta}\\
&+\rho(f_1+f_2,g_1+g_2)\rho(f_1+f_2,g_3)\{\beta^2(g_1),\beta^2(g_2),\{\beta(g_3),\beta(f_1),\alpha(f_2)\}^{\alpha,\beta}\}^{\alpha,\beta}\\
&+\rho(f_1+f_2,g_2)\rho(g_1,g_2+g_3)\rho(f_1+f_2,g_3)\\
&\cdot\{\beta^2(g_2),\{\alpha^{-1}\beta^2(g_3),\alpha^{-1}\beta^2(f_1),\beta(f_2)\}^{\alpha,\beta},\alpha\beta(g_1)\}^{\alpha,\beta}\\
&+\rho(f_1+f_2+g_1+g_2,g_3)\{\{\alpha^{-1}\beta^2(g_3),\alpha^{-1}\beta^2(f_1),\beta(f_2)\}^{\alpha,\beta},\beta^2(g_1),\alpha\beta(g_2)\}^{\alpha,\beta}.
\end{aligned}
$$

Since $(\mathcal{B},\{\cdot,\cdot,\cdot\}^{\alpha,\beta},\rho,\alpha,\beta)$ is a 3-pre-Bihom-ρ-Lie algebra, by (3.1)–(3.3) we have

$$
\begin{aligned}
&[\beta^2(f_1),\beta^2(f_2),[\beta(g_1),\beta(g_2),\alpha(g_3)]_c^{\alpha,\beta}]_c^{\alpha,\beta}\\
=&\rho(f_1+f_2+g_1,g_2+g_3)[\beta^2(g_2),\beta^2(g_3),[\beta(f_1),\beta(f_2),\alpha(g_1)]_c^{\alpha,\beta}]_c^{\alpha,\beta}\\
&-\rho(f_1+f_2,g_1+g_3)\rho(g_2,g_3)[\beta^2(g_1),\beta^2(g_3),[\beta(f_1),\beta(f_2),\alpha(g_2)]_c^{\alpha,\beta}]_c^{\alpha,\beta}\\
&+\rho(f_1+f_2,g_1+g_2)[\beta^2(g_1),\beta^2(g_2),[\beta(f_1),\beta(f_2),\alpha(g_3)]_c^{\alpha,\beta}]_c^{\alpha,\beta}.
\end{aligned}
$$

Lemma 3.1 *Consider the following assumptions*:

(i) $(\mathcal{B},\{\cdot,\cdot,\cdot\},\rho)$ *is a 3-pre-ρ-Lie algebra.*

(ii) $\alpha,\beta:\mathcal{B}\to\mathcal{B}$ *are two even linear maps satisfying*

$$
\begin{aligned}
&\alpha\circ\beta=\beta\circ\alpha,\\
&\alpha\{f_1,f_2,f_3\}^{\alpha,\beta}=\{\alpha(f_1),\alpha(f_2),\alpha(f_3)\}^{\alpha,\beta},\\
&\beta\{f_1,f_2,f_3\}^{\alpha,\beta}=\{\beta(f_1),\beta(f_2),\beta(f_3)\}^{\alpha,\beta}.
\end{aligned}
$$

If we define a multiplication $\{\cdot,\cdot,\cdot\}_\dagger^{\alpha,\beta}$ *on* $\mathcal{B}$ *by*

$$
\{f_1,f_2,f_3\}_\dagger^{\alpha,\beta}=\{\alpha(f_1),\alpha(f_2),\beta(f_3)\}, \tag{3.4}
$$

then $(\mathcal{B},\{\cdot,\cdot,\cdot\}_\dagger^{\alpha,\beta},\rho,\alpha,\beta)$ *is a 3-pre-Bihom-ρ-Lie algebra.*

Proof By Definitions 3.1–3.2, we prove (3.2) and the other assertions are similar. The left hand side of (3.2) is equivalent to

$$
\begin{aligned}
&\{[\beta(g_1),\beta(g_2),\alpha(g_3)]_c^{\alpha,\beta},\alpha\beta(f_1),\beta(f_2)\}_\dagger^{\alpha,\beta}\\
=&\{\{\beta(g_1),\beta(g_2),\alpha(g_3)\}_\dagger^{\alpha,\beta},\alpha\beta(f_1),\beta(f_2)\}_\dagger^{\alpha,\beta}\\
&+\rho(g_1,g_2+g_3)\{\{\beta(g_2),\beta(g_3),\alpha(g_1)\}_\dagger^{\alpha,\beta},\alpha\beta(f_1),\beta(f_2)\}_\dagger^{\alpha,\beta}\\
&+\rho(g_1+g_2,g_3)\{\{\beta(g_3),\beta(g_1),\alpha(g_2)\}_\dagger^{\alpha,\beta},\alpha\beta(f_1),\beta(f_2)\}_\dagger^{\alpha,\beta}\\
=&\{\{\alpha^2\beta(g_1),\alpha^2\beta(g_2),\alpha^2\beta(g_3)\},\alpha^2\beta(f_1),\beta^2(f_2)\}\\
&+\rho(g_1,g_2+g_3)\{\{\alpha^2\beta(g_2),\alpha^2\beta(g_3),\alpha^2\beta(g_1)\},\alpha^2\beta(f_1),\beta^2(f_2)\}\\
&+\rho(g_1+g_2,g_3)\{\{\alpha^2\beta(g_3),\alpha^2\beta(g_1),\alpha^2\beta(g_2)\},\alpha^2\beta(f_1),\beta^2(f_2)\}\\
=&\{[\alpha^2\beta(g_1),\alpha^2\beta(g_2),\alpha^2\beta(g_3)]_c,\alpha^2\beta(f_1),\beta^2(f_2)\}.
\end{aligned}
$$

The right hand side is equivalent to

$$
\rho(g_1,g_2+g_3)\{\alpha\beta(g_2),\alpha\beta(g_3),\{\alpha(g_1),\alpha(f_1),f_2\}_\dagger^{\alpha,\beta}\}_\dagger^{\alpha,\beta}
$$

$$
\begin{aligned}
&+\{\alpha\beta(g_1),\alpha\beta(g_2),\{\alpha(g_3),\alpha(f_1),f_2\}_{\dagger}^{\alpha,\beta}\}_{\dagger}^{\alpha,\beta}\\
&-\rho(g_2,g_3)\{\alpha\beta(g_1),\alpha\beta(g_3),\{\alpha(g_2),\alpha(f_1),f_2\}_{\dagger}^{\alpha,\beta}\}_{\dagger}^{\alpha,\beta}\\
=&\ \rho(g_1,g_2+g_3)\{\alpha^2\beta(g_2),\alpha^2\beta(g_3),\{\alpha^2\beta(g_1),\alpha^2\beta(f_1),\beta^2(f_2)\}^{\alpha,\beta}\}^{\alpha,\beta}\\
&-\rho(g_2,g_3)\{\alpha^2\beta(g_1),\alpha^2\beta(g_3),\{\alpha^2\beta(g_2),\alpha^2\beta(f_1),\beta^2(f_2)\}^{\alpha,\beta}\}^{\alpha,\beta}\\
&+\{\alpha^2\beta(g_1),\alpha^2\beta(g_2),\{\alpha^2\beta(g_3),\alpha^2\beta(f_1),\beta^2(f_2)\}^{\alpha,\beta}\}^{\alpha,\beta}.
\end{aligned}
$$

Since $\{\cdot,\cdot,\cdot\}$ is a 3-pre-ρ-Lie algebra, we have

$$
\begin{aligned}
&\{[\alpha^2\beta(g_1),\alpha^2\beta(g_2),\alpha^2\beta(g_3)]_c,\alpha^2\beta(f_1),\beta^2(f_2)\}\\
=&\ \rho(g_1,g_2+g_3)\{\alpha^2\beta(g_2),\alpha^2\beta(g_3),\{\alpha^2\beta(g_1),\alpha^2\beta(f_1),\beta^2(f_2)\}^{\alpha,\beta}\}^{\alpha,\beta}\\
&-\rho(g_2,g_3)\{\alpha^2\beta(g_1),\alpha^2\beta(g_3),\{\alpha^2\beta(g_2),\alpha^2\beta(f_1),\beta^2(f_2)\}^{\alpha,\beta}\}^{\alpha,\beta}\\
&+\{\alpha^2\beta(g_1),\alpha^2\beta(g_2),\{\alpha^2\beta(g_3),\alpha^2\beta(f_1),\beta^2(f_2)\}^{\alpha,\beta}\}^{\alpha,\beta}.
\end{aligned}
$$

Therefore, the proof is done.

Remark 3.1 If $(\mathcal{B},\{\cdot,\cdot,\cdot\}^{\alpha,\beta},\rho,\alpha,\beta)$ is a 3-pre-Bihom-ρ-Lie algebra and $\widehat{\alpha},\widehat{\beta}$ are two even morphisms of 3-pre-Bihom-ρ-Lie algebras which all of the maps $\alpha,\beta,\widehat{\alpha},\widehat{\beta}$ commute with each other, then $(\mathcal{B},\langle\cdot,\cdot,\cdot\rangle^{\widehat{\alpha},\widehat{\beta}},\rho,\alpha\circ\widehat{\alpha},\beta\circ\widehat{\beta})$ is a 3-pre-Bihom-ρ-Lie algebra, where

$$\langle f_1,f_2,f_3\rangle^{\widehat{\alpha},\widehat{\beta}}=\{\widehat{\alpha}(f_1),\widehat{\alpha}(f_2),\widehat{\beta}(f_3)\}^{\alpha,\beta}.$$

Lemma 3.2 *Consider the following assumptions*:
(i) $(\mathcal{B},[\cdot,\cdot,\cdot]_{\mathcal{B}},\rho,\alpha,\beta)$ *is a 3-Bihom-ρ-Lie algebra,*
(ii) $P:\mathcal{B}\to\mathcal{B}$ *is a Rota-Baxer operator of weight* 0*,*
(iii) $P\circ\alpha=\alpha\circ P$, $P\circ\beta=\beta\circ P$.
If we define a multiplication $\{\cdot,\cdot,\cdot\}^{\alpha,\beta}$ on $\mathcal{B}$ by

$$\{f_1,f_2,f_3\}^{\alpha,\beta}=[P(f_1),P(f_2),f_3]_{\mathcal{B}},\tag{3.5}$$

then $(\mathcal{B},\{\cdot,\cdot,\cdot\}^{\alpha,\beta},\rho,\alpha,\beta)$ is a 3-pre-Bihom-ρ-Lie algebra.

Proof We prove (3.2) and the others are similar. We have

$$
\begin{aligned}
&\{[\beta(g_1),\beta(g_2),\alpha(g_3)]_c^{\alpha,\beta},\alpha\beta(f_1),\beta(f_2)\}^{\alpha,\beta}\\
=&\ \{\{\beta(g_1),\beta(g_2),\alpha(g_3)\}^{\alpha,\beta},\alpha\beta(f_1),\beta(f_2)\}^{\alpha,\beta}\\
&+\rho(g_1,g_2+g_3)\{\{\beta(g_2),\beta(g_3),\alpha(g_1)\}^{\alpha,\beta},\alpha\beta(f_1),\beta(f_2)\}^{\alpha,\beta}\\
&+\rho(g_1+g_2,g_3)\{\{\beta(g_3),\beta(g_1),\alpha(g_2)\}^{\alpha,\beta},\alpha\beta(f_1),\beta(f_2)\}^{\alpha,\beta}.
\end{aligned}
$$

By (3.5), we can rewrite the above relation as follows

$$
\begin{aligned}
&\{[\beta(g_1),\beta(g_2),\alpha(g_3)]_c^{\alpha,\beta},\alpha\beta(f_1),\beta(f_2)\}^{\alpha,\beta}\\
=&\ [P[P\beta(g_1),P\beta(g_2),\alpha(g_3)]_{\mathcal{B}},P\alpha\beta(f_1),\beta(f_2)]_{\mathcal{B}}\\
&+\rho(g_1,g_2+g_3)[P[P\beta(g_2),P\beta(g_3),\alpha(g_1)]_{\mathcal{B}},P\alpha\beta(f_1),\beta(f_2)]_{\mathcal{B}}
\end{aligned}
$$

$$+ \rho(g_1 + g_2, g_3)[P[P\beta(g_3), P\beta(g_1), \alpha(g_2)]_{\mathcal{B}}, P\alpha\beta(f_1), \beta(f_2)]_{\mathcal{B}}.$$

Since P is a Rota-Baxer operator, we have

$$\{[\beta(g_1), \beta(g_2), \alpha(g_3)]_c^{\alpha,\beta}, \alpha\beta(f_1), \beta(f_2)\}^{\alpha,\beta} = [[P\beta(g_1), P\beta(g_2), P\alpha(g_3)]_{\mathcal{B}}, P\alpha\beta(f_1), \beta(f_2)]_{\mathcal{B}}.$$

The ρ-Bihom-skew symmetry condition and relations $P \circ \alpha = \alpha \circ P$ and $P \circ \beta = \beta \circ P$ help us find the following relation

$$\begin{aligned}&\{[\beta(g_1), \beta(g_2), \alpha(g_3)]_c^{\alpha,\beta}, \alpha\beta(f_1), \beta(f_2)\}^{\alpha,\beta}\\ &= \rho(g_1 + g_2 + g_3, f_1 + f_2)[\alpha\beta P(f_1), \alpha^{-1}\beta^2(f_2), [\alpha P(g_1), \alpha P(g_2), \alpha^2\beta^{-1}P(g_3)]_{\mathcal{B}}]_{\mathcal{B}}.\end{aligned}$$

On the other hand, Proposition 2.3 allows us to consider $\mathcal{B}$ as a 3-Bihom-ρ-Leibniz algebra, so

$$\begin{aligned}&\{[\beta(g_1), \beta(g_2), \alpha(g_3)]_c^{\alpha,\beta}, \alpha\beta(f_1), \beta(f_2)\}^{\alpha,\beta}\\ &= \rho(g_1 + g_2 + g_3, f_1 + f_2)[[\beta P(f_1), \alpha^{-2}\beta^2(f_2), \alpha P(g_1)]_{\mathcal{B}}, \alpha\beta P(g_2), \alpha^2 P(g_3)]_{\mathcal{B}}\\ &\quad + \rho(g_1 + g_2 + g_3, f_1 + f_2)\rho(f_1 + f_2, g_1)[\alpha\beta P(g_1), [\beta P(f_1), \alpha^{-2}\beta^2(f_2), \alpha P(g_2)]_{\mathcal{B}}, \alpha^2 P(g_3)]_{\mathcal{B}}\\ &\quad + \rho(g_1 + g_2 + g_3, f_1 + f_2)\rho(f_1 + f_2, g_1 + g_2)\\ &\quad \cdot [\alpha\beta P(g_1), \alpha\beta P(g_2), [\alpha P(f_1), \alpha^{-1}\beta(f_2), \alpha^2\beta^{-1}P(g_3)]_{\mathcal{B}}]_{\mathcal{B}}.\end{aligned}$$

Again, by the ρ-Bihom-skew symmetry condition, we find

$$\begin{aligned}&\{[\beta(g_1), \beta(g_2), \alpha(g_3)]_c^{\alpha,\beta}, \alpha\beta(f_1), \beta(f_2)\}^{\alpha,\beta}\\ &= \rho(g_1 + g_2, g_3)[\alpha\beta P(g_2), \alpha\beta P(g_3), [\alpha P(g_1), \alpha P(f_1), f_2]_{\mathcal{B}}]_{\mathcal{B}}\\ &\quad - \rho(g_2, g_3)[\alpha\beta P(g_1), , \alpha\beta P(g_3), [\alpha P(g_2), \alpha P(f_1), f_2]_{\mathcal{B}}]_{\mathcal{B}}\\ &\quad + [\alpha\beta P(g_1), \alpha\beta P(g_2), [\alpha P(g_3), \alpha P(f_1), f_2]_{\mathcal{B}}]_{\mathcal{B}}.\end{aligned}$$

Therefore, (3.5) gives us

$$\begin{aligned}&\{[\beta(g_1), \beta(g_2), \alpha(g_3)]_c^{\alpha,\beta}, \alpha\beta(f_1), \beta(f_2)\}^{\alpha,\beta}\\ &= \rho(g_1 + g_2, g_3)\{\alpha\beta(g_2), \alpha\beta(g_3), \{\alpha(g_1), \alpha(f_1), f_2\}^{\alpha,\beta}\}^{\alpha,\beta}\\ &\quad - \rho(g_2, g_3)\{\alpha\beta(g_1), \alpha\beta(g_3), \{\alpha(g_2), \alpha(f_1), f_2\}^{\alpha,\beta}\}^{\alpha,\beta}\\ &\quad + \{\alpha\beta(g_1), \alpha\beta(g_2), \{\alpha(g_3), \alpha(f_1), f_2\}^{\alpha,\beta}\}^{\alpha,\beta}.\end{aligned}$$

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