

Long Time Asymptotics Behavior of the Focusing Nonlinear Kundu-Eckhaus Equation

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Abstract The authors study the Cauchy problem for the focusing nonlinear Kundu-Eckhaus (KE for short) equation and construct the long time asymptotic expansion of its solution in fixed space-time cone with $C(x_1, x_2, v_1, v_2) = \{(x, t) \in \mathbb{R}^2 : x = x_0 + vt, x_0 \in [x_1, x_2], v \in [v_1, v_2]\}$. By using the inverse scattering transform, Riemann-Hilbert approach and $\bar{\partial}$ steepest descent method, they obtain the long time asymptotic behavior of the solution, at the same time, they obtain the solitons in the cone compare with the all N -soliton the residual error up to order $\mathcal{O}(t^{-\frac{3}{4}})$.

Keywords Focusing Kundu-Eckhaus equation, Riemann-Hilbert problem, $\bar{\partial}$ steepest descent method

2000 MR Subject Classification O175.24

1 Introduction

We study the long time asymptotic behavior of the focusing nonlinear Kundu-Eckhaus (fKE for short) equation on $\mathbb{R} \times \mathbb{R}_+$:

$$\begin{aligned} i q_t + q_{xx} + 2|q|^2 q + 4\beta^2 |q|^4 q - 4i\beta(|q|)_x^2 q &= 0, \\ q(x, 0) &= q_0(x). \end{aligned} \quad (1.1)$$

In the defocusing case with the sign of cubic term reversed and initial value $q_0(x)$ in Schwarz space, it has known that (see [15]) as $t \rightarrow \infty$,

$$q(x, t) = \sqrt{\frac{\nu}{2t}} e^{i(4tk_0^2 - \nu(k_0)\log 8t)} e^{i\phi(k_0)} e^{-\frac{2}{\pi}i\beta \int_{k_0}^{\infty} \log(1 - |r'(k')|^2) dk'} + \mathcal{O}(t^{-1} \log t), \quad (1.2)$$

where

$$k_0 = -\frac{x}{4t}, \quad \nu(k_0) = -\frac{1}{2\pi} \log(1 - |r(k_0)|^2), \quad (1.3)$$

$$\phi(k_0) = \frac{1}{\pi} \int_{-\infty}^{k_0} \log(k_0 - k') d \log(1 - |r(k')|^2) + \frac{\pi}{4} - \arg r(k_0) + \arg \Gamma(i\nu), \quad (1.4)$$

where Γ is Gamma function, $r(z)$ is the reflection coefficient. Recently [11] applied Riemann-Hilbert approach to N -soliton solutions for the fKE equation (1.1) with nonzero boundary

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conditions. Wang et al. obtained long-time asymptotics of the fKE equation (1.2) with nonzero boundary conditions (see [12]). In this paper, we consider a much weaker boundary condition that supposes $q_0(x)$ in the weighted Sololev space

$$H^{1,1} = \{f \in L^2(\mathbb{R}) : xf, f' \in L^2(\mathbb{R})\}. \quad (1.5)$$

There exists a nonzero complex number c_k called a norming constant associated with any point in the simple discrete spectrum $z_k \in \mathbb{C}^+$. Define reflection coefficient $r : \mathbb{R} \rightarrow \mathbb{C}$ (where in the ZS-AKNS operator we know that the real axis is the continuous spectrum, and reflection coefficient r may take any value in $\mathbb{C}$ in the focusing case (see [12]), if r has singularities along the real line, call it spectral singularities. If there exist spectral singularities, it is possible that infinite discrete spectrum accumulate at a spectral singularity (see [2]). In this paper, we only consider that no spectral singularities exist so the discrete spectrum is finite.

If the spectrum consists of a single point $\sigma_d = \{(\xi + i\eta)\}$, the corresponding solution of (1.1) the one soliton

$$q_{\text{sol}}(x, t) = q_{\text{sol}}(x, t; \{\xi + i\eta\}) = 2ia_1 e^{\Omega_1 - \Omega_1^*} (P^{-1}) e^{8i\beta f |ae^{\Omega_1 - \Omega_1^*} (P^{-1})^2| dx}, \quad (1.6)$$

$$v_1 = e^{-iz_1 \sigma_3 x - 2iz_1^2 \sigma_3 t v} v_{10}, \quad (1.7)$$

$$\tilde{v}_1 = \tilde{v}_{10} e^{i\tilde{z}_1 \sigma_3 x + 2i\tilde{z}_1^2 \sigma_3 t v}, \quad (1.8)$$

$$P = \frac{\tilde{v}_1 v_1}{z_1^* - z_1}, \quad (1.9)$$

where $\Omega_1 = -iz_1 x - 2iz_1^2 t$, a_1 and z_1 are complex constants and v_{10} is initial speed. Let $q_{\text{sol}}(x, t; \sigma_d)$ stand for N -soliton solution with scattering data $\{r \equiv 0, \sigma_d = \{(z_k, c_k)_{k=1}^N\}\}$. Generally, the solution breaks apart into N independent one-soliton, each traveling at initial speed v_k (see [5]).

1.1 Main results and remark

In order to describe the asymptotic behavior of the solution of (1.1) as $t \rightarrow \infty$, for generic initial data $q_0 \in H^{1,1}(\mathbb{R})$. Define the discrete scattering data $\{r, \{(z_k, c_k)\}_{k=1}^N\}$. Let $\mathcal{Z} = \{z_k\}_{k=1}^N \subset \mathbb{C}^+$. Define

$$\kappa(s) = -\frac{1}{2\pi} \log(1 + |r(s)|^2), \quad (1.10)$$

and for any number ξ , let

$$\begin{aligned} \Delta_\xi^- &= \{k \in \{1, 2, \dots, N\} : \text{Re } z_k < \xi\}, \\ \Delta_\xi^+ &= \{k \in \{1, 2, \dots, N\} : \text{Re } z_k > \xi\}. \end{aligned} \quad (1.11)$$

Given any real interval $\mathcal{I} = [a, b]$, let

$$\begin{aligned} \mathcal{I} &= |\mathcal{Z}(\mathcal{I})|, \\ \mathcal{Z}(\mathcal{I}) &= \{z_k \in \mathcal{Z} : \text{Re } z_k \in \mathcal{I}\}, \\ \mathcal{Z}^-(\mathcal{I}) &= \{z_k \in \mathcal{Z} : \text{Re } z_k < a\}, \\ \mathcal{Z}^+(\mathcal{I}) &= \{z_k \in \mathcal{Z} : \text{Re } z_k > b\}. \end{aligned} \quad (1.12)$$

For $\xi \in \mathcal{I}$ let

$$\begin{aligned}\Delta_{\xi}^{-} &= \{k \in \{1, 2, \dots, N\} : a \leq \operatorname{Re} z_k < \xi\}, \\ \Delta_{\xi}^{+} &= \{k \in \{1, 2, \dots, N\} : \xi < \operatorname{Re} z_k \leq b\}, \\ \sigma_d^{\pm} &= \{(z_k, c_k^{\pm}(\mathcal{I}) : z_k \in \mathcal{I})\}, \\ c_k^{\pm} &= c_k \prod_{z_j \in \mathcal{Z}^{\pm}(\mathcal{I})} \left(\frac{z_k - z_j}{z_k - z_j^*} \right)^2 \exp\left(\pm 2i \int_{\xi}^{\mp \infty} \frac{\kappa(s)}{s - z_k} ds \right).\end{aligned}\tag{1.13}$$

Finally, given pairs of velocities $v_1 \leq v_2$ and points $x_1 \leq x_2$, we define the cone

$$C(x_1, x_2, v_1, v_2) := \{(x, t) \in \mathbb{R}^2 : x = x_0 + vt \text{ with } x_0 \in [x_1, x_2], v \in [v_1, v_2]\}.\tag{1.14}$$

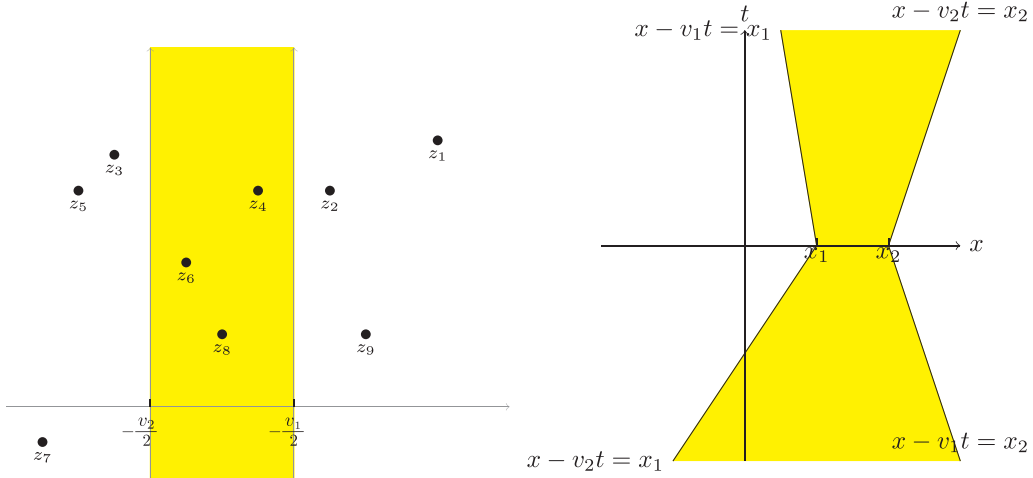


Figure 1 Using soliton contained in the cone C associated with its reflectionless scattering data to describe the asymptotic behavior of $q(x, t)$ as $|t| \rightarrow \infty$. In the example here, we use 3-soliton $\mathcal{Z}(\mathcal{I}) = \{z_4, z_6, z_8\}$ inside the cone C to asymptotically describe the discrete spectrum instead of 9-soliton.

Assumption 1.1 The initial data $q_0(x, t)$ for the Cauchy problem of fKE satisfies:

- (1) Every $z_k \in \mathbb{C}^+$ satisfied $a(z_k) = 0$ is simple, that is, the discrete spectrum is simple.
- (2) There exists a constant $c > 0$ such that $|a(z)| \geq 0$, that is, no spectral singularities exist.

According to above assumptions which guarantee that the discrete spectrum is finite.

Theorem 1.1 Let $q(x, t)$ be the solution of (1.1) with initial data $q_0(x) \in H^{1,1}(\mathbb{R})$ satisfying Assumption 1.1 and generating the scattering data $\{r, \{z_k, c_k\}_{k=1}^N\}$. Fix $x_1, x_2, v_1, v_2 \in \mathbb{R}$ with $x_1 < x_2$ and $v_1 < v_2$. Let $\mathcal{I} = [-\frac{v_2}{2}, -\frac{v_1}{2}]$ and $\xi = -\frac{x}{4t}$, when $t \rightarrow \pm\infty$ with $(x, t) \in C(x_1, x_2, v_1, v_2)$, which is defined in (1.14), we have

$$q(x, t) = (q_{\text{sol}}(x, t; \sigma^{\pm}(\mathcal{I})) + t^{-\frac{1}{2}} f^{\pm}(x, t) + O(t^{-\frac{3}{4}})) e^{-8i\beta \int_{(-\infty, t)}^{\frac{x}{4t}} |m|^2 dx} + \mathcal{O}(t^{-\frac{3}{4}}),\tag{1.15}$$

where

$$f^{\pm}(x, t) = m_{11}(\xi; x, t)^2 \alpha(\xi, \pm) e^{\frac{i\pi}{2t} \mp i\kappa(\xi) \log|4t|},$$

$$+ m_{12}(\xi; x, t)^2 \alpha(\xi, \pm)^* e^{-\frac{ix^2}{2t} \pm i\kappa(\xi) \log|4t|}$$

with

$$|\alpha(\xi, \pm)|^2 = |\kappa(\xi)| \quad (1.16)$$

satisfying

$$\begin{aligned} \arg \alpha(\xi, \pm) &= \pm \frac{\pi}{4} \pm \arg \Gamma(i\kappa(\xi)) - \arg r(\xi) \\ &\quad - 4 \sum_{k \in \Delta_{\xi}^{\mp}} \arg(\xi - z_k) \mp 2 \int_{\mp\infty}^{\xi} \log|\xi - s| d_s \kappa(s), \end{aligned}$$

and

$$|m(x, t)|^2 = \left| \frac{1}{2i} (q_{\text{sol}}(x, t; \sigma^{\pm}(I)) + t^{-\frac{1}{2}} f^{\pm}(x, t) + \mathcal{O}(t^{-\frac{3}{4}})) \right|^2, \quad (1.17)$$

the coefficients $m_{11}(\xi; x, t)$ and $m_{12}(\xi; x, t)$ are the entries in the first row of the solution of RHP A.2 with discrete scattering data $\sigma_d^{\pm}(I)$ and $\Delta = \Delta_{\xi}^{\mp}(I)$ evaluated at $a = \xi$.

1.2 Organization of the rest of the paper and notation

In Section 2 we construct the RHP 2.1 associated with initial-value problem (1.1), and then we work out the steepest descent analysis of RHP 2.1 for $t \rightarrow \infty$ from Section 3 to Section 7. In Section 3 we introduce the matrix $T(z)$ to separate the jump matrix defined in RHP 2.1 at $\xi = -\frac{x}{4t}$. Section 4 introduces the $\bar{\partial}$ analysis to define extensions of the jump matrix for the non-linear steepest descent method. In Section 5 we construct a global model solution which captures the leading order asymptotic behavior of the solution. Removing this component of the solution results in a small norm $\bar{\partial}$ -problem which is analyzed in Section 6. The proof of Theorem 1.1 is given in Section 7.

2 Results of Scattering Theory for Focusing KE

Our calculations are based on the Lax pair of the focusing KE equation (1.1),

$$\psi_x + iz\sigma_3\psi = Q_1\psi, \quad (2.1)$$

$$\psi_t + 2iz^2\sigma_3\psi = Q_2\psi, \quad (2.2)$$

where z is a spectral parameter and

$$\begin{aligned} Q &= \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ Q_1 &= Q - i\beta Q^2 \sigma_3, \\ Q_2 &= 4i\beta^2 Q^4 \sigma_3 - 2\beta Q^3 - iQ^2 \sigma_3 + 2zQ - iQ_x \sigma_3 + \beta(Q_x Q - Q Q_x). \end{aligned} \quad (2.3)$$

Specially, $z \in \mathbb{R}$, then the eigenfunction $\psi(x, t, z)$ is the Lax pair. Denote

$$\Psi = \psi e^{i(zx + 2z^2 t)\sigma_3}.$$

We obtain the equivalent Lax pair

$$\begin{aligned}\Psi_x + iz[\sigma, \Psi] &= Q_1 \Psi, \\ \Psi_t + 2iz^2[\sigma_3, \psi] &= Q_2 \Psi,\end{aligned}\tag{2.4}$$

its full derivative form is

$$d e^{i(zx+2z^2t)\widehat{\sigma}_3} \Psi(x, , t, k) = e^{i(zx+2z^2t)\widehat{\sigma}_3} U \Psi,\tag{2.5}$$

$$U = Q_1 dx + Q_2 dt.\tag{2.6}$$

Consider the form of solution (2.5),

$$\Psi = D + \frac{\Psi_1}{z} + \frac{\Psi_2}{z^2} + O\left(\frac{1}{z^3}\right), \quad z \rightarrow \infty,\tag{2.7}$$

where D, Ψ_1, Ψ_2 are independent of z , substituting above expansion into two equations of (2.4), and comparing the same order of z' frequency, we find the following equations

$$D_x = -i\beta Q^2 \sigma_3 D,\tag{2.8}$$

$$D_t = \beta(q\bar{q}_x - q_x\bar{q} + 4i\beta^2|q|^4)\sigma_3 D,\tag{2.9}$$

we get (1.1) has the conservation law

$$(i\beta|q|^2)_t = (\beta(q\bar{q}_x - q_x\bar{q}) + 4i\beta^2|q|^4)_x.$$

(2.8) and (2.9) for D are consistent and are both satisfied if we define

$$D(x, t) = e^{i \int_{(x,t)}^{(+\infty, t)} \Delta \sigma_3},\tag{2.10}$$

where Δ is

$$\Delta(x, t) = \beta|q|^2 dx + (-i\beta(q\bar{q}_x - q_x\bar{q}) + 4\beta^2|q|^4) dt.\tag{2.11}$$

According to asymptotic analysis we introduce a new function μ by

$$\Psi(x, t, z) = e^{-i \int_{(-\infty, t)}^{(x, t)} \Delta \widehat{\sigma}} \mu(x, t, z) D(x, t).\tag{2.12}$$

Thus, we have

$$\mu = I + \mathcal{O}\left(\frac{1}{z}\right), \quad z \rightarrow \infty,\tag{2.13}$$

and (2.5) becomes

$$d(e^{i(zx+2z^2t)\widehat{\sigma}_3} \mu(x, t, k)) = W(x, t, z) = e^{i(zx+2z^2t)\widehat{\sigma}_3} V(x, t, z) \mu,\tag{2.14}$$

in which

$$V = V_1 dx + V_2 dt\tag{2.15}$$

with

$$V_1 = \begin{pmatrix} 2i\beta|u|^2 & ue^{2i\int_{(-\infty,t)}^{(x,t)} \Delta} \\ -\bar{u}e^{-2i\int_{(-\infty,t)}^{(x,t)} \Delta} & -2i\beta|u|^2 \end{pmatrix}, \quad (2.16)$$

$$V_2 = \begin{pmatrix} i|u|^2 & (2\beta u|u|^2 + 2zu + iu_x)e^{2i\beta\int_{(-\infty,t)}^{(x,t)} \Delta} \\ (-2\beta\bar{u}|u|^2 - 2z\bar{u} - i\bar{u}_x)e^{-2i\beta\int_{(-\infty,t)}^{(x,t)} \Delta} & -i|u|^2 \end{pmatrix}. \quad (2.17)$$

The Lax pair (2.3) can be changed into

$$\begin{aligned} \mu_x + iz[\sigma_3, \mu] &= V_1\mu, \\ \mu_t + 2iz^2[\sigma_3, \mu] &= V_2\mu. \end{aligned} \quad (2.18)$$

We assume that $\mu(x, t)$ is sufficiently smooth, we define two solutions of (2.14) by

$$\mu_j(x, t, z) = I + \int_{(x_j, t_j)}^{(x, t)} e^{-i(zx+2z^2t)\widehat{\sigma}_3} W(y, \tau, z) dy, \quad j = 1, 2, \quad (2.19)$$

where $(x_1, t_1) = (-\infty, t)$, $(x_2, t_2) = (+\infty, t)$, it follows that $\det \Psi_{(1,2)} = \det \mu_{(1,2)} \equiv 1$, and it satisfies

$$\mu_x + iz[e^{-i\int_{(x,t)}^{(+\infty,t)} \Delta \sigma_3} \sigma_3, \mu] = Q\mu, \quad (2.20)$$

that is

$$\mu_x + iz[D^{-1}\sigma_3, \mu] = Q\mu. \quad (2.21)$$

Its Volterra type integral is

$$\mu_{(1,2)}(x, z) = I + \int_{\pm\infty}^x e^{izD^{-1}(x-y)\sigma_3} Q\mu_{(1,2)}(y, z) e^{-izD^{-1}(x-y)\sigma_3} dy. \quad (2.22)$$

Also, if $\mu(x, z)$ is any solution of (2.18), then $\tilde{\mu}(x, z) = \sigma_2 \overline{\mu(x, \bar{z})} \sigma_2$ (complex conjugate but no transpose) also solves (2.18). For $z \in \mathbb{R}$, $\sigma_2 \overline{\mu_{(1,2)}(x, \bar{z})} \sigma_2$ also satisfies (2.18) and it follows that

$$\Psi_{(1,2)}(x, z) = \sigma_2 \overline{\Psi_{(1,2)}(x, \bar{z})} \sigma_2, \quad z \in \mathbb{R}. \quad (2.23)$$

Since the eigenfunctions $\mu_1(x, t, z)$ and $\mu_2(x, t, z)$ satisfy both equations of the Lax pair (2.18), there exists a continuous scattering matrix function $S(z)$ satisfying

$$\mu_1(x, t, z) = \mu_2(x, t, z) e^{-i(zx+2z^2t)\widehat{\sigma}_3} S(z), \quad z \in \mathbb{R}, \quad (2.24)$$

where

$$\begin{aligned} S(z) &= \begin{pmatrix} a(z) & -\overline{b(\bar{z})} \\ b(z) & \overline{a(\bar{z})} \end{pmatrix}, \\ \det S(z) &= |a(z)|^2 + |b(z)|^2 = 1. \end{aligned} \quad (2.25)$$

Define

$$\mu_1 = (\mu_1^{(1)}, \mu_1^{(2)}) = \begin{pmatrix} \mu_1^{(11)} & \mu_1^{(12)} \\ \mu_1^{(21)} & \mu_1^{(22)} \end{pmatrix}, \quad \mu_2 = (\mu_2^{(1)}, \mu_2^{(2)}) = \begin{pmatrix} \mu_2^{(11)} & \mu_2^{(12)} \\ \mu_2^{(21)} & \mu_2^{(22)} \end{pmatrix}, \quad (2.26)$$

where $\mu_j^1(x, t, z)$ and $\mu_j^2(x, t, z)$ are the first and second columns of $\mu_j(x, t; z)$, $j = 1, 2$.

Remark 2.1 • $\mu_1^{(1)}, \mu_2^{(2)}$ and $a(z)$ extend analytically to $z \in \mathbb{C}^+$ with continuous boundary values on $\mathbb{R}$, and $\mu_1^{(1)} \rightarrow e_1, \mu_2^{(2)} \rightarrow e_2$ and $a(z) \rightarrow 1$ when $z \rightarrow \infty$, similar consequences hold for $z \in \mathbb{C}^-$, however, $b(z)$ is defined only for $z \in \mathbb{R}$.

• The solutions $\Psi_1^{(1)}(x, z_k)$ and $\Psi_2^{(2)}(x, z_k)$ are linearly dependent when $a(z_k) = 0$ for $z_k \in \mathbb{C}$. So there exists a norming constant c_k satisfying

$$\Psi_1^{(1)}(x, z_k) = c_k \Psi_2^{(2)}(x, z_k). \quad (2.27)$$

The symmetry (2.20) implies that

$$\Psi_2^{(1)}(x, z_k^*) = c_k^* \Psi_1^{(2)}(x, z_k^*). \quad (2.28)$$

• The reflection coefficient r and transmission coefficient τ are defined by

$$r(z) = \frac{b(z)}{a(z)}, \quad \tau = \frac{1}{a(z)}, \quad (2.29)$$

and it follows from (2.22) that $1 + |r(z)|^2 = |\tau(z)|^2$ for each $z \in \mathbb{R}$.

We construct the function

$$M(z) = M(z; x, t) : \begin{cases} \begin{bmatrix} \frac{\mu_1^{(1)}(x, t; z)}{a(z)} & \mu_2^{(2)}(x, t; z) \end{bmatrix}, & z \in \{z \in \mathbb{C} \mid \text{Im } z > 0\}, \\ \begin{bmatrix} \mu_2^{(1)}(x, t; z) & \frac{\mu_1^{(2)}(x, t; z)}{\overline{a(\bar{z})}} \end{bmatrix}, & z \in \{z \in \mathbb{C} \mid \text{Im } z < 0\}. \end{cases} \quad (2.30)$$

The matrix M defined above is the solution of the following Riemann-Hilbert problem.

Next, we consider the characteristic function $\mu_1^{(1)}, \mu_2^{(2)}$ and analytic properties of spectral matrix $S(z)$. For the integral equation, let the integral variable $y \leq x$. We calculate directly to obtain:

$$\begin{aligned} & e^{-i(zx+2z^2t)\sigma_3} W(y, t, z) \\ &= \begin{pmatrix} 2i\beta|u|^2 & ue^{2i\int_{(-\infty, t)}^{(x, t)} \Delta} e^{2i(zx+2z^2t)} \\ -\bar{u}e^{-2i\int_{(-\infty, t)}^{(x, t)} \Delta} e^{-2i(zx+2z^2t)} & -2i\beta|u|^2 \end{pmatrix} dx \\ &+ \begin{pmatrix} i|u|^2 & (2\beta u|u|^2 + 2zu + iu_x)e^{2i\int_{(-\infty, t)}^{(x, t)} \Delta} e^{2i(zx+2z^2t)} \\ (-2\beta\bar{u}|u|^2 - 2z\bar{u} - i\bar{u}_x)e^{-2i\int_{(-\infty, t)}^{(x, t)} \Delta} e^{-2i(zx+2z^2t)} & -i|u|^2 \end{pmatrix} dt \end{aligned}$$

and

$$e^{2i(zx+2z^2t)} = e^{4iz^2t} e^{2ix\text{Re } z} e^{-2x\text{Im } z}, \quad e^{-2i(zx+2z^2t)} = e^{-4iz^2t} e^{-2ix\text{Re } z} e^{-2x\text{Im } z}.$$

Therefore, the first and second columns of μ_1 are analyzed separately in the upper half plane $\mathbb{C}_+$ and lower half plane $\mathbb{C}_-$, and we record them as

$$\mu_1 = (\mu_1^+, \mu_1^-) = \begin{pmatrix} \mu_1^{(11)} & \mu_1^{(12)} \\ \mu_1^{(21)} & \mu_1^{(22)} \end{pmatrix}.$$

The same can be proved, the first and second columns of μ_2 are analyzed separately in $\mathbb{C}_-$ and $\mathbb{C}_+$, we record as

$$\mu_2 = (\mu_2^-, \mu_2^+) = \begin{pmatrix} \mu_2^{(11)} & \mu_2^{(12)} \\ \mu_2^{(21)} & \mu_2^{(22)} \end{pmatrix}.$$

Note that ψ_1, ψ_2 are the solutions of Lax (2.1)–(2.2), so, we obtain

$$\text{tr}(P - iz\sigma_3) = \text{tr}(Q - 2iz^2\sigma_3) = 0.$$

According to Abel formula, we have

$$(\det \psi_j)_x = (\det \psi_j)_t = 1. \quad (2.31)$$

From the transformation (2.12), we get

$$\det \mu_j = \det \psi_j, \quad \det(e^{-i \int_{(-\infty, t)}^{\langle x, t \rangle} \Delta \hat{\sigma}}) = \det \Psi_j = \det \psi_j.$$

Using Abel formula above, we obtain

$$(\det \mu_j)_x = (\det \mu_j)_t = 1.$$

This means $\det \mu_j$ has no relationship with x, t . Then asymptotically $\mu_j \rightarrow I$ as $|x| \rightarrow \infty$. Take the determinant on both sides of this relationship to get

$$\det S(z) = 1.$$

So μ_1, μ_2 are reversible, and their inverse matrix is the corresponding adjoint matrix. In addition, it is based on the analyticity of the column vector of μ_1, μ_2 . It can be deduced that the first and second lines of μ_1^{-1} are analyzed separately in $\mathbb{C}_-$ and $\mathbb{C}_+$ and are recorded as

$$\mu_2^{-1} = \begin{pmatrix} \mu_2^{(22)} & -\mu_2^{(12)} \\ -\mu_2^{(21)} & \mu_2^{(22)} \end{pmatrix} = \begin{pmatrix} \widehat{\mu}_2^+ \\ \widehat{\mu}_2^- \end{pmatrix}.$$

It can be seen that the spectral function is analytical

$$e^{-i(zx+2z^2t)\widehat{\sigma}_3} S(z) = \mu_2^{-1} \mu_1 = \begin{pmatrix} \widehat{\mu}_2^+ \\ \widehat{\mu}_2^- \end{pmatrix} (\mu_1^+ \mu_1^-) = \begin{pmatrix} \widehat{\mu}_2^+ \mu_1^+ & \widehat{\mu}_2^+ \mu_1^- \\ \widehat{\mu}_2^- \mu_1^+ & \widehat{\mu}_2^- \mu_1^- \end{pmatrix}.$$

We get that $\mathbb{C}_+$ is analyzed in the $\mathbb{C}_+$, $s_{22}(z)$ is analyzed in the $\mathbb{C}_-$, $s_{12}(z)$ and $s_{21}(z)$ are not analyzed in the lower half plane, but are continuous to the real axis $\mathbb{R}$.

Lemma 2.1 *The characteristic function μ_1, μ_2 constructed above and spectral function $S(z)$ have the following symmetry*

$$\begin{aligned} \mu_j^H(x, t, \bar{z}) &= \mu_j^{-1}(x, t, z), \quad j = 1, 2, \\ S^H(\bar{z}) &= S^{-1}(z). \end{aligned}$$

Here, the superscript H represents the conjugate transpose of the matrix.

Proof Because of

$$\mu_{j,x}(x, t, z) + iz[D^{-1}\sigma_3, \mu_j(x, t, z)] = Q\mu_j(x, t, z), \quad j = 1, 2.$$

Replace z with $\bar{z}$ and then take the conjugate transpose of the equation to obtain

$$\mu_{j,x}^H(x, t, \bar{z}) + iz[\sigma_3^H(D^{-1})^H, \mu_j^H(x, t, \bar{z})] = \mu_j^H(x, t, \bar{z})Q^H, \quad j = 1, 2.$$

Note that Q is an inverse Hermite matrix, that is $P^H = -P$, so

$$\mu_{j,x}^H(x, t, \bar{z}) + iz[\sigma_3^H(D^{-1})^H, \mu_j^H(x, t, \bar{z})] = -\mu_j^H(x, t, \bar{z})Q, \quad j = 1, 2.$$

In addition, the derivative of $u_j(x, t, z)\mu_j^{-1}(x, t, z) = I$ with respect to x implies

$$\mu_{j,x}^{-1}(x, t, z) = -\mu_j^{-1}(x, t, z)\mu_{j,x}(x, t, z)\mu_j^{-1}(x, t, z), \quad j = 1, 2.$$

Bring the equation into the above formula to obtain:

$$\begin{aligned} \mu_{j,x}^{-1}(x, t, z) &= -\mu_j^{-1}(x, t, z)(Q\mu_j(x, t, z) - iz[D^{-1}\sigma_3, \mu_j(x, t, z)])\mu_j^{-1}(x, t, z) \\ &= -\mu_j^{-1}(x, t, z)Q - iz[D^{-1}\sigma_3, \mu_j^{-1}(x, t, z)], \quad j = 1, 2, \end{aligned}$$

that is

$$\mu_{j,x}^{-1}(x, t, z) + iz[D^{-1}\sigma_3, \mu_j(x, t, z)]\mu_j^{-1}(x, t, z) = -\mu_j^{-1}(x, t, z)Q.$$

μ_j satisfy the same differential equation and have the same asymptotic behavior:

$$\mu_j^H(x, t, \bar{z}), \quad \mu_j^{-1}(x, t, z) \rightarrow I, \quad |x| \rightarrow \infty.$$

So the two are equal, and we get a symmetric relationship

$$\mu_j^H(x, t, \bar{z}) = \mu_j^{-1}(x, t, z), \quad j = 1, 2.$$

Consider the symmetry of $S(z)$, we have

$$S(z) = e^{i(zx+2z^2t)\widehat{\sigma_3}}[\mu_2^{-1}(x, t, z)\mu_1(x, t, z)],$$

then from $\overline{\bar{z}x + 2\bar{z}^2t} = zx + 2z^2t$, we get

$$\begin{aligned} S(\bar{z})^H &= [e^{i(\bar{z}x+2\bar{z}^2t)\sigma_3}[\mu_2^{-1}(x, t, \bar{z})\mu_1(x, t, \bar{z})]e^{i(\bar{z}x+2\bar{z}^2t)\sigma_3}]^H \\ &= e^{i(zx+2z^2t)\sigma_3}([\mu_1^H(x, t, \bar{z})\mu_2^{-1}(x, t, \bar{z})])^H e^{i(zx+2z^2t)\sigma_3} \\ &= e^{i(zx+2z^2t)\sigma_3}([\mu_1^{-1}(x, t, z)\mu_2(x, t, z)])^H e^{i(zx+2z^2t)\sigma_3} \\ &= S^{-1}(z). \end{aligned}$$

Comparing the corresponding elements of the matrix on both sides, we obtain

$$\overline{s_{11}(\bar{z})} = s_{22}(z), \quad \overline{s_{12}(\bar{z})} = -s_{21}(z).$$

Riemann-Hilbert Problem 2.1 Find an analytic function $M : \mathbb{C} \setminus (\mathbb{R} \cup \mathcal{Z} \cup \mathcal{Z}^*) \rightarrow SL_2(\mathbb{C})$ with the following properties

- (1) $M(z) = I + O(z^{-1})$ as $z \rightarrow \infty$.
- (2) The continuous boundary values $M_{\pm}(z)$ satisfy the jump relation $M_+(x, t, z) = M_-(x, t, z), \dots, J(x, t, z), z \in \mathbb{R}$, where

$$J(z) = \begin{pmatrix} 1 + |r(z)|^2 & r^*(z)e^{-2it\theta(z)} \\ r(z)e^{2it\theta(z)} & 1 \end{pmatrix}, \quad (2.32)$$

$$\theta(z) := 2z^2 + z\frac{x}{t} = 2(z - \xi)^2 - 2\xi^2, \quad \xi = -\frac{x}{4t}. \quad (2.33)$$

(3) $M(z)$ has simple poles at each $z_k \in \mathbb{R}$ and $z_k^* \in \mathbb{R}^*$ at which

$$\begin{aligned} \operatorname{Res}_{z_k} M &= \lim_{z \rightarrow z_k} M \begin{pmatrix} 0 & 0 \\ c_k e^{2it\theta} & 0 \end{pmatrix}, \\ \operatorname{Res}_{z_k^*} M &= \lim_{z \rightarrow z_k^*} M \begin{pmatrix} 0 & -c_k^* e^{-2it\theta} \\ 0 & 0 \end{pmatrix}. \end{aligned} \quad (2.34)$$

Consider (2.7), we get $\Psi_1 = \frac{i}{2} Q D \sigma_3$, $i[\sigma_3, \Psi_1] = Q D$. The existence of solutions of RHP 2.1 for all $(x, t) \in \mathbb{R}^2$ follows by means of expanding this solution as $z \rightarrow \infty$, $M = I + \frac{M^{(1)}(x, t)}{z} + o(z^{-1})$, one finds that

$$q(x, t) = \lim_{z \rightarrow \infty} (2izM(x, t, z))_{12} e^{-2i \int_{(-\infty, t)}^{(x, t)} \Delta} = 2im(x, t) e^{-2i \int_{(-\infty, t)}^{(x, t)} \Delta}, \quad (2.35)$$

where

$$m(x, t) = \lim_{z \rightarrow \infty} (zM(x, t, z))_{12} \quad (2.36)$$

and

$$\mu = I + \frac{\mu^{(1)}}{z} + \frac{\mu^{(2)}}{z^2} + \mathcal{O}\left(\frac{1}{z^3}\right), \quad z \rightarrow \infty \quad (2.37)$$

is the corresponding solution of (2.14) related to Ψ via (2.12), moreover, from its complex conjugate, we obtain

$$\begin{aligned} |q|^2 &= 4|m|^2, \\ u\bar{u}_x - u_x\bar{u} &= 4(m\bar{m}_x - \bar{m}m_x) + 64i\beta|m|^4. \end{aligned}$$

Thus, we are able to express the one-form Δ defined in (2.35) in terms of m as

$$\Delta = 4\beta|m|^2 dx + [4i\beta(\bar{m}m_x - m\bar{m}_x) + 128\beta^2|m|^4]dt. \quad (2.38)$$

3 Conjugation

In this section, we introduce the function $T(z)$ to renormalize the Riemann-Hilbert problem with ξ fixed

$$\begin{aligned} T(z) &= T(z, \xi) = \prod_{k \in \Delta_\xi^-} \left(\frac{z - z_k^*}{z - z_k} \right) \exp\left(i \int_{-\infty}^{\xi} \frac{\kappa(s)}{s - z} ds\right), \\ \kappa(s) &= -\frac{1}{2\pi} \log(1 + |r(s)|^2), \end{aligned} \quad (3.1)$$

we can also get the standard result of the transmission coefficient

$$\frac{1}{a(z)} = \prod_{k=1}^N \left(\frac{z - z_k^*}{z - z_k} \right) \exp\left(i \int_{-\infty}^{\infty} \frac{\kappa(s)}{s - z} ds\right), \quad (3.2)$$

and we can find $T(z; \xi) \rightarrow \frac{1}{a(z)}$ when $\xi \rightarrow \infty$.

Proposition 3.1 *The function $T(z)$ defined by (3.1) has the following properties:*

(a) T is nonzero and meromorphic in $\mathbb{C} \setminus (-\infty, \xi]$. For each k in Δ_ξ^- , $T(z)$ has a simple pole at z_k and a simple zero at $\overline{z_k}$.

(b) For $z \in \mathbb{C} \setminus (-\infty, \xi]$, $\overline{T(\overline{z})} = \frac{1}{T(z)}$.

(c) For $z \in (-\infty, \xi]$, the boundary values $T_\pm$ satisfy

$$\frac{T_+(z)}{T_-(z)} = 1 + |r(z)|^2, \quad z \in (-\infty, \xi). \quad (3.3)$$

(d) As $|z| \rightarrow \infty$ with $|\arg(z)| \leq c \leq \pi$,

$$T(z) = 1 + \frac{i}{z} \left[2 \sum_{k \in \Delta_\xi^-} \operatorname{Im} z_k - \int_{-\infty}^{\xi} \kappa(s) ds \right] + O(z^{-2}). \quad (3.4)$$

(e) As $z \rightarrow \xi$ along any ray $\xi + e^{i\phi} \mathbb{R}_+$ with $|\phi| \leq c \leq \pi$ (see [1]),

$$|T(z, \xi) - T_0(\xi)(z - \xi)^{i\kappa(\xi)}| \leq C \|r\|_{H^1(\mathbb{R})} |z - \xi|^{\frac{1}{2}}, \quad (3.5)$$

where $T_0(\xi)$ is the complex unit

$$T_0(\xi) = \prod_{k \in \Delta_\xi^-} \left(\frac{\xi - z_k^*}{\xi - z_k} \right) e^{i\beta(\xi, \xi)} = \exp \left[i \left(\beta(\xi, \xi) - 2 \sum_{k \in \Delta_\xi^-} \arg(\xi - z_k) \right) \right],$$

$$\beta(z, \xi) = -\kappa(\xi) \log(z - \xi + 1) + \int_{-\infty}^{\xi} \frac{\kappa(s) - \chi(s)\kappa(\xi)}{s - z} ds,$$

and $\chi(s)$ is the characteristic function of the interval $(\xi - 1, \xi)$ and the logarithm is principally branched along $(-\infty, \xi - 1]$.

Proof For parts (a)–(d) we can prove them directly by using the definition and the Sokhotski-Plemelj formula (see [7, 14]). For part (e) we write

$$\begin{aligned} T(z, \xi) &= \prod_{k \in \Delta_\xi^-} \left(\frac{z - z_k^*}{z - z_k} \right) \exp \left(i \int_{\xi-1}^{\xi} \frac{\kappa(s)}{s - z} ds + i \int_{-\infty}^{\xi} \frac{\kappa(s) - \chi(s)\kappa(\xi)}{s - z} ds \right) \\ &= \prod_{k \in \Delta_\xi^-} \left(\frac{z - z_k^*}{z - z_k} \right) (z - \xi)^{i\kappa(\xi)} \exp(i\beta(z, \xi)). \end{aligned} \quad (3.6)$$

The result then follows from the facts that

$$|(z - \xi)^{i\kappa(\xi)}| \leq e^{-\pi\kappa(\xi)} = \sqrt{1 + |r(\xi)|^2}, \quad (3.7)$$

and using [3, Lemma 23.3],

$$|\beta(z, \xi) - \beta(\xi, \xi)| \leq C \|r\|_{H^1(\mathbb{R})} |z - \xi|^{\frac{1}{2}}. \quad (3.8)$$

Define a new function $M^{(1)}$,

$$M^{(1)}(z) = M(z)T(z)^{-\sigma_3}, \quad (3.9)$$

we can prove the function $M^{(1)}$ satisfies the following Riemann-Hilbert problem 3.1.

Riemann-Hilbert Problem 3.1 Find an analysis function $M^{(1)} : \mathbb{C} \setminus (\mathbb{R} \cup \mathcal{Z} \cup \mathcal{Z}^*) \rightarrow SL_2(\mathbb{C})$ with the following properties:

- (1) $M^{(1)}(z) = I + O(z^{-1})$ as $z \rightarrow \infty$.
 (2) For each $z \in \mathbb{R}$, the boundary values $M_{\pm}^{(1)}(z)$ satisfy the jump relationship $M_+^{(1)}(z) = M_-^{(1)}(z)J^{(1)}(z)$ where

$$J^{(1)}(z) = \begin{cases} \begin{pmatrix} 1 & r^*(z)T(z)^2e^{-2it\theta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r(z)T(z)^{-2}e^{2it\theta} & 1 \end{pmatrix}, & z \in (\xi, \infty), \\ \begin{pmatrix} 1 & 0 \\ \frac{r(z)T_-(z)^{-2}}{1+|r(z)|^2}e^{2it\theta} & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{r^*(z)T_+(z)^2}{1+|r(z)|^2}e^{-2it\theta} \\ 0 & 1 \end{pmatrix}, & z \in (-\infty, \xi). \end{cases} \quad (3.10)$$

- (3) $M^{(1)}(z)$ has simple poles at each $z_k \in \mathbb{R}$ and $z_k^* \in \mathbb{R}^*$ at which

$$\begin{aligned} \text{Res}_{z_k} M^{(1)} &= \begin{cases} \lim_{z \rightarrow z_k} M^{(1)} \begin{pmatrix} 0 & c_k^{-1}(\frac{1}{T})'(z_k)^{-2}e^{-2it\theta} \\ 0 & 0 \end{pmatrix}, & k \in \Delta_{\xi}^-, \\ \lim_{z \rightarrow z_k} M^{(1)} \begin{pmatrix} 0 & 0 \\ c_k^{-1}T(z_k)^{-2}e^{2it\theta} & 0 \end{pmatrix}, & k \in \Delta_{\xi}^+, \end{cases} \\ \text{Res}_{z_k^*} M^{(1)} &= \begin{cases} \lim_{z \rightarrow z_k^*} M^{(1)} \begin{pmatrix} 0 & 0 \\ -(c_k^*)^{-1}T'(z_k^*)^{-2}e^{-2it\theta} & 0 \end{pmatrix}, & k \in \Delta_{\xi}^-, \\ \lim_{z \rightarrow z_k^*} M^{(1)} \begin{pmatrix} 0 & -c_k^*T(z_k^*)^2e^{-2it\theta} \\ 0 & 0 \end{pmatrix}, & k \in \Delta_{\xi}^+. \end{cases} \end{aligned} \quad (3.11)$$

Proof From above definition, we can get that $M^{(1)}$ is unimodular, analytic in $\mathbb{C} \setminus (\mathbb{R} \cup \mathcal{Z} \cup \mathcal{Z}^*)$, and approaches identity as $z \rightarrow \infty$ and we factorize jump (3.10) as following

$$\begin{aligned} &J^{(1)}(z) \\ &= \begin{cases} T(z)^{\sigma_3} \begin{pmatrix} 1 & r^*(z)e^{-2it\theta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r(z)e^{2it\theta} & 1 \end{pmatrix} T(z)^{-\sigma_3}, & z \in (\xi, \infty), \\ T_-(z)^{\sigma_3} \begin{pmatrix} 1 & 0 \\ \frac{r(z)}{1+|r(z)|^2}e^{2it\theta} & 1 \end{pmatrix} \left(\frac{T_+(z)}{T_-(z)}\right)^{\sigma_3} \begin{pmatrix} 1 & \frac{r^*(z)}{1+|r(z)|^2}e^{-2it\theta} \\ 0 & 1 \end{pmatrix}, & z \in (-\infty, \xi). \end{cases} \end{aligned} \quad (3.12)$$

For $k \in \Delta_{\xi}^+$, $T(z)$ has zero at z_k^* and a pole at z_k , so that $M_1^{(1)} = M_1(z)T(z)^{-1}$ has a removable singularity at z_k and a pole at z_k^* . For $M_2^{(1)}$ the situation is reversed; we have

$$\begin{aligned} M_1^{(1)}(z_k) &= \lim_{z \rightarrow z_k} M_1(z)T(z)^{-1} = \text{Res}_{z_k} M_1(z) \left(\frac{1}{T}\right)'(z_k) = c_k e^{2it\theta_k} M_2(z_k) \left(\frac{1}{T}\right)'(z_k); \\ \text{Res}_{z_k} M_2^{(1)}(z) &= \text{Res}_{z=z_k} M_2(z_k)T(z) = M_2(z_k) \left[\left(\frac{1}{T}\right)'(z_k)\right]^{-1} \\ &= c_k^{-1} \left[\left(\frac{1}{T}\right)'(z_k)\right]^{-2} e^{-2it\theta} M_1^{(1)}(z_k), \end{aligned} \quad (3.13)$$

from which the first formula in (3.11) clearly follows. The computation of the residue at z_k^* for $k \in \Delta_{\xi}^-$ is similar.

4 Introducing $\bar{\partial}$ Extensions of Jump Factorization

In these section, our work is to extend the jump matrix off the real axis to new contours whose factors satisfy continuous, decaying but not analytic, we define an unknown non-analytic transformation increasing nonzero $\bar{\partial}$ derivatives insides the regions.

Define the new contours

$$\sum_k = \xi + e^{i(2k-1)\frac{\pi}{4}} \mathbb{R}_+, \quad k = 1, 2, 3, 4. \quad (4.1)$$

Additionally, let

$$\Sigma_R = \mathbb{R} \cup \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Sigma_4, \quad (4.2)$$

$$\rho = \frac{1}{2} \min_{\lambda, \mu \in \mathcal{Z} \cup \mathcal{Z}^*, \lambda \neq \mu} |\lambda - \mu|. \quad (4.3)$$

According to our assumption, there is no pole lying on the real axis and all poles are in conjugate pairs. We have $\rho \leq \text{dist}(\mathcal{Z}, \mathbb{R})$, define $\chi_{\mathcal{Z}} \in C_0^\infty(\mathbb{C}, [0, 1])$ as characteristic function:

$$\chi_{\mathcal{Z}}(z) = \begin{cases} 1, & \text{dist}(z, \mathcal{Z} \cup \mathcal{Z}^*) < \frac{\rho}{3}, \\ 0, & \text{dist}(z, \mathcal{Z} \cup \mathcal{Z}^*) > \frac{2\rho}{3}. \end{cases} \quad (4.4)$$

Lemma 4.1 Define function $R_j \rightarrow \mathbb{C}$, $j = 1, 3, 4, 6$ with boundary values satisfying

$$R_1(z) = \begin{cases} r(z)T(z)^{-2}, & z \in (\xi, \infty); \\ r(\xi)T_0(\xi)^{-2}(z - \xi)^{-2i\kappa(\xi)}(1 - \chi_{\mathcal{Z}}(z)), & z \in \Sigma_1; \end{cases} \quad (4.5)$$

$$R_3(z) = \begin{cases} \frac{r(z)^*}{1 + |r(z)|^2} T_+(z)^2, & z \in (-\infty, \xi); \\ \frac{r(\xi)^*}{1 + |r(\xi)|^2} T_0^2(\xi)^2 (z - \xi)^{2i\kappa(\xi)} (1 - \chi_{\mathcal{Z}}(z)), & z \in \Sigma_2; \end{cases} \quad (4.6)$$

$$R_4(z) = \begin{cases} \frac{r(z)}{1 + |r(z)|^2} T_-(z)^{-2}, & z \in (-\infty, \xi); \\ \frac{r(\xi)}{1 + |r(\xi)|^2} T_0^{-2}(\xi)^2 (z - \xi)^{-2i\kappa(\xi)} (1 - \chi_{\mathcal{Z}}(z)), & z \in \Sigma_3; \end{cases} \quad (4.7)$$

$$R_6(z) = \begin{cases} r(z)^* T(z)^2, & z \in (\xi, \infty); \\ r(\xi)^* T_0(\xi)^2 (z - \xi)^{2i\kappa(\xi)} (1 - \chi_{\mathcal{Z}}(z)), & z \in \Sigma_4, \end{cases} \quad (4.8)$$

such that for a fixed constant $c_1 = c_1(q_0)$, and a characteristic function $\chi_{\mathcal{Z}} \in C_0^\infty(\mathbb{C}, [0, 1])$ satisfying (4.4), we have

$$\begin{aligned} |R_j(z)| &\leq c_1 \sin^2(\arg(z - \xi)) + c_1 \langle \text{Re } z \rangle^{-\frac{1}{2}}, \\ |\bar{\partial} R_j(z)| &\leq c_1 \bar{\partial} \chi_{\mathcal{Z}}(z) + c_1 |r'(\text{Re } z)| + c_1 |z - \xi|^{-\frac{1}{2}}, \\ \bar{\partial} R_j(z) &= 0 \quad \text{if } \text{dist}(z, \mathcal{Z} \cup \mathcal{Z}^*) \leq \frac{\rho}{3}. \end{aligned} \quad (4.9)$$

Moreover, if we set $R : (\mathbb{C} \setminus \sum_{\mathbb{R}}) \rightarrow \mathbb{C}$ by $\mathbb{R}(z)|_{z \in \Omega_j} = \mathbb{R}_j(z)$ (with $\mathbb{R}_2(z) = \mathbb{R}_5(z) = 0$), the extension can be made such that $\overline{\mathbb{R}(\bar{z})} = \mathbb{R}(z)$.

Next we construct the $M^{(2)}$ which is continuous to the real axis and deform its jump matrix into the Σ_k , let

$$M^{(2)}(z) = M^{(1)}(z)\mathcal{R}^{(2)}(z), \quad (4.10)$$

$$\mathcal{R}^{(2)}(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ -R_1(z)e^{2it\theta} & 1 \end{pmatrix}, & z \in \Omega_1, \\ \begin{pmatrix} 1 & -R_3(z)e^{-2it\theta} \\ 0 & 1 \end{pmatrix}, & z \in \Omega_3, \\ \begin{pmatrix} 1 & 0 \\ -R_4(z)e^{2it\theta} & 1 \end{pmatrix}, & z \in \Omega_4, \\ \begin{pmatrix} 1 & -R_6(z)e^{-2it\theta} \\ 0 & 1 \end{pmatrix}, & z \in \Omega_6, \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & z \in \Omega_2 \cup \Omega_5. \end{cases} \quad (4.11)$$

Let $\Sigma^{(2)} = \bigcup_{j=1}^4 \Sigma_k$. Then $M^{(2)}$ satisfies the following $\bar{\partial}$ -Riemann-Hilbert problem.

$\bar{\partial}$ -Riemann-Hilbert Problem 4.1 Find a function $M^{(2)} : \mathbb{C} \setminus (\Sigma^{(2)} \cup \mathcal{Z} \cup \mathcal{Z}^*) \rightarrow SL_2(\mathbb{C})$ with the following properties.

- (1) $M^{(2)}$ is continuous and its first derivatives is sectionally continuous in $\mathbb{C} \setminus (\Sigma^{(2)} \cup \mathcal{Z} \cup \mathcal{Z}^*)$.
- (2) $M^{(2)}(z) = I + O(z^{-1})$ as $z \rightarrow \infty$.
- (3) For each $z \in \Sigma^{(2)}$, the boundary values satisfy the jump relationship $M_+^{(2)}(z) = M_-^{(2)}(z)J^{(2)}(z)$ where

$$J^{(2)}(z) = I + (1 - \chi_{\mathcal{Z}}(z))\delta J^{(2)},$$

$$\delta J^{(2)}(z) = \begin{cases} \begin{pmatrix} 0 & 0 \\ r(\xi)T_0(\xi)^{-2}(z - \xi)^{-2i\kappa(\xi)}e^{2it\theta} & 0 \end{pmatrix}, & z \in \Sigma_1, \\ \begin{pmatrix} 0 & \frac{r(\xi)^*T_0(\xi)^2}{1+|r(\xi)|^2}(z - \xi)^{2i\kappa(\xi)}e^{-2it\theta} \\ 0 & 0 \end{pmatrix}, & z \in \Sigma_2, \\ \begin{pmatrix} 0 & 0 \\ \frac{r(\xi)T_0(\xi)^{-2}}{1+|r(\xi)|^2}(z - \xi)^{-2i\kappa(\xi)}e^{2it\theta} & 0 \end{pmatrix}, & z \in \Sigma_3, \\ \begin{pmatrix} 0 & r(\xi)^*T_0(\xi)^2(z - \xi)^{2i\kappa(\xi)}e^{-2it\theta} \\ 0 & 0 \end{pmatrix}, & z \in \Sigma_4. \end{cases} \quad (4.12)$$

(4) For $\mathbb{C} \setminus (\Sigma^{(2)} \cup \mathcal{Z} \cup \mathcal{Z}^*)$ we have $\bar{\partial}M^{(2)} = M^{(2)}\bar{\partial}\mathcal{R}^{(2)}(z)$, where

$$\bar{\partial}\mathcal{R}^{(2)}(z) = \begin{cases} \begin{pmatrix} 0 & 0 \\ \bar{\partial}R_1(z)e^{2it\theta} & 0 \end{pmatrix}, & z \in \Omega_1, \\ \begin{pmatrix} 0 & \bar{\partial}R_3(z)e^{-2it\theta} \\ 0 & 0 \end{pmatrix}, & z \in \Omega_3, \\ \begin{pmatrix} 0 & 0 \\ \bar{\partial}R_4(z)e^{2it\theta} & 0 \end{pmatrix}, & z \in \Omega_4, \\ \begin{pmatrix} 0 & \bar{\partial}R_6(z)e^{-2it\theta} \\ 0 & 0 \end{pmatrix}, & z \in \Omega_6, \\ 0, & \text{elsewhere.} \end{cases} \quad (4.13)$$

(5) $M^{(2)}$ has simple poles at each $z_k \in \mathbb{R}$ and $z_k^* \in \mathbb{R}^*$ at which

$$\begin{aligned} \text{Res}_{z_k} M^{(2)} &= \begin{cases} \lim_{z \rightarrow z_k} M^{(2)} \begin{pmatrix} 0 & c_k^{-1}(\frac{1}{T})'(z_k)^{-2}e^{-2it\theta} \\ 0 & 0 \end{pmatrix}, & k \in \Delta_\xi^-, \\ \lim_{z \rightarrow z_k} M^{(2)} \begin{pmatrix} 0 & 0 \\ c_k^{-1}T(z_k)^{-2}e^{2it\theta} & 0 \end{pmatrix}, & k \in \Delta_\xi^+, \end{cases} \\ \text{Res}_{z_k^*} M^{(2)} &= \begin{cases} \lim_{z \rightarrow z_k^*} M^{(2)} \begin{pmatrix} 0 & 0 \\ -(c_k^*)^{-1}T'(z_k^*)^{-2}e^{-2it\theta} & 0 \end{pmatrix}, & k \in \Delta_\xi^-, \\ \lim_{z \rightarrow z_k^*} M^{(2)} \begin{pmatrix} 0 & -c_k^*T(z_k^*)^2e^{-2it\theta} \\ 0 & 0 \end{pmatrix}, & k \in \Delta_\xi^+. \end{cases} \end{aligned} \quad (4.14)$$

Remark 4.1 Considering the $\bar{\partial}$ -RHP for $M^{(2)}$ above, though (4.13) suggests that $M^{(2)}$ is non-analytic near the small neighborhoods at each point of discrete spectrum, we consider $M^{(2)}$ is analytic in $\mathbb{C}$ as its $\bar{\partial}$ -derivative vanishes in small neighborhoods of the each point of the discrete spectrum. And we also get its jump matrices approach identity point-wise. The final two sections construct the solution $M^{(2)}$ as follows:

(1) We prove the existence of the solution of the pure Riemann-Hilbert problem which the $\bar{\partial}$ component of nonanalytic $\bar{\partial}$ -RHP 4.1 is ignored and compute its asymptotic expansion.

(2) We consider the existence of the solution of the $\bar{\partial}$ problem and prove that the solution is bounded.

5 Removing the Riemann-Hilbert Component of the Solution and Analysis of the Remaining $\bar{\partial}$ -Problem

In this section, we define $M_{RHP}^{(2)}$ as the pure Riemann-Hilbert problem of $\bar{\partial}$ -RHP 4.1 when $\bar{\partial}R^{(2)} \equiv 0$, we will prove that the solution of $M_{RHP}^{(2)}$ exists and construct its asymptotic expansion for large t , and we will prove when reducing $M_{RHP}^{(2)}$ the $\bar{\partial}$ -RHP 4.1 becomes a pure $\bar{\partial}$ problem.

Proposition 5.1 Suppose that $M_{RHP}^{(2)}$ is a solution of pure Riemann-Hilbert problem. Define a continuously differentiable function

$$M^{(3)}(z) := M^{(2)}(z)M_{RHP}^{(2)}(z)^{-1} \quad (5.1)$$

satisfying the following $\bar{\partial}$ -problem.

$\bar{\partial}$ Problem 5.1 Find a function $M^{(3)} : \mathbb{C} \rightarrow SL_2(\mathbb{C})$ with the following properties:

- (1) $M^{(3)}$ is continuous and its first derivative is sectionally continuous in $\mathbb{C} \setminus (\mathbb{R} \cup \Sigma^{(2)})$,
- (2) $M^{(3)} = I + \mathcal{O}(z^{-1})$,
- (3) for $z \in \mathbb{C}$, we have

$$\bar{\partial}M^{(3)} = M^{(3)}(z)W^{(3)}, \quad (5.2)$$

where $W^{(3)} := M_{RHP}^{(2)}(z)\bar{\partial}R^2(z)M_{RHP}^{(2)}(z)^{-1}$ and $\bar{\partial}R^2$ is defined above.

Proof From (5.1) we know that the properties of $M^{(3)}$ inherit from $M^{(2)}$ and $M_{RHP}^{(2)}$, both of them are continuously differentiable in $\mathbb{C} \setminus \Sigma^{(2)}$, unimodular and approach identity as $z \rightarrow \infty$ according to jump relationship

$$\begin{aligned} M_-^{(3)-1}M_+^{(3)} &= M_{(RHP-)}^{(2)}(z)M_-^{(2)}(z)^{-1}M_+^{(2)}(z)M_{RHP+}^{(2)}(z)^{-1} \\ &= M_{(RHP-)}^{(2)}(z)J^{(2)}(z)(M_{RHP-}^{(2)}(z)J^{(2)}(z))^{-1} = I. \end{aligned} \quad (5.3)$$

As both $M^{(2)}$ and $M_{RHP}^{(2)}$ can be regarded as analytic function when deleting neighborhood of each point of discrete spectrum z_k and they satisfy the residue relation (4.14), we denote constant nilpotent matrix N_k then get the Laurent expansions

$$\begin{aligned} M^{(2)}(z) &= C_0 \left[\frac{N_k}{z - z_k} + I \right] + \mathcal{O}(z - z_k), \\ M_{RHP}^{(2)}(z)^{-1} &= \left[\frac{-N_k}{z - z_k} + I \right] \hat{C}_0 + \mathcal{O}(z - z_k), \end{aligned} \quad (5.4)$$

where C_0 and $\hat{C}_0$ are the constant terms, this implies that

$$M^{(2)}(z)M_{RHP}^{(2)}(z)^{-1} = \mathcal{O}(1), \quad (5.5)$$

we know that $M^{(3)}$ has only removable singularities at each z_k ,

$$\bar{\partial}M^{(3)}(z) = \bar{\partial}M^{(2)}(z)M_{RHP}^{(2)}(z)^{-1} = M^{(2)}\bar{\partial}R^2M_{RHP}^{(2)}(z)^{-1} = M^{(3)}W^{(3)}(z). \quad (5.6)$$

The existence of $M^{(3)}(z)$ is proved in the next section, so $\bar{\partial}$ -Problem 5.1 is equivalent to the integral equation

$$M^{(3)}(z) = I - \frac{1}{\pi} \iint_{\mathbb{C}} \frac{\bar{\partial}M^{(3)}(s)}{s - z} dA(s) = I - \frac{1}{\pi} \iint_{\mathbb{C}} \frac{M^{(3)}(s)W^{(3)}(s)}{s - z} dA(s), \quad (5.7)$$

where $dA(s)$ is Lebesgue measure.

Using operator notation, (5.7) can be written as

$$(1 - S)[M^{(3)}(z)] = I, \quad (5.8)$$

where S is the solid Cauchy operator

$$S[f](z) = -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{f(s)W^{(3)}(s)}{s - z} dA(s). \quad (5.9)$$

In the next we will show that when t is sufficiently large, S is a small-norm operator, so $(1 - S)^{-1}$ exists and can be expressed as a Neumann series.

Proposition 5.2 *There exists a constant C such that for all $t > 0$, the operator (5.9) satisfies the inequality*

$$\|S\|_{L^\infty \rightarrow L^\infty} \leq Ct^{-\frac{1}{4}}. \quad (5.10)$$

Proof We only discuss the matrix function in the region Ω_1 . Let $A \in L^\infty(\Omega_1)$ and $s = u + iv$,

$$\begin{aligned} |S[A](z)| &\leq \iint_{\Omega_1} \frac{|A(s)M_{RHP}^{(2)}(s)W^{(2)}(s)M_{RHP}^{(2)}(s)^{-1}|}{|s - z|} dA(s), \\ &\leq \|A\|_{L^\infty(\Omega_1)} \|M_{RHP}^{(2)}\|_{L^\infty(\Omega'_1)} \iint_{\Omega_1} \frac{|\bar{\partial}R_1(s)| |e^{-4tv(u-\xi)}|}{|s - z|} dA(s), \end{aligned} \quad (5.11)$$

where $\Omega'_1 := \Omega_1 \cap (1 - \chi_{\mathcal{Z}})$ is bounded away from the poles z_k of $M_{RHP}^{(2)}$, so that

$$\|(M_{RHP}^{(2)})^{\pm 1}\|_{L^\infty(\Omega'_1)}$$

are finite. Using the Appendix B we get:

$$\|S\|_{L^\infty \rightarrow L^\infty} \leq C(I_1 + I_2 + I_3) \leq Ct^{-\frac{1}{4}}, \quad (5.12)$$

where

$$\begin{aligned} I_1 &= \iint_{\Omega_1} \frac{|\chi_{\mathcal{Z}}(s)| |e^{-4tv(u-\xi)}|}{|s - z|} dA(s), \\ I_2 &= \iint_{\Omega_1} \frac{|r'(u)| |e^{-4tv(u-\xi)}|}{|s - z|} dA(s), \\ I_3 &= \iint_{\Omega_1} \frac{|s - \xi|^{-\frac{1}{2}} |e^{-4tv(u-\xi)}|}{|s - z|} dA(s). \end{aligned} \quad (5.13)$$

Given z^{-1} in the laurent expansion of $M^{(3)}$ at infinity, we consider the asymptotic behavior of $q(x, t)$,

$$M^{(3)} = I - \frac{1}{\pi} \iint_{\mathbb{C}} \frac{M^{(3)}(s)W^{(3)}(s)}{s - z} dA(s) = I + \frac{M_1}{z} + \frac{1}{\pi} \iint_{\mathbb{C}} \frac{sM^{(3)}(s)W^{(3)}(s)}{z(s - z)} dA(s), \quad (5.14)$$

where

$$M_1^{(3)} = \frac{1}{\pi} \iint_{\mathbb{C}} M^{(3)} W^{(3)}(s) dA(s). \quad (5.15)$$

Proposition 5.3 *For all $t > 0$, there exists a constant c such that*

$$|M_1^{(3)}| \leq ct^{-\frac{3}{4}}. \quad (5.16)$$

Proof of Proposition 5.2 is detailed in Appendix B.

6 Existence of the Pure Riemann-Hilbert Problem

6.1 Constructing the model problems

In this section, recall the definition of ρ , we restrict the N -soliton in $\mathbb{C} \setminus U_\xi$,

$$U_\xi = \left\{ z : |z - \xi| < \frac{\rho}{2} \right\}, \quad (6.1)$$

and construct solution of the form:

$$M_{RHP}^{(2)}(z) = \begin{cases} E(z)M^{(\text{out})}(z), & |z - \xi| > \frac{\rho}{2}, \\ E(z)M^{(\xi)}(z), & |z - \xi| < \frac{\rho}{2}, \end{cases} \quad (6.2)$$

where $M^{(\text{out})}$ is the solution of RHP which only concerns the N -soliton, the error $E(z)$ is a small-norm Riemann-Hilbert problem. $M^{(\xi)}$ concerns the jump relation between $M^{(\text{out})}(z)$ and $M_{RHP}^{(2)}(z)$.

6.1.1 The outer model: An N -soliton potential

The matrix $M_{RHP}^{(2)}$ is pure-RHP, it is meromorphic away from the contour $\Sigma^{(2)}$, and its boundary values satisfy the jump relation (4.12) on $\Sigma^{(2)}$, moreover, the jump is uniformly near identity at any distance from ξ , and the norm

$$\|J^{(2)} - I\|_{L^\infty(\Sigma^{(2)})} = \mathcal{O}(\rho^{-2}e^{-4t|z-\xi|^2}) \quad (6.3)$$

shows the jump is exponentially small outside U_ξ , so we construct the model outside U_ξ which ignores the jump completely.

Riemann-Hilbert Problem 6.1 Find an analytic function $M^{(\text{out})} : (\mathbb{C} \setminus \mathbb{R} \cup \mathbb{R}^*) \rightarrow SL_2(\mathbb{C})$ such that

(1) $M^{(\text{out})}(z) = I + O(z^{-1})$ as $z \rightarrow \infty$.

(2) $M^{(\text{out})}$ has simple poles at each $z_k \in \mathbb{R}$ and $z_k^* \in \mathbb{R}^*$ satisfying the residue relations in (4.13) with $M^{(\text{out})}(z)$ replacing $M^{(2)}(z)$.

Proposition 6.1 *There exists a unique solution $M^{(\text{out})}$ of RHP 6.1, specifically,*

$$M^{(\text{out})}(z) = m^{\Delta_\xi^-}(z|\sigma_d^{\text{out}}), \quad (6.4)$$

where $m^{\Delta_\xi^-}$ is the solution of RHP A.2 with $\Delta = \Delta_\xi^-$ and $\sigma_d^{(\text{out})} := \{z_k, \tilde{c}_k(\xi)\}_{k=1}^N$ with

$$\tilde{c}_k(\xi) = c_k \exp\left(\frac{i}{\pi} \int_{-\infty}^{\xi} \log(1 + |r(s)|^2) \frac{ds}{s - z_k}\right). \quad (6.5)$$

Moreover,

$$\lim_{z \rightarrow \infty} 2iz M_{12}^{(\text{out})}(z; x, t) = q_{\text{sol}}(x, t; \sigma_d^{\text{out}}),$$

where $q_{\text{sol}}(x, t; \sigma_d^{\text{out}})$ is the N -soliton solution of (1.1) corresponding to the discrete scattering data $\sigma_d^{(\text{out})}$.

6.1.2 Local model near the saddle point $z = \xi$

According to the analysis of the jump relation (4.13), it shows that at any distance from the saddle point $z = \xi$, the jump is uniformly near identity, so we construct $M^{(\text{out})}$ only considering its N solitons without any jump. Considering (6.3), it shows when $z \in U_\xi$, the bound gives a point-wise, but not uniform on the decay of the jump $J^{(2)}$ to identity. In order to make the jump uniformly, we introduce the function $E(z)$. At first, we introduce $M^{(\xi)}$ to make $M_{RHP}^{(\text{out})}$ math the jump on $\Sigma^{(2)} \cap U_\xi$.

In order to use the jumps of the parabolic cylinder model problem (C.3), We define $\zeta = \zeta(z)$,

$$\zeta = \zeta(z) = 2\sqrt{t}(z - \xi) \Rightarrow 2t\theta = \frac{\zeta^2}{2} - 2t\xi^2, \quad (6.6)$$

which maps U_ξ to an expanding neighborhood of $\zeta = 0$. Additionally, let

$$r_\xi = r(\xi)T_0(\xi)^{-2}e^{2i\kappa(\xi)\log(2\sqrt{t}-t\xi^2)}. \quad (6.7)$$

Since $1 - \chi_{\mathbb{R}}(z) \equiv 1$ for $z \in U_\xi$, the jumps of $M_{RHP}^{(2)}$ in U_ξ can be expressed as

$$J^{(2)} \Big|_{z \in U_\xi} = \begin{cases} \begin{pmatrix} 1 & 0 \\ r_\xi \zeta(z)^{-2i\kappa(\xi)} e^{\frac{i\zeta(z)^2}{2}} & 1 \end{pmatrix}, & z \in \Sigma_1, \\ \begin{pmatrix} 1 & \frac{r_\xi^*}{1+|r_\xi|^2} \zeta(z)^{2i\kappa(\xi)} e^{-\frac{i\zeta(z)^2}{2}} \\ 0 & 1 \end{pmatrix}, & z \in \Sigma_2, \\ \begin{pmatrix} 1 & 0 \\ \frac{r_\xi}{1+|r_\xi|^2} \zeta(z)^{-2i\kappa(\xi)} e^{\frac{i\zeta(z)^2}{2}} & 1 \end{pmatrix}, & z \in \Sigma_3, \\ \begin{pmatrix} 1 & r_\xi^* \zeta(z)^{2i\kappa(\xi)} e^{-\frac{i\zeta(z)^2}{2}} \\ 0 & 1 \end{pmatrix}, & z \in \Sigma_4. \end{cases} \quad (6.8)$$

We calculate the solution in Appendix C, and define the local model $M^{(\xi)}$ in (6.2) by

$$M^{(\xi)}(z) = M^{(\text{out})}(z)M^{(PC)}(\zeta(z), r_\xi), \quad z \in U_\xi. \quad (6.9)$$

Then we know that $M^{(\text{out})}$ is an analytic and bounded function in U_ξ so that $M^{(\xi)}$ inherits the jump $J^{(2)}$ of $M_{RHP}^{(2)}$.

6.2 The small-norm Riemann-Hilbert problem for $E(z)$

Recall the definition of (6.2), the unknown function $E(z)$ is analytic in $\mathbb{C} \setminus \Sigma^{(E)}$,

$$\Sigma^{(E)} = \partial U_\xi \cup (\Sigma^{(2)} \setminus U_\xi), \quad (6.10)$$

where we orient ∂U_ξ in clockwise and $E(z)$ satisfies the following small-norm Riemann-Hilbert problem.

Riemann-Hilbert Problem 6.2 Find a holomorphic function $E : \mathbb{C} \setminus \Sigma^{(E)} \rightarrow SL_2(\mathbb{C})$ with the following properties:

- (1) $E(z) = I + \mathcal{O}(z^{-1})$ as $z \rightarrow \infty$.
- (2) For each $z \in \Sigma^{(E)}$, the boundary values $E_\pm(z)$ satisfy $E_+(z) = E_-(z)J^{(E)}(z)$ where

$$J^{(E)} = \begin{cases} M^{(\text{out})}(z)J^{(2)}(z)M^{(\text{out})}(z)^{-1}, & z \in \Sigma^{(2)} \setminus U_\xi, \\ M^{(\text{out})}(z)J^{(2)}(z)M^{(PC)}(\zeta(z), r_\xi)M^{(\text{out})}(z)^{-1}, & z \in \partial U_\xi, \end{cases} \quad (6.11)$$

and we can find its uniformly vanishing bound on $J_E - I$ as

$$|J_E(z) - I| = \begin{cases} \mathcal{O}(\rho^{-2}e^{-4t|z-\xi|^2}), & z \in \Sigma^{(E)} \setminus U_\xi; \\ \mathcal{O}(t^{-\frac{1}{2}}), & z \in \partial U_\xi. \end{cases} \quad (6.12)$$

Then

$$\|\langle \cdot \rangle^k (J_E - I)\|_{L^p(\Sigma^E)} = \mathcal{O}(t^{-\frac{1}{2}}), \quad p \in [1, +\infty], \quad k \geq 0. \quad (6.13)$$

The RHP 6.2 as a small-norm Riemann-Hilbert problem has a well known existence and uniqueness theorem, we may write

$$E(z) = I + \frac{1}{2\pi i} \int_{\Sigma^E} \frac{(I + \eta(s))(J_E(s) - I)}{s - z} ds, \quad (6.14)$$

where $\eta \in L^2(\Sigma^{(E)})$ is the unique solution of

$$(I - C_{J^{(E)}})\eta = C_{J^{(E)}}I. \quad (6.15)$$

Define $C_{V^{(E)}} =: L^2(\Sigma^{(E)}) \rightarrow L^2(\Sigma^{(E)})$ by

$$C_{J^{(E)}}f = C_-(f(J_E - I)), \quad (6.16)$$

$$C_-f(z) = \lim_{z \rightarrow \Sigma_-^{(E)}} \frac{1}{2\pi i} \int_{\Sigma^{(E)}} f(s) \frac{ds}{s - z}, \quad (6.17)$$

where C_- is the Cauchy operator, then we know that

$$\|C_{V^{(E)}}\|_{L^2_{\text{op}}(\Sigma^{(E)})} \lesssim \|C_-\|_{L^2_{\text{op}}(\Sigma^{(E)})} \|V^{(E)} - I\|_{L^\infty(\Sigma^{(E)})} \lesssim \mathcal{O}(t^{-\frac{1}{2}}). \quad (6.18)$$

The operator $(1 - C_{V^{(E)}})^{-1}$ guarantees the existence of both η and E , so it is reasonable to define $M_{RHP}^{(2)}(z)$ given by (6.2), and we can solve Proposition 5.1 to an unknown $M^{(3)}$ which satisfies the pure $\bar{\partial}$ -Problem 5.1.

We analyse the asymptotic behavior for large z of the solution of RHP 2.1, we construct the function $E(z)$ of the form

$$E(z) = I + z^{-1}E_1 + \mathcal{O}(z^{-2}), \quad (6.19)$$

where

$$E_1 = -\frac{1}{2\pi i} \int_{\Sigma^{(E)}} (I + \eta(s))(V^{(E)} - I) ds, \quad (6.20)$$

$$E_1 = -\frac{1}{2\pi i} \oint_{\partial U_\xi} (V^E(s) - I) ds + \mathcal{O}(t^{-1}), \quad (6.21)$$

$$E_1(x, t) = \frac{1}{2i\sqrt{t}} M^{(\text{out})}(\xi; x, t) \begin{pmatrix} 0 & \beta_{12}(r_\xi) \\ -\beta_{12}(r_\xi) & 0 \end{pmatrix} M^{(\text{out})}(\xi; x, t)^{-1} + \mathcal{O}(t^{-1}). \quad (6.22)$$

We have

$$\beta_{12}(r_\xi) = \beta_{21}(r_\xi)^* = \alpha(\xi, +) e^{i\frac{\pi^2}{2t} - i\kappa(\xi)\log|4t|}, \quad (6.23)$$

here

$$|\alpha(\xi, +)|^2 = |\kappa(\xi)|, \quad (6.24)$$

$$\begin{aligned} \arg \alpha(\xi, +) &= \frac{\pi}{4} + \arg \Gamma(i\kappa(\xi)) - \arg r(\xi) - 4 \sum_{k \in \Delta_\xi^-} \arg(\xi - z_k) \\ &\quad - 2 \int_{-\infty}^{\xi} \log|\xi - s| d_s \kappa(s). \end{aligned} \quad (6.25)$$

7 Long Time Asymptotics for Focusing KE

In this section, we will give the details of the proof for Theorem 1.1 as $t \rightarrow +\infty$.

Proof of Theorem 1.1 According to transformations, we know that the solution of (1.1) can be expressed as

$$M(z) = M^{(3)}(z)E(z)M^{(\text{out})}(z)R^{(2)}(z)T(z)^{\sigma_3}, \quad z \in \mathbb{C} \setminus U_\xi. \quad (7.1)$$

Let $z \rightarrow \infty$ eventually $z \in \Omega_2$ so that $R^{(2)} = I$, we have

$$T(z)^{\sigma_3} = I + \frac{T_1 \sigma_3}{z} + \mathcal{O}(z^{-2}), \quad T_1 = 2 \sum_{\Delta^-} \text{Im } z_k - \int_{-\infty}^{\xi} \kappa(s) ds. \quad (7.2)$$

Now

$$M = \left(I + \frac{M_1^{(3)}}{z} + \cdots\right) \left(I + \frac{E_1}{z} + \cdots\right) \left(I + \frac{M_1^{(\text{out})}}{z} + \cdots\right) \left(I + \frac{T_1 \sigma_3}{z} + \cdots\right), \quad (7.3)$$

the coefficient of the z^{-1} in the Laurent expansion of M is

$$M_1 = M_1^{(3)} + E_1 + M_1^{(\text{out})} + \mathcal{O}(t^{-\frac{3}{4}}). \quad (7.4)$$

We know that

$$q(x, t) = 2i(M_1^{(\text{out})})_{12} + 2i(E_1)_{12} + \mathcal{O}(t^{-\frac{3}{4}}). \quad (7.5)$$

Applying Proposition 6.1 to the first term and using (6.20)–(6.25) to evaluate the second term we have

$$q(x, t) = q_{\text{sol}}(x, t; \sigma_d^{\text{out}}) + t^{-\frac{1}{2}} f^+(x, t) + \mathcal{O}(t^{-\frac{3}{4}}). \quad (7.6)$$

We know that $q_{\text{sol}}(x, t; \sigma_d^{\text{out}})$ is the solution of N -soliton generated from Proposition 6.2, we will give the relationship concerning $q_{\text{sol}}(x, t; \sigma_d^{\text{out}})$ with $q_{\text{sol}}(x, t; \sigma_d^+(\mathcal{I}))$ which are contained in the cone $C(x_1, x_2, v_1, v_2)$ as defined in Theorem 1.1. Using the appendix A, we know that replacing $q_{\text{sol}}(x, t; \sigma_d^{\text{out}})$ with $q_{\text{sol}}(x, t; \sigma_d^+(\mathcal{I}))$ there exist exponential errors which are absorbed into the $\mathcal{O}(t^{-\frac{3}{4}})$ term.

The long-time asymptotics As $t \rightarrow \infty$ such that $|\xi| = \left| -\frac{x}{4t} \right| < M$,

$$2im(x, t) = (q_{\text{sol}}(x, t; \sigma^\pm(I)) + t^{-\frac{1}{2}} f^\pm(x, t) + \mathcal{O}(t^{-\frac{3}{4}})), \quad (7.7)$$

where q_{sol} and $f^\pm$ are shown above. In order to get the asymptotics of $q(x, t)$, we also need to calculate $e^{-2i \int_{(-\infty, t)}^{(x, t)} \Delta}$.

Proposition 7.1 As $t \rightarrow \infty$,

$$e^{-2i \int_{(-\infty, t)}^{(x, t)} \Delta} = e^{-2i\beta \int_{-\infty}^x |q(x', t)|^2 dx'} = e^{-8i\beta \int_{(-\infty, t)}^{(x, t)} |m(x', t)|^2 dx'} + \mathcal{O}(t^{-\frac{3}{2}}), \quad (7.8)$$

where

$$|m(x, t)|^2 = \frac{1}{2} |(q_{\text{sol}}(x, t; \sigma^\pm(I)) + t^{-\frac{1}{2}} f^\pm(x, t) + \mathcal{O}(t^{-\frac{3}{4}}))|^2. \quad (7.9)$$

A Appendix: Meromorphic Solutions of the Focusing KE Riemann-Hilbert Problem

In this section, we consider the unknown meromorphic function (with the reflection coefficient $r(z) \equiv 0$) only has a finite discrete spectrum, we will prove the existence and uniqueness of this problem and discuss its asymptotic behavior as $t \rightarrow \infty$.

Riemann-Hilbert Problem A.1 Given discrete data $\sigma_d = \{(z_k, c_k)\}_{k=1}^N \in \mathbb{C}^+ \times \mathbb{C}_*$, let $\mathcal{Z} = \{z_k\}_{k=1}^N$. Find an analytic function $m : \mathbb{C} \setminus (\mathcal{Z} \cup \mathcal{Z}^*) \rightarrow SL_2(\mathbb{C})$ with the following properties:

1. $m(z; x, t | \sigma_d) = I + O(z^{-1})$ as $z \rightarrow \infty$.
2. Each point of $\mathcal{Z} \cup \mathcal{Z}^*$ is a simple pole of $m(z; x, t | \sigma_d)$. They satisfy the residue conditions

$$\begin{aligned} \operatorname{Res}_{z=z_k^*} m(z; x, t | \sigma_d) &= \lim_{z=z_k^*} m(z; x, t | \sigma_d) \sigma_2 n_k^* \sigma_2, \\ \operatorname{Res}_{z=z_k} m(z; x, t | \sigma_d) &= \lim_{z=z_k} m(z; x, t | \sigma_d) n_k, \end{aligned} \quad (\text{A.1})$$

where n_k is the nilpotent matrix,

$$n_k = \begin{pmatrix} 0 & 0 \\ \gamma_k(x, t) & 0 \end{pmatrix}, \quad \gamma_k(x, t) := c_k \exp(2i(tz_k^2 + xz_k)). \quad (\text{A.2})$$

Using the Liouville's theorem to get the uniqueness of the solution and we can prove the symmetry $m(z | \sigma_d) = \sigma_2 m(z^* | \sigma_d)^* \sigma_2$. It follows that any solution of RHP A.1 has the solution of the following form:

$$m(z; x, t | \sigma_d) = I + \sum_{k=1}^N \frac{1}{z - z_k} \begin{pmatrix} \alpha_k(x, t) & 0 \\ \beta_k(x, t) & 0 \end{pmatrix} + \frac{1}{z - z_k^*} \begin{pmatrix} 0 & -\beta_k(x, t)^* \\ 0 & \alpha_k(x, t)^* \end{pmatrix} \quad (\text{A.3})$$

for coefficients $\alpha_k(x, t), \beta_k(x, t)$ to be determined.

Proposition A.1 Given data $\sigma_d = \{(z_k, c_k)\}_{k=1}^N \in \mathbb{C} \times \mathbb{C}_*$ such that $z_j \neq z_k$ for $j \neq k$, there exists a unique solution of RHP B.1 for each $(x, t) \in \mathbb{R}^2$.

Proof The proof can be found in [4].

A.1 Renormalizations of the reflectionless Riemann-Hilbert problem

Define the N -soliton solutions of RHP A.1 with $r(z) = 0$, and $\frac{1}{a(z)}$ being the transmission coefficient of the reflectionless initial data

$$m(z; x, t | \sigma_d) = \left[\frac{\phi_1^{(-)}(x, t; z)}{a(z)} \middle| \phi_2^{(+)}(x, t; z) \right] e^{i(tz^2 + xz)\sigma_3}, \quad a(z) = \prod_{k=1}^N \left(\frac{z - z_k}{z - z_k^*} \right). \quad (\text{A.4})$$

Let $\Delta \subset \{1, 2, \dots, N\}$ and $\nabla = \Delta^c = \{1, \dots, N\} \setminus \Delta$. Define

$$a_\Delta(z) = \prod_{k \in \Delta} \left(\frac{z - z_k}{z - z_k^*} \right) \quad \text{and} \quad a_\nabla(z) = \frac{a(z)}{a_\Delta(z)} = \prod_{k \in \nabla} \left(\frac{z - z_k}{z - z_k^*} \right). \quad (\text{A.5})$$

The renormalization

$$m^\Delta(z | \sigma_d) = m(z | \sigma_d) a_\Delta(z)^{\sigma_3} = \left[\frac{\phi_1^{(-)}(x, t; z)}{a_\nabla(z)} \middle| \frac{\phi_2^{(+)}(x, t; z)}{a_\Delta(z)} \right] e^{i(tz^2 + xz)\sigma_3}, \quad (\text{A.6})$$

it is obvious that by choice of Δ we split the poles between the columns, and m^Δ satisfies the followed modified discrete Riemann-Hilbert problem.

Riemann-Hilbert Problem A.2 Given discrete data $\sigma_d = \{(z_k, c_k)\}_{k=1}^N \subset \mathbb{C}^+ \times \mathbb{C}_*$ and $\Delta \subset \{1, \dots, N\}$. Find an analytic function $m^\Delta : \mathbb{C} \setminus (\mathcal{Z} \cup \mathcal{Z}^*) \rightarrow SL_2(\mathbb{C})$ with the following properties:

1. $m^\Delta(z; x, t|\sigma_d) = I + O(z^{-1})$, $z \rightarrow \infty$.
2. Each point of $\mathcal{Z} \cup \mathcal{Z}^*$ is a simple pole of $m^\Delta(z; x, t|\sigma_d)$, they satisfy the residue conditions

$$\begin{aligned} \operatorname{Res}_{z=z_k} m^\Delta(z; x, t|\sigma_d) &= \lim_{z=z_k} m^\Delta(z; x, t|\sigma_d) n_k^\Delta, \\ \operatorname{Res}_{z=z_k^*} m^\Delta(z; x, t|\sigma_d) &= \lim_{z=z_k^*} m^\Delta(z; x, t|\sigma_d) \sigma_2 (n_k^\Delta)^* \sigma_2, \end{aligned} \quad (\text{A.7})$$

where n_k is the nilpotent matrix,

$$n_k^\Delta = \begin{cases} \begin{pmatrix} 0 & 0 \\ \gamma_k(x, t) a_\Delta(z_k)^2 & 0 \\ 0 & \gamma_k(x, t)^{-1} a'_\Delta(z_k)^{-2} \\ 0 & 0 \end{pmatrix}, & \gamma_k(x, t) := c_k \exp(2i(tz_k^2 + xz_k)) \end{cases} \quad (\text{A.8})$$

and a_Δ is defined in (A.5).

When the poles $z_k \in \mathbb{R}$ are distinct, we know that the RHP B.2 has a unique solution because it is a transformation of $m(z; x, t|\sigma_d)$, the advantage of this method we will prove above that by choosing the Δ correctly, other soliton asymptotic behavior are under better control when $t \rightarrow \infty$, $-\frac{x}{4t} = \xi$.

A.2 Long time behavior of the soliton solutions

If there is only a single solution $\sigma_d = \{(\xi + i\eta, c_1)\}$, we know that

$$q_{\text{sol}}(x, t) = q_{\text{sol}}(x, t; \{\xi + i\eta\}) = 2ia_1 e^{\Omega_1 - \Omega_1^*} (P^{-1}) e^{8i\beta \int |ae^{\Omega_1 - \Omega_1^*} (P^{-1})^2| dx}. \quad (\text{A.9})$$

When there are N -solitons ($N > 1$), we know that the N -solitons asymptotically separate into N single-soliton as $t \rightarrow \infty$.

Define

$$\mu = \mu(\mathcal{I}) = \min_{z_k \in \mathcal{Z} \setminus \mathcal{Z}(\mathcal{I})} \{\operatorname{Im}(z_k) \operatorname{dist}(\operatorname{Re} z_k, \mathcal{I})\}. \quad (\text{A.10})$$

Proposition A.2 Given discrete scattering data $\sigma_d = \{(z_k, c_k)\}_{k=1}^N \subset \mathbb{C}^+ \times (\mathbb{C} \setminus \{0\})$, fix $x_1, x_2, v_1, v_2 \in \mathbb{R}$ with $x_1 \leq x_2$ and $v_1 \leq v_2$. Let $\mathcal{I} = [-\frac{v_2}{2}, -\frac{v_1}{2}]$. Then, as $t \rightarrow \pm\infty$ with $(x, t) \in C(x_1, x_2, v_1, v_2)$, we have

$$m^{\Delta_\xi^\mp}(z; x, t|\sigma_d) = (I + \mathcal{O}(e^{-4\mu|t|})) m^{\Delta_\xi^\mp(\mathcal{I})}(z; x, t|\sigma_d^\pm(\mathcal{I})) \quad (\text{A.11})$$

for all z bounded away from $\mathcal{Z} \cup \mathcal{Z}^*$.

Here $\sigma_d^\pm(\mathcal{I})$ is the scattering data for the $N(\mathcal{I}) \leq N$ soliton given by

$$\sigma_d^\pm(\mathcal{I}) = \{(z_k, c_k^\pm(\mathcal{I})) : z_k \in \mathcal{Z}(\mathcal{I})\}, \quad c_k^\pm(\mathcal{I}) = c_k \prod_{z_j \in \mathcal{Z}^\mp(\mathcal{I})} \left(\frac{z_k - z_j}{z_k - z_j^*} \right)^2. \quad (\text{A.12})$$

Corollary A.3 Let $q_{\text{sol}}(x, t; \sigma_d)$ be the N -soliton of the fKE equation (1.1) with its discrete scattering data $\sigma_d = \{(z_k, c_k)\}_{k=1}^N \in \mathbb{C}^+ \times (\mathbb{C} \setminus \{0\})$ and let $\mathcal{I}, C(x_1, x_2, v_1, v_2)$ and $\sigma_d^\pm(\mathcal{I})$ be as given in Proposition A.2. Then as $t \rightarrow \pm\infty$ with $(x, t) \in C(x_1, x_2, v_1, v_2)$,

$$q_{\text{sol}}(x, t; \sigma_d) = q_{\text{sol}}(x, t; \sigma_d^\pm(\mathcal{I})) + \mathcal{O}(e^{-4\mu t}), \quad (\text{A.13})$$

where $q_{\text{sol}}(x, t; \sigma_d^\pm(\mathcal{I}))$ is the solution of the fKE with $N(\mathcal{I})$ -soliton and its scattering data is $\sigma_d^\pm(\mathcal{I})$.

Proof of Proposition A.2 Observe that

$$|\gamma_k(x_0 + vt, t)| = |c_k| |\exp[-2x_0 \text{Im}(z_k)] \exp\left[-4t \text{Im}(z_k) \text{Re}\left(z_k + \frac{v}{2}\right)\right]|. \quad (\text{A.14})$$

The choice of normalization $\Delta = \Delta_\xi^\mp$ in RHP A.2 ensures when $|t| \rightarrow \infty$ with $(x, t) \in C(x_1, x_2, v_1, v_2)$ that

$$\|n_k^{\Delta_\xi^\mp}\| = \begin{cases} \mathcal{O}(1), & z_k \in \mathcal{Z}(\mathcal{I}), \\ \mathcal{O}(\exp(-4\mu|t|)), & z_k \in \mathcal{Z} \setminus \mathcal{Z}(\mathcal{I}). \end{cases} \quad (\text{A.15})$$

This suggests that the residues with $z_k \in \mathcal{Z} \setminus \mathcal{Z}(\mathcal{I})$ contribute to the solution $m^{\Delta_\xi^\pm}$ insignificantly. Around each $z_k \in \mathcal{Z} \setminus \mathcal{Z}(\mathcal{I})$ we trade its residue for a near identity jump by introducing small disk D_k whose radii are chosen sufficiently small that they are non-overlapping. We make the change of variables

$$m^{\Delta_\xi^\mp}(z|\sigma_d) = \begin{cases} \hat{m}^{\Delta_\xi^\mp}(z) \left(I + \frac{n_k}{z - z_k} \right), & z \in D_k, \\ \hat{m}^{\Delta_\xi^\mp}(z) \left(I + \frac{\sigma_2 n_k^* \sigma_2}{z - z_k^*} \right), & z \in D_k^*, \\ \hat{m}^{\Delta_\xi^\mp}(z), & \text{elsewhere.} \end{cases} \quad (\text{A.16})$$

The new unknown $\hat{m}^{\Delta_\xi^\mp}(z)$ has jumps across each disk boundary which, by (A.15), satisfy

$$\hat{m}_+^{\Delta_\xi^\mp}(z) = m^{\Delta_\xi^\mp}(z) \hat{v}(z), \quad z \in \partial D_k \cup \partial D_k^* \quad (\text{A.17})$$

with

$$\|\hat{v} - I\| = \mathcal{O}(\exp(-4\mu|t|)), \quad z \in \partial D_k \cup \partial D_k^*. \quad (\text{A.18})$$

The $m^{\Delta_\xi^\mp(\mathcal{I})}(z|\sigma_d^\pm(\mathcal{I}))$ has the same poles as $\hat{m}^{\Delta_\xi^\mp(\mathcal{I})}(z|\sigma)$ with the same residue conditions, that

$$e(z) = \hat{m}^{\Delta_\xi^\mp}(z|\sigma_d) [m^{\Delta_\xi^\mp(\mathcal{I})}(z|\sigma_d^\pm(\mathcal{I}))]^{-1} \quad (\text{A.19})$$

has no poles, and its jumps satisfy estimates identically to (A.18).

We show that $e(z)$ exists and that $e(z) = I + \mathcal{O}(e^{-4\mu|t|})$ for all sufficiently large $|t|$ by using the small-norm Riemann-Hilbert problems. From (A.16) and (A.19) that $m^{\Delta_{\xi}^{\mp}}(z; x, t|\sigma_d) = e(z)m^{\sigma_{\xi}^{\mp}(\mathcal{I})}(z; x, t|\sigma_d^{\pm}(\mathcal{I}))$ for z outside each dist D_k and D_k^* , the result follows immediately.

B Appendix: Detail of Calculations for the $\bar{\partial}$ Problem

Proposition B.1 *There exist constants c_1, c_2 and c_3 such that for all $t > 0$, the integrals $I_j, j = 1, 2, 3$, defined by (6.7)–(6.8) satisfy the bound*

$$|I_j| \leq \frac{c_j}{t^{\frac{1}{4}}}, \quad j = 1, 2, 3. \quad (\text{B.1})$$

Proof Our proof follows [3]. Let $s = u + iv$ and $z = \alpha + i\beta$. We use the elementary fact

$$\left\| \frac{1}{s - z} \right\|_{L_u^2(v+\xi, \infty)}^2 = \left(\int_{v+\xi}^{\infty} \frac{1}{(u - \alpha)^2 + (v - \beta)^2} ds \right)^{\frac{1}{2}} \leq \int_{\mathbb{R}} \frac{1}{u^2 + (v - \beta)^2} du = \frac{\pi}{v - \beta}$$

to show that

$$\begin{aligned} |I_1| &\leq \int_0^{\infty} \int_{v+\xi}^{\infty} \frac{|\chi z(s)|}{|s - z|} e^{-4tv(u-\xi)} du dv \leq \int_0^{\infty} e^{-4tv^2} \int_{v+\xi}^{\infty} \frac{|\chi z(s)|}{|s - z|} du dv \\ &\leq \int_0^{\infty} e^{-4tv^2} \|\chi z(s)\|_{L_u^2(v+\xi, \infty)} \left\| \frac{1}{s - z} \right\|_{L_u^2(v+\xi, \infty)} dv \\ &\leq c_1 \int_0^{\infty} \frac{e^{-4tv^2}}{|v - \beta|^{\frac{1}{2}}} dv \leq c_1 t^{-\frac{1}{4}} \int_{\mathbb{R}} \frac{e^{-4(w+\sqrt{t}\beta)^2}}{|w|^{\frac{1}{2}}} dw \leq c_1 t^{-\frac{1}{4}}. \end{aligned} \quad (\text{B.2})$$

The bound for I_2 is similar to I_1 . Recalling that $r \in H^{1,1}(\mathbb{R})$,

$$\begin{aligned} |I_2| &\leq \int_0^{\infty} e^{-4tv^2} \int_{v+\xi}^{\infty} \frac{|r'(u)|}{|s - z|} du dv \\ &\leq \|r'(u)\|_{L^2(\mathbb{R})} \int_0^{\infty} e^{-4tv^2} \left\| \frac{1}{s - z} \right\|_{L_u^q(v+\xi, \infty)} dv \leq \frac{c_2}{t^{\frac{1}{4}}}. \end{aligned} \quad (\text{B.3})$$

For I_3 , choose $p > 2$ and q Hölder conjugate to p , then

$$\begin{aligned} |I_3| &\leq \int_0^{\infty} e^{-4tv^2} \|(s - \xi)^{-\frac{1}{2}}\|_{L_u^p(v+\xi, \infty)} \|(s - z)^{-1}\|_{L_u^q(v+\xi, \infty)} dv \\ &\leq c_p \int_0^{\infty} e^{-4tv^2} v^{\frac{1}{p}-\frac{1}{2}} |v - \beta|^{\frac{1}{q}-1} dv. \end{aligned} \quad (\text{B.4})$$

To bound this last integral, we observe that

$$\begin{aligned} \int_0^{\beta} e^{-tv^2} v^{\frac{1}{p}-\frac{1}{2}} (\beta - v)^{\frac{1}{q}-1} dv &= \int_0^1 \beta^{\frac{1}{2}} e^{-t\beta^2 w^2} w^{\frac{1}{p}-\frac{1}{2}} (1 - w)^{\frac{1}{q}-1} dw \\ &\leq ct^{-\frac{1}{4}} \int_0^1 \int_0^1 w^{\frac{1}{p}-1} (1 - w)^{\frac{1}{q}-1} dw \leq Ct^{-\frac{1}{4}}, \end{aligned} \quad (\text{B.5})$$

where we used the bound $e^{-m} \leq m^{-\frac{1}{4}}$ for $m \geq 0$ to replace the exponential factor in the second integral. Finally

$$\int_{\beta}^{\infty} e^{-tv^2} v^{\frac{1}{p}-\frac{1}{2}} (v-\beta)^{\frac{1}{q}-1} dv \leq \int_0^{\infty} e^{-tw^2} w^{-\frac{1}{2}} dw \leq Ct^{-\frac{1}{4}}. \quad (\text{B.6})$$

The result is confirmed.

Proposition B.2 *For all $t > 0$, there exists a constant c such that*

$$|M_1^{(3)}| \leq ct^{-\frac{3}{4}}. \quad (\text{B.7})$$

Proof The proof given here follows calculations that can be found in [2, 10]. Recalling that the set $\Omega'_1 = \Omega_1 \cup \text{supp}(1 - \chi_{\mathcal{Z}})$ is bounded away from the poles of $M_{RHP}^{(2)}$, we have

$$\begin{aligned} |M_1^{(3)}| &\leq \iint_{\Omega_1} |M^{(3)}(s) M_{RHP}^{(2)}(s) M_{RHP}^{(2)}(s)^{-1}| dA \\ &\leq \frac{1}{\pi} \|M^{(3)}\|_{L^\infty(\Omega)} \|M_{RHP}^{(2)}\|_{L^\infty(\Omega')} \iint_{\Omega} |\bar{\partial} R_1 e^{2it\theta}| dA \\ &\leq C \left(\iint_{\Omega_1} |\chi_{\mathcal{Z}}(s)| e^{-4tv(u-\xi)} dA + \iint_{\Omega_1} |r'(u)| e^{-4tv(u-\xi)} dA \right. \\ &\quad \left. + \iint_{\Omega_1} \frac{1}{|s-\xi|} e^{-4tv(u-\xi)} dA \right) \\ &\leq C(I_4 + I_5 + I_6). \end{aligned} \quad (\text{B.8})$$

We bound I_4 by applying the Cauchy-Schwarz inequality

$$\begin{aligned} |I_4| &\leq \int_0^{\infty} \|\chi_{\mathcal{Z}}\|_{L^2_u(u+\xi, \infty)} \left(\int_v^{\infty} e^{-8uv} ds \right)^{\frac{1}{2}} dv \\ &\leq ct^{-\frac{1}{2}} \int_0^{\infty} \frac{e^{-4tv^2}}{\sqrt{v}} dv \leq \int_0^{\infty} \frac{e^{-4tw^2}}{\sqrt{w}} dw \leq ct^{-\frac{3}{4}}. \end{aligned} \quad (\text{B.9})$$

For $2 < p < 4$,

$$\begin{aligned} |I_6| &\leq c \int_0^{\infty} v^{\frac{1}{p}-\frac{1}{2}} \left(\int_v^{\infty} e^{-4qtuv} du \right)^{\frac{1}{q}} dv \leq ct^{-\frac{1}{q}} \int_0^{\infty} v^{\frac{1}{p}-\frac{3}{2}} e^{-4tv^2} dv \\ &\leq ct^{-\frac{3}{4}} \int_0^{\infty} w^{\frac{2}{p}-\frac{3}{2}} e^{-4w^2} dw \leq ct^{-\frac{3}{4}}, \end{aligned} \quad (\text{B.10})$$

where we have used the substitution $w = t^{\frac{1}{2}}v$ and the fact that $-1 < \frac{2}{p} - \frac{3}{2} < -\frac{1}{2}$.

C Appendix: The Parabolic Cylinder Model Problem

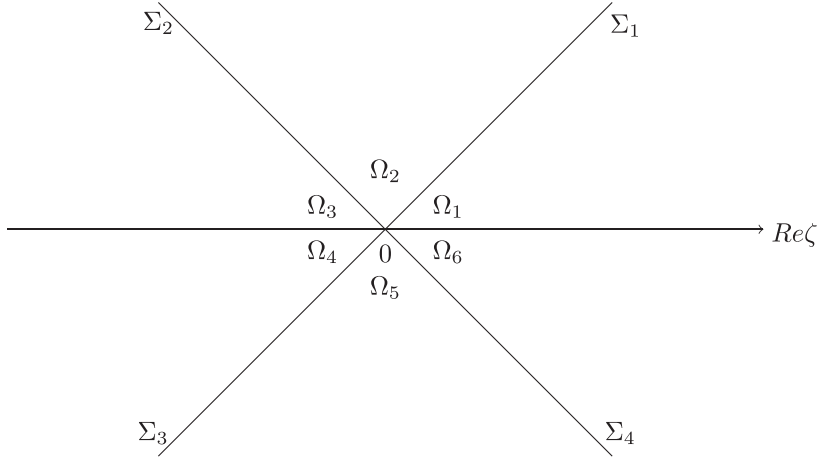
In this section we describe the long-time asymptotic calculations of the integrable nonlinear waves of the parabolic cylinder model problem (see [9]). Define

$$\Sigma_j = \left\{ \zeta \in \mathbb{C} \mid \arg \zeta = \frac{2j-1}{4} \pi \right\}, \quad j = 1, \dots, 4. \quad (\text{C.1})$$

Fix $r \in \mathbb{C}$ and define

$$\kappa = \kappa(r) := -\frac{1}{2\pi} \log(1 + |r|^2). \quad (\text{C.2})$$

And we define six connected open sectors in $\mathbb{C} \setminus (\Sigma^{(PC)} \cup \mathbb{R})$, the sequence of region is in a counterclockwise.



Parabolic Cylinder Model Riemann-Hilbert Problem A.1 The analytic function $M^{(PC)}(\cdot, r) : \mathbb{C} \setminus \Sigma^{(PC)} \rightarrow SL_2(\mathbb{C})$, where $r \in \mathbb{C}$ is fixed, satisfies

- (1) $M^{(PC)}(\zeta, r) = I + \frac{M^{(PC)}(r)}{\zeta} + O(\zeta^{-2})$.
- (2) For $\zeta \in \Sigma^{(PC)}$, the continuous boundary values $M_{\pm}^{(PC)}(\zeta, r)$ satisfy the jump relation $M_{+}^{(PC)}(\zeta, r) = M_{+}^{(PC)}(\zeta, r)V^{(PC)}(\zeta, r)$, where

$$V^{(PC)}(\zeta, r) = \begin{cases} \begin{pmatrix} 1 & 0 \\ r\zeta^{-2i\kappa}e^{i\frac{\zeta^2}{2}} & 1 \end{pmatrix}, & \arg \zeta = \frac{\pi}{4}, \\ \begin{pmatrix} 1 & r^*\zeta^{2i\kappa}e^{-i\frac{\zeta^2}{2}} \\ 0 & 1 \end{pmatrix}, & \arg \zeta = -\frac{\pi}{4}, \\ \begin{pmatrix} 1 & 0 \\ \frac{r^*}{1+|r|^2}\zeta^{2i\kappa}e^{-i\frac{\zeta^2}{2}} & 1 \end{pmatrix}, & \arg \zeta = \frac{3\pi}{4}, \\ \begin{pmatrix} 1 & 0 \\ \frac{r}{1+|r|^2}\zeta^{-2i\kappa}e^{i\frac{\zeta^2}{2}} & 1 \end{pmatrix}, & \arg \zeta = -\frac{3\pi}{4}. \end{cases} \quad (\text{C.3})$$

According to the solutions of the parabolic cylinder equation $(\frac{\partial^2}{\partial z^2} + (\frac{1}{2} - \frac{z^2}{2} + a))D_a(z) = 0$ (see [1, 10]), we have an explicit solution of the $M^{(PC)}(\zeta, r)$:

$$M^{(PC)}(\zeta, r) = \Phi(\zeta, r)\mathcal{P}(\zeta, r)e^{i\frac{\zeta^2\sigma_3}{4}}\zeta^{-i\kappa\sigma_3}, \quad (\text{C.4})$$

where

$$\mathcal{P}(\zeta, r) = \begin{cases} \begin{pmatrix} 1 & 0 \\ -r & 1 \end{pmatrix}, & \zeta \in \Omega_1, \\ \begin{pmatrix} 1 & \frac{-r^*}{1+|r|^2} \\ 0 & 1 \end{pmatrix}, & \zeta \in \Omega_3, \\ \begin{pmatrix} 1 & 0 \\ \frac{r}{1+|r|^2} & 1 \end{pmatrix}, & \zeta \in \Omega_4, \\ \begin{pmatrix} 1 & r^* \\ 0 & 1 \end{pmatrix}, & \zeta \in \Omega_6 \end{cases} \quad (\text{C.5})$$

and

$$\Phi(\zeta, r) = \begin{cases} \begin{pmatrix} e^{-\frac{3\pi\kappa}{4}} D_{i\kappa}(e^{-\frac{3i\pi}{4}} \zeta) & -i\beta_{12} e^{\frac{\pi}{4}} D_{-i\kappa-1}(e^{-\frac{i\pi}{4}} \zeta) \\ i\beta_{21} e^{-\frac{3\pi}{4}} D_{-i\kappa+1}(e^{-\frac{i\pi}{4}} \zeta) & e^{\frac{\pi\kappa}{4}} D_{-i\kappa}(e^{-\frac{-i\pi}{4}} \zeta) \end{pmatrix}, & \zeta \in \mathbb{C}^+, \\ \begin{pmatrix} e^{\frac{\pi\kappa}{4}} D_{i\kappa}(e^{-\frac{-i\pi}{4}} \zeta) & -i\beta_{12} e^{-\frac{3\pi}{4}(\kappa-i)} D_{-i\kappa-1}(e^{\frac{3i\pi}{4}} \zeta) \\ i\beta_{21} e^{\frac{\pi}{4}(\kappa+i)} D_{i\kappa-1}(e^{\frac{i\pi}{4}} \zeta) & e^{-\frac{3\pi\kappa}{4}} D_{-i\kappa}(e^{\frac{3i\pi}{4}} \zeta) \end{pmatrix}, & \zeta \in \mathbb{C}^-, \end{cases} \quad (\text{C.6})$$

in which β_{12} and β_{21} are the complex constants

$$\beta_{12} = \beta_{12}(r) = \frac{\sqrt{2\pi} e^{i\frac{\pi}{4}} e^{-\frac{\pi\kappa}{2}}}{r\Gamma(-i\kappa)}, \quad \beta_{21} = \beta_{21}(r) = \frac{-\sqrt{2\pi} e^{-i\frac{\pi}{4}} e^{-\frac{\pi\kappa}{2}}}{r^* \Gamma(i\kappa)}. \quad (\text{C.7})$$

We use the result given in [6] and get the result as

$$M^{(PC)}(\zeta, r) = I + \frac{1}{\zeta} \begin{pmatrix} 0 & -i\beta_{12}(r) \\ i\beta_{21}(r) & 0 \end{pmatrix} + O(\zeta^{-2}). \quad (\text{C.8})$$

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