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The Inviscid Limit for the Steady Incompressible Navier-Stokes Equations in the Three Dimension*

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Abstract In this paper, the authors consider the zero-viscosity limit of the three dimensional incompressible steady Navier-Stokes equations in a half space $\mathbb{R}^+ \times \mathbb{R}^2$. The result shows that the solution of three dimensional incompressible steady Navier-Stokes equations converges to the solution of three dimensional incompressible steady Euler equations in Sobolev space as the viscosity coefficient going to zero. The method is based on a new weighted energy estimates and Nash-Moser iteration scheme.

Keywords Navier-Stokes equations, Euler equations, Zero viscosity limit
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1 Introduction and Main Results

In this paper, we consider the vanishing viscosity limit of steady incompressible Navier-Stokes equations:

$$\begin{cases} -\nu\triangle U + U\cdot\nabla U + \nabla P = g^\nu, \\ \nabla\cdot U = 0, \end{cases} \tag{1.1}$$

where $x\in\Omega$, and $\Omega := \mathbb{R}^+\times\mathbb{R}^2$ is a half space, $U:\mathbb{R}^+\times\Omega\to\mathbb{R}^3$ is the fluid velocity, and it is of the form $U(x) = (U_1(x), U_2(x), U_3(x))$, $P(x):\mathbb{R}^+\times\Omega\to\mathbb{R}$ stands for the pressure in the fluid, and the constant ν is the viscosity. The vector field g^ν is an external force and $g^\nu|_{x\in\partial\Omega} = 0$. The divergence free condition in second equation of (1.1) guarantees the incompressibility of the fluid.

We supplement the steady incompressible Navier-Stokes equations (1.1) with the boundary condition

$$U(x)|_{x\in\partial\Omega} = 0, \tag{1.2}$$

that is, for $i = 1, 2, 3$, in x_1 direction

$$U_i(x)|_{x_1=0} = 0, \quad \lim_{x_1\to+\infty} U_i(t,x) = 0,$$

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and the vanishing boundary condition in $\overline{x} := (x_2, x_3)$ direction

$$\lim_{|\overline{x}|\to+\infty} U_i(x) = 0.$$

The pressure takes the form

$$\triangle P(x) = -\sum_{i,j=1}^{n} \frac{\partial U_i}{\partial x_j}\frac{\partial U_j}{\partial x_i}. \tag{1.3}$$

In particularly, when the viscosity coefficient $\nu = 0$ in (1.1), it is the steady equation of incompressible three dimensional Euler equations

$$\begin{cases} U\cdot\nabla U + \nabla P = g^\nu, & x\in\Omega,\\ \nabla\cdot U = 0, & x\in\Omega,\\ U|_{x\in\partial\Omega} = 0, \end{cases} \tag{1.4}$$

where the external force g^ν is the same with the force given in (1.1). One can check that there exist two functions $P(x)$ and g^ν such that the following function is an exact solution of (1.4):

$$U(x) = (U_1(x), U_2(x), U_3(x)), \quad \forall x\in\mathbb{R}^+\times\mathbb{R}^2,$$

where

$$\begin{aligned} U_1(x) &= x_1^q x_2^{2p+1} x_3^{2p+1} \mathrm{e}^{-(x_1^{2(p+1)}+x_2^{2(p+1)}+x_3^{2(p+1)})},\\ U_2(x) &= 2^{-1}(1+p)^{-1}(q-2(p+1)x_1^{2(p+1)-q})x_1^{q-1}x_3^{2p+1}\mathrm{e}^{-(x_1^{2(p+1)}+x_2^{2(p+1)}+x_3^{2(p+1)})},\\ U_3(x) &= -(1+p)^{-1}(q-2(p+1)x_1^{2(p+1)-q})x_1^{q-1}x_2^{2p+1}\mathrm{e}^{-(x_1^{2(p+1)}+x_2^{2(p+1)}+x_3^{2(p+1)})}. \end{aligned}$$

An interesting problem in fluid mechanics is the study of the zero-viscosity limit in the presence of a boundary with certain boundary conditions, such as the non-slip boundary condition and the Dirichlet boundary condition. A nature problem is to study the convergence of the steady solution of 3-d incompressible Navier-Stokes equations (1.1) to the solution of steady Euler equations (1.4) as the viscosity going to zero.

The relationship between the solution of Navier-Stokes equations and the solution of Euler equations is a challenging problem due to the formation of a boundary layer whose thickness is proportional to the square root of the viscosity. For the unsteady equations and in the absence of the boundary, it has been proved that the Navier-Stokes equations converge to the Euler equations in various functional settings (see [1, 3, 14, 28]). However, in the presence of the boundary, the inviscid limit problem will become very complicated due to the appearance of boundary layer. Masmoudi & Rousset [18] introduced the conormal functional space to justify the limit from the incompressible Navier-Stokes equations to the incompressible Euler equations for the Navier slip boundary condition. One can see [11–12, 19] for more results on this boundary condition. For the non-slip boundary condition, Sammartino & Caflisch [26–27] proved the inviscid limit of the incompressible Navier-Stokes equations for well-prepared data with analytic regularity in the half-space. Wang, Wang & Zhang [32] developed an energy method for the inviscid limit problem in the analytic setting to deal with the inviscid limit problem in general domain. Nguyen & Nguyen [23] gave a direct proof of the inviscid limit for general analytic data without having to construct Prandtl's boundary layer correctors. Very recently, Kukavica, Vicol & Wang [15] obtained that the inviscid limit for the Navier-Stokes equations in a half space, and they only required with initial datum that is analytic only close to the boundary of the domain. Meanwhile, it has Sobolev regularity in the complement. We refer to [4, 17, 24–25, 29–31, 33] and references therein for more relevant results.

For the two dimensional stationary equations, Iyer [13] considered the validity of the Prandtl boundary layer theory for steady incompressible Navier-Stokes flows over a rotating disk. Guo & Nguyen [9] constructed general boundary layer expansions to the steady Navier-Stokes equations in a half plane, it required a positive Dirichlet datum for the horizontal velocity. Gerard-Varet & Maekawa [7] gave the inviscid limit problem in Sobolev regularity (H^1-regularity) for a non-trivial class of steady two dimensional Navier-Stokes flows with no-slip boundary condition. Recently, Li, Li & Yan [16] showed the vanishing viscosity limit for homogeneous axisymmetric no-swirl solutions of stationary Navier-Stokes equations. To our knowledge, there is few result on the three dimensional vanishing viscosity limit for the steady Navier-Stokes equations in general case. In this paper, we will deal with this problem.

Let the smooth function $(U^e(x), P^e(x))$ be a solution of the incompressible three dimensional steady Euler equations (1.4) with the external force g^ν (see [5–6] for the existence of smooth solutions of steady 3-d Euler equations). Here the smooth vector function $U^e(x) = (U_1^e(x), U_2^e(x), U_3^e(x))$. Assume that

$$\begin{aligned} \sum_{k=0}^{s} \|\partial_{x_i}^k U_j^e\|_{\mathbb{L}^\infty(\Omega)} &\lesssim c_0, \quad \forall i,j = 1,2,3, \\ \|U^e\|_{H^{s+2}(\Omega)} &\lesssim c_0, \\ \partial_{x_i}^l U^e|_{x\in\partial\Omega} &= 0, \quad 0 \le l \le s \end{aligned} \tag{1.5}$$

for a fixed positive constant c_0.

We now state the main result in this paper.

Theorem 1.1 *Let ε be a small positive constant. Assume that* (1.5) *and the external force $g^\nu|_{x\in\partial\Omega} = 0$ and $\|g\|_{H^s(\Omega)} < \varepsilon$ hold. For any fixed constant $s > 1$, the steady incompressible Navier-Stokes equations* (1.1) *with the boundary conditions* (1.2) *admits Sobolev regular solutions with finite energy of the form*

$$\begin{aligned} U(x) &= U^e(x) + w(x), \\ P(x) &= P^e(x) + \overline{P}(x), \end{aligned}$$

where the reminder term $w(x)$, $\overline{P} \in H^s(\Omega)$.

Moreover, it holds

$$\begin{aligned} &\|w\|_{H^s(\Omega)} \sim \mathcal{O}(\nu), \\ &\|\overline{P}(x)\|_{H^s(\Omega)} \sim \mathcal{O}(\nu) \end{aligned}$$

for any $x \in \Omega$.

Remark 1.1 Let the parameter λ satisfy $1 < \max\{\nu^{-1}, c_0\} \le \lambda < \varepsilon^{-1}$. We will construct the small Sobolev regular solutions of (1.1) by means of the explicit representation formula as follows

$$U^{(\infty)}(x) = U^{(e)}(x) + w^{(0)}(x) + \sum_{m=1}^{\infty} \mathbf{h}^{(m)}(\lambda x) = U^{(e)}(x) + \mathcal{O}(\nu), \tag{1.6}$$

where the function $w^{(0)}(x)$ satisfies the assumption

$$\begin{cases} \nabla \cdot w^{(0)}(x) = 0, \\ \|w^{(0)}\|_{H^s(\Omega)} \lesssim \varepsilon < \nu^{s+2} < \nu^{\frac{1}{2}}, & \text{for } 0 < \nu \ll 1, \\ \partial_{x_i}^l w^{(0)}(x)|_{x\in\partial\Omega} = 0, & 0 \le l \le s \end{cases} \tag{1.7}$$

and

$$\sum_{k=0}^{s} \|\partial_{x_i}^k w_j^{(0)}(x)\|_{\mathbb{L}^\infty} \lesssim \varepsilon_0 < \varepsilon, \quad \forall i, j = 1, 2, 3, \tag{1.8}$$

and $\mathbf{h}^{(m)}(\lambda x)$ $(m = 1, 2, 3, \cdots)$ is obtained by solving the linearized problem with the Dirichlet boundary condition in Sobolev space $H^s(\Omega)$ with $s > 1$,

$$\begin{cases} \mathcal{L}[w^{m-1}]\mathbf{h}^{(m)} = E^{(m-1)}(x), \\ \nabla \cdot \mathbf{h}^{(m)} = 0, \\ \mathbf{h}^{(m)}(\lambda x)|_{x\in\partial\Omega} = 0, \end{cases}$$

and $E^{(m-1)}(x)$ denotes the error term, the linear operator $\mathcal{L}[w^{m-1}]\mathbf{h}^{(m)}$ is defined in (2.9). The index s of Sobolev regularity depends on the higher derivative estimate of solution for the linearized equations. From (1.6), one can see the solution depends on the initial approximation function $w^{(0)}(x)$ strongly. Our proof is based on Nash-Moser iteration scheme, it has been used in [34–39]. For general Nash-Moser implict function theorem, one can see the celebrated work of Nash [22], Moser [20–21], Hörmander [10].

Remark 1.2 When we deal with the higher regularity of linearized equations (2.28), we should notice that $\mathbf{h}^{(m)}(\lambda x)$ is an approximation function which will satisfy the boundary condition $\partial_{x_i}^l \mathbf{h}^{(m)}(\lambda x)|_{x\in\partial\Omega} = 0$ with $i = 1, 2, 3$ and a fixed integer s $(\ge l > 0)$. This is because the external force (error term) $E^{(m-1)}(x)$ satisfies $\partial_{x_i}^l E^{(m-1)}(\lambda x)|_{x\in\partial\Omega} = 0$. We give an exact example to explain it. Let us consider the linear elliptic equation with an external force:

$$-\triangle u = f(x) \tag{1.9}$$

with the non-slip boundary condition (1.2). There exists an external force being of the form

$$f(x) = x_1^p(p(p-1)x_1^{-2} - 4 + 4(x_1^2 + x_2^2))\mathrm{e}^{-x_2^2 - x_3^2}, \quad \forall p > s > 1,$$

such that the linear elliptic equation admits an exact solution

$$u^*(x) = x_1^p \mathrm{e}^{-x_2^2 - x_3^2},$$

which satisfies the non-slip boundary condition (1.2). Moreover, it holds

$$\partial_{x_i}^l u^*(x)|_{x\in\partial\Omega} = 0, \quad \forall 1 \le l \le s.$$

Notations Throughout this paper, let $\Omega := \mathbb{R}^+ \times \mathbb{R}^2$, we denote the usual norm of $\mathbb{L}^2(\Omega)$ and Sobolev space $\mathbb{H}^s(\Omega)$ by $\|\cdot\|_{\mathbb{L}^2}$ and $\|\cdot\|_{\mathbb{H}^s}$, respectively. The norm of Sobolev space $H^s(\Omega) := (\mathbb{H}^s(\Omega))^3$ is denoted by $\|\cdot\|_{H^s}$. The symbol $a \lesssim b$ means that there exists a positive constant C such that $a \le Cb$. $(x_1, x_2, x_3)^{\mathrm{T}}$ denotes the column vector in $\mathbb{R}^3$. The letter C with subscripts denoting dependencies stands for a positive constant that might change its value at each occurrence.

The organization of this paper is as follows. In Section 2, we first give a class of initial approximation functions, then the Carleman-type estimate of solution for the linearized equations about the initial approximation functions is shown. After that, we prove the existence of Sobolev regular solutions for the linearized equations. In Section 3, we establish the general approximation step for the construction of Nash-Moser iteration scheme. This last section shows the convergence of Nash-Moser iteration scheme.

2 The First Approximation Step

We denote the solution of the steady Euler equation (1.4) by $(U^e(x), P^e(x))$, and we set the solution of incompressible steady Navier-Stokes equations (1.1) by

$$U(x) = U^e(x) + w(x), \quad P(x) = P^e(x) + \overline{P}(x),$$

then it holds

$$\begin{cases} -\nu\triangle w + U^e \cdot \nabla w + w \cdot \nabla U^e + w \cdot \nabla w + \nabla\overline{P} = f^\nu, \\ \nabla \cdot w = 0, \\ w|_{x\in\partial\Omega} = 0, \end{cases} \tag{2.1}$$

where

$$f^\nu = \nu\triangle U^e,$$

which satisfies $\nabla \cdot f^\nu = 0$ by $\nabla \cdot U^e = 0$.

The pressure takes the form

$$\overline{P}(x) = -\triangle^{-1}\nabla(U^e \cdot \nabla w + w \cdot \nabla U^e + w \cdot \nabla w). \tag{2.2}$$

We introduce a family of smooth operators possessing the following properties.

Lemma 2.1 (see [2, 10]) *There is a family $\{\Pi_\theta\}_{\theta\ge 1}$ of smoothing operators in the space $H^s(\Omega)$ acting on the class of functions such that*

$$\begin{aligned} &\|\Pi_\theta U\|_{H^{s_1}(\Omega)} \le C\theta^{(s_1-s_2)_+}\|U\|_{H^{k_2}(\Omega)}, \quad \forall s_1,\ s_2 \ge 0, \\ &\|\Pi_\theta U - U\|_{H^{s_1}(\Omega)} \le C\theta^{s_1-s_2}\|U\|_{H^{s_2}(\Omega)}, \quad 0 \le s_1 \le s_2, \\ &\Big\|\frac{\mathrm{d}}{\mathrm{d}\theta}\Pi_\theta U\Big\|_{H^{s_1}(\Omega)} \le C\theta^{(s_1-s_2)_+-1}\|U\|_{H^{s_2}(\Omega)}, \quad \forall s_1,\ s_2 \ge 0, \end{aligned} \tag{2.3}$$

where C is a positive constant and $(s_1 - s_2)_+ := \max(0, s_1 - s_2)$.

In our iteration scheme, we set

$$\theta = N_m = N_0^m, \quad \forall m = 0, 1, 2, \cdots,$$

where N_0 is a fixed positive constant, then by (2.3), it holds

$$\|\Pi_{N_m} U\|_{H^{s_1}(\Omega)} \lesssim N_m^{s_1-s_2} \|U\|_{H^{s_2}(\Omega)}, \quad \forall s_1 \geq s_2. \tag{2.4}$$

We consider the approximation problem of nonlinear equations (2.1) as follows

$$\mathcal{J}(U) := -\nu \triangle w + \Pi_{N_m}(U^e \cdot \nabla w + w \cdot \nabla U^e + w \cdot \nabla w + \nabla \overline{P}) - f^\nu \tag{2.5}$$

with the boundary condition (1.2) and the incompressible condition

$$\nabla \cdot w = 0.$$

2.1 The initial approximation function

Let $s \geq 1$ be a fixed finite constant and $0 < \varepsilon_0 < \varepsilon < \nu^{s+2} \ll 1$. For any $x \in \Omega$, we choose the initial approximation functions

$$w^{(0)}(x) = (w_1^{(0)}(x), w_2^{(0)}(x), w_3^{(0)}(x)) \in H^s(\Omega).$$

Meanwhile, we require

$$\begin{cases} \nabla \cdot w^{(0)}(x) = 0, \\ \|w^{(0)}\|_{H^s} \lesssim \varepsilon_0, \\ \partial_{x_i}^l w^{(0)}(x)|_{x \in \partial\Omega} = 0, \quad 0 \leq l \leq s. \end{cases} \tag{2.6}$$

Moreover, for any fixed constant $s \geq 1$ and $x \in \Omega$ and $i, j = 1, 2, 3$, it also needs the condition

$$\sum_{k=0}^{s} \|\partial_{x_i}^k w_j^{(0)}(x)\|_{\mathbb{L}^\infty} \lesssim \varepsilon_0, \quad \forall i, j = 1, 2, 3 \tag{2.7}$$

and the initial error term

$$\begin{aligned} &\partial_{x_i}^l E^{(0)}(x)|_{x \in \partial\Omega} = 0, \quad 0 \leq l \leq s, \\ &\|E^{(0)}\|_{H^s} \lesssim \varepsilon_0, \end{aligned} \tag{2.8}$$

where $E^{(0)}$ denotes the error term taking the form

$$E^{(0)} := \mathcal{J}(w^{(0)})$$

with

$$P^{(0)}(x) = -\triangle^{-1} \nabla (U^e \cdot \nabla w^{(0)} + w^{(0)} \cdot \nabla U^e + w^{(0)} \cdot \nabla w^{(0)})$$

and

$$E^{(0)} = (E_1^{(0)}, E_2^{(0)}, E_3^{(0)}).$$

2.2 A priori estimate of linearized equations

Let λ be a positive constant, it satisfies

$$1 < \max\{\nu^{-1}, c_0\} \leq \lambda < \varepsilon^{-1}.$$

We now construct the first approximation solution denoted by $w^{(1)}(x)$ of (2.5). The first approximation step between the initial approximation function and first approximation solution is denoted by

$$\mathbf{h}^{(1)}(\lambda x) := w^{(1)}(\lambda x) - w^{(0)}(x),$$

then we linearize nonlinear system (2.5) around $w^{(0)}$ to get the linearized operators as follows

$$\begin{aligned}\mathcal{L}[w^0]\mathbf{h}^{(1)} := & -\nu\lambda^2\triangle\mathbf{h}^{(1)} + \Pi_{N_1}[\lambda((w^{(0)} + U^e)\cdot\nabla)\mathbf{h}^{(1)} \\ & + (\mathbf{h}^{(1)}\cdot\nabla)(w^{(0)} + U^e) + \nabla(\mathcal{D}_{w^{(0)}}P)\mathbf{h}^{(1)}],\end{aligned} \tag{2.9}$$

where $\mathcal{D}_{w^{(0)}}$ denotes the Fréchet derivatives on $w^{(0)}$, and by (2.2), it takes the form

$$\nabla(\mathcal{D}_{w^{(0)}}P)\mathbf{h}^{(1)} := -\triangle^{-1}\nabla(\lambda(U^e + w^{(0)})\cdot\nabla\mathbf{h}^{(1)} + \mathbf{h}^{(1)}\cdot\nabla(U^e + w^{(0)})). \tag{2.10}$$

We now consider the linear system

$$\begin{aligned}&\mathcal{L}[w^0]\mathbf{h}^{(1)} = \Pi_{N_1}E^{(0)}, \\ &\nabla\cdot\mathbf{h}^{(1)} = 0\end{aligned} \tag{2.11}$$

and the boundary condition

$$\mathbf{h}^{(1)}(\lambda x)|_{x\in\partial\Omega} = 0, \tag{2.12}$$

from which, the solution of it gives the first approximation step of nonlinear equations (2.1).

Before we carry out some priori estimates, for $j = 1, 2, 3$, we rewrite equations of (2.11) into the following coupled system

$$\begin{aligned}& -\nu\lambda^2\triangle h_j^{(1)} + \lambda\Pi_{N_1}\sum_{i=1}^{3}(w_i^{(0)} + U_i^e)\partial_{x_i}h_j^{(1)} \\ & + \Pi_{N_1}\sum_{i=1}^{3}h_i^{(1)}\partial_{x_i}(w_j^{(0)} + U_i^e) + \Pi_{N_1}\partial_{x_j}((\mathcal{D}_{w^{(0)}}P)\mathbf{h}^{(1)}) = \Pi_{N_1}E_j^{(0)}\end{aligned} \tag{2.13}$$

with the boundary condition

$$h_j^{(1)}(\lambda x)|_{x\in\partial\Omega} = 0. \tag{2.14}$$

The idea of following estimate of solution for the linear system (2.13) inspired by the Carleman-type estimate.

Lemma 2.2 *Let* $0 < \nu \ll 1$. *Assume that* (1.5) *holds, and the initial approximation function* $w^{(0)}$ *satisfies* (2.6)–(2.8). *Then the solution* $\boldsymbol{h}^{(1)}(\lambda x)$ *of the linear system* (2.13) *satisfies*

$$\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega(\partial_{x_i}h_j^{(1)})^2\mathrm{d}x + \Pi_{N_1}\sum_{j=1}^{3}\int_\Omega(h_j^{(1)})^2\mathrm{d}x \lesssim \Pi_{N_1}\sum_{j=1}^{3}\int_\Omega(E_j^{(0)})^2\mathrm{d}x. \tag{2.15}$$

Proof Let $\psi(x_1)$ be a function defined in $(0, +\infty)$ such that

$$0 < \kappa \le \psi''(x_1) - (\psi'(x_1))^2 < +\infty, \tag{2.16}$$

and $\mathrm{e}^{-\psi(x_1)}$ is bounded in $(0,+\infty)$. The condition (2.16) implies $\psi''(x_1)\geq\kappa$. In fact, there are many functions can satisfy above conditions. For a simple example, we take the function as the form

$$\psi(x_1)=-\ln|\cos(\sqrt{\kappa}x_1)|,\quad x_1\neq 2i\pi+\frac{\pi}{2}\quad\text{for } i\in\mathbb{Z}.$$

Multiplying both sides of equations in (2.13) by $\mathrm{e}^{-\psi(x_1)}h_j^{(1)}$, respectively, then integrating over Ω, by noticing the boundary condition (2.12), for $j=1,2,3$, it holds

$$\begin{aligned}&\nu\lambda^2\sum_{i=1}^{3}\int_\Omega(\partial_{x_i}h_j^{(1)})^2\mathrm{e}^{-\psi(x_1)}\mathrm{d}x\\&+\frac{\nu\lambda^2}{2}\int_\Omega(\psi''(x_1)-(\psi'(x_1))^2)(h_j^{(1)})^2\mathrm{e}^{-\psi(x_1)}\mathrm{d}x\\&+\lambda\Pi_{N_1}\sum_{i=1}^{3}\int_\Omega((w_i^{(0)}+U_i^e)\partial_{x_i}h_j^{(1)})h_j^{(1)}\mathrm{e}^{-\psi(x_1)}\mathrm{d}x\\&+\Pi_{N_1}\sum_{i=1}^{3}\int_\Omega(h_i^{(1)}\partial_{x_i}(w_j^{(0)}+U_j^e))h_j^{(1)}\mathrm{e}^{-\psi(x_1)}\mathrm{d}x\\&+\Pi_{N_1}\int_\Omega\partial_{x_j}((\mathcal{D}_{w^{(0)}}P)\mathbf{h}^{(1)})h_j^{(1)}\mathrm{e}^{-\psi(x_1)}\mathrm{d}x\\=\;&\Pi_{N_1}\int_\Omega E_j^{(0)}h_j^{(1)}\mathrm{e}^{-\psi(x_1)}\mathrm{d}x.\end{aligned}\tag{2.17}$$

We sum up (2.17) from $j=1$ to $j=3$, then it holds

$$\begin{aligned}&\nu\lambda^2\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega(\partial_{x_i}h_j^{(1)})^2\mathrm{e}^{-\psi(x_1)}\mathrm{d}x\\&+\frac{\nu\lambda^2}{2}\sum_{j=1}^{3}\int_\Omega(\psi''(x_1)-(\psi'(x_1))^2)(h_j^{(1)})^2\mathrm{e}^{-\psi(x_1)}\mathrm{d}x\\&+\lambda\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega((w_i^{(0)}+U_i^e)\partial_{x_i}h_j^{(1)})h_j^{(1)}\mathrm{e}^{-\psi(x_1)}\mathrm{d}x\\&+\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega(h_i^{(1)}\partial_{x_i}(w_j^{(0)}+U_j^e))h_j^{(1)}\mathrm{e}^{-\psi(x_1)}\mathrm{d}x\\&+\Pi_{N_1}\sum_{j=1}^{3}\int_\Omega\partial_{x_j}((\mathcal{D}_{w^{(0)}}P)\mathbf{h}^{(1)})h_j^{(1)}\mathrm{e}^{-\psi(x_1)}\mathrm{d}x\\=\;&\Pi_{N_1}\int_\Omega E_j^{(0)}h_j^{(1)}\mathrm{e}^{-\psi(x_1)}\mathrm{d}x.\end{aligned}\tag{2.18}$$

On one hand, note that we have chosen the initial approximation function $w^{(0)}$ satisfying (2.6)–(2.8). We integrate by parts to get

$$\begin{aligned}&\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega((w_i^{(0)}+U_i^e)\partial_{x_i}h_j^{(1)})h_j^{(1)}\mathrm{e}^{-\psi(x_1)}\mathrm{d}x\\=&-\frac{1}{2}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega\partial_{x_i}(w_i^{(0)}+U_i^e)(h_j^{(1)})^2\mathrm{e}^{-\psi(x_1)}\mathrm{d}x\end{aligned}$$

$$+\frac{1}{2}\sum_{j=1}^{3}\int_{\Omega}\psi'(x_1)(w_1^{(0)}+U_1^e)(h_j^{(1)})^2\mathrm{e}^{-\psi(x_1)}\mathrm{d}x, \tag{2.19}$$

since the initial approximation function $w^{(0)}$ satisfies $\nabla\cdot w^{(0)}$, inequality (2.19) is reduced into

$$\begin{aligned}&\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}((w_i^{(0)}+U_i^e)\partial_{x_i}h_j^{(1)})h_j^{(1)}\mathrm{e}^{-\psi(x_1)}\mathrm{d}x\\ =&\frac{1}{2}\sum_{j=1}^{3}\int_{\Omega}\psi'(x_1)(w_1^{(0)}+U_1^e)(h_j^{(1)})^2\mathrm{e}^{-\psi(x_1)}\mathrm{d}x,\end{aligned} \tag{2.20}$$

and direct computation gives that

$$\begin{aligned}&\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}h_i^{(1)}\partial_{x_i}(w_j^{(0)}+U_j^e)h_j^{(1)}\mathrm{e}^{-\psi(x_1)}\mathrm{d}x\\ =&\sum_{j=1}^{3}\int_{\Omega}\partial_{x_j}(w_j^{(0)}+U_j^e)(h_j^{(1)})^2\mathrm{e}^{-\psi(x_1)}\mathrm{d}x\\ &+\sum_{j=1}^{3}\sum_{i\neq j}\int_{\Omega}h_i^{(1)}\partial_{x_i}(w_j^{(0)}+U_j^e)h_j^{(1)}\mathrm{e}^{-\psi(x_1)}\mathrm{d}x,\end{aligned} \tag{2.21}$$

and noticing the incompressible condition

$$\nabla\cdot\mathbf{h}^{(1)}=0,$$

it holds

$$\begin{aligned}&\sum_{j=1}^{3}\int_{\Omega}\partial_{x_j}((\mathcal{D}_{w^{(0)}}P)\mathbf{h}^{(1)})h_j^{(1)}\mathrm{e}^{-\psi(x_1)}\mathrm{d}x\\ =&-\sum_{j=1}^{3}\int_{\Omega}((\mathcal{D}_{w^{(0)}}P)\mathbf{h}^{(1)})\partial_{x_j}h_j^{(1)}\mathrm{e}^{-\psi(x_1)}\mathrm{d}x\\ &+\int_{\Omega}\psi'(x_1)((\mathcal{D}_{w^{(0)}}P)\mathbf{h}^{(1)})h_1^{(1)}\mathrm{e}^{-\psi(x_1)}\mathrm{d}x\\ =&\int_{\Omega}\psi'(x_1)((\mathcal{D}_{w^{(0)}}P)\mathbf{h}^{(1)})h_1^{(1)}\mathrm{e}^{-\psi(x_1)}\mathrm{d}x,\end{aligned} \tag{2.22}$$

furthermore, from (2.10), using the standard Calderon-Zygmund theory and Young's inequality, it holds

$$\begin{aligned}&\Big|\int_{\Omega}\psi'(x_1)((\mathcal{D}_{w^{(0)}}P)\mathbf{h}^{(1)})h_1^{(1)}\mathrm{e}^{-\psi(x_1)}\mathrm{d}x\Big|\\ =&\Big|\triangle^{-1}\sum_{i=1}^{3}\sum_{j=1}^{3}\int_{\Omega}\psi'(x_1)(\partial_{x_j}h_i^{(1)}\partial_{x_i}\partial_{x_i}(w_j^{(0)}+U_j^e)+\partial_{x_j}(w_i^{(0)}+U_i^e)\partial_{x_i}h_j^{(1)})h_1^{(1)}\mathrm{e}^{-\psi(x_1)}\mathrm{d}x\Big|\\ \lesssim&\frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\int_{\Omega}|\psi'(x_1)(\partial_{x_i}(w_j^{(0)}+U_j^e)+\partial_{x_j}(w_i^{(0)}+U_i^e))|(h_1^{(1)})^2\mathrm{e}^{-\psi(x_1)}\mathrm{d}x\\ &+\frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\int_{\Omega}|\psi'(x_1)|(|\partial_{x_i}(w_j^{(0)}+U_j^e)|(\partial_{x_j}h_i^{(1)})^2\end{aligned}$$

$$+|\partial_{x_j}(w_i^{(0)}+U_i^e)|(\partial_{x_i}h_j^{(1)})^2)\mathrm{e}^{-\psi(x_1)}\mathrm{d}x. \tag{2.23}$$

On the other hand, by Young's inequality, we derive

$$\begin{aligned}&\sum_{j=1}^{3}\sum_{i\neq j}\int_{\Omega}h_i^{(1)}\partial_{x_i}(w_j^{(0)}+U_j^e)h_j^{(1)}\mathrm{e}^{-\psi(x_1)}\mathrm{d}x\\ &\le\frac{1}{2}\sum_{j=1}^{3}\sum_{i\neq j}\int_{\Omega}|\partial_{x_i}(w_j^{(0)}+U_j^e)|((h_i^{(1)})^2+(h_j^{(1)})^2)\mathrm{e}^{-\psi(x_1)}\mathrm{d}x,\end{aligned} \tag{2.24}$$

and $(\psi''(x_1))^{-1}$ being bounded in $\mathbb{R}^+$, it holds

$$\sum_{j=1}^{3}\int_{\Omega}E_j^{(0)}h_j^{(1)}\mathrm{e}^{-\psi(x_1)}\mathrm{d}x\le\frac{1}{2}\sum_{j=1}^{3}\int_{\Omega}((E_j^{(0)})^2+|\psi''(x_1)|(h_j^{(1)})^2)\mathrm{e}^{-\psi(x_1)}\mathrm{d}x. \tag{2.25}$$

Thus we substitute (2.19)–(2.25) into (2.18) to get

$$\begin{aligned}&\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}\Big(\nu\lambda^2-\frac{\lambda}{2}|\psi'(x_1)|(|\partial_{x_i}(w_j^{(0)}+U_j^e)|\\ &+|\partial_{x_j}(w_j^{(0)}+U_j^e)|)\Big)(\partial_{x_i}h_j^{(1)})^2\mathrm{e}^{-\psi(x_1)}\mathrm{d}x+\Pi_{N_1}\sum_{j=1}^{3}\int_{\Omega}A_j(x)(h_j^{(1)})^2\mathrm{e}^{-\psi(x_1)}\mathrm{d}x\\ &\lesssim\Pi_{N_1}\sum_{j=1}^{3}\int_{\Omega}|(\psi''(x_1))^{-1}|(E_j^{(0)})^2\mathrm{e}^{-\psi(x_1)}\mathrm{d}x,\end{aligned} \tag{2.26}$$

where the coefficients

$$\begin{aligned}A_1(x):=&\frac{\nu\lambda^2}{2}(\psi''(x_1)-(\psi'(x_1))^2)-\frac{1}{2}|\psi''(x_1)|\\ &+\frac{\lambda}{2}(w_1^{(0)}+U_1^e)\psi'(x_1)+\partial_{x_1}(w_1^{(0)}+U_1^e)\\ &-\frac{\lambda}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}|\psi'(x_1)(\partial_{x_i}(w_j^{(0)}+U_j^e)+\partial_{x_j}(w_i^{(0)}+U_i^e))|\\ &-\frac{1}{2}\sum_{j=1}^{3}\sum_{i\neq j}|\partial_{x_i}(w_j^{(0)}+U_j^e)|,\\ A_2(x):=&\frac{\nu\lambda^2}{2}(\psi''(x_1)-(\psi'(x_1))^2)-\frac{1}{2}|\psi''(x_1)|+\frac{\lambda}{2}(w_1^{(0)}+U_1^e)\psi'(x_1)\\ &+\partial_{x_2}(w_2^{(0)}+U_2^e)-\frac{1}{2}\sum_{j=1}^{3}\sum_{i\neq j}|\partial_{x_i}(w_j^{(0)}+U_j^e)|,\\ A_3(x):=&\frac{\nu\lambda^2}{2}(\psi''(x_1)-(\psi'(x_1))^2)-\frac{1}{2}|\psi''(x_1)|+\frac{\lambda}{2}(w_1^{(0)}+U_1^e)\psi'(x_1)\\ &+\partial_{x_3}(w_3^{(0)}+U_3^e)-\frac{1}{2}\sum_{j=1}^{3}\sum_{i\neq j}|\partial_{x_i}(w_j^{(0)}+U_j^e)|.\end{aligned}$$

Since the weighted funtion $\psi(x_1)$ satisfies (2.16), the main term of $A_1(x)$ $(i=1,2,3,\cdots,n)$ is

$$\frac{\nu\lambda^2}{2}(\psi''(x_1)-(\psi'(x_1))^2)-\frac{1}{2}|\psi''(x_1)|.$$

Thus, by noticing (1.5) and the term $w_1^{(0)} + U_1^e$ decay faster than the function $\psi(x_1)$, there exists a sufficient big constant $\lambda \geq \max\{\nu^{-1}, c_0\} > 1$ such that

$$
\begin{aligned}
A_1(x) \geq & \frac{\nu\lambda^2}{2}(\psi''(x_1) - (\psi'(x_1))^2) - \frac{1}{2}|\psi''(x_1)| - \frac{\lambda}{2}|\psi'(x_1)|\|(w_1^{(0)} + U_1^e)\|_{\mathbb{L}^\infty(\Omega)} \\
& - \|\partial_{x_1}(w_1^{(0)} + U_1^e)\|_{\mathbb{L}^\infty(\Omega)} - \frac{\lambda}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}|\psi'(x_1)|(\|\partial_{x_i}(w_j^{(0)} + U_j^e)\|_{\mathbb{L}^\infty} \\
& + \|\partial_{x_j}(w_i^{(0)} + U_i^e)\|_{\mathbb{L}^\infty}) - \frac{1}{2}\sum_{j=1}^{3}\sum_{i\neq j}\|\partial_{x_i}(w_j^{(0)} + U_j^e)\|_{\mathbb{L}^\infty} \\
\gtrsim & \frac{\nu\lambda^2}{2}(\psi''(x_1) - (\psi'(x_1))^2) - \frac{1}{2}|\psi''(x_1)| - 3(\varepsilon_0 + c_0),
\end{aligned}
$$

from which and (2.16) and $\lambda \geq \max\{\nu^{-1}, c_0\} > 1$, one can see that there exists a positive constant C_ν depending on ν such that

$$
A_1(x) \gtrsim \frac{\nu\lambda^2}{2}(\psi''(x_1) - (\psi'(x_1))^2) - 3(\varepsilon_0 + c_0) \geq \frac{\nu\kappa\lambda^2}{2} - 3(\varepsilon_0 + c_0) \geq C_\nu > 0,
$$

where $\kappa \in \left(0, \frac{1}{4}\right)$. Similarly, it holds

$$
A_2(x),\ A_3(x) \gtrsim C_\nu,
$$

and

$$
\nu\lambda^2 - \frac{\lambda}{2}|\psi'(x_1)|(|\partial_{x_i}(w_j^{(0)} + U_j^e)| + |\partial_{x_j}(w_j^{(0)} + U_j^e)|) \gtrsim C_\nu.
$$

Thus it follows from (2.26) that

$$
\begin{aligned}
& \sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega(\partial_{x_i}h_j^{(1)})^2 \mathrm{e}^{-\psi(x_1)}\mathrm{d}x + C_\nu\Pi_{N_1}\sum_{j=1}^{3}\int_\Omega(h_j^{(1)})^2\mathrm{e}^{-\psi(x_1)}\mathrm{d}x \\
\lesssim & \Pi_{N_1}\sum_{j=1}^{3}\int_\Omega(E_j^{(0)})^2\mathrm{e}^{-\psi(x_1)}\mathrm{d}x,
\end{aligned}
\tag{2.27}
$$

which combining with $\mathrm{e}^{-\psi(x_1)}$ being bounded function to obtain

$$
\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega(\partial_{x_i}h_j^{(1)})^2\mathrm{d}x + \Pi_{N_1}\sum_{j=1}^{3}\int_\Omega(h_j^{(1)})^2\mathrm{d}x \lesssim \Pi_{N_1}\sum_{j=1}^{3}\int_\Omega(E_j^{(0)})^2\mathrm{d}x.
$$

Furthermore, we derive the higher order derivatives estimates of elliptic equations. For a fixed constant $s \geq 1$, we apply $D_i^s := \partial_{x_i}^s$ $(\forall i = 1, 2, 3)$ to both sides of (2.13), it holds

$$
\begin{aligned}
& -\nu\lambda^2\triangle D_i^s h_j^{(1)} + \lambda\Pi_{N_1}\sum_{i=1}^{3}(w_i^{(0)} + U_i^e)\partial_{x_i}D_i^s h_j^{(1)} + \Pi_{N_1}\sum_{i=1}^{3}D_i^s h_i^{(1)}\partial_{x_i}(w_j^{(0)} + U_j^e) \\
& + \Pi_{N_1}\partial_{x_j}D_i^s((\mathcal{D}_{w^{(0)}}P)\mathbf{h}^{(1)}) = F_j \quad \text{for } j = 1, 2, 3, \cdots, n
\end{aligned}
\tag{2.28}
$$

with the boundary condition

$$
D_i^l h_j^{(1)}(\lambda x)|_{x\in\partial\Omega} = 0, \tag{2.29}
$$

where the constant $1 \le l \le s$, and

$$F_j := \Pi_{N_1} D_i^s E_j^{(0)} - \Pi_{N_1} \sum_{s_1+s_2=s,\ 0\le s_2\le s-1} \sum_{i=1}^{3} D_i^{s_1}(w_i^{(0)}+U_i^e)\partial_{x_i} D_i^{s_2} h_j^{(1)}$$
$$- \Pi_{N_1} \sum_{\substack{s_1+s_2=s\\ 0\le s_2\le s-1}} \sum_{i=1}^{3} (D_i^{s_2} h_i^{(1)})(D_i^{s_1}\partial_{x_i}(w_j^{(0)}+U_j^e)).$$

Here it should be noticed that we denote $D_i^l(h_j^{(1)}(\lambda x))$ by $D_i^l h_j^{(1)}$ for convenience.

Remark 2.1 It should notice that we will construct the first approximation step $h_j^{(1)}(\lambda x)$ satisfying the boundary condition (2.29). It depends on the initial approximation function $w^{(0)}(x)$ satisfying

$$\partial_{x_i}^l w^{(0)}(x)|_{x\in\partial\Omega} = 0 \quad \text{for} \quad 0 \le l \le s.$$

Next we derive higher derivative estimate of solution for (2.13).

Lemma 2.3 *Let* $0<\nu\ll 1$. *Assume that* (1.5) *holds, and the initial approximation function* $w^{(0)}$ *satisfying* (2.6)–(2.8). *Then the solution* $\boldsymbol{h}^{(1)}(\lambda x)$ *of the linear system* (2.13) *satisfies*

$$\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega (\partial_{x_j} D_i^s h_j^{(1)})^2 \mathrm{d}x + \Pi_{N_1}\sum_{i=1}^{3}\sum_{j=1}^{3}\int_\Omega (D_i^s h_j^{(1)})^2\mathrm{d}x$$
$$\lesssim \Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\sum_{l_0=0}^{s}\int_\Omega (D_i^{l_0}E_j^{(0)})^2\mathrm{d}x. \tag{2.30}$$

Proof This proof is based on the induction. Let $s=1$, by (2.28), it holds

$$-\nu\lambda^2\triangle D_i^1 h_j^{(1)} + \lambda\Pi_{N_1}\sum_{i=1}^{3}(w_i^{(0)}+U_i^e)\partial_{x_i}D_i^1 h_j^{(1)} + \Pi_{N_1}\sum_{i=1}^{3} D_i^1 h_i^{(1)}\partial_{x_i}(w_j^{(0)}+U_j^e)$$
$$+\Pi_{N_1}\partial_{x_j}D_i^1((\mathcal{D}_{w^{(0)}}P)\mathbf{h}^{(1)}) + \lambda\Pi_{N_1}\sum_{i=1}^{3} D_i^1(w_i^{(0)}+U_i^e)\partial_{x_i}h_j^{(1)}$$
$$+\Pi_{N_1}\sum_{i=1}^{3} h_i^{(1)} D_i^1\partial_{x_i}(w_j^{(0)}+U_j^e) = \Pi_{N_1}D_i^1 E_j^{(0)} \quad \text{for } j=1,2,3 \tag{2.31}$$

with the boundary condition

$$D_i^1 h_j^{(1)}(\lambda x)|_{x\in\partial\Omega} = 0. \tag{2.32}$$

Let us choose the weighted function that satisfies (2.16). We multiply both sides of (2.31) by $D_i^1 h_j^{(1)}\mathrm{e}^{-\psi(x_1)}$, then integrate over Ω by noticing (2.32), and sum up those equalities from $j=1$ to $j=3$, it holds

$$\nu\lambda^2\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega (\partial_{x_j}D_i^1 h_j^{(1)})^2\mathrm{e}^{-\psi(x_1)}\mathrm{d}x$$
$$+\frac{\nu\lambda^2}{2}\sum_{j=1}^{3}\int_\Omega (\psi''(x_1)-(\psi'(x_1))^2)(D_i^1 h_j^{(1)})^2\mathrm{e}^{-\psi(x_1)}\mathrm{d}x$$

$$
\begin{aligned}
&+\lambda\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega (w_i^{(0)}+U_i^e)(\partial_{x_i}D_i^1h_j^{(1)})(D_i^1h_j^{(1)})\mathrm{e}^{-\psi(x_1)}\mathrm{d}x\\
&+\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega D_i^1h_i^{(1)}\partial_{x_i}(w_j^{(0)}+U_j^e)D_i^1h_j^{(1)}\mathrm{e}^{-\psi(x_1)}\mathrm{d}x\\
&+\Pi_{N_1}\sum_{j=1}^{3}\int_\Omega \partial_{x_j}D_i^1((\mathcal{D}_{w^{(0)}}P)\mathbf{h}^{(1)})D_i^1h_j^{(1)}\mathrm{e}^{-\psi(x_1)}\mathrm{d}x\\
&+\Pi_{N_1}\sum_{a=1}^{3}I_a=0,
\end{aligned}
\tag{2.33}
$$

where

$$
\begin{aligned}
\mathrm{I}_1&:=\lambda\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega D_i^1(w_i^{(0)}+U_i^e)\partial_{x_i}h_j^{(1)}D_i^1h_j^{(1)}\mathrm{e}^{-\psi(x_1)}\mathrm{d}x,\\
\mathrm{I}_2&:=\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega h_i^{(1)}D_i^1\partial_{x_i}(w_j^{(0)}+U_j^e)D_i^1h_j^{(1)}\mathrm{e}^{-\psi(x_1)}\mathrm{d}x,\\
\mathrm{I}_3&:=\Pi_{N_1}\sum_{j=1}^{3}\int_\Omega D_i^1E_j^{(0)}D_i^1h_j^{(1)}\mathrm{e}^{-\psi(x_1)}\mathrm{d}x.
\end{aligned}
$$

We now estimate each terms in (2.33). On one hand, since we have chosen the initial approximation function $w^{(0)}$ satisfying (2.6)–(2.8), using the similar method of getting (2.19)–(2.22), we get

$$
\begin{aligned}
&\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega (w_i^{(0)}+U_i^e)(\partial_{x_i}D_i^1h_j^{(1)})(D_i^1h_j^{(1)})\mathrm{e}^{-\psi(x_1)}\mathrm{d}x\\
&=\frac12\sum_{j=1}^{3}\int_\Omega \psi'(x_1)(w_1^{(0)}+U_1^e)(D_i^1h_j^{(1)})^2\mathrm{e}^{-\psi(x_1)}\mathrm{d}x,
\end{aligned}
\tag{2.34}
$$

$$
\begin{aligned}
&\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega D_i^1h_i^{(1)}\partial_{x_i}(w_j^{(0)}+U_j^e)D_i^1h_j^{(1)}\mathrm{e}^{-\psi(x_1)}\mathrm{d}x\\
&=\sum_{j=1}^{3}\int_\Omega \partial_{x_j}(w_j^{(0)}+U_j^e)(D_j^1h_j^{(1)})^2\mathrm{e}^{-\psi(x_1)}\mathrm{d}x\\
&\quad+\sum_{j=1}^{3}\sum_{i\neq j}\int_\Omega \partial_{x_i}(w_j^{(0)}+U_j^e)(D_i^1h_i^{(1)})(D_i^1h_j^{(1)})\mathrm{e}^{-\psi(x_1)}\mathrm{d}x,
\end{aligned}
\tag{2.35}
$$

and by the incompressible condition $\nabla\cdot\mathbf{h}^{(1)}=0$, we integrate by parts to get

$$
\begin{aligned}
&\sum_{j=1}^{3}\int_\Omega \partial_{x_j}D_i^1((\mathcal{D}_{w^{(0)}}P)\mathbf{h}^{(1)})D_i^1h_j^{(1)}\mathrm{e}^{-\psi(x_1)}\mathrm{d}x\\
&=\int_\Omega \psi'(x_1)D_i^1((\mathcal{D}_{w^{(0)}}P)\mathbf{h}^{(1)})(D_i^1h_1^{(1)})\mathrm{e}^{-\psi(x_1)}\mathrm{d}x,
\end{aligned}
\tag{2.36}
$$

from which, we use the standard Calderon-Zygmund theory and Young's inequality to derive

$$\begin{aligned}
&\Big|\int_\Omega \psi'(x_1)D_i^1((\mathcal{D}_{w^{(0)}}P)\mathbf{h}^{(1)})(D_i^1h_1^{(1)})\mathrm{e}^{-\psi(x_1)}\mathrm{d}x\Big|\\
=&\Big|\sum_{i=1}^3\sum_{j=1}^3\int_\Omega \psi'(x_1)D_i^1(\partial_{x_j}h_i^{(1)}\partial_{x_i}(w_j^{(0)}+U_j^e)\\
&+\partial_{x_j}(w_i^{(0)}+U_i^e)\partial_{x_i}h_j^{(1)})(D_i^1h_1^{(1)})\mathrm{e}^{-\psi(x_1)}\mathrm{d}x\Big|\\
\lesssim&\frac{1}{2}\sum_{i=1}^3\sum_{j=1}^3\int_\Omega |\psi'(x_1)(\partial_{x_i}(w_j^{(0)}+U_j^e)+\partial_{x_j}(w_i^{(0)}+U_i^e)+\partial_{x_i}D_i^1(w_j^{(0)}+U_j^e)\\
&+\partial_{x_j}D_i^1(w_i^{(0)}+U_i^e))|(D_i^1h_1^{(1)})^2\mathrm{e}^{-\psi(x_1)}\mathrm{d}x\\
&+\frac{1}{2}\sum_{i=1}^3\sum_{j=1}^3\int_\Omega |\psi'(x_1)|(|\partial_{x_i}(w_j^{(0)}+U_j^e)|(\partial_{x_j}D_i^1h_i^{(1)})^2\\
&+|\partial_{x_j}(w_i^{(0)}+U_i^e)|(\partial_{x_i}D_i^1h_j^{(1)})^2)\mathrm{e}^{-\psi(x_1)}\mathrm{d}x\\
&+\frac{1}{2}\sum_{i=1}^3\sum_{j=1}^3\int_\Omega |\psi'(x_1)|(|\partial_{x_i}D_i^1(w_j^{(0)}+U_j^e)|(\partial_{x_j}h_i^{(1)})^2\\
&+|\partial_{x_j}D_i^1(w_i^{(0)}+U_i^e)|(\partial_{x_i}h_j^{(1)})^2)\mathrm{e}^{-\psi(x_1)}\mathrm{d}x,
\end{aligned}\tag{2.37}$$

and

$$\begin{aligned}
&\sum_{j=1}^3\sum_{i\neq j}\int_\Omega \partial_{x_i}(w_j^{(0)}+U_j^e)(D_i^1h_i^{(1)})(D_i^1h_j^{(1)})\mathrm{e}^{-\psi(x_1)}\mathrm{d}x\\
\leq&\frac{1}{2}\sum_{j=1}^3\sum_{i\neq j}\int_\Omega |\partial_{x_i}(w_j^{(0)}+U_j^e)|((D_i^1h_i^{(1)})^2+(D_i^1h_j^{(1)})^2)\mathrm{e}^{-\psi(x_1)}\mathrm{d}x.
\end{aligned}$$

On the other hand, it holds

$$\mathrm{I}_1=\lambda\sum_{j=1}^3\sum_{i=1}^3\int_\Omega D_i^1(w_i^{(0)}+U_i^e)(D_i^1h_j^{(1)})^2\mathrm{e}^{-\psi(x_1)}\mathrm{d}x,\tag{2.38}$$

$$\begin{aligned}
\mathrm{I}_2\leq&\frac{1}{2}\sum_{j=1}^3\int_\Omega\Big(\sum_{i=1}^3|D_i^1\partial_{x_i}(w_j^{(0)}+U_j^e)|\Big)(h_j^{(1)})^2\mathrm{e}^{-\psi(x_1)}\mathrm{d}x\\
&+\sum_{j=1}^3\int_\Omega\Big(\sum_{i=1}^3|D_i^1\partial_{x_i}(w_j^{(0)}+U_j^e)|\Big)(D_i^1h_j^{(1)})^2\mathrm{e}^{-\psi(x_1)}\mathrm{d}x\\
\lesssim&(\nu_0+c_0)\sum_{j=1}^3\int_\Omega(h_j^{(1)})^2\mathrm{e}^{-\psi(x_1)}\mathrm{d}x\\
&+\sum_{j=1}^3\int_\Omega\Big(\sum_{i=1}^3|D_i^1\partial_{x_i}(w_j^{(0)}+U_j^e)|\Big)(D_i^1h_j^{(1)})^2\mathrm{e}^{-\psi(x_1)}\mathrm{d}x,
\end{aligned}\tag{2.39}$$

$$\mathrm{I}_3\leq\frac{1}{2}\sum_{j=1}^3\int_\Omega((D_i^1E_j^{(0)})^2+|\psi''(x_1)|(D_i^1h_j^{(1)})^2)\mathrm{e}^{-\psi(x_1)}\mathrm{d}x.\tag{2.40}$$

Thus summing up (2.33) from $i = 1$ to $i = 3$, we use (2.34)–(2.40) to derive

$$\begin{aligned}
&\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}\Big(\nu\lambda^2 - \frac{\lambda}{2}|\psi'(x_1)|(|\partial_{x_i}(w_j^{(0)} + U_j^e)| \\
&+ |\partial_{x_j}(w_i^{(0)} + U_i^e)|)\Big)(\partial_{x_j} D_i^1 h_j^{(1)})^2 \mathrm{e}^{-\psi(x_1)}\mathrm{d}x \\
&+ \Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}\overline{A}_j(x)(D_i^1 h_j^{(1)})^2 \mathrm{e}^{-\psi(x_1)}\mathrm{d}x \\
\lesssim\ & \Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}(D_i^1 E_j^{(0)})^2 \mathrm{e}^{-\psi(x_1)}\mathrm{d}x + (\varepsilon_0 + c_0)\sum_{j=1}^{3}\int_{\Omega}(h_j^{(1)})^2 \mathrm{e}^{-\psi(x_1)}\mathrm{d}x \\
&+ \frac{1}{2}\Pi_{N_1}\sum_{i=1}^{3}\sum_{j=1}^{3}\int_{\Omega}|\psi'(x_1)|(|\partial_{x_i} D_i^1(w_j^{(0)} + U_j^e)|(\partial_{x_j} h_i^{(1)})^2 \\
&+ |\partial_{x_j} D_i^1(w_i^{(0)} + U_i^e)|(\partial_{x_i} h_j^{(1)})^2)\mathrm{e}^{-\psi(x_1)}\mathrm{d}x,
\end{aligned} \tag{2.41}$$

where

$$\begin{aligned}
\overline{A}_1(x) :=\ & \frac{\nu\lambda^2}{2}(\psi''(x_1) - (\psi'(x_1))^2) + \frac{\lambda}{2}(w_1^{(0)} + U_1^e)\psi'(x_1) + \partial_{x_1}(w_1^{(0)} + U_1^e) \\
&- \frac{\lambda}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}|\psi'(x_1)(\partial_{x_i}(w_j^{(0)} + U_j^e) + \partial_{x_j}(w_i^{(0)} + U_i^e) \\
&+ \partial_{x_i} D_i^1(w_j^{(0)} + U_j^e) + \partial_{x_j} D_i^1(w_i^{(0)} + U_i^e))| \\
&- \frac{1}{2}\sum_{j=1}^{3}\sum_{i\neq j}|\partial_{x_i}(w_j^{(0)} + U_j^e)| - \frac{1}{2}|\psi''(x_1)| \\
&+ \sum_{i=1}^{3} D_i^1(w_i^{(0)} + U_i^e) - \sum_{i=1}^{3}|D_i^1\partial_{x_i}(w_j^{(0)} + U_j^e)|, \\
\overline{A}_2(x) :=\ & \frac{\nu\lambda^2}{2}(\psi''(x_1) - (\psi'(x_1))^2) + \frac{\lambda}{2}(w_1^{(0)} + U_1^e)\psi'(x_1) + \partial_{x_2}(w_2^{(0)} + U_2^e) \\
&- \frac{1}{2}\sum_{j=1}^{3}\sum_{i\neq j}|\partial_{x_i}(w_j^{(0)} + U_j^e)| - \frac{1}{2}|\psi''(x_1)| \\
&+ \sum_{i=1}^{3} D_i^1(w_i^{(0)} + U_i^e) - \sum_{i=1}^{3}|D_i^1\partial_{x_i}(w_j^{(0)} + U_j^e)|, \\
\overline{A}_3(x) :=\ & \frac{\nu\lambda^2}{2}(\psi''(x_1) - (\psi'(x_1))^2) + \frac{\lambda}{2}(w_1^{(0)} + U_1^e)\psi'(x_1) + \partial_{x_3}(w_3^{(0)} + U_3^e) \\
&- \frac{1}{2}\sum_{j=1}^{3}\sum_{i\neq j}|\partial_{x_i}(w_j^{(0)} + U_j^e)| - \frac{1}{2}|\psi''(x_1)| \\
&+ \sum_{i=1}^{3} D_i^1(w_i^{(0)} + U_i^e) - \sum_{i=1}^{3}|D_i^1\partial_{x_i}(w_j^{(0)} + U_j^e)|.
\end{aligned}$$

We notice that the main term of $\overline{A}_i(x)$ $(i = 1, 2, 3)$ is also the following terms

$$\frac{\nu\lambda^2}{2}(\psi''(x_1) - (\psi'(x_1))^2) - \frac{1}{2}|\psi''(x_1)|.$$

which combining with the assumption (2.7) gives that

$$\overline{A}_1(x) \gtrsim \frac{\nu\lambda^2}{2}(\psi''(x_1) - (\psi'(x_1))^2) - \frac{1}{2}|\psi''(x_1)| - (\varepsilon_0 + c_0),$$

from which and (2.16), one can see that there exists a positive constant C_{ν,ε_0} depending on ν, ε_0 such that

$$\overline{A}_1(x) \gtrsim \frac{\nu\lambda^2}{2}(\psi''(x_1) - (\psi'(x_1))^2) - (\varepsilon_0 + c_0) \geq C_{\nu,\varepsilon_0} > 0,$$

where $\kappa \in \left(0, \frac{1}{4}\right)$. Similarly, it holds

$$\overline{A}_2(x),\ \overline{A}_3(x) \gtrsim C_{\nu,\varepsilon_0},$$

and

$$\nu\lambda^2 - \frac{\lambda}{2}|\psi'(x_1)|(|\partial_{x_i}(w_j^{(0)} + U_j^e)| + |\partial_{x_j}(w_i^{(0)} + U_i^e)|) \gtrsim C_\nu.$$

Thus, it holds

$$\begin{aligned}
&\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega (\partial_{x_j} D_i^1 h_j^{(1)})^2 \mathrm{e}^{-\psi(x_1)}\mathrm{d}x + C_{\nu,\varepsilon_0}\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega (D_i^1 h_j^{(1)})^2\mathrm{e}^{-\psi(x_1)}\mathrm{d}x\\
&\lesssim \Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega (D_i^1 E_j^{(0)})^2\mathrm{e}^{-\psi(x_1)}\mathrm{d}x + (\varepsilon_0 + c_0)\sum_{j=1}^{3}\int_\Omega (h_j^{(1)})^2\mathrm{e}^{-\psi(x_1)}\mathrm{d}x\\
&\quad + \Pi_{N_1}\sum_{i=1}^{3}\sum_{j=1}^{3}\int_\Omega ((\partial_{x_j} h_i^{(1)})^2 + (\partial_{x_i} h_j^{(1)})^2)\mathrm{e}^{-\psi(x_1)}\mathrm{d}x,
\end{aligned} \tag{2.42}$$

furthermore, one can see the last two terms in the right-hand side of (2.42) can be controlled by using (2.15), thus, it holds

$$\begin{aligned}
&\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega (\partial_{x_j} D_i^1 h_j^{(1)})^2\mathrm{e}^{-\psi(x_1)}\mathrm{d}x + \Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega (D_i^1 h_j^{(1)})^2\mathrm{e}^{-\psi(x_1)}\mathrm{d}x\\
&\lesssim \Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega (D_i^1 E_j^{(0)})^2\mathrm{e}^{-\psi(x_1)}\mathrm{d}x + \Pi_{N_1}\sum_{j=1}^{3}\int_\Omega (E_j^{(0)})^2\mathrm{e}^{-\psi(x_1)}\mathrm{d}x.
\end{aligned} \tag{2.43}$$

Assume that the $2 \leq l \leq s-1$ derivative case holds, i.e.,

$$\begin{aligned}
&\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega (\partial_{x_j} D_i^l h_j^{(1)})^2\mathrm{e}^{-\psi(x_1)}\mathrm{d}x + \Pi_{N_1}\sum_{i=1}^{3}\sum_{j=1}^{3}\int_\Omega (D_i^l h_j^{(1)})^2\mathrm{e}^{-\psi(x_1)}\mathrm{d}x\\
&\lesssim \Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\sum_{l_0=0}^{l}\int_\Omega (D_i^{l_0} E_j^{(0)})^2\mathrm{e}^{-\psi(x_1)}\mathrm{d}x.
\end{aligned} \tag{2.44}$$

We now prove the sth derivative case holds. Multiplying both sides of equations (2.28) by $D_i^s h_j^{(1)}\mathrm{e}^{-\psi(x_1)}$, then integrating over Ω by using boundary condition (2.29), and summing up those equalities from $j = 1$ to $j = n$, it holds

$$\nu\lambda^2\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega (\partial_{x_j} D_i^s h_j^{(1)})^2\mathrm{e}^{-\psi(x_1)}\mathrm{d}x$$

$$
\begin{aligned}
&+\frac{\nu\lambda^2}{2}\sum_{j=1}^{3}\int_\Omega(\psi''(x_1)-(\psi'(x_1))^2)(D_i^s h_j^{(1)})^2\mathrm{e}^{-\psi(x_1)}\mathrm{d}x\\
&+\lambda\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega(w_i^{(0)}+U_i^e)(\partial_{x_i}D_i^s h_j^{(1)})(D_i^s h_j^{(1)})\mathrm{e}^{-\psi(x_1)}\mathrm{d}x\\
&+\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega D_i^s h_i^{(1)}\partial_{x_i}(w_j^{(0)}+U_j^e)D_i^s h_j^{(1)}\mathrm{e}^{-\psi(x_1)}\mathrm{d}x\\
&+\Pi_{N_1}\sum_{j=1}^{3}\int_\Omega\partial_{x_j}D_i^s((\mathcal{D}_{w^{(0)}}P)\mathbf{h}^{(1)})D_i^s h_j^{(1)}\mathrm{e}^{-\psi(x_1)}\mathrm{d}x\\
=&\sum_{j=1}^{3}\int_\Omega F_j D_i^s h_j^{(1)}\mathrm{e}^{-\psi(x_1)}\mathrm{d}x. \qquad (2.45)
\end{aligned}
$$

We notice that

$$
\begin{aligned}
&\Big|\int_\Omega\psi'(x_1)D_i^s((\mathcal{D}_{w^{(0)}}P)\mathbf{h}^{(1)})(D_i^s h_1^{(1)})\mathrm{e}^{-\psi(x_1)}\mathrm{d}x\Big|\\
=&\Big|\sum_{i=1}^{3}\sum_{j=1}^{3}\int_\Omega\psi'(x_1)D_i^s(\partial_{x_j}h_i^{(1)}\partial_{x_i}(w_j^{(0)}+U_j^e)\\
&+\partial_{x_j}(w_i^{(0)}+U_i^e)\partial_{x_i}h_j^{(1)})(D_i^s h_1^{(1)})\mathrm{e}^{-\psi(x_1)}\mathrm{d}x\Big|\\
=&\Big|\sum_{i=1}^{3}\sum_{j=1}^{3}\int_\Omega\psi'(x_1)\sum_{\substack{j_1+j_2=s\\0\le j_1,j_2\le s}}(D_i^{j_2}\partial_{x_j}h_i^{(1)}D_i^{j_1}\partial_{x_i}(w_j^{(0)}+U_j^e)\\
&+D_i^{j_1}\partial_{x_j}(w_i^{(0)}+U_i^e)D_i^{j_2}\partial_{x_i}h_j^{(1)})(D_i^s h_1^{(1)})\mathrm{e}^{-\psi(x_1)}\mathrm{d}x\Big|\\
\lesssim&\frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{j_1=1}^{s}\int_\Omega|\psi'(x_1)(\partial_{x_i}D_i^{j_1}(w_j^{(0)}+U_j^e)+\partial_{x_j}D_i^{j_1}(w_i^{(0)}+U_i^e))|(D_i^s h_1^{(1)})^2\mathrm{e}^{-\psi(x_1)}\mathrm{d}x\\
&+\frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{\substack{j_1+j_2=s\\0\le j_2\le s-1}}\int_\Omega|\psi'(x_1)|(|\partial_{x_i}D_i^{j_1}(w_j^{(0)}+U_j^e)|(\partial_{x_j}D_i^{j_2}h_i^{(1)})^2\\
&+|\partial_{x_j}D_i^{j_1}(w_i^{(0)}+U_i^e)|(\partial_{x_i}D_i^{j_2}h_j^{(1)})^2)\mathrm{e}^{-\psi(x_1)}\mathrm{d}x\\
&+\frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\int_\Omega|\psi'(x_1)|(|\partial_{x_i}(w_j^{(0)}+U_j^e)|(\partial_{x_j}D_i^s h_i^{(1)})^2\\
&+|\partial_{x_j}(w_i^{(0)}+U_i^e)|(\partial_{x_i}D_i^s h_j^{(1)})^2)\mathrm{e}^{-\psi(x_1)}\mathrm{d}x \qquad (2.46)
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{\substack{s_1+s_2=s\\0\le s_2\le s-1}}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega D_i^{s_1}(w_i^{(0)}+U_i^e)(\partial_{x_i}D_i^{s_2}h_j^{(1)})D_i^s h_j^{(1)}\mathrm{e}^{-\psi(x_1)}\mathrm{d}x\\
=&\sum_{\substack{s_1+s_2=s,\\0\le s_2\le s-2}}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega D_i^{s_1}(w_i^{(0)}+U_i^e)(\partial_{x_i}D_i^{s_2}h_j^{(1)})D_i^s h_j^{(1)}\mathrm{e}^{-\psi(x_1)}\mathrm{d}x
\end{aligned}
$$

$$+\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega D_i^1(w_i^{(0)}+U_i^e)(D_i^s h_j^{(1)})^2 \mathrm{e}^{-\psi(x_1)}\mathrm{d}x, \tag{2.47}$$

thus, similar to get (2.41), we can obtain

$$\begin{aligned}
&\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega \Big(\nu\lambda^2-\frac{\lambda}{2}|\psi'(x_1)|(|\partial_{x_i}(w_j^{(0)}+U_j^e)| \\
&+|\partial_{x_j}(w_i^{(0)}+U_i^e)|)\Big)(\partial_{x_j}D_i^s h_j^{(1)})^2\mathrm{e}^{-\psi(x_1)}\mathrm{d}x \\
&+\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega C_j(x)(D_i^s h_j^{(1)})^2\mathrm{e}^{-\psi(x_1)}\mathrm{d}x \\
\lesssim\, &\Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\int_\Omega (D_i^s E_j^{(0)})^2\mathrm{e}^{-\psi(x_1)}\mathrm{d}x \\
&+\Pi_{N_1}\sum_{\substack{s_1+s_2=s\\0\le s_2\le s-1}}\sum_{i=1}^{3}\int_\Omega (D_i^{s_1}\partial_{x_i}(w_j^{(0)}+U_j^e))(D_i^{s_2}h_i^{(1)})^2\mathrm{e}^{-\psi(x_1)}\mathrm{d}x \\
&+\frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{\substack{j_1+j_2=s\\0\le j_2\le s-1}}\int_\Omega |\psi'(x_1)|(|\partial_{x_i}D_i^{j_1}(w_j^{(0)}+U_j^e)|(\partial_{x_j}D_i^{j_2}h_i^{(1)})^2 \\
&+|\partial_{x_j}D_i^{j_1}(w_i^{(0)}+U_i^e)|(\partial_{x_i}D_i^{j_2}h_j^{(1)})^2)\mathrm{e}^{-\psi(x_1)}\mathrm{d}x,
\end{aligned} \tag{2.48}$$

where

$$\begin{aligned}
C_1(x) :=\, &\frac{\nu\lambda^2}{2}(\psi''(x_1)-(\psi'(x_1))^2)+\frac{\lambda}{2}(w_1^{(0)}+U_1^e)\psi'(x_1) \\
&+\partial_{x_1}(w_1^{(0)}+U_1^e)+\sum_{i=1}^{3}D_i^1(w_i^{(0)}+U_i^e) \\
&-\frac{\lambda}{2}\sum_{j_1=1}^{3}|\psi'(x_1)(\partial_{x_i}D_i^{j_1}(w_j^{(0)}+U_j^e)+\partial_{x_j}D_i^{j_1}(w_i^{(0)}+U_i^e))| \\
&-\frac{1}{2}|\psi'(x_1)|\sum_{i=1}^{3}|D_i^2(w_i^{(0)}+U_i^e)| \\
&-\frac{1}{2}\sum_{j=1}^{3}\sum_{i\ne j}|\partial_{x_i}(w_j^{(0)}+U_j^e)|-\frac{1}{2}|\psi''(x_1)|-\sum_{i=1}^{3}\sum_{j=1}^{3}|D_i^j\partial_{x_i}(w_j^{(0)}+U_j^e)|, \\
C_2(x) :=\, &\frac{\nu\lambda^2}{2}(\psi''(x_1)-(\psi'(x_1))^2)+\frac{\lambda}{2}(w_1^{(0)}+U_1^e)\psi'(x_1) \\
&+\partial_{x_2}(w_2^{(0)}+U_2^e)+\sum_{i=1}^{3}D_i^1(w_i^{(0)}+U_i^e) \\
&-\frac{1}{2}\sum_{j=1}^{3}\sum_{i\ne j}|\partial_{x_i}(w_j^{(0)}+U_j^e)|-\frac{1}{2}|\psi''(x_1)|-\frac{1}{2}|\psi'(x_1)|\sum_{i=1}^{3}|D_i^2(w_i^{(0)}+U_i^e)| \\
&-\sum_{i=1}^{3}\sum_{j=1}^{3}|D_i^j\partial_{x_i}(w_j^{(0)}+U_j^e)|,
\end{aligned}$$

$$
\begin{aligned}
C_3(x) := {}& \frac{\nu\lambda^2}{2}(\psi''(x_1) - (\psi'(x_1))^2) + \frac{\lambda}{2}(w_1^{(0)} + U_1^e)\psi'(x_1) \\
& + \partial_{x_3}(w_3^{(0)} + U_3^e) + \sum_{i=1}^{3} D_i^1(w_i^{(0)} + U_i^e) \\
& - \frac{1}{2}\sum_{j=1}^{3}\sum_{i\neq j} |\partial_{x_i}(w_j^{(0)} + U_j^e)| - \frac{1}{2}|\psi''(x_1)| - \frac{1}{2}|\psi'(x_1)|\sum_{i=1}^{3}|D_i^2(w_i^{(0)} + U_i^e)| \\
& - \sum_{i=1}^{3}\sum_{j=1}^{3}|D_i^j\partial_{x_i}(w_j^{(0)} + U_j^e)|.
\end{aligned}
$$

By asssumptions of (2.6)–(2.7), one can see that the coefficients $C_i(x)$ $(i = 1, 2, 3)$ have the same main terms with $\overline{A}_i(x)$, and it holds

$$
C_1(x) \gtrsim \frac{\nu\lambda^2}{2}(\psi''(x_1) - (\psi'(x_1))^2) - \frac{1}{2}|\psi''(x_1)| - (\varepsilon_0 + c_0),
$$

from which and (2.16), one can see that there exists a positive constant C_{ν,ε_0} depending on ν, ε_0 such that

$$
C_1(x) \gtrsim \frac{\nu\lambda^2}{2}((\psi''(x_1))^2 - |\psi'(x_1)|) - (\varepsilon_0 + c_0) \geq C_{\nu,\varepsilon_0} > 0,
$$

where $\kappa \in \left(0, \frac{1}{4}\right)$. Similarly, it holds

$$
C_2(x),\ C_3(x) \gtrsim C_{\nu,\varepsilon_0}.
$$

Thus, we can reduce (2.48) into

$$
\begin{aligned}
& \sum_{i=1}^{3}\int_{\Omega}(\partial_{x_j} D_i^s h_j^{(1)})^2 \mathrm{e}^{-\psi(x_1)}\mathrm{d}x + C_{\nu,\varepsilon_0}\Pi_{N_1}\sum_{j=1}^{3}\int_{\Omega}(D_i^s h_j^{(1)})^2\mathrm{e}^{-\psi(x_1)}\mathrm{d}x \\
\lesssim {}& \Pi_{N_1}\int_{\Omega}(D_i^s E_j^{(0)})^2 \mathrm{e}^{-\psi(x_1)}\mathrm{d}x \\
& + \Pi_{N_1}\sum_{\substack{s_1+s_2=s\\ 0\leq s_2\leq s-1}}\sum_{i=1}^{3}\int_{\Omega}(D_i^{s_1}\partial_{x_i}(w_j^{(0)} + U_j^e))(D_i^{s_2}h_i^{(1)})^2\mathrm{e}^{-\psi(x_1)}\mathrm{d}x \\
& + \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{\substack{j_1+j_2=s\\ 0\leq j_2\leq s-1}}\int_{\Omega}|\psi'(x_1)|(|\partial_{x_i}D_i^{j_1}(w_j^{(0)} + U_j^e)|(\partial_{x_j}D_i^{j_2}h_i^{(1)})^2 \\
& + |\partial_{x_j}D_i^{j_1}(w_i^{(0)} + U_i^e)|(\partial_{x_i}D_i^{j_2}h_j^{(1)})^2)\mathrm{e}^{-\psi(x_1)}\mathrm{d}x.
\end{aligned}
\tag{2.49}
$$

Hence, similarly to get (2.43), we use (2.44) to derive

$$
\begin{aligned}
& \sum_{j=1}^{3}\sum_{i=1}^{3}\int_{\Omega}(\partial_{x_j}D_i^s h_j^{(1)})^2\mathrm{e}^{-\psi(x_1)}\mathrm{d}x \\
& + \Pi_{N_1}\sum_{i=1}^{3}\sum_{j=1}^{3}\int_{\Omega}(D_i^s h_j^{(1)})^2\mathrm{e}^{-\psi(x_1)}\mathrm{d}x \\
\lesssim {}& \Pi_{N_1}\sum_{j=1}^{3}\sum_{i=1}^{3}\sum_{l_0=0}^{s}\int_{\Omega}(D_i^{l_0}E_j^{(0)})^2\mathrm{e}^{-\psi(x_1)}\mathrm{d}x,
\end{aligned}
$$

which combining with $\mathrm{e}^{-\psi(x_1)}$ bounded gives (2.30).

2.3 The existence of first approximation step

Based on above priori estimates, we are ready to prove the existence of first approximation step by the classic theory of elliptic equations [8, 30].

Proposition 2.1 *Let* $0 < \nu \ll 1$. *Assume that* (1.5) *holds, and the initial approximation function* $w^{(0)}$ *satisfies* (2.6)–(2.8). *Then the linearized elliptic system*

$$\begin{cases} \mathcal{L}[w^0]\boldsymbol{h}^{(1)} = \Pi_{N_1} E^{(0)}, \\ \nabla \cdot \boldsymbol{h}^{(1)} = 0, \\ \boldsymbol{h}^{(1)}(\lambda x)|_{x\in\partial\Omega} = 0 \end{cases}$$

admits Sobolev regular solutions $\boldsymbol{h}^{(1)}(\lambda x) \in H^s(\Omega)$.

Moreover, it holds

$$\|\boldsymbol{h}^{(1)}\|_{H^s}^2 \lesssim \|\Pi_{N_1} E^{(0)}\|_{H^s}^2, \quad \forall t > 0 \tag{2.50}$$

and

$$\partial_{x_i}^l \boldsymbol{h}^{(1)}(\lambda x)|_{x\in\partial\Omega} = 0. \tag{2.51}$$

Proof Let $\mathbb{P}$ the Leray projector onto the space of divergence free functions. We apply the Leray projector to equations (2.11), it holds

$$-\nu\lambda^2 \triangle \mathbf{h}^{(1)} + \mathbb{P}\Pi_{N_1}[\lambda((w^{(0)} + U^e) \cdot \nabla)\mathbf{h}^{(1)} + (\mathbf{h}^{(1)} \cdot \nabla)(w^{(0)} + U^e)] = \mathbb{P}\Pi_{N_1} E^{(0)}. \tag{2.52}$$

By (2.15) in Lemma 2.2 and (2.30) in Lemma 2.3, we can get the uniform bound estimate

$$\|\mathbf{h}^{(1)}\|_{H^s}^2 \lesssim \|\Pi_{N_1} E^{(0)}\|_{H^s}^2.$$

From the standard theory of elliptic equations of the general order [8, 30], the linear elliptic equations (2.52) admit a unique weak solution $\mathbf{h}^{(1)} \in H^1$ if $E^{(0)} \in H^1$. Since the error term $E^{(0)} \in H^s(\Omega)$ for $s > 1$, we obtain the solution $\mathbf{h}^{(1)} \in H^s(\Omega)$.

Noticing that

$$\partial_{x_i}^l E^{(0)}(x)|_{x\in\partial\Omega} = 0,$$

it holds

$$\partial_{x_i}^l \mathbf{h}^{(1)}(x)|_{x\in\partial\Omega} = 0.$$

Remark 2.2 To see the boundary condition (2.51), we first use the idea of getting L^2-estimate of solutions in Lemma 2.2 to show that the linear elliptic operator can generate a Green function $\mathcal{S}_0(x)$ in L^2 space, then we get the solution

$$\mathbf{h}^{(1)}(\lambda x) = \mathcal{S}_0(x)\mathbb{P}\Pi_{N_1} E^{(0)}.$$

which satisfies the non-slip boundary condition $\mathbf{h}^{(1)}(x)|_{x\in\partial\Omega} = 0$, then by the first condition (2.8), it is straightforward to get

$$\partial_{x_i} \mathbf{h}^{(1)}(\lambda x)|_{x\in\partial\Omega} = 0.$$

Using this condition, we follow the method given in Lemma 2.3 to derive H^1-estimate of solutions. Furthermore, by above steps, one can obtain H^s-estimate of solutions and the boundary condition (2.51).

3 The m-th Approximation Step

We define

$$\mathcal{B}_\varepsilon := \{w^{(k)}(x) : \|w^{(k)}\|_{H^s} \lesssim \varepsilon < 1\} \tag{3.1}$$

with the integer $2 \leq k \leq m-1$ and the constant $s \geq 1$.

Assume that the m-th approximation solutions of (2.5) is denoted by $\mathbf{h}^{(m)}(\lambda x)$ with $m = 2, 3, \cdots$. Let

$$\mathbf{h}^{(m)}(\lambda x) := w^{(m)}(\lambda x) - w^{(m-1)}(\lambda x),$$

then we have

$$w^{(m)}(x) = w^{(0)}(x) + \mathbf{h}^{(1)}(\lambda x) + \sum_{i=2}^{m} \mathbf{h}^{(i)}(\lambda x).$$

We linearize nonlinear equations (2.5) around $w^{(m-1)}(x)$ to get the following boundary value problem

$$\begin{cases} \mathcal{L}[w^{m-1}](\mathbf{h}^{(m)}) = \Pi_{N_m} E^{(m-1)}, \\ \nabla \cdot \mathbf{h}^{(m)} = 0 \end{cases} \tag{3.2}$$

with the boundary conditions

$$\mathbf{h}^{(m)}(\lambda x)|_{x \in \partial\Omega} = 0, \tag{3.3}$$

where the error term

$$E^{(m-1)} := \mathcal{J}[w^{m-1}(x)] = \mathcal{R}(\mathbf{h}^{(m)}(\lambda x)), \tag{3.4}$$

and

$$\begin{aligned} \mathcal{R}(\mathbf{h}^{(m)}) &:= \mathcal{J}(w^{(m-1)} + \mathbf{h}^{(m)}) - \mathcal{J}(w^{(m-1)}) - \mathcal{L}[w^{(m-1)}](\mathbf{h}^{(m)}) \\ &= \Pi_{N_m}(\lambda \mathbf{h}^{(m)} \cdot \nabla \mathbf{h}^{(m)} + \nabla P^{(m)}), \end{aligned} \tag{3.5}$$

where

$$P^{(m)} = -\lambda^2 \triangle^{-1} \sum_{i,j=1}^{3} \frac{\partial h_i^{(m)}}{\partial x_j} \frac{\partial h_j^{(m)}}{\partial x_i}.$$

It is also the nonlinear term in approximation problem (2.5) at $w^{(m-1)}(x)$. The following result is to show how to construct the m-th approximation solution.

Proposition 3.1 *Let $0 < \nu \ll 1$ and a fixed constant $s > 1$. Assume that* (1.5) *holds, and the initial approximation function $w^{(0)}$ satisfies* (2.6)–(2.8)*, $w^{(m-1)}(x) \in \mathcal{B}_\varepsilon$ and $\sum_{i=1}^{m-1} \|\boldsymbol{h}^{(i)}\|^2_{H^s} \lesssim \varepsilon^2$. Then the linearized problem* (3.2) *with the boundary condition* (3.3) *admits Sobolev regular solutions $\boldsymbol{h}^{(m)}(\lambda x) \in H^s(\Omega)$, it satisfies*

$$\|\boldsymbol{h}^{(m)}\|^2_{H^s} \lesssim \|\Pi_{N_m} E^{(m-1)}\|^2_{H^s}, \quad \forall t > 0, \tag{3.6}$$

and

$$\boldsymbol{h}^{(m)}(\lambda x)|_{x \in \partial\Omega} = 0,$$

where the error term satisfies

$$\|E^{(m)}\|_{H^s} = \|\mathcal{R}(\boldsymbol{h}^{(m)})\|_{H^s} \lesssim \lambda^{2+s} N_m^2 \|\boldsymbol{h}^{(m)}\|^2_{H^s}. \tag{3.7}$$

Proof Direct computation gives that

$$\partial_{x_i} w_j^{(m-1)}(x) = \partial_{x_i} w_j^{(0)}(x) + \lambda \partial_{x_i} \mathbf{h}^{(1)}(\lambda x) + \lambda \sum_{i=2}^{m-1} \partial_{x_i} \mathbf{h}^{(i)}(\lambda x). \tag{3.8}$$

By the assumption $\sum\limits_{i=1}^{m-1} \|\mathbf{h}^{(i)}\|_{H^s}^2 \lesssim \varepsilon^2$ and $1 < \lambda < \varepsilon^{-1}$, it is easy to see

$$\partial_{x_i} w_j^{(m-1)}(x) \sim \partial_{x_i} w_j^{(0)}(x) + \mathcal{O}(\varepsilon^2),$$

thus, note that $w^{(0)}(x)$ satisfies (2.6)–(2.8), by small modification of $\partial_{x_i} w_j^{(0)}(x)$, it holds

$$\sum_{k=0}^{s} \|\Pi_{N_m} \partial_{x_i}^k w_j^{(m-1)}(x)\|_{\mathbb{L}^\infty} \lesssim \varepsilon_0, \quad \forall i, j = 1, 2, 3. \tag{3.9}$$

Moreover, we notice that the $(m-1)$-th approximation solution is

$$w^{(m-1)}(x) = w^{(0)}(x) + \mathbf{h}^{(1)}(\lambda x) + \sum_{i=2}^{m-1} \mathbf{h}^{(i)}(\lambda x)$$

and

$$\nabla \cdot \mathbf{h}^{(m-1)} = 0,$$

thus, it holds

$$\begin{cases} \nabla \cdot w^{(m-1)}(x) = 0, \\ \|w^{(m-1)}\|_{H^s} \lesssim \varepsilon, \\ \partial_{x_i}^l w^{(m-1)}(x)|_{x \in \partial\Omega} = 0, \quad 0 \le l \le s. \end{cases}$$

Then we will find the m-th ($m \ge 2$) approximation solution $w^{(m)}(x)$, which is equivalent to find $\mathbf{h}^{(m)}(x)$ such that

$$w^{(m)}(x) = w^{(m-1)}(x) + \mathbf{h}^{(m)}(\lambda x). \tag{3.10}$$

Substituting (3.10) into (2.5), it holds

$$\mathcal{J}(w^{(m)}) = \mathcal{J}(w^{(m-1)}) + \mathcal{L}[w^{(m-1)}]\mathbf{h}^{(m)} + \mathcal{R}(\mathbf{h}^{(m)}).$$

Set

$$\mathcal{L}[w^{(m-1)}]\mathbf{h}^{(m)} = -\mathcal{J}(w^{(m-1)}) = -E^{(m-1)},$$

we supplement it with the boundary conditions (3.3).

Since we assume $w^{(m-1)}(x) \in \mathcal{B}_\varepsilon$, there is the same structure between the linear system (2.11) and the linear system of m-th approximation solutions. Thus by means of the same proof process in Proposition 2.1, we can show above problem admits a solution $\mathbf{h}^{(m)}(\lambda x) \in H^s(\Omega)$. Here we should use (2.4). Furthermore, similar to (2.50), we can use (3.8)–(3.9) to derive

$$\|\mathbf{h}^{(m)}\|_{H^s}^2 \lesssim \|E^{(m-1)}\|_{H^s}^2, \quad \forall t > 0$$

and

$$\mathbf{h}^{(m)}(\lambda x)|_{x \in \partial\Omega} = 0,$$

where one can see the $(m-1)$-th error term $E^{(m-1)}$ such that

$$E^{(m-1)} := \mathcal{J}(w^{(m-1)}) = \mathcal{R}(\mathbf{h}^{(m)}).$$

Moreover, by (3.5) and the standard Calderon-Zygmund theory, it holds

$$\|E^{(m)}\|_{H^s} = \lambda \|\Pi_{N_m}(\mathbf{h}^{(m)} \cdot \nabla \mathbf{h}^{(m)} + \nabla P^{(m)})\|_{H^s} \lesssim \lambda^{s+2} N_m^2 \|\mathbf{h}^{(m)}\|_{H^s}^2.$$

4 Convergence of the Approximation Scheme

Our target is to prove that $w^{(\infty)}(x)$ is a global solution of nonlinear equations (1.1). It is equivalent to show the series $\sum\limits_{i=1}^{m} \mathbf{h}^{(i)}(x)$ is convergence.

For a fixed constant $s > 1$, let $1 < s = \overline{k} < k_0 \leq k$ and

$$k_m := \overline{k} + \frac{k - \overline{k}}{2^m}, \quad k_{+\infty} = \overline{k},$$
$$\alpha_{m+1} := k_m - k_{m+1} = \frac{k - \overline{k}}{2^{m+1}},$$

which gives that

$$k_0 > k_1 > \cdots > k_m > k_{m+1} > \cdots . \tag{4.1}$$

Proposition 4.1 *Let* $0 < \nu \ll 1$. *Assume that* (1.5) *holds, and the initial approximation function* $w^{(0)}$ *satisfies* (2.6)–(2.8). *Then there exists a small positive constant* $\varepsilon < \nu^{s+2}$ *such that the nonlinear problem* (2.1) *admits global Sobolev solutions*

$$w^{(\infty)}(x) = w^{(0)}(x) + \sum_{m=1}^{\infty} \boldsymbol{h}^{(m)}(\lambda x) \in H^s(\Omega).$$

Moreover, it holds

$$\|w^{(\infty)}\|_{H^s} \lesssim \varepsilon < \nu^{s+2}.$$

Proof The proof is based on the induction. Note that $N_m = N_0^m$ with $N_0 > 1$. For any $m = 1, 2, \cdots$, we claim that there exists a sufficient small positive constant ε such that

$$\begin{aligned} &\|\mathbf{h}^{(m)}\|_{H^{k_{m-1}}} \lesssim \varepsilon^{2^{m-1}}, \\ &\|E^{(m)}\|_{H^{k_{m-1}}} \lesssim \varepsilon^{2^m}, \\ &w^{(m)} \in \mathcal{B}_\varepsilon. \end{aligned} \tag{4.2}$$

For the case of $m = 1$, we recall that the assumptions (2.6)–(2.8) on the initial approximation function $w^{(0)}(x)$. By (2.50), let $0 < \varepsilon_0 < \lambda^{-(s+2)} N_0^{-(8+k-\overline{k})} \varepsilon^2 \ll 1$, it derives

$$\|\mathbf{h}^{(1)}\|_{H^{k_0}} \lesssim \|E^{(0)}\|_{H^{k_0}} < \varepsilon_0 < \varepsilon^2.$$

Moreover, by (3.7) and the above estimate, it holds

$$\|E^{(1)}\|_{H^{k_0}} \lesssim \|\mathcal{R}(\mathbf{h}^{1})\|_{H^{k_0}} \lesssim \lambda^{s+2} N_1^2 \|\mathbf{h}^{(1)}\|_{H^{k_0}}^2 \lesssim \varepsilon^2$$

and

$$\|w^{(1)}\|_{H^{k_0}} \lesssim \|w^{(0)}\|_{H^{k_0}} + \|\mathbf{h}^{(1)}\|_{H^{k_0}} \lesssim \varepsilon,$$

which means that $w^{(1)} \in \mathcal{B}_\varepsilon$.

Assume that the case of $(m-1)$ holds, i.e.,

$$\begin{aligned} &\|\mathbf{h}^{(m-1)}\|_{H^{k_m}} < \varepsilon^{2^{m-2}}, \\ &\|E^{(m-1)}\|_{H^{k_m}} < \varepsilon^{2^{m-1}}, \\ &w^{(m-1)} \in \mathcal{B}_\varepsilon, \end{aligned} \tag{4.3}$$

then we prove the case of m holds. Using (2.4), (3.6) and the second inequality of (4.3), we derive

$$\begin{aligned} \|\mathbf{h}^{(m)}\|_{H^{k_{m-1}}} &\lesssim \|\Pi_{N_m} E^{(m-1)}\|_{H^{k_{m-1}}} \\ &\lesssim N_m^{\alpha_m} \|E^{(m-1)}\|_{H^{k_m}} \\ &< \varepsilon^{2^{m-2}}, \end{aligned} \tag{4.4}$$

which combining with (2.4), (3.7) and (4.1), it holds

$$\begin{aligned} \|E^{(m)}\|_{H^{k_m}} &\lesssim \lambda^{s+2} N_m^2 \|\mathbf{h}^{(m)}\|_{H^{k_m}}^2 \\ &\lesssim \lambda^{2(s+2)} N_m^{2+\alpha_{m+1}} (\|E^{(m-1)}\|_{H^{k_{m+1}}})^2 \\ &\lesssim (\lambda^{s+2} N_0)^{(2+\alpha_{m+1})m + 2(2+\alpha_{m+2})(m-1)} (\|E^{(m-2)}\|_{H^{k_{m+2}}})^{2^2} \\ &\lesssim \cdots \\ &\lesssim (\lambda^{s+2} N_0^{8+k-\overline{k}} \|E^{(0)}\|_{H^{k_{2m}}})^{2^m}. \end{aligned} \tag{4.5}$$

We choose a sufficient small positive constant ε_0 such that

$$0 < \lambda^{s+2} N_0^{8+k-\overline{k}} \|E^{(0)}\|_{H^{\overline{k}}} \lesssim \varepsilon^2 \quad \text{for a fixed } s > 1.$$

Thus, by (4.5) we have

$$\|E^{(m)}\|_{H^{k_m}} \lesssim \varepsilon^{2^m}$$

and

$$0 \le \lim_{m\to+\infty} \|E^{(m)}\|_{H^{k_m}} \lesssim (\lambda^{s+2} N_0^{8+k-\overline{k}} \|E^{(0)}\|_{H^{k_{+\infty}}})^{2^{+\infty}} \to 0.$$

So the error term goes to 0 as $m \to \infty$, that is,

$$\lim_{m\to\infty} \|E^{(m)}\|_{H^{k_m}} = 0.$$

On the other hand, note that $N_m = N_0^m$, by (4.3)–(4.4), it holds

$$\begin{aligned} \|w^{(m)}\|_{H^{k_m}} &\lesssim \|w^{(m-1)}\|_{H^{k_m}} + \|\mathbf{h}^{(m)}\|_{H^{k_m}} \\ &\lesssim \varepsilon + N_m^3 \varepsilon^{2^m} \lesssim \varepsilon. \end{aligned}$$

This means that $w^{(m)} \in \mathcal{B}_\varepsilon$. Hence we conclude that (4.2) holds.

Therefore, the nonlinear problem (2.1) admits global solutions

$$w^{(\infty)}(x) = w^{(0)}(x) + \sum_{m=1}^{\infty} \mathbf{h}^{(m)}(\lambda x) = w^{(0)}(x) + \mathcal{O}(\varepsilon^2),$$

from which, one can see the solution depends on the initial approximation function $w^{(0)}(x)$ strongly. For two different $w^{(0)}(x)$, one can obtain two different Sobolev regular solutions $w^{(\infty)}(x)$. This gives the non-uniqueness of Sobolev regular solutions for equations (2.1).

At last, by (1.3) and the standard Calderon-Zygmund theory, i.e., for Riesz operator $\mathcal{R}$, there is $\|\mathcal{R}w\|_{\mathbb{L}^{s_0}} \leq \|w\|_{\mathbb{L}^{s_0}}$ with $1 < s_0 < \infty$, we obtain

$$\|P\|_{H^s} \lesssim \varepsilon.$$

This completes the proof.

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