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# The Refined Schwarz-Pick Estimates for Positive Real Part Holomorphic Functions in Several Complex Variables*

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**Abstract** In this article, the refined Schwarz-Pick estimates for positive real part holomorphic functions

$$p(x) = p(0) + \sum_{m=k}^{\infty} \frac{D^m p(0)(x^m)}{m!} : G \to \mathbb{C}$$

are given, where $k$ is a positive integer, and $G$ is a balanced domain in complex Banach spaces. In particular, the results of first order Fréchet derivative for the above functions and higher order Fréchet derivatives for positive real part holomorphic functions

$$p(x) = p(0) + \sum_{s=1}^{\infty} \frac{D^{sk} p(0)(x^{sk})}{(sk)!} : G \to \mathbb{C}$$

are sharp for $G = B$, where $B$ is the unit ball of complex Banach spaces or the unit ball of complex Hilbert spaces. Their results reduce to the classical result in one complex variable, and generalize some known results in several complex variables.

**Keywords** Refined Schwarz-Pick estimate, Positive real part holomorphic function, First order Fréchet derivative, Higher order Fréchet derivatives
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## 1 Introduction

In the case of one complex variable, it is well known that the classical result as follows.

**Theorem 1.1** (see [1]) *Let $p(z)$ be a holomorphic function on the unit disk $U$, and $\Re\, p(z) > 0, z \in U$. Then*

$$|p'(z)| \leq \frac{2\Re p(z)}{1 - |z|^2}, \quad z \in U.$$

In 2008, Dai and Pan [2] obtained the following interesting result concerning the estimate of higher derivative for the above function.

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**Theorem 1.2** (see [2]) *Let $p(z)$ be a holomorphic function on the unit disk $U$, and $\Re p(z) > 0, z \in U$. Then*

$$|p^{(m)}(z)| \leq \frac{2 \cdot m! \Re p(z)(1+|z|)^{m-1}}{(1-|z|^2)^m}, \quad m = 1, 2, \cdots, \ z \in U.$$

The related results in one complex variable can consult (see [3–4]).

A natural question arouses many people to ask whether the above theorem hold for positive real part holomorphic functions with respect to all homogeneous expansions in several complex variables or not? Our answer is completely affirmative, and we shall provide a refinement version of Schwarz-Pick estimates for positive real part holomorphic functions. In the case of several complex variables, the Schwarz-Pick estimates of the higher order partial derivatives for positive real part holomorphic functions with multi-index in various complete Reinhardt domains may consult (see [5–6]).

Let $X$ be complex Banach spaces, and let $B$ be the unit ball of $X$. Let $X^H$ denote a complex Hilbert spaces, let $\langle \cdot, \cdot \rangle$ be the inner product of $X^H$, and let $B^H$ denote the unit ball of $X^H$. Let $U$ be the unit disc of $\mathbb{C}$. We denote by $\mathbb{N}^+$ the set of all positive integers. For domains $G \subset X$, we denote by $H(G, \mathbb{C})$ the set of holomorphic mappings from $G$ into $\mathbb{C}$. For $p \in H(G, \mathbb{C})$ and $x \in G$, let $D^l p(x)$ denote the the $l$-th Fréchet derivative of $p$ at $x$. Each function $p \in H(G, \mathbb{C})$ possesses the following Taylor expansion series

$$p(x) = \sum_{l=0}^{\infty} \frac{1}{n!} D^n p(0)(x^n)$$

in a neighbourhood of the origin. Let $L(X, Y)$ be the set of continuous linear operators from $X$ into $Y$ and $I$ be the identity in $L(X, X)$. For arbitrary $x \in X \backslash \{0\}$,

$$T(x) = \{T_x \in L(X, \mathbb{C}) : \|T_x\| = 1, T_x(x) = \|x\|\}$$

is well defined. Let $H, K$ be complex Hilbert spaces. For each operator $A \in L(H, K)$, there exists a uniquely determined operator $A^* \in L(K, H)$ such that

$$\langle Ax, y \rangle = \langle x, A^* y \rangle$$

for every $x \in H$ and $y \in K$, where $\langle \cdot, \cdot \rangle$ is the inner product in a complex Hilbert space. For each $x \in G, \xi \in X$, we denote by

$$F_C^G(x, \xi) = \sup\{|Df(x)\xi| : f \in H(G, U), f(x) = 0\}$$

the infinitesimal Carathéodory pseudometric on $G$. We say that a domain $G \subset X$ is balanced if $\lambda G \subset G$ for all $\lambda \in \overline{U}$, its Minkowski functional $\rho(x)$ is defined by $\rho(x) = \inf\left\{t > 0 : \frac{x}{t} \in G\right\}$ for $x \in X$. It is clear that $G = \{x : \rho(x) < 1\}$. In this article, let $G \subset X$ always be the balanced domain of $X$.

## 2 Some Lemmas

It is necessary to provide the following lemmas in order to present our main result in this section.

**Lemma 2.1** *Let* $k \in \mathbb{N}^+, p(z) = p(0) + \sum\limits_{m=k}^{\infty} \frac{p^{(m)}(0)}{m!} z^m \in H(U, \mathbb{C})$ *and* $\Re p(z) > 0, z \in U$. *Then*

$$\frac{\Re p(0)(1 - |z|^k)}{1 + |z|^k} \leq \Re p(z) \leq \frac{\Re p(0)(1 + |z|^k)}{1 - |z|^k}.$$

*The above estimate is sharp.*

**Proof** Let $p_1(z) = \frac{p(z) - \Im p(0)}{\Re p(0)}, z \in U$. Then $\Re p_1(z) > 0, z \in U$ and $p_1(0) = 1$. It is shown that

$$\frac{1 - |z|^k}{1 + |z|^k} \leq \Re p_1(z) = \frac{\Re p(z)}{\Re p(0)} \leq \frac{1 + |z|^k}{1 - |z|^k}$$

from [7]. Hence the desired result holds. This completes the proof.

**Lemma 2.2** (see [8]) *Let* $k \in \mathbb{N}^+, \varphi(z) = \varphi(0) + \sum\limits_{m=k}^{\infty} \frac{\varphi^{(m)}(0)}{m!} z^m \in H(U, U)$. *Then*

$$|\varphi'(z)| \leq \frac{k|z|^{k-1}(1 - |\varphi(z)|^2)}{1 - |z|^{2k}}.$$

*The above estimate is sharp.*

**Lemma 2.3** *Let* $k \in \mathbb{N}^+, p(z) = p(0) + \sum\limits_{m=k}^{\infty} \frac{p^{(m)}(0)}{m!} z^m \in H(U, \mathbb{C})$ *and* $\Re p(z) > 0, z \in U$. *Then*

$$|p'(z)| \leq \frac{2k|z|^{k-1}\Re p(z)}{1 - |z|^{2k}} \leq \frac{2k\Re p(0)|z|^{k-1}}{(1 - |z|^k)^2}.$$

**Proof** Define

$$\varphi(z) = \frac{p(z) - p_0}{p(z) + \overline{p_0}}, \quad z \in U.$$

Then $\varphi \in H(U, U)$ and $\varphi(0) = 0$. A direct calculation shows that

$$\varphi'(z) = \frac{2\Re p(0) \cdot p'(z)}{(p(z) + \overline{p(0)})^2}$$

and

$$1 - |\varphi(z)|^2 = \frac{4\Re p(0) \cdot \Re p(z)}{|p(z) + \overline{p(0)}|^2}.$$

Therefore, we conclude that

$$|p'(z)| \leq \frac{2k|z|^{k-1}\Re p(z)}{1 - |z|^{2k}} \leq \frac{2k\Re p(0)|z|^{k-1}}{(1 - |z|^k)^2}$$

from Lemmas 2.1–2.2. This completes the proof.

**Lemma 2.4** *Let* $k \in \mathbb{N}^+, p(x) = p(0) + \sum\limits_{s=k}^{\infty} \frac{D^s p(0)(x^s)}{s!} \in H(G, \mathbb{C})$ *and* $\Re\, p(x) > 0, x \in G$. *Then*

$$\frac{|D^s p(0)(x^s)|}{s!} \leq 2\Re\, p(0)(\rho(x))^s, \quad s = k, k+1, \cdots, \ x \in G.$$

**Proof** Let $h(\lambda) = p(\lambda x_0), \lambda \in U$, where $x_0 = \frac{x}{\rho(x)}$, $x \in X \setminus \{0\}$ which is fixed. Then $h \in H(U, \mathbb{C})$ and $\Re h(\lambda) > 0, \lambda \in U$. Thus, we show that

$$\frac{|D^s p(0)(x_0^s)|}{s!} = \frac{|h^{(s)}(0)|}{s!} \leq 2\Re\, h(0), \quad s = k, k+1, \cdots$$

from the fact $\frac{|h^{(m)}(0)|}{m!} \leq 2\Re\, h(0), m = 1, 2, \cdots$ (see [1]). It follows the desired result. This completes the proof.

**Lemma 2.5** *Let* $k \in \mathbb{N}^+, q(z) = 1 + \sum\limits_{s=1}^{\infty} a_{sk} z^{sk} \in H(U, \mathbb{C})$ *and* $\Re\, q(z) > 0, z \in U$. *Then there exists a non-decreasing function* $\mu$ *on* $[0, 2\pi]$ *with* $\mu(2\pi) - \mu(0) = 1$ *and such that*

$$q(z) = \int_0^{2\pi} \frac{1 + z^k \mathrm{e}^{-\mathrm{i}kt}}{1 - z^k \mathrm{e}^{-\mathrm{i}kt}} \mathrm{d}\mu(t), \quad z \in U.$$

**Proof** Define

$$\mu(r, t) = \frac{1}{2\pi} \int_0^t \Re\, q(r\mathrm{e}^{\mathrm{i}\theta}) \mathrm{d}\theta, \quad 0 < r < 1, \ 0 \leq \theta \leq 2\pi.$$

Then $\mu(r, \cdot)$ is a non-decreasing function on $[0, 2\pi]$ with $\mu(r, 2\pi) = 1$. It is known that

$$\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}skt} \mathrm{d}\mu(r, t) = \begin{cases} a_{sk} \dfrac{r^{sk}}{2}, & s = 1, 2, \cdots, \\ 1, & s = 0 \end{cases}$$

by a direct calculation. Therefore,

$$q(z) = \int_0^{2\pi} \mathrm{d}\mu(r, t) + \sum_{s=1}^{\infty} \Big( \frac{2}{r^{sk}} \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}skt} \mathrm{d}\mu(r, t) \Big) z^{sk}.$$

For $|z| < r$, we have

$$q(z) = \int_0^{2\pi} \Big( 1 + 2\sum_{s=1}^{\infty} \Big( \frac{\mathrm{e}^{-\mathrm{i}t} z}{r} \Big)^{sk} \Big) \mathrm{d}\mu(r, t).$$

Note that the series in the last integrand converges uniformly with respect to $t$. Hence we conclude that

$$q(z) = \int_0^{2\pi} \frac{r^k \mathrm{e}^{\mathrm{i}kt} + z^k}{r^k \mathrm{e}^{\mathrm{i}kt} - z^k} \mathrm{d}\mu(r, t).$$

With analogous arguments in the proof of [1], we derive the desired result. This completes the proof.

**Lemma 2.6**

$$\sum_{s=[\frac{m-1}{k}]+1}^{\infty} sk(sk-1)\cdots(sk-m+1)|z|^{sk-m} = \Big(\frac{kr^{k-1}}{(1-r^k)^2}\Big)^{(m-1)}\Big|_{r=|z|}, \quad z \in U$$

*for* $m = 1, 2, \cdots$.

**Proof** A short calculation shows that

$$\begin{aligned}&\sum_{s=[\frac{m-1}{k}]+1}^{\infty} sk(sk-1)\cdots(sk-m+1)|z|^{sk-m}\\ &= \Big(\sum_{s=[\frac{m-1}{k}]+1}^{\infty} \int_0^{|z|} sk(sk-1)\cdots(sk-m+1)r^{sk-m}\mathrm{d}t\Big)'\\ &= \Big(\sum_{s=[\frac{m-1}{k}]+1}^{\infty} sk(sk-1)\cdots(sk-m+2)r^{sk-m+1}\Big)'\Big|_{r=|z|}\\ &\quad\vdots\\ &= \Big(\frac{kr^{k-1}}{(1-r^k)^2}\Big)^{(m-1)}\Big|_{r=|z|}, \quad m = 1, 2, \cdots, \ z \in U.\end{aligned}$$

It follows that the desired result holds. This completes the proof.

**Lemma 2.7** *Let* $k \in \{2, 3, \cdots\}$. *Then*

$$(\nu+1)k((\nu+1)k-1)\cdots((\nu+1)k-m+1) - 2\nu k(\nu k-1)\cdots(\nu k-m+1) \geq 0$$

*for* $(\nu-1)k+1 \leq m \leq \nu k, \ \nu = 1, 2, \cdots$.

**Proof** If $\nu = 1$ and $m = 1$, then we deduce that

$$(\nu+1)k - 2\nu k = 2k - 2k = 0.$$

Also if $2 \leq m \leq k, \ (\nu-1)k+1 \leq m \leq \nu k, \ \nu = 2, 3, \cdots$, then we write as

$$w_1(t) = (2k-t+2)(2k-t+1) - 2(k-t+2)(k-t+1), \quad t \in [2, k]$$

and

$$w_2(t) = ((\nu+1)k-t+2)((\nu+1)k-t+1) - 2(\nu k-t+2)(\nu k-t+1)$$

for $t \in [(\nu-1)k+1, \nu k]$. A brief calculation shows that

$$w_1'(t) = -2t+3, \quad w_2'(t) = 2(\nu-1)k-2t+3, \quad \nu = 2, 3, \cdots.$$

Hence, we obtain that

$$w_1'(t) \leq 0, \quad t \in [2, k]$$

and

$$w_2'(t) \geq 0, \quad t \in \Big[(\nu-1)k+1, (\nu-1)k+\frac{3}{2}\Big]; \quad w_2'(t) \leq 0, \quad t \in \Big[(\nu-1)k+\frac{3}{2}, \nu k\Big]$$

for $\nu = 2, 3, \cdots$. We pay attention to the fact that

$$w_1(k) = (k+2)(k+1) - 4 > 0$$

and

$$w_2((\nu-1)k+1) = (2k+1)2k - 2(k+1)k > 0; \quad w_2(\nu k) = (k+2)(k+1) - 4 > 0$$

for $\nu = 2, 3, \cdots$. Therefore, we conclude that

$$w_1(t) = (2k-t+2)(2k-t+1) - 2(k-t+2)(k-t+1) > 0, \quad t \in [2, k]$$

and

$$\begin{aligned} &w_2(t) = ((\nu+1)k-t+2)((\nu+1)k-t+1) - 2(\nu k-t+2)(\nu k-t+1) > 0, \\ &t \in [(\nu-1)k+1, \nu k] \end{aligned}$$

for $\nu = 2, 3, \cdots$. Hence, we show that

$$\begin{aligned} &2k(2k-1)\cdots(2k-m+2)(2k-m+1) - 2k(k-1)\cdots(k-m+2)(k-m+1) > 0, \\ &m = 2, \cdots, k \end{aligned}$$

and

$$\begin{aligned} &(\nu+1)k((\nu+1)k-1)\cdots((\nu+1)k-m+2)((\nu+1)k-m+1) \\ &- 2\nu k(\nu k-1)\cdots(\nu k-m+2)(\nu k-m+1) > 0 \end{aligned}$$

for $(\nu-1)k+1 \leq m \leq \nu k, \nu = 2, 3, \cdots$. Thus, we derive the desired result. This completes the proof.

**Lemma 2.8** *Let* $k \in \{2, 3, \cdots\}$. *Then*

$$\begin{aligned} &(\nu+2)k((\nu+2)k-1)\cdots((\nu+2)k-m+1) \\ &- 2(\nu+1)k((\nu+1)k-1)\cdots((\nu+1)k-m+1) \\ &+ \nu k(\nu k-1)\cdots(\nu k-m+1) \geq 0 \end{aligned}$$

*for* $(\nu-1)k+1 \leq m \leq \nu k, \nu = 1, 2, \cdots$.

**Proof** For $(\nu-1)k+1 \leq m \leq \nu k, \nu = 1, 2, \cdots$, it is shown that

$$\begin{aligned} &(\nu+2)k((\nu+2)k-1)\cdots((\nu+2)k-m+1) + \nu k(\nu k-1)\cdots(\nu k-m+1) \\ = &((\nu+1)k+k)((\nu+1)k-1+k)\cdots((\nu+1)k-m+1+k) \end{aligned}$$

$$
\begin{aligned}
&+((\nu+1)k-k)((\nu+1)k-1-k)\cdots((\nu+1)k-m+1-k)\\
&\geq 2(\nu+1)k((\nu+1)k-1)\cdots((\nu+1)k-m+1)
\end{aligned}
$$

by a direct calculation. Hence, it follows the desired result. This completes the proof.

**Lemma 2.9** *Let $k\in\{2,3,\cdots\}$. Then*

$$
\begin{aligned}
&\Big|(1-\mathrm{e}^{-\mathrm{i}kt}z^k)^2\sum_{s=[\frac{m-1}{k}]+1}^{\infty}\mathrm{e}^{-\mathrm{i}skt}sk(sk-1)\cdots(sk-m+1)z^{sk-m}\Big|\\
&\leq(1-|z|^k)^2\Big(\frac{kr^{k-1}}{(1-r^k)^2}\Big)^{(m-1)}\Big|_{r=|z|}
\end{aligned}
$$

*for $m=1,2,\cdots$, where $\mathrm{i}=\sqrt{-1}, t\in[0,2\pi]$.*

**Proof** We denote by

$$
P(z)=\sum_{s=\nu}^{\infty}\mathrm{e}^{-\mathrm{i}skt}sk(sk-1)\cdots(sk-m+1)z^{sk-m},
$$

where $\nu=\left[\frac{m-1}{k}\right]+1$. Then we show that $(\nu-1)k+1\leq m\leq\nu k, k=1,2,\cdots$. Thus, we conclude that

$$
\begin{aligned}
&(1-\mathrm{e}^{-\mathrm{i}kt}z^k)^2P(z)=(1-2\mathrm{e}^{-\mathrm{i}kt}z^k+\mathrm{e}^{-\mathrm{i}2kt}z^{2k})P(z)\\
&=(1-2\mathrm{e}^{-\mathrm{i}kt}z^k+\mathrm{e}^{-\mathrm{i}2kt}z^{2k})\sum_{s=\nu}^{\infty}\mathrm{e}^{-\mathrm{i}skt}sk(sk-1)\cdots(sk-m+1)z^{sk-m}\\
&=\Big\{\sum_{s=\nu}^{\infty}\mathrm{e}^{-\mathrm{i}skt}sk(sk-1)\cdots(sk-m+1)z^{sk-m}\\
&\quad-2\sum_{s=\nu}^{\infty}\mathrm{e}^{-\mathrm{i}(s+1)kt}sk(sk-1)\cdots(sk-m+1)z^{(s+1)k-m}\\
&\quad+\sum_{s=\nu}^{\infty}\mathrm{e}^{-\mathrm{i}(s+2)kt}sk(sk-1)\cdots(sk-m+1)z^{(s+2)k-m}\Big\}\\
&=\Big\{\mathrm{e}^{-\mathrm{i}\nu kt}\nu k(\nu k-1)\cdots(\nu k-m+1)z^{\nu k-m}\\
&\quad+\mathrm{e}^{-\mathrm{i}(\nu+1)kt}\{(\nu+1)k((\nu+1)k-1)\cdots((\nu+1)k-m+1)\\
&\quad-2\nu k(\nu k-1)\cdots(\nu k-m+1)\}z^{(\nu+1)k-m}\\
&\quad+\sum_{s=\nu}^{\infty}\mathrm{e}^{-\mathrm{i}(s+2)kt}[(s+2)k((s+2)k-1)\cdots((s+2)k-m+1)\\
&\quad-2(s+1)k((s+1)k-1)\cdots((s+1)k-m+1)\\
&\quad+sk(sk-1)\cdots(sk-m+1)]z^{(s+2)k-m}\Big\}.
\end{aligned}
$$

Hence, we deduce that

$$
|(1-\mathrm{e}^{-\mathrm{i}kt}z^k)^2P(z)|=|(1-2\mathrm{e}^{-\mathrm{i}kt}z^k+\mathrm{e}^{-\mathrm{i}2kt}z^{2k})P(z)|
$$

$$
\begin{aligned}
&= \Big| \mathrm{e}^{-\mathrm{i}\nu kt}\nu k(\nu k-1)\cdots(\nu k-m+1)z^{\nu k-m} \\
&\quad + \mathrm{e}^{-\mathrm{i}(\nu+1)kt}\{(\nu+1)k((\nu+1)k-1)\cdots((\nu+1)k-m+1) \\
&\quad - 2\nu k(\nu k-1)\cdots(\nu k-m+1)\}z^{(\nu+1)k-m} \\
&\quad + \sum_{s=\nu}^{\infty}\mathrm{e}^{-\mathrm{i}(s+2)kt}[(s+2)k((s+2)k-1)\cdots((s+2)k-m+1) \\
&\quad - 2(s+1)k((s+1)k-1)\cdots((s+1)k-m+1) \\
&\quad + sk(sk-1)\cdots(sk-m+1)]z^{(s+2)k-m}\Big| \\
&\leq \Big\{\nu k(\nu k-1)\cdots(\nu k-m+1)|z|^{\nu k-m} \\
&\quad + \{(\nu+1)k((\nu+1)k-1)\cdots((\nu+1)k-m+1) \\
&\quad - 2\nu k(\nu k-1)\cdots(\nu k-m+1)\}|z|^{(\nu+1)k-m} \\
&\quad + \sum_{s=\nu}^{\infty}\{(s+2)k((s+2)k-1)\cdots((s+2)k-m+1) \\
&\quad - 2(s+1)k((s+1)k-1)\cdots((s+1)k-m+1) \\
&\quad + sk(sk-1)\cdots(sk-m+1)\}|z|^{(s+2)k-m}\Big\} \\
&= (1-|z|^k)^2\sum_{s=\nu}^{\infty}sk(sk-1)\cdots(sk-m+1)|z|^{sk-m} \\
&= (1-|z|^k)^2\Big(\frac{kr^{k-1}}{(1-r^k)^2}\Big)^{(m-1)}\Big|_{r=|z|}
\end{aligned}
$$

from Lemmas 2.6–2.8.

**Lemma 2.10** *Let* $k \in \mathbb{N}^+, p(z) = p(0) + \sum\limits_{s=1}^{\infty}\frac{p^{(sk)}(0)}{(sk)!}z^{sk} \in H(U,\mathbb{C})$ *and* $\Re\, p(z) > 0, z \in U$. *Then*

$$
|p^{(m)}(z)| \leq 2\Re\, p(z)\frac{1-|z|^k}{1+|z|^k}\Big(\frac{kr^{k-1}}{(1-r^k)^2}\Big)^{(m-1)}\Big|_{r=|z|}, \quad m = 1, 2, \cdots, \ z \in U.
$$

**Proof** Let $q(z) = \frac{p(z)-\mathrm{i}\Im p(0)}{\Re p(0)}, z \in U$, where $\mathrm{i} = \sqrt{-1}$. Then $q(z) = 1 + \sum\limits_{s=1}^{\infty}\frac{q^{(sk)}(0)}{(sk)!}z^{sk} \in H(U,\mathbb{C})$, and $\Re\, q(z) > 0, z \in U$. Hence, there exists a non-decreasing function $\mu$ on $[0, 2\pi]$ with $\mu(2\pi) - \mu(0) = 1$ and such that

$$
q(z) = \int_0^{2\pi}\frac{1+z^k\mathrm{e}^{-\mathrm{i}kt}}{1-z^k\mathrm{e}^{-\mathrm{i}kt}}\mathrm{d}\mu(t), \quad z \in U
$$

from Lemma 2.5. Note that

$$
\Re\, q(z) = \int_0^{2\pi}\frac{1-|z|^{2k}}{|1-z^k\mathrm{e}^{-\mathrm{i}kt}|^2}\mathrm{d}\mu(t), \quad z \in U
$$

and

$$
\Re\, q(z) = \frac{\Re\, p(z)}{\Re p(0)}, \quad q^{(m)}(z) = \frac{p^{(m)}(z)}{\Re p(0)}.
$$

If $k=1$, then we show that

$$q^{(m)}(z)=2\int_0^{2\pi}\frac{m!\mathrm{e}^{-\mathrm{i}mt}}{(1-z\mathrm{e}^{-\mathrm{i}t})^{m+1}}\mathrm{d}\mu(t),\quad z\in U$$

by a short calculation. Hence, we conclude that

$$\begin{aligned}|q^{(m)}(z)|=\frac{|p^{(m)}(z)|}{\Re p(0)}&=2\Big|\int_0^{2\pi}\frac{m!\mathrm{e}^{-\mathrm{i}mt}}{(1-z\mathrm{e}^{-\mathrm{i}t})^{m+1}}\mathrm{d}\mu(t)\Big|\\&\le 2\int_0^{2\pi}\frac{m!}{|1-z\mathrm{e}^{-\mathrm{i}t}|^{m+1}}\mathrm{d}\mu(t)\\&=\frac{2}{1-|z|^2}\int_0^{2\pi}\frac{m!}{|1-z\mathrm{e}^{-\mathrm{i}t}|^{m-1}}\frac{1-|z|^2}{|1-z\mathrm{e}^{-\mathrm{i}t}|^2}\mathrm{d}\mu(t)\\&\le\frac{2\cdot m!\Re\, p(z)(1+|z|)^{m-1}}{\Re p(0)(1-|z|^2)^m}.\end{aligned}$$

Also if $k\in\{2,3,\cdots\}$, a direct computation shows that

$$\begin{aligned}q^{(m)}(z)&=\int_0^{2\pi}\Big(1+2\sum_{s=1}^{\infty}\mathrm{e}^{-\mathrm{i}\nu kt}z^{\nu k}\Big)^{(m)}\mathrm{d}\mu(t)\\&=2\int_0^{2\pi}\sum_{s=[\frac{m-1}{k}]+1}^{\infty}\mathrm{e}^{-\mathrm{i}skt}sk(sk-1)\cdots(sk-m+1)z^{sk-m}\mathrm{d}\mu(t).\end{aligned}$$

Therefore, by Lemma 2.9, we deduce that

$$|q^{(m)}(z)|=\frac{|p^{(m)}(z)|}{\Re p(0)}=2\Big|\int_0^{2\pi}\sum_{s=[\frac{m-1}{k}]+1}^{\infty}\mathrm{e}^{-\mathrm{i}skt}sk(sk-1)\cdots(sk-m+1)z^{sk-m}\mathrm{d}\mu(t)\Big|$$

$$\begin{aligned}&\le 2\int_0^{2\pi}\Big|\sum_{s=[\frac{m-1}{k}]+1}^{\infty}\mathrm{e}^{-\mathrm{i}skt}sk(sk-1)\cdots(sk-m+1)z^{sk-m}\Big|\mathrm{d}\mu(t)\\&=\frac{2}{1-|z|^{2k}}\int_0^{2\pi}\Big|(1-z^k\mathrm{e}^{-\mathrm{i}kt})^2\sum_{s=[\frac{m-1}{k}]+1}^{\infty}\mathrm{e}^{-\mathrm{i}skt}sk(sk-1)\cdots(sk-m+1)z^{sk-m}\Big|\\&\quad\cdot\frac{1-|z|^{2k}}{|1-z^k\mathrm{e}^{-\mathrm{i}kt}|^2}\mathrm{d}\mu(t)\\&\le\frac{2(1-|z|^k)^2}{1-|z|^{2k}}\Big(\frac{kr^{k-1}}{(1-r^k)^2}\Big)^{(m-1)}\Big|_{r=|z|}\int_0^{2\pi}\frac{1-|z|^{2k}}{|1-z^k\mathrm{e}^{-\mathrm{i}kt}|^2}\mathrm{d}\mu(t)\\&=2\frac{\Re\, p(z)}{\Re p(0)}\frac{1-|z|^k}{1+|z|^k}\Big(\frac{kr^{k-1}}{(1-r^k)^2}\Big)^{(m-1)}\Big|_{r=|z|},\quad m=1,2,\cdots,\ z\in U.\end{aligned}$$

It follows the result, as desired. This completes the proof.

## 3 Main Results

We show the desired theorems in this section.

**Theorem 3.1** *Let* $k\in\mathbb{N}^+, p(x)=p(0)+\sum\limits_{m=k}^{\infty}\frac{D^m p(0)(x^m)}{m!}\in H(G,\mathbb{C})$ *and* $\Re p(x)>0, x\in G$. *Then*

$$|Dp(x)x|\le\frac{2k\Re p(x)(\rho(x))^k}{1-(\rho(x))^{2k}}\le\frac{2k\Re p(0)(\rho(x))^k}{(1-(\rho(x))^k)^2}.$$

*The above estimate is sharp for* $G=B$.

**Proof** For an arbitrary fixed $x_0=\frac{x}{\rho(x)}, x\in X\setminus\{0\}$,

$$h(\lambda)=p(\lambda x_0),\quad \lambda\in U.$$

Then $h(\lambda)=p(0)+\sum\limits_{m=k}^{\infty}\frac{D^m p(0)(x_0^m)}{m!}\lambda^m$ and $\Re h(\lambda)>0, \lambda\in U$. A straightforward computation shows that

$$h'(\lambda)=Dp(\lambda x_0)x_0,\quad \Re h(\lambda)=\Re p(\lambda x_0).$$

Putting $\lambda=\rho(x)$, then we deduce that

$$|Dp(x)x|\le\frac{2k\Re p(x)(\rho(x))^k}{1-(\rho(x))^{2k}}\le\frac{2k\Re p(0)(\rho(x))^k}{(1-(\rho(x))^k)^2}$$

from Lemmas 2.1 and 2.3. We show that the desired result holds.

Let

$$p(x)=\Re p(0)\frac{1+T_u^k(x)}{1-T_u^k(x)}+\mathrm{i}\Im p(0),\quad x\in B,\ k\in\mathbb{N}^+, \tag{3.1}$$

where $u$ which satisfies $\|u\|=1$ is fixed. Then it is not difficult to verify that $p$ satisfies the hypothesis of Theorem 3.1. It is shown that

$$Dp(x)x=\frac{2k\Re p(0)T_u^k(x)}{(1-T_u^k(x))^2},\quad \Re p(x)=\Re p(0)\frac{1-|T_u^k(x)|^2}{|1-T_u^k(x)|^2}$$

by a simple calculation. Set $x=ru(0\le r\le 1)$. We see that

$$|Dp(x)x|=\frac{2k\Re p(x)\|x\|^k}{1-\|x\|^{2k}}=\frac{2k\Re p(0)\|x\|^k}{(1-\|x\|^k)^2}.$$

Hence we show that the estimate of Theorem 3.1 is sharp. This completes the proof.

**Theorem 3.2** *Let* $k\in\mathbb{N}^+, p(x)=p(0)+\sum\limits_{s=1}^{\infty}\frac{D^{sk}p(0)(x^{sk})}{(sk)!}\in H(G,\mathbb{C})$ *and* $\Re p(x)>0, x\in B$. *Then*

$$|D^m p(x)(x^m)|\le 2\Re p(x)\rho^m(x)\frac{1-(\rho(x))^k}{1+(\rho(x))^k}\Big(\frac{kr^{k-1}}{(1-r^k)^2}\Big)^{(m-1)}\Big|_{r=\rho(x)}$$
$$\le 2\Re p(0)\rho^m(x)\Big(\frac{kr^{k-1}}{(1-r^k)^2}\Big)^{(m-1)}\Big|_{r=\rho(x)},\quad m=1,2,\cdots,\ x\in G.$$

*The above estimate is sharp for* $G=B$.

**Proof** We denote by $x_0 = \frac{x}{\rho(x)}, x \in X \setminus \{0\}$,

$$h(\lambda) = p(\lambda x_0), \quad \lambda \in U.$$

Then $h(\lambda) = h(0) + \sum_{s=1}^{\infty} \frac{D^{sk}p(0)(x_0^{sk})}{(sk)!}\lambda^{sk}$ and $\Re h(\lambda) > 0, \lambda \in U$. A brief calculation show that

$$h^{(m)}(\lambda) = D^{(m)}p(\lambda x_0)(x_0^m), \quad \Re h(\lambda) = \Re p(\lambda x_0).$$

Set $\lambda = \rho(x)$. Then we conclude that

$$\begin{aligned}|D^m p(x)(x^m)| &\le 2\Re p(x)(\rho(x))^m \frac{1-(\rho(x))^k}{1+(\rho(x))^k}\Big(\frac{kr^{k-1}}{(1-r^k)^2}\Big)^{(m-1)}\Big|_{r=\rho(x)} \\ &\le 2\Re p(0)(\rho(x))^m \Big(\frac{kr^{k-1}}{(1-r^k)^2}\Big)^{(m-1)}\Big|_{r=\rho(x)}, \quad m = 1, 2, \cdots, \ x \in G\end{aligned}$$

from Lemmas 2.1 and 2.10. We derive the desired result.

Let $p(x)$ be the same as that of (3.1). A direct calculation shows that

$$D^m p(x)(x^m) = 2\Re p(0)(T_u(x))^m \Big(\frac{kr^{k-1}}{(1-r^k)^2}\Big)^{(m-1)}\Big|_{\lambda=T_u(x)}, \quad m = 1, 2, \cdots, \ x \in B.$$

Putting $x = ru (0 \le r \le 1)$, this gives that

$$\begin{aligned}|D^m p(x)(x^m)| &= 2\Re p(x)\|x\|^m \frac{1-\|x\|^k}{1+\|x\|^k}\Big(\frac{kr^{k-1})}{(1-r^k)^2}\Big)^{(m-1)}\Big|_{r=\|x\|} \\ &= 2\Re p(0)\|x\|^m \Big(\frac{kr^{k-1}}{(1-r^k)^2}\Big)^{(m-1)}\Big|_{r=\|x\|}, \quad m = 1, 2, \cdots, \ x \in B.\end{aligned}$$

Then it is shown that the estimates of Theorem 3.2 are sharp. This completes the proof.

When $k = 1$, we have the following corollary.

**Corollary 3.1** *Let* $p(x) = p(0) + \sum_{s=1}^{\infty} \frac{D^s p(0)(x^s)}{s!} \in H(G, \mathbb{C})$ *and* $\Re p(x) > 0, x \in B$. *Then*

$$\begin{aligned}|D^m p(x)(x^m)| &\le 2 \cdot m!\Re p(x)(\rho(x))^m \frac{(1+\rho(x))^{m-1}}{(1-\rho^2(x))^m} \\ &\le \Re p(0)\rho^m(x)\frac{2 \cdot m!(1+\rho(x))^{m+1}}{(1-\rho^2(x))^{m+1}}, \quad m = 1, 2, \cdots, \ x \in G.\end{aligned}$$

*The above estimate is sharp for* $G = B$.

**Remark 3.1** Corollary 3.1 reduces to Theorem 1.2 if $G = U$.

**Theorem 3.3** *Let* $k \in \mathbb{N}^+, p(x) = p(0) + \sum_{s=1}^{\infty} \frac{D^s p(0)(x^s)}{s!} \in H(B^H, \mathbb{C})$ *and* $\Re p(x) > 0, x \in B^H$. *Then*

$$|D^m p(\xi^m)| \le 2m!\Re p(x)\Big(1 + \frac{|\langle x, \xi\rangle|}{\sqrt{(1-\|x\|^2_{B^H})\|\xi\|^2_{B^H} + |\langle x, \xi\rangle|^2}}\Big)^{m-1}(F_C^{B^H}(x, \xi))^m,$$

$$m = 1, 2, \cdots$$

*for* $x \in B^H, \xi \in X^H \setminus \{0\}$. *The above estimate is sharp.*

**Proof** Let $x \in B^H, \xi \in X\backslash\{0\}$ , and denote $\xi_0 = \frac{\xi}{\|\xi\|}$. A short calculation shows that

$$|z + \langle x, \xi_0\rangle| < \sqrt{1 - \|x\|^2_{B^H} + |\langle x, \xi_0\rangle|^2}$$

if $\|x + z\xi_0\|_{B^H_X} < 1$. For brevity, we write as $z_0 = -\langle x, \xi_0\rangle, \sigma = \sqrt{1 - \|x\|^2_{B^H} + |\langle x, \xi_0\rangle|^2}$. Hence,

$$g(z) = p(x + z\xi_0), \quad z \in U(z_0, \sigma)$$

is well defined. We also let

$$\varphi(w) = g(\sigma w + z_0), \quad w \in U.$$

Then it is known that $\varphi \in H(U, \mathbb{C})$ and $\Re\,\varphi(w) > 0$. Note that

$$g(0) = p(x), \quad g^{(m)}(0) = D^m p(x)(\xi_0^m) \tag{3.2}$$

and

$$\sigma^m g^{(m)}(z) = \varphi^{(m)}\Big(\frac{z - z_0}{\sigma}\Big), \quad m = 1, 2, \cdots. \tag{3.3}$$

Thus, by Theorem 1.2 and (3.3), we conclude that

$$|g^{(m)}(z)| \le \frac{2 \cdot m!\big(1 + \big|\frac{z-z_0}{\sigma}\big|\big)^{m-1}\Re\,\varphi(\frac{z-z_0}{\sigma})}{\sigma^m\big(1 - \big|\frac{z-z_0}{\sigma}\big|^2\big)^m}. \tag{3.4}$$

We also pay attention to the fact that

$$F_C^{B^H}(x, \xi) = \sqrt{\frac{|\langle x, \xi\rangle|^2}{(1 - \|x\|^2_{B^H})^2} + \frac{\|\xi\|^2_{B^H}}{1 - \|x\|^2_{B^H}}} \quad \text{(see [9])}. \tag{3.5}$$

Taking $z = 0$ in (3.4), it follows that the desired result holds from (3.2) and (3.5).

Considering

$$p(x) = \Re\,p(0)\frac{1 + \langle x, u\rangle}{1 - \langle x, u\rangle} + \Im\,p(0), \quad x \in B^H,$$

where $u$ which satisfies $\|u\|_{B^H} = 1$ is fixed, then it is easy to verify that $p \in H(B^H, \mathbb{C})$ and $\Re\,p(x) > 0, x \in B^H$. A straightforward calculation shows that

$$D^m p(x)(\xi^m) = \frac{2m!\Re\,p(0)(\langle \xi, u\rangle)^m}{(1 - \langle x, u\rangle)^{m+1}}, \quad \Re\,p(x) = \Re\,p(0)\frac{1 - |\langle x, u\rangle|^2}{|1 - \langle x, u\rangle|^2}.$$

Taking $x = ru (0 \le r < 1), \xi = Ru (R > 0)$, then

$$\frac{|\langle ru, Ru\rangle|}{\sqrt{(1 - \|ru\|^2_{B^H})\|Ru\|^2_{B^H} + |\langle ru, Ru\rangle|^2}} = r, \quad F_C^{B^H_X}(ru, Ru) = \frac{R}{1 - r^2}.$$

Hence,

$$\begin{aligned}|D^m p(x)(\xi^m)| &= \frac{2m!\Re\,p(0)R^m}{(1 - r)^{m+1}} \\ &= 2m!\Re\,p(x)\Big(1 + \frac{|\langle x, \xi\rangle|}{\sqrt{(1 - \|x\|^2_{B^H})\|\xi\|^2_{B^H} + |\langle x, \xi\rangle|^2}}\Big)^{m-1}(F_C^{B^H}(x, \xi))^m.\end{aligned}$$

Therefore, we show that the estimate of Theorem 3.3 is sharp. This completes the proof.

**Remark 3.2** When $B^H = U$, Theorem 3.3 reduces to Theorem 1.2. Moreover, Theorem 3.3 reflects the feature of several complex variables due to the fact that one variable of $z$ in one complex variable is replaced with two variables of $x$ and $\xi$ in several complex variables.

**Theorem 3.4** *Let* $k \in \mathbb{N}^+, p(x) = p(0) + \sum\limits_{s=k}^{\infty} \frac{D^s p(0)(x^s)}{s!} \in H(G, \mathbb{C})$ *and* $\Re p(x) > 0, x \in G$. *Then*

$$|D^m p(x)(x^m)| \leq 2\Re p(0)\rho^m(x)\Big(\frac{kr^{k-1} + (1-k)r^k}{(1-r)^2}\Big)^{(m-1)}\Big|_{r=\rho(x)}, \quad m = 1, 2, \cdots$$

*for* $x \in G$.

**Proof** It is shown that

$$D^m p(x)(x^m) = \sum_{s=\max\{m,k\}}^{\infty} \frac{s(s-1)\cdots(s-m+1)D^s p(0)(x^s)}{s!}, \quad x \in G \tag{3.6}$$

by a simple calculation. Therefore, we deduce that

$$\begin{aligned}
|D^m p(x)(x^m)| &\leq \sum_{s=\max\{m,k\}}^{\infty} \frac{s(s-1)\cdots(s-m+1)|D^s p(0)(x^s)|}{s!} \\
&\leq 2\Re p(0) \sum_{s=\max\{m,k\}}^{\infty} s(s-1)\cdots(s-m+1)(\rho(x))^s \\
&= 2\Re p(0)(\rho(x))^m \sum_{s=\max\{m,k\}}^{\infty} s(s-1)\cdots(s-m+1)(\rho(x))^{s-m} \\
&= 2\Re p(0)(\rho(x))^m \Big(\sum_{s=\max\{m,k\}}^{\infty} \int_0^{\rho(x)} s(s-1)\cdots(s-m+1)r^{s-m}\mathrm{d}r\Big)' \\
&= 2\Re p(0)(\rho(x))^m \Big(\sum_{s=\max\{m,k\}}^{\infty} s(s-1)\cdots(s-m+2)r^{s-m+1}\Big)'\Big|_{r=\rho(x)} \\
&\ \ \vdots \\
&= 2\Re p(0)(\rho(x))^m \Big(\frac{kr^{k-1} + (1-k)r^k}{(1-r)^2}\Big)^{(m-1)}\Big|_{r=\rho(x)}, \quad x \in G
\end{aligned}$$

from Lemma 2.4 and (3.6). It is shown that the desired result holds. This completes the proof.

When $k = 1$, we have the following corollary.

**Corollary 3.2** *Let* $p(x) = p(0) + \sum\limits_{s=1}^{\infty} \frac{D^s p(0)(x^s)}{s!} \in H(G, \mathbb{C})$ *and* $\Re p(x) > 0, x \in G$. *Then*

$$|D^m p(x)(x^m)| \leq \Re p(0)\rho^m(x)\frac{2 \cdot m!}{(1-\rho(x))^{m+1}}, \quad m = 1, 2, \cdots, \ x \in G.$$

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